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Corpuscular description of black hole interiors

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Irrtum verläßt uns nie; doch ziehet ein höher Bedürfnis,
immer den strebenden Geist leise zur Wahrheit hinan.

Error never leaves us, yet a higher need
always draws the striving mind gently towards the truth.

Gedichte - "Vier Jahreszeiten (Herbst)"
JOHANN WOLFGANG VON GOETHE, 1827

Sommario

La motivazione di questo lavoro trova origine nell'idea di Dvali e Gomez che lo stato finale del collasso gravitazionale sia un condensato di Bose-Einstein al punto critico, costituito da un gran numero N di gravitoni soft, off-shell e in una condizione di massima occupazione del condensato. Questo approccio è innovativo perché va oltre la trattazione semiclassica e considera i buchi neri come oggetti puramente quantistici. Il risultato è un modello privo della singolarità centrale, che riesce a esprimere le quantità caratterizzanti, come la costante di auto-accoppiamento dei gravitoni e la massa del buco nero, in termini dell'unico parametro N. Inoltre, si riesce a dare una giustificazione per l'entropia di Bekenstein e la radiazione di Hawking (interpretata come un effetto di impoverimento del condensato). Di recente, è stata costruita una descrizione quantistica effettiva del potenziale gravitazionale statico per un sistema a simmetria sferica fino all'ordine post-Newtoniano, sulla base di un toy model di gravitoni scalari. Questo modello permette di riprodurre il potenziale Newtoniano classico tramite l'uso di uno stato coerente, dunque stabilendo una connessione tra il modello corpuscolare e la correzione post-Newtoniana. Tali lavori costituiscono il punto di partenza di questa tesi. Dopo aver recuperato le unità fisiche nella Lagrangiana del campo gravitazionale con correzioni post-Newtoniane e averne trovato l'equazione del moto, si procede alla sua linearizzazione, che modella il campo come un background Newtoniano più una piccola perturbazione. La perturbazione è parametrizzata da un ansatz di onda sferica nel contesto dell'approssimazione WKB, in cui la lunghezza d'onda dell'onda è considerata molto minore della scala su cui varia il potenziale di background. In seguito, dopo aver ricavato le equazioni del moto per la perturbazione, si trova la corrispondente relazione di dispersione, fino al prim'ordine nell'auto-accoppiamento dei gravitoni, e si fanno considerazioni sul comportamento delle perturbazioni. Il risultato finale è che le perturbazioni vengono riassorbite nel background e decadono al passare del tempo. Se invece si considera il limite opposto, di grandi lunghezze d'onda, le perturbazioni si amplificano e potrebbero segnalare un'instabilità del sistema di gravitoni.

Abstract

The motivation from this work stems from the idea of Dvali and Gomez that the end-state of the gravitational collapse is a Bose-Einstein condensate at the critical point, constituted by a large number of soft, off-shell and maximally packed gravitons. This approach is innovative since it goes beyond the semiclassical picture and considers black holes as purely quantum objects. The result is a model without a central singularity that expresses crucial quantities, such as the self-coupling constant of gravitons and the mass of the black hole in terms of only one parameter, the number of gravitons N, and is able to account for Bekenstein entropy and Hawking radiation (the latter seen as the depletion effect of the condensate). Recently, an effective quantum description of the static gravitational potential for a spherically symmetric system up to post-Newtonian order has been constructed, relying on a toy model of scalar gravitons. This model allows to reproduce the classical Newtonian potential by employing a coherent state, thus establishing a connection between the corpuscular model and post-Newtonian corrections. These works constitute the starting point of this thesis. After recovering physical units in the Lagrangian for the gravitational field up to post-Newtonian order and finding its equations of motion, we move on to its linearisation, that models the field as a Newtonian background plus a small perturbation. The perturbation is parametrized by a spherical wave ansatz in the WKB approximation, where the wavelength of the wave is thought to be much smaller than the scale on which the background potential varies. After writing the equations of motion for the perturbation, we then find the correspondent dispersion relation, up to first order in the graviton self-coupling, and make considerations about the behaviour of the perturbations. The end result is that the perturbations decay in time and end up being reabsorbed in the background. If, on the other hand, the opposite, long-wavelength limit is considered, the perturbations get amplified and might signal an instability of the system of gravitons.

Contents

Introduction and outline

At present and for many decades in the past, one of the most pressing problems in theoretical physics has been that of finding a consistent and complete theory of Quantum Gravity. While all the other forces of Nature, unified in the Standard Model, are understood in terms of Quantum Field Theories and thus in a framework that combines Quantum Mechanics and Special Relativity, the gravitational force seems to escape such a treatment.

The elegant revolutionary insights of Albert Einstein allowed the formulation of one of the most successful theories of the history of physics, that brilliantly describes a plethora of phenomena and received countless experimental confirmations, but also presents many puzzles, not only from the theoretical point of view, such as the lacking unification with the other forces of Nature, but from the physical one as well.

Arguably, black holes constitute one of the most striking of these puzzles, since their fundamental structure still proves to be a conundrum for physicists. Black holes are also the natural arena for the study of Quantum Gravity, since we have come to think that quantum effects become non-negligible in their vicinity. Their extreme gravitational field implies that no signal can escape from their event horizon, the boundary that separates the events happening inside the black hole from being causally connected to the rest of the Universe.

The celebrated Einstein field equations read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{0.0.1}$$

and, as famously stated by J. A. Wheeler, they elucidate that "spacetime tells matter how to move and matter tells spacetime how to curve." The presence of matter as the source of the gravitational field is encoded in the energy-momentum tensor $T_{\mu\nu}$, while the curvature of spacetime is contained in the metric $g_{\mu\nu}$, the Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\lambda\rho}\Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\rho}{}_{\lambda\mu}, \tag{0.0.2}$$

and the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$. The Christoffel symbols $\Gamma^{\lambda}{}_{\mu\nu}$ are related to the metric by means of

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}). \tag{0.0.3}$$

The Schwarzschild solution In order to introduce the concept of black holes, let us consider the gravitational field of a spherical source, so that the spacetime around it has spherical symmetry [1]. Additionally, we restrict ourselves to a static source. Under such conditions, a particularly important solution to the Einstein field equations in vacuum (where $R_{\mu\nu} = 0$) was found in 1916, shortly after the publication of Einstein's field equations in 1915 [2], by the astrophysicist K. Schwarzschild. The solution reads

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{0.0.4}$$

and it is named after its discoverer. In this expression, $0 < r < \infty$ is defined so that the area of a sphere of given r is $4\pi r^2$ and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, with $0 < \theta < \pi$ and $0 < \phi < 2\pi$, is the curvilinear metric on S^2. This solution describes the spacetime around a non rotating and non-charged black hole and is endowed with asymptotic flatness, since in the limit $r \to \infty$, the Minkowski metric in spherical coordinates can be recovered. In the region $r < 2m$, the temporal coordinate t becomes space-like and the radial one r becomes time-like, thus exhibiting a peculiar exchange of the roles of these variables. Moreover, the metric element (0.0.4) is independent of t, so that it is static for $r > 2m$. This property is related to Birkhoff's theorem, which states that (0.0.4) is the unique spherically symmetric solution to the Einstein equations, and implies that the spacetime around a spherical source is time-independent even if the source itself depends on time. Therefore, even during the collapse of baryonic matter that forms the black hole, the spacetime around it will be described by the Schwarzschild metric.

The elegant Schwarzschild solution allows the recognition of numerous peculiar features of black holes. Firstly, the presence of a horizon: a one-way membrane that constitutes the boundary of the black hole. From inside the horizon, no signal can be sent to an external observer, or more specifically, no signal can be sent from a point with $r < 2m$ to a point with $r > 2m$, as calculations show that such a signal will reach the horizon only after an infinitely long time. In practice, this is due to the fact that the redshift of the signal increases so much that the star that collapsed to a black hole is stripped of all its luminosity an thus becomes a black hole. For a Schwarzschild black hole, the horizon is located at $r = 2m$ and since motions with decreasing r only are possible beyond this buondary, it becomes clear that the collapse of a star becomes inevitable once its radius has shrunk to $r < 2m$.

It is straightforward to notice that (0.0.4) exhibits a singular behaviour for $r = 2m$. A singularity can be of two types: either due to the choice of coordinates or to an actual physical singular behaviour. In the first case, the singularity is simply a mathematical quirk and can be eliminated by a suitable coordinate transformation. An observer on the surface of a collapsing star will not notice anything peculiar when the horizon is crossed because, locally, the spacetime geometry is the same as it is elsewhere. The Schwarzschild singularity is precisely of this sort, as it can be checked that switching to Kruskal-Szekeres coordinates eliminates the singularity.

However, in the second case, when the singularity persists even after suitable changes of coordinates and cannot be eliminated, it has to be regarded as a physical, "true", or curvature singularity. This is the case of the singularity at $r = 0$, at the "center" of the black hole, where the curvature of spacetime diverges to infinity and spacetime itself is no longer well-defined (as shown by means of the Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, where $R_{\mu\nu\alpha\beta}$ is the Riemann curvature tensor).

This singularity is behind the horizon, and hence has no causal connection to an external observer. According to Penrose's Cosmic Censorship Hypothesis [3], singularities are always hidden within a horizon and thus are not visible to the rest of spacetime. The prediction of singularities in GR is regarded as a signal of the inadequacy of the theory in the extreme conditions that black holes present. At some point in the vicinity of the singularity, GR will no longer be valid as quantum effects start to dominate, and an account of the physics in that context could probably only come from a full-fledged theory of Quantum Gravity.

It is also worth remarking that Hawking-Penrose showed in their theorems [4] that the occurrence of singularities does not depend on exact symmetries. In the language of differential geometry, singularities are caused by geodesic incompleteness, that can be expressed in a simplified way by saying that there are geodesics that can only be extended for a finite time as measured by an observer traveling along one of them.

Semiclassical treatment of black holes The first attempt to reconcile Quantum Mechanics and General Relativity was made in the context of the so-called Quantum Field Theory on curved spaces, in which the curved spacetime represents a classical background and the quantized fields are thought to act on it without perturbing it. This framework has allowed a breakthrough to be made, that is the discovery that something can, after all, escape a black hole. The latter are not exactly blackbodies, but rather emit thermal radiation (negligible on astrophysical scales), slowly evaporating away, as discovered by Hawking in [5].

The crucial point of the semiclassical treatment is that, in curved spacetimes, not all observers agree on the choice of the vacuum state and thus on the definition of one- and many-particle states, so that the particle number depends on the observer. It is thus crucial to understand how two sets of field operators defined with respect to different vacua can be related to one another: this task is accomplished by the Bogolyubov transformations.

It has to be remarked that the semiclassical treatment opened up further questions that are left unanswered, notably the origin of the black hole entropy and the so-called information paradox. Bekenstein's argumentation that black holes had to carry entropy, according to the Second Law of Thermodynamics, seemed to be in contradiction with the No-hair Theorem, that states that stationary black holes are characterized by only their mass, charge and angular momentum. Based on Hawking's work, Bekenstein argued that the problem would be solved with an entropy proportional to the horizon area of the black hole. At present, still, no precise microscopical explanation of this entropy in the language of Statistical Mechanics has been found, even though some results have been obtained in higher dimensions and in the context of String Theory.

The information paradox, on the other hand, consists of the contradiction between the amount of information that the black hole absorbs during the collapse of baryonic matter and the fact that this information is not released in the later evaporation, since only thermal radiation is emitted. The information encoded in all the matter that has been devoured by the black hole seems to be lost in the process of collapse.

From the point of view of Quantum Mechanics, the collapsing matter can be represented by a pure state, although the final state that emerges from the evaporation of the black hole would be a mixed state, since Hawking radiation is thermal. Therefore, such a process from a pure state to a mixed one would violate the fundamental postulate of unitarity in the time evolution of a quantum system.

Furthermore, the semiclassical approach does not focus on the treatment of singularities at all.

These issues show that the semiclassical paradigm is, however extremely insightful, not the ultimate approach that can unveil the most fundamental structure of black holes.

Motivations and outline Since General Relativity predicts the existence of singularities as an inevitable occurrence, the theory is not complete without a specification for what happens to matter that falls into a black hole and the understanding of what a singularity physically is in the first place.

Recently, an alternative model [6], [7] for fully quantum black holes has been theorized. Its most striking feature is that the model only relies on a single parameter for the description of black holes, that do not contain any singularity at all. The model interprets black holes as a Bose-Einstein condensate of virtual, non-propagating gravitons at the critical point. Moreover, the absence of singularities in this treatment allows the investigation on the properties of the interior of black holes and therefore facilitates the analysis of the matter collapse. Such questions were addressed in [8], [9], leading to a better understanding of the self-sustained quantum state that the black holes constitute and to an analysis of the potential generated by the gravitons, up to post-Newtonian corrections.

This thesis will expand on these findings and recover a dispersion relation for the field describing the gravitons up to post-Newtonian order, with the aim of investigating the internal structure of black holes, since the starting point is a model without the central singularity.

In the first Chapter 1 we will present the problem of the non-renormalizability of GR and show how the theory can nonetheless be read in the language of QFTs, as long as we restrict ourselves to its low-energy limit. Moreover, the concept of effective theories will be clarified and classicalization, an alternative solution to the usual UV-completion of non-renormalizable theories will be hinted at. Finally, we will briefly present the problem of an unambiguous definition of mass in GR, since the concept of ADM mass will be relied upon in the following Chapters.

In the second Chapter 2, we will present the details of classicalization and explain how it naturally leads to a corpuscular model of black holes, highlighting its main features and especially understanding the importance of the single parameter N. Additionally, we will analyse the role played by the Planck scale in the search for a quantum theory of gravity.

In the third Chapter 3, we will see how this framework constitutes the starting point for a more rigorous treatment of the matter collapse, that relies on the introduction of a toy model of scalar gravitons. Furthermore, the refinement of those results will lead to the construction of an effective theory for the Newtonian potential of static spherically symmetric sources, up to the post-Newtonian order.

In the fourth and final Chapter 4, we will consider the quantum field analyzed in the previous Chapter, linearising it by splitting it into a Newtonian background and a perturbation parametrized by a WKB-like ansatz. We will then recover a dispersion relation from the equations of motion of the perturbation and understand its behaviour with respect to the background field.

In the Appendix A, some basic properties of Bose-Einstein condensation will be provided, with particular attention to those features that are relevant in the corpuscular model of black holes.

Chapter 1

General Relativity as an Effective Theory

The statement that General Relativity seems to be incompatible with Quantum Mechanics is widely mentioned in uncountable popular science publications. However, as always, such a statement is not completely correct. The issue with GR lies in its non-renormalizability, that expresses the fact that perturbative calculations of physical quantities yield infinities of ever-increasing orders. This strips the theory of gravity of its predictive power at such energies, and implies that a "completion" at high energies needs to be found.

Nonetheless, this does *not* mean that the theory does not have predictive power at all, since GR can be seen as an effective theory at low energies and it is well-known that it can make incredibly accurate predictions within its domain of applicability. Therefore, GR can actually be made compatible with Quantum Mechanics, or more specifically, can be thought of as a Quantum Field Theory, — just not at arbitrarily high energies, where, still, a full-fledged theory of Quantum Gravity is needed. The goal of this Chapter is to clarify how this happens, and to further discuss the use of effective theories in physics.

1.1 GR as a QFT

The theory of gravity has been formulated in a very different way with respect to the theories that describe the other fundamental forces of Nature, now unified in the Standard Model of particle physics. The reason for this difference becomes clear if we consider the historical development the theories underwent. Einstein formulated General Relativity in the second decade of the twentieth century: although many aspects of GR have been understood or unveiled much later, the theory was more or less firmly established by the time the Standard Model was formulated by means of Quantum Field Theory. The development of QFT and the Standard Model almost proceeded hand in hand, in an interplay of theory and experiment that culminated in the experimental confirmation of the existence of the Higgs boson in 2012 [10].

Thus, contrary to the theories describing the other fundamental interactions, the interpretation of the theory of gravity as a QFT could not come naturally. However, in the past few decades, much effort has been devoted to make GR fit as much as possible in the well-established framework of Quantum Field Theories. At a later time, once the problem of non-renormalizability was encountered, it became clear that the theory of gravity could only be understood as an Effective Field Theory, although only in the 1990s it became possible to reformulate it in the light of such developments.

At present, we have come to understand gravity as mediated by a spin-2 field sourced by an energy-momentum tensor $T_{\mu\nu}$. Let us justify this claim by means of qualitative considerations.

If we were to build a QFT of gravity from scratch, we would of course need it to recover the Newtonian potential in the low-energy limit

$$V_{\mathrm{N}}(r) = -G_{\mathrm{N}}\frac{m_1 m_2}{r}. \tag{1.1.1}$$

The similarity with the Coulomb potential might lead us to think that a spin-1 field is appropriate to describe gravity, if it were not for the fact that the electromagnetic potential can be attractive or repulsive, whereas the gravitational potential is only attractive. Thus, neglecting higher spins, the only two choices left are a spin-0 and a spin-2 field. However, a spin-0 field would violate the Equivalence Principle, since the only allowed coupling of the would-be gravitational field φ to the source in the Lagrangian would be φT^{μ}_{μ}, which is zero for a purely electromagnetic source. This way, light could never be considered in gravitational interactions. The only remaining choice is that of a spin-2 field.

We will briefly mention that the theory of gravity can also be understood as a gauge theory in a way that resembles Yang-Mills theory: gravity can be thought of as a gauge field resulting from gauging the global symmetry that has the stress-energy tensor as its conserved charge. The global symmetry considered is, of course, the symmetry under general coordinate transformations

$$x^{\mu} \to x'^{\mu}. \tag{1.1.2}$$

However, leaving the gauge theory formulation aside, we want to briefly present the key features of General Relativity as a Quantum Field Theory, largely following the clear exposition in [11].

The dynamics of the theory is given by the Einstein-Hilbert action coupled to matter:

$$S_{\mathrm{EH}} + S_m = \int d^4x\sqrt{-g}\left(-\frac{2}{\kappa^2}R + \mathcal{L}_{\mathrm{M}}\right), \tag{1.1.3}$$

where $\kappa^2 = 32\pi G_{\mathrm{N}}$. By means of the principle of least action $\delta S = 0$, the celebrated

Einstein field equations (0.0.1) can be straightforwardly be derived. We remark that they can also be written in the form that makes use of the so-called Einstein tensor $G_{\mu\nu} = 8\pi T_{\mu\nu}$, so that they read

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.1.4}$$

We might wonder the deep reason why the gravitational action is simply proportional to R and not any other terms, since it is not a symmetry requirement and cannot be inferred from the renormalizability argument. The answer is that the Einstein-Hilbert action is the simplest possible one but it is not unique. However, we can also state that the curvature is supposed to be physically small, so that R^2 terms would be even smaller, and thus, with an argument that resembles those of effective field theory, for most low-energy applications, this term is not relevant. Nonetheless, there are infinitely many terms allowed by invariance under general coordinate transformations, so that the action (1.1.3) could be rewritten as [12]

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left[\Lambda + \frac{2}{\kappa} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + ... \right], \tag{1.1.5}$$

where Λ is of order ∂^0, R of order ∂^2, $R_{\mu\nu}R^{\mu\nu}$ of order ∂^4 and so on. Due to a result known as the Gauss-Bonnet theorem, contributions of the form $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ can be expressed in terms of R^2 and $R_{\mu\nu}R^{\mu\nu}$. The first term in (1.1.5) is proportional to the cosmological constant, but due to its well-known smallness, it can be neglected. As for the R^2 terms, it can be shown that they lead to a very small and short-range modification of the gravitational interaction.

As a result of these considerations, we can say that in any low-energy test of General Relativity, only the effect of the Einstein-Hilbert action proportional to R is visible. This argument is, of course, in perfect agreement to the Effective Field Theory approach.

Weak-field limit

It is evident that the Einstein field equations are highly non-linear, which creates obvious problems in finding exact solutions for them, in situations where we cannot rely on specific symmetries. A widely-used approach is thus that of performing a weak-field approximation on the equations, leading to the so-called linearised form of the equations. Probably the most relevant application of this approach, and the one that has brought the most astounding results, is that to gravitational radiation. Because of the straightforwardness of the weak-field approach, gravitational waves had been already predicted by Einstein shortly after the formulation of General Relativity [2].

The weak-field limit is essentially the perturbative expansion of the metric around the Minkowski background:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.1.6}$$

where $\eta_{\mu\nu}$ is the Minkowskian metric with signature $(-,+,+,+)$ that will be used throughout this work (unless otherwise specified) and $|h_{\mu\nu}| \ll 1$ is the fundamental assumption of the weak-field approximation. In this linearised version of the theory, self-interactions are neglected and it is technically in this limit that General Relativity can be said to be the theory of the massless spin-2 graviton, which is described, in its linearised form, exactly by $h_{\mu\nu}$. If we restrict ourselves to first order in $h_{\mu\nu}$, indices can be raised and lowered with the Minkowskian metric $\eta_{\mu\nu}$. The relevant quantities in the field equations, up to $\mathcal{O}(h^2)$ are

$$\Gamma^{(1)}{}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}(\partial_\nu h_{\mu\rho} + \partial_\mu h_{\nu\rho} - \partial_\rho h_{\mu\nu}) \tag{1.1.7}$$

$$R^{(1)}{}_{\mu\nu} = \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\lambda} = \frac{1}{2}\left(\partial_\lambda\partial_\mu h^\lambda_\nu - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial_\lambda h^\lambda_\mu\right) \tag{1.1.8}$$

and

$$R^{(1)} = \partial_\mu\partial_\lambda h^{\mu\lambda} - \Box h, \tag{1.1.9}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

Newtonian limit Another important feature of the weak-field limit is that it can be used for finding the Newtonian limit of GR and recovering Newton's theory of gravity, requirement which is clearly crucial for any theory of gravity. However, the weak-field assumption is necessary but not sufficient to recover Newton's limit: additionally, we have to require the field to be static (so that the time derivatives of the metric will be neglected) and matter is characterized by non-relativistic velocities, i.e. $v \ll c$ This is tantamount to only considering one component of the energy-momentum tensor, T_{00}, that has the physical meaning of rest mass density, while $|T_{ij}| \ll T_{00}$ and $|G_{ij}| \ll G_{00}$. Moreover, $R_{ij} \simeq \frac{1}{2}g_{ij}R$ and thus we obtain $R = 2R_{00}$, which leads to the component

$$R_{00} = 8\pi G_{\mathrm{N}} T_{00} \tag{1.1.10}$$

of the Einstein equations.

Of, course, due to the form of the Ricci scalar (1.1.9), we get

$$\triangle h_{00} = -8\pi G_{\mathrm{N}}\rho, \tag{1.1.11}$$

which allows the important identification

$$h_{00} = -2V_{\rm N}, \tag{1.1.12}$$

where $V_{\rm N}$ is obviously the Newtonian potential satisfying the Poisson equation

$$\triangle V_{\rm N} = 4\pi G_{\rm N}\rho. \tag{1.1.13}$$

Post-Newtonian approximation The idea underlying the post-Newtonian approximation is the same of the Newtonian limit; however, the perturbative expansion of the weak field limit is carried out at one order higher, the so-called next-to-leading order. The small parameter that drives the expansion is the characteristic relative velocity of the system we are analysing, which can be roughly considered to be, by the virial theorem

$$\frac{v}{c} \sim \sqrt{\frac{G_{\rm N}Mc^2}{R}} \sim \epsilon, \tag{1.1.14}$$

if M is the mass and R the typical size of the body or system of bodies in consideration, and of course $\epsilon \ll 1$. The post-Newtonian approximation will be crucial in the results of [9] that provide the starting point for the present thesis, but we also remark their fundamental importance in the applications to the study of gravitational radiation. The first use of the approximation was made by Einstein himself, in the calculation of the perihelion of Mercury in 1915 [2]. Following the conventions in [13] (so that the metric is now $(-,+,+,+)$, as it will also be employed in Chapter 3, based on [9]) the metric is expanded as

$$g_{00} = -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + \dots \tag{1.1.15}$$

$$g_{ij} = \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \dots \tag{1.1.16}$$

$$g_{0i} = {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \dots \tag{1.1.17}$$

where the last expression is justified from the requirement that g_{0i} changes sign under time reversal. Similar expressions can be found for the inverse metric tensor $g^{\mu\nu}$ and for all the other tensor quantities involved in the field equations, that have to be rewritten at the different orders in the expansion. We avoid this tedious task and simply trace the general procedure [13]: the expansion of the metric is considered as an ansatz which will have to be shown to satisfy the field equations. Indeed, it still needs to be verified that higher orders are actually smaller than the previous ones, since this is not ensured due to the high non-linearity of the Einstein equations.

In the end it will be found that a consistent post-Newtonian approximation determines g_{00} up to $\mathcal{O}(\epsilon^4)$, g_{0i} up to $\mathcal{O}(\epsilon^3)$ and g_{ij} up to $\mathcal{O}(\epsilon^2)$. It has to be remarked that space and time derivatives belong to different orders of the expansion, since

$$\frac{\partial}{\partial x^i} \sim \frac{1}{r} \quad \frac{\partial}{\partial t} \sim \frac{v}{r}. \tag{1.1.18}$$

Keeping this in mind, the Christoffel symbols can be computed by performing derivatives of the metric, and the Ricci tensor can be found consequently. The chosen gauge is generally either the harmonic or the so-called standard post-Newtonian gauge, that plays the same role of the Coulomb gauge in electrodynamics. The latter implies

$$g_{0j,j} - \frac{1}{2} g_{jj,0} = \mathcal{O}(c^{-5}) \tag{1.1.19}$$
$$g_{ij,j} - \frac{1}{2}(g_{jj} - g_{00})_{,i} = \mathcal{O}(c^{-4}), \tag{1.1.20}$$

which translates to

$${}^{(3)}g_{0k,k} - \frac{1}{2}{}^{(2)}g_{kk,0} = 0, \tag{1.1.21}$$
$$\frac{1}{2}{}^{(2)}g_{00,i} + {}^{(2)}g_{ij,j} - \frac{1}{2}{}^{(2)}g_{jj,i} = 0. \tag{1.1.22}$$

After imposing gauge-fixing conditions, the Einstein equations are found, and they can be recast in the following form

$$R_{\mu\nu} = 8\pi G_{\rm N}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) = 8\pi G_{\rm N} S_{\mu\nu}, \tag{1.1.23}$$

thus defining the new tensor $S_{\mu\nu}$ and considering $T = T^\mu_\mu$. We remark that, of course, the energy-momentum tensor has to be expanded in a similar way to the other quantities, even if we will not provide the explicit calculation. The final equations in the standard post-Newtonian gauge are:

$$\triangle\, {}^{(2)}g_{00} = -8\pi G_{\rm N}\, {}^{(0)}T^{00}, \tag{1.1.24a}$$
$$\begin{aligned}\triangle\, {}^{(4)}g_{00} = \; & {}^{(2)}g_{ij}\, {}^{(2)}g_{00,ij} + {}^{(2)}g_{ij,j}\, {}^{(2)}g_{00,i} - \frac{1}{4}{}^{(2)}g_{00,i}\, {}^{(2)}g_{00,i} - \frac{1}{4}{}^{(2)}g_{00,i}\, {}^{(2)}g_{jj,i} \\ & - 8\pi G_{\rm N}\left({}^{(2)}T^{00} - 2\, {}^{(2)}g_{00}\, {}^{(0)}T^{00} + {}^{(2)}T^{ii}\right)\end{aligned} \tag{1.1.24b}$$
$$\triangle\, {}^{(3)}g_{0i} = -\frac{1}{2}{}^{(2)}g_{jj,0i} + {}^{(2)}g_{ij,0j} + 16\pi G_{\rm N}\, {}^{(1)}T^{i0} \tag{1.1.24c}$$
$$\triangle\, {}^{(2)}g_{ij} = -8\pi G_{\rm N} \delta_{ij}\, {}^{(0)}T^{00}. \tag{1.1.24d}$$

The first (1.1.24a) and the second (1.1.24b) of these equations yield

$${}^{(2)}g_{00} = -2V_{\rm N} \quad {}^{(2)}g_{ij} = -2\delta_{ij} V_{\rm N} \tag{1.1.25}$$

and it is clear that the first reproduces the expected Newtonian result. We also have

$$V_{\mathrm{N}}(\mathbf{x}, t) = -G_{\mathrm{N}} \int \mathrm{d}^3 x' \frac{{}^{(0)}T_{00}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}. \tag{1.1.26}$$

The solutions to (1.1.24c) and (1.1.24d) require the definitions of three more potentials, but we will not mention them here for the sake of brevity. We will just present the final form of the metric, which is a consistent solution of the Einstein field equation in the post-Newtonian approximation and in the simplest case of a distribution of mass at rest and without angular momentum:

$$\begin{aligned} g_{00} &= -1 + \frac{2G_{\mathrm{N}}M}{r} - 2\frac{G_{\mathrm{N}}^2 M^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \\ g_{0i} &= 0 + \mathcal{O}\left(\frac{1}{r^3}\right) \\ g_{ij} &= \left(1 + \frac{2G_{\mathrm{N}}M}{r}\right)\delta_{ij} + \mathcal{O}\left(\frac{1}{r^2}\right). \end{aligned} \tag{1.1.27}$$

Employing the linearised version of the theory under the weak-field assumption allows the second quantization of General Relativity to be performed. The field $h_{\mu\nu}$ will be the one to be quantized, even if its decomposition is not unique because the theory is invariant with respect to general coordinate transformations. In the linearised version of the theory, this invariance must satisfy the additional requirement of still leaving the field weak. The most general form for such transformations is

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{1.1.28}$$

where $\epsilon^\mu(x)$ has derivatives of the same order as $h_{\mu\nu}$. These requirements are meant to ensure that, if $h_{\mu\nu}$ is a solution of the linearised Einstein field equations,

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \epsilon_\mu - \partial_\mu \epsilon_\nu \tag{1.1.29}$$

is one as well. In order for the decomposition of $h_{\mu\nu}$ to be unique, a gauge fixing should be performed, and the most convenient choice is the harmonic or de Donder gauge, given by

$$g^{\mu\nu}\Gamma^\rho{}_{\mu\nu} = 0. \tag{1.1.30}$$

In the weak field limit, it becomes

$$\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda = 0. \tag{1.1.31}$$

If we write Einstein's field equations in the form that makes use of the Einstein tensor

(1.1.4), we can perform the decomposition

$$G_{\mu\nu} \approx {G^{(1)}}_{\mu\nu} + {G^{(2)}}_{\mu\nu}, \tag{1.1.32}$$

where ${G^{(i)}}_{\mu\nu}$ is the part that contains the power of $h_{\mu\nu}$ of order i, up to the second order. For the second order

$$t_{\mu\nu} = -\frac{1}{8\pi G}{G^{(2)}}_{\mu\nu}, \tag{1.1.33}$$

so that the substitution of (1.1.32) and (1.1.33) in (1.1.4) yields the Einstein equations in the form

$$\Box h_{\mu\nu} \approx 8\pi G(T_{\mu\nu} + t_{\mu\nu}), \tag{1.1.34}$$

where ${G^{(1)}}_{\mu\nu} = \Box h_{\mu\nu}$ has been used. The non-linear nature of gravity is thus clearly displayed: higher-order powers of $h_{\mu\nu}$ are a source of $h_{\mu\nu}$ itself.

In order to perform the second quantization procedure for $h_{\mu\nu}$, we should find the general solution of the linearised equations of motion above, in the case where matter is absent.

The two possible polarizations of the gravitons are accounted for by the the polarization tensor $\epsilon_{\mu\nu}$, constituted by the usual polarization vectors

$$\epsilon_{\mu}(\lambda) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \tag{1.1.35}$$

with $\lambda = \pm$. These vectors satisfy

$${\epsilon^*}_{\mu}\epsilon^{\mu}(\lambda) = -1 \quad \epsilon^{\mu}(\lambda)\epsilon_{\mu}(\lambda) = 0. \tag{1.1.36}$$

The polarization tensor is thus

$$\epsilon_{\mu\nu}(\lambda_1\lambda_2) = \epsilon_{\mu}(\lambda_1)\epsilon_{\nu}(\lambda_2). \tag{1.1.37}$$

We can now decompose the tensor $h_{\mu\nu}$ in plane waves as

$$h_{\mu\nu} = \sum_{\lambda=++,--}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a(p,\lambda)\epsilon_{\mu\nu}(p,\lambda)e^{-ipx} + h.c.]. \tag{1.1.38}$$

Therefore, in order to treat $h_{\mu\nu}$ as a quantum field, the coefficients $a(p,\lambda)$ and $a^*(p,\lambda)$ are promoted to distribution-valued operators $\hat{a}(p,\lambda)$ and $\hat{a}^\dagger(p,\lambda)$, which satisfy the canonical commutation relations

$$[\hat{a}(p,\lambda), \hat{a}^\dagger(p',\lambda')] = \delta(p-p')\delta_{\lambda\lambda'}. \tag{1.1.39}$$

In order to find the propagator of the theory, we expand the action (1.1.3) up to second order in $h_{\mu\nu}$, by making use of the quantity

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{1.1.40}$$

The Lagrangian is thus

$$\sqrt{-g}\mathcal{L} = \sqrt{-g}\left(-\frac{2}{\kappa^2}R + \mathcal{L}_m + \mathcal{L}_{\text{GF}}\right), \tag{1.1.41}$$

where the last one is the gauge-fixing term. that allows the propagator to be computed. We remark that, up to the second order in $h_{\mu\nu}$,

$$-\sqrt{-g}\frac{2}{\kappa^2}R = -\frac{2}{\kappa^2}(\partial_\mu\partial_\nu h^{\mu\nu} - \Box h) + \frac{1}{2}\left[\partial_\lambda h_{\mu\nu}\partial^\lambda \bar{h}^{\mu\nu} - 2\partial^\lambda \bar{h}_{\mu\lambda}\partial_\sigma \bar{h}^{\mu\sigma}\right] \tag{1.1.42}$$

and

$$\mathcal{L}_{\text{GF}} = \partial_\mu \bar{h}^{\mu\nu}\partial^\lambda \bar{h}_{\lambda\nu} \tag{1.1.43}$$

in the harmonic gauge. Therefore, the Lagrangian can be rewritten as

$$\sqrt{-g}\mathcal{L} = \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\lambda h\partial^\lambda h - \frac{\kappa}{2}h^{\mu\nu}T_{\mu\nu}. \tag{1.1.44}$$

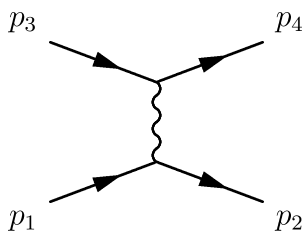
and the solution of the gauge-fixed equations of motion allows us to find the graviton propagator under the usual Feynman-Stückelberg boundary conditions, which results in

$$iD^{\alpha\beta\gamma\delta}(x) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{i}{q^2 + i\epsilon}e^{-iqx}P^{\alpha\beta\gamma\delta}, \tag{1.1.45}$$

where

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2}\left[\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta}\right]. \tag{1.1.46}$$

After deriving the Feynman rules for the graviton, we can compute the scattering of two scalar particles mediated by the exchange of a virtual graviton (diagram below), in order to see how the Newtonian potential can be recovered in the non-relativistic limit.



The amplitude of the process is given by

$$
\begin{aligned}
& i\mathcal{M} \\
= \; & \frac{i\kappa}{2}[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \eta^{\mu\nu}(p_1 \cdot p_2 - m^2)] \cdot \frac{i}{q(p_i)^2} P_{\mu\nu\alpha\beta} \frac{i\kappa}{2}[p_3^\mu p_2^\nu + p_4^\mu p_3^\nu - \eta^{\mu\nu}(p_3 \cdot p_4 - m^2)].
\end{aligned} \tag{1.1.47}
$$

Taking the non-relativistic limit, in which $p^\mu \approx (m, \vec{0})$, the amplitude becomes

$$
\mathcal{M} = -\frac{\kappa^2}{4}\frac{m_1^2 m_2^2}{q(p_i)^2} = -16\pi G_\mathrm{N}\frac{m_1^2 m_2^2}{q(p_i)^2} \tag{1.1.48}
$$

By means of a Fourier transform, we obtain the non-relativistic Newtonian potential

$$
V(r) = -\frac{G_\mathrm{N} m_1 m_2}{r}, \tag{1.1.49}
$$

thus completing the picture of GR as a QFT at tree level.

Schrödinger equation Another interesting application of the weak-field limit is the possibility to recover the Schrödinger equation for a particle in an external gravitational field, in a non-relativistic scenario.

If we consider the Klein-Gordon equation

$$
(\Box + m^2)\phi = 0 \tag{1.1.50}
$$

and the weak field expansion performed as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, the d'Alembertian can be rewritten as

$$
\Box = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) = g^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu})\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu, \tag{1.1.51}
$$

where also the definition of the harmonic gauge

$$
\partial_\mu(\sqrt{-g}g^{\mu\nu}) = -\kappa\partial_\mu\left(h^{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_\lambda^\lambda + \mathcal{O}(h^2)\right) \simeq 0. \tag{1.1.52}
$$

has been used. Considering the metric for a static external gravitational field,

$$g_{00} = 1 - 2\Phi_{\mathrm{N}}, \quad g_{ij} = -(1 + 2\Phi_{\mathrm{N}})\delta_{ij}, \quad \Phi_{\mathrm{N}} \ll 1. \tag{1.1.53}$$

If we perform a non-relativistic limit on the scalar field ϕ (see e.g. [14]), we obtain

$$\phi = e^{-imt}\psi(t, \mathbf{x}), \tag{1.1.54}$$

that can be inserted into the field equation (1.1.50) to yield

$$[(1 + 2\Phi_{\mathrm{N}})(-m^2 - 2im\partial_0 + \partial_0^2) - \delta^{ij}\partial_i\partial_j + m^2]\psi(t, \mathbf{x}) = 0. \tag{1.1.55}$$

The mass term can cancel to the leading order in Φ_{N}, while the term containing the second derivative can be dropped as it is higher order in the momentum. The result is the well-known Schrödinger equation for a particle in an external gravitational field

$$i\partial_0\psi = \left[-\frac{\triangle}{2m} + m\Phi_{\mathrm{N}}\right]\psi. \tag{1.1.56}$$

Non-renormalizability

However, the most fundamental problem with gravity as a QFT, before the understanding of effective theories, came from its non-renormalizability, that can be proven even with simple dimensional analysis. A somewhat heuristic argument to determine the renormalizability of a theory is the so-called power-counting criterion. In natural units, the action of every consistent theory has to be dimensionless, so the lagrangian needs to have dimensions of $[\mathcal{L}] = L^{-4}$ or $[\mathcal{L}] = M^4$. If the coupling constant of a theory is not dimensionless, such as in Fermi's electroweak theory, the result of a scattering amplitude depends on the chosen cut-off of regularization and unitarity is violated in its calculation. Moreover, the perturbative expansion is not valid at arbitrarily high energies. These are all the signals of the theory's inability to describe phenomena beyond a certain scale.

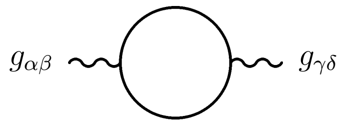
If we take the power-counting renormalizability criterion into account, we can immediately recognize that gravity is not renormalizable because its coupling constant G_{N} is not dimensionless. Instead, in natural units, it would have mass dimensions of -2. Calculating the graviton-graviton scattering amplitude at energy E, we would obtain $\mathcal{M} \sim 1 + G_{\mathrm{N}}E^2 + (G_{\mathrm{N}}E^2)^2 + \dots$.

This signals that the breakdown energy scale is $E = (1/G_{\mathrm{N}})^{1/2}$ (in natural units) because when this condition is met, the second term in the expansion becomes of the same order of the first, thus undermining any attempt to a perturbative expansion. In natural units,

$$(1/G_{\mathrm{N}})^{1/2} = m_{\mathrm{p}} = 1.22 \cdot 10^{19} \quad \text{GeV}, \tag{1.1.57}$$

where m_{p} is the Planck mass. Beyond the energy scale dictated by the Planck mass, new physics must appear and the still unknown theory of Quantum Gravity should describe it.

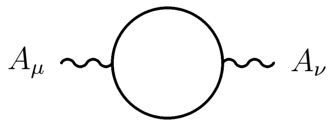
Another, more formal, way of showing the non-renormalizability of gravity is in terms of one-loop diagrams, that we will only schematically present. Even simply considering the one-loop matter correction to the graviton propagator and comparing it to the one in QED (which is, of course, a renormalizable theory), it becomes clear that the two theories are fundamentally different. For the graviton propagator, the diagram is



The computation of the loop gives, simplistically, an expression of the form

$$\frac{\kappa^2}{16\pi}(q_\gamma q_\delta q_\alpha q_\beta)\left(\frac{1}{\epsilon}+\ln q^2\right), \tag{1.1.58}$$

which is of course different from the QED result



for which the diagram is given by

$$\frac{e^2}{16\pi^2}(q_\mu q_\nu - \eta_{\mu\nu}q^2)\left(\frac{1}{\epsilon}+\ln q^2\right). \tag{1.1.59}$$

Both these expressions present divergences: for QED it can be renormalized by a term of the form $\frac{1}{\epsilon}F_{\mu\nu}F^{\mu\nu}$, since QED is a renormalizable theory. On the other hand, expression (1.1.58) would need terms containing four derivatives of $h_{\mu\nu}$ to allow the renormalization, but there are no such terms in the Einstein-Hilbert action. Therefore, it has been alternatively shown that GR is a non-renormalizable theory. While the problem of finding a quantum theory of gravity still remains open, however, the modern approach sees the theory as effective, ensuring its validity within a certain energy range and clarifying how it still retains predictive power regardless of the presence of divergences.

1.2 UV-completion and classicalization

The aim of this section is to clarify the concept of classicalization as an alternative to the UV-completion of a non-renormalizable field theory. We will briefly and qualitatively review some concepts in QFT in order to understand the difference between renormalizable and non-renormalizable theories, then move on to present the standard way to deal with the problem of non-renormalizability and the alternative path laid down by classicalization.

In Quantum Field Theory (QFT), the strength of an interaction taking place among elementary particles is determined by a coupling "constant" that we can call α. The process is described by a perturbative expansion controlled by α as an expansion parameter, which must then satisfy $\alpha \ll 1$ in order to guarantee the validity of the perturbative approach itself.

About fifty years ago, great conceptual advances led to the development of the renormalization group formalism. We will not dive into this topic in the present work, but we briefly mention the basic underlying idea, which is that the value of the coupling constant depends on the energy at which it is employed, so $\alpha = \alpha(E)$ and it is called a "running" coupling. The renormalization group approach provides a framework to understand how the *physical* coupling constants that can be measured in experiments vary with the energy scale at which we are operating. This approach has been developed at the same time by the high-energy and condensed matter physics communities, and has become an important result of both, since it allows the description of the physical phenomena at different energy scales (in the high-energy approach) or length scales (in the condensed matter approach). This idea is important for our purposes, since understanding the dependence of the coupling constants from the energy is crucial to formulate the concept of an effective field theory, as we will see in the following.

Another basic concept in QFT is that the usual interactions that we consider are said to be *weak*, in contrast with interactions where $\alpha \sim 1$ or $\alpha > 1$. In the latter cases, we say we enter the *strong*-coupling regime, whose boundary can be represented by a certain energy threshold Λ. In this regime, the theory ceases to provide a valid description of phenomena because the perturbative approach breaks down. If we were to calculate the scattering amplitude of a process in such a regime, it would result in a violation of unitarity, one of the basic guiding principles of Quantum Mechanics, whose validity is in general always assumed.

Many techniques can be employed to deal with this serious problem, and in many cases they lead to a so-called renormalization of the theory. In particular, one first needs to *regularize* the theory, or to render divergent quantities finite, by making use of a suitable parameter that works as a cut-off. The cut-off indicates the presence of "new physics" beyond the energy scale Λ, that the current theory is not able to properly describe. There are several different techniques to be employed in different contexts, such as Pauli-Villars regularization and dimensional regularization, developed by 't Hooft and Veltman. After this procedure, the regularized theory needs to be rewritten so that the physical observables that it yields are only in terms of physically measurable quantities that do not depend on the chosen regulators; such a theory is said to have been renormalized.

Unfortunately, some theories, the most striking example of all being gravity itself, resist the renormalization procedure and must then be regarded as only valid up to an energy scale Λ. These theories are said to be non-renormalizable and the standard approach to deal with them is finding an UV-completion for them. This translates to shifting to a higher energy scale and reconstruct a sector of the theory in this regime. In particular, the so-called Wilsonian UV-completion relies on the introduction of new, weakly-coupled degrees of freedom above the breakdown threshold to accomplish this task. More specifically, two scenarios can appear: either some new degrees of freedom are introduced in addition to the already-existing ones (such as in the case of the Higgs boson in the Standard Model, that resolves the strong coupling in the interactions of the longitudinal W-bosons), or there is a total renewal of the degrees of freedom, such as in the case of quantum chromodynamics (QCD), where quarks and gluons get replaced by mesons, glueballs and baryons.

1.2.1 Effective theories

Fundamental physics at any energy scale can be described in terms of some propagating degrees of freedom, namely particles. At different scales, the elementary degrees of freedom might change, as the more fundamental ones could become composite. An effective theory is a description of Nature in terms of certain degrees of freedom that are appropriate at a specific length scale, or energy scale, below which any substructure is ignored. The theory then comes, as it ought to, with a domain of applicability.

The degrees of freedom that are used in physical theories are weakly interacting, but when we try to use the theory in a domain that is not appropriate, beyond a certain threshold (at a scale that is too small or an energy that is too high), the theory itself signals its breakdown. This is marked by the weakly interacting degrees of freedom becoming strongly interacting. The validity of the perturbative approach is thus undermined, which translates in nonphysical results, such as observables that become infinite and scattering amplitudes that violate the principle of unitarity.

For example, the theory developed by Fermi to describe β-decay is an effective theory that works well at low interaction energies, but has been replaced by the more complete Glashow-Weinberg-Salam Theory of Electro-Weak Interactions, via the introduction of the new degrees of freedom corresponding to the massive $W^{\pm}$ and Z^0 bosons. It is important to point out that even the Standard Model itself is not valid at arbitrarily high energies and at present there are several instances of physics beyond the SM, an example being neutrino oscillations.

As we will clarify later on, General Relativity can also be considered as a low-energy approximation of a more fundamental theory of gravity, since it loses its validity beyond the Planck scale, where it should be substituted by a complete theory of Quantum Gravity.

In the modern perspective, the whole Quantum Field Theory itself is seen as an effective theory, that must be replaced by a more fundamental theory at very high energies. Such a theory is claimed to also unify gravity with the other fundamental forces described by the Standard Model. A candidate for such a theory is arguably said to be String Theory, but until a theory of everything is found (if ever), all of our physical theories should, at least technically, be regarded as effective.

In the earlier times of QFT, a non-renormalizable theory was doomed: since it could not be employed at all energies, it was thought to be devoid of any predictive power and was considered useless. However, in the modern perspective, renormalizability is no longer a sacred criterion to be satisfied by all meaningful theories. Since experiments can only make measurements up to a certain accuracy and a certain energy scale anyway, as long as a theory comes with its own regime of validity, there is no reason why it should not be employed to make predictions within its limits. Moreover, effective theories allow to avoid the complications of a full theory and simplify the calculations so that physicists are nowadays used to extract reliable predictions at a specific energy scale from a non-renormalizable theory.

For example, in a low-energy setting, Fermi's theory of β-decay can still be used to compute the cross-section of processes in a way that is fairly reliable, if compared to the more accurate results coming from the Electro-Weak Theory.

It should also be pointed out that it is not accurate to say that the low-energy dynamics is not affected by the high-energy behaviour *at all* [11]. Physics at high energies can, in particular cases, influence that at low energies. For example, if restricting to low energies implies a spontaneous symmetry breaking, the symmetric part of the theory at high energies can manifest itself in the low energy interactions. Computing loop corrections in effective field theory, the UV-dynamics manifests itself in the running of coupling constants.

However, the influence of the UV sector is limited because the effects of high energy particles appear to be local when viewed at low energies: the higher the energy, the smaller the distance, in a manifestation of the uncertainty principle, so these particles do not propagate very far. In contrast, the light particles in the theory can propagate long distances, and the local Lagrangians are written in terms of the light fields, so that a full quantum treatment can be considered.

If the high-energy effects are known, because one has the full theory at hand, such as in the case of Fermi's effective theory and the full Electro-Weak Theory, the coefficients of the terms in the Lagrangian can be directly computed. If the high-energy effects are unknown because we do not have the overarching full theory, like in the case of Quantum Gravity, these coefficients will be free parameters to be fixed by experiments at the available energy scale. Of course, if the full theory is known, a precise matching of predictions with the effective theory should be achieved for the effective theory to be correct. Since the predictions at low energies have to match those of the known full theory, the reliability of the effective theory can be ensured.

Although it is common practice to make reliable physical predictions from non-renormalizable theories, the search for Quantum Gravity still represents a fundamental problem in Theoretical Physics. The scale where we should expect new physics (the Planck scale) is endowed within the theory itself, and even if it seems unreachable for experimental purposes in the near future, the very existence of objects like black holes, where extreme gravity takes place and quantum effects cannot be ignored, cries out for a more complete theory.

Even if we are far from the development of such a theory, major progress has been made towards an effective theory of gravity, as we have previously seen in this Chapter, so that the classic popular science argument about the incompatibility between General Relativity and Quantum Mechanics no longer holds. The theory of gravity we know works at ordinary energies as an effective theory, and it is possible that without new experimental input, it will be difficult to decide between different proposed UV-completions of Quantum Gravity (or lack thereof, as we will see in the following).

1.2.2 Classicalization

In [15] and [16], Dvali, Gomez et al. suggested an alternative approach for the UV-completion of non-renormalizable theories and gravity in particular, which differs from the Wlsonian one that has been previously described. While the standard Wilsonian procedure aims to reconstruct a weakly coupled quantum field theory above the considered energy scale, the alternative approach suggests that the problematic theory can *self-complete* in the UV, saving itself from breaking down, by a process called classicalization.

In a nutshell, we can say that classicalization converts high-energy physics into high-multiplicity physics. In contrast with the usual approaches, in classicalization there are no new postulated degrees of freedom above the threshold scale. Instead, it is hypothesized that the usual low-energy degrees of freedom that are present in the IR physics simply acquire a higher multiplicity above the energy threshold. Therefore, the high energy of the few initial "hard" quanta becomes redistributed among a large number N of "soft", low-energy ones (the crucial role of the number N will be explored in more detail in the following).

Let us exemplify the main claim of classicalization from a particle physics point of view. A high-energy 2-particle scattering that needs to be described in a strong-coupling regime,

$$2X \to 2X \tag{1.2.60}$$

can become, due to classicalization, a multi-particle scattering

$$2X \to NX, \tag{1.2.61}$$

consisting of many elementary processes. Each of these processes involves low-energies and thus requires a weak coupling, that does not create any problems within the usual perturbative approach.

The higher the energy of the center of mass of the process, the higher the number N of elementary processes needed to redistribute the energy. We can acquire a more quantitative understanding by considering the collision of two quanta with center of mass energy $\sqrt{s} \gg \Lambda$. The entire energy of the collision should be redistributed among soft quanta in such a way that $\sqrt{s}/N < \Lambda$, and $\alpha(\sqrt{s}/N) < 1$, as it ought to be. Therefore, though this process, the theory would *prevent itself* from entering the strong-coupling regime. Within the framework of classicalization, the breakdown of the theory at high energies just seems an artifact of the perturbative approach and it does not appear to have a deep physical reason.

The large-N limit

The number N represents the occupation number of a generic boson field and is used as a measure of "classicality". The key concept to be clarified here is that a state with large occupation number behaves effectively classically.

This stems from the idea of the so-called large-N expansion, a concept used in several different areas of physics and reviewed in [17]. Not long after the formulation of Quantum Mechanics, it was noticed that the increase of the number of degrees of freedom led to a simplification in the analysis of many distinct quantum theories, allowing to more easily extract from them some physical observables to be confronted with experiments.

Let us consider a theory with N degrees of freedom. If such a theory can be generalized to the limit of $N \to \infty$, and is solvable in this very limit, then $1/N$ (required to be small) can be introduced as a new parameter for an "alternative" perturbation theory. The expansion in powers of $1/N$ can provide a way to compute the physical observables with increasing precision, even in the case of finite N. This expansion can come into play in contexts in which the usual perturbation theory is not applicable, for example where dimensionless ratios of physical quantities are involved, since the latter are pure numbers and don't depend on the coupling constant of the theory. Another example is the study of critical phenomena, in which critical exponents are again pure numbers. The large-N expansion thus qualifies as an alternative method to be used when perturbation theory fails.

The large-N limit is a different sort of classical limit of a quantum theory, in which the dynamics of the quantum system tends to that of the classical system. It should be pointed out that this limit is distinct from the classical limit in which $\hbar \to 0$. Solving the quantum theory in the limit of $N \to \infty$ just reduces to the minimization of the correspondent classical Hamiltonian. To demonstrate that it is indeed a classical limit, the same steps used for showing the classical limit of $\hbar \to 0$ can be used. A basis of coherent quantum states should be constructed (as we know that coherent states are the one that better mimic the behaviour of classical states, since they are states that minimize uncertainty). We then need to show that the coherent superposition of these states becomes effectively indistinguishable (in the considered limit) from an incoherent mixture of the same states, at least in any measurable way. In the previously-mentioned paper, this is verified in the large-N limit as well, thus justifying that the latter can be considered as a classical limit.

The large-N limit technique is usually employed in the context of QCD (where it was introduced by 't Hooft) even if N is only 3, but also in the study of AdS/CFT dualities and in condensed matter physics, where it can justify the use of mean field theory. For our purposes, the classical behaviour of a state with a large number of degrees of freedom will be one of the underlying assumptions that guides the work by Dvali and Gomez that we are reviewing. As we will see in the next sections, the corpuscular model of black holes is a large-N system in the sense specified by 't Hooft in his famous article [18], as pointed out by Dvali and Gomez in [19].

1.3 Mass and energy in GR

In this section, we will divert for a moment from the problem of the UV-completion of gravity (or lack thereof), and briefly analyze the problem of the definition of mass in General Relativity, shedding light on some of the subtleties that this task entails. More specifically, we will provide the definition of mass devised by R. Arnowitt, S. Deser and C. W. Misner (ADM) in their canonical reformulation of GR, since it will be heavily relied upon in the following sections. However, due to the scope of this work, we will not present the detailed ADM formalism, for which we recommend the original references reprinted in [20].

It is well-known that formulating an *unambiguous* definition of mass in GR proves problematic. In Newtonian gravity, the mass can be obtained by performing an integration over a volume, whereas this is not possible in GR since the energy density, unlike the material density, is not a total divergence. While in Special Relativity, the invariant mass is simply defined in terms of energy and momentum, in General Relativity, mass is just one contribution to the energy-momentum tensor. Moreover, the energy of the gravitational field contributes non-locally to the total energy, as shown by the fact that it does not give rise to a conservation law. Therefore, the separation of the gravitational energy from the energy-momentum tensor is not agreed upon by all observers and there is no general definition of energy.

The only case where non-local gravitational masses are unambiguous is at infinity, whether spatial or null. For asymptotically flat spacetimes, a clear-cut definition of mass is given by the ADM mass in the context of the Hamiltonian formalism for GR: the time direction is singled out from the spatial ones and has an associated energy which can be integrated to yield the ADM mass, as will be clarified in the following. Another definition of mass useful in a different context and defined at null infinity is the Bondi mass, which does not take into account the contribution carried away by gravitational waves (whereas this is included in the ADM mass).

Nonetheless, since these masses are non-local, considerable effort has been more recently devoted to the definition of a quasi-local mass, that makes use of quantities defined only in a finite region of space, such as the Hawking mass.

We can get a first intuition of the physical meaning behind the ADM mass by considering spherically symmetric spacetimes. In this case, the most general form for the metric element is

$$\mathrm{d}s^2 = g_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j + r^2(x^k)(d\theta^2 + sin^2\theta\mathrm{d}\phi^2), \tag{1.3.62}$$

where r is the areal coordinate and the $x^i = (x^1, x^2)$ are coordinates on surfaces where the angles θ and ϕ are constant. If we set $x^1 = t$ and $x^2 = r$, the field equations imply

$$g^{rr} = 1 - \frac{2l_{\mathrm{p}}(M/m_{\mathrm{p}})}{r}, \tag{1.3.63}$$

where

$$m(r) = 4\pi \int_0^r \mathrm{d}r'\, \rho(r')r'^2, \tag{1.3.64}$$

$\rho(r)$ being the static matter density. $m(r)$ is called the Misner-Sharp mass and is calculated under the condition that the space inside the spherical region is flat. It represents the gravitational energy (taking in consideration both the contribution of matter and gravitational potential energy) inside a sphere of radius r. It can also be used to determine the position of trapping surfaces, located where the escape velocity equals the speed of light.

Under the assumption of static spacetime, the Misner-Sharp mass is related to the ADM mass M by:

$$\lim_{r\to\infty} m(r) = M. \tag{1.3.65}$$

The Hamiltonian formulation of General Relativity Let us now provide the basics of the ADM formalism, in order to better understand the context in which the ADM mass is formulated. In the historical development of the quantum theory of gravity as presented in [21], the approach with the canonical formalism was one of the first to be employed, in search of a Hamiltonian for General Relativity. The canonical formalism was chosen because it only employs a minimal set of variables to specify the state of the system. Arnowitt, Deser and Misner were the first to give a physical interpretation of the canonical quantization method applied to gravity, which allowed them to provide a rigorous definition of energy.

The ADM formalism relies on a foliation of spacetime into a family of space-like hypersurfaces of constant "time", so that fixing $t = const$ is tantamount to selecting one of the hypersurfaces. This means that the variable corresponding to time is singled out and the theory is recast in a 3+1 dimensional form, suitable for the canonical formulation, in which the field equations have to be first-order in time. This re-parametrization is also extremely useful for applications to the study of gravitational radiation, since the canonical variables will represent the independent excitations of the field, allowing gravitational radiation to be defined in a coordinate-independent way.

The crucial assumption underlying the whole formalism is the asymptotic flatness of spacetime: in a simple coordinate-dependent definition, this means that, if

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.3.66}$$

(where $\eta_{\mu\nu}$ is the Minkowski background metric and $h_{\mu\nu}$ is a perturbation tensor) and $r^2 = x^2 + y^2 + z^2$, we require

$$\lim_{r\to\infty} h_{\mu\nu} = \mathcal{O}(1/r). \tag{1.3.67}$$

This requirement comes down to the possibility of neglecting boundary terms in integrations by parts: standard field theory always entails the assumption that the field vanishes outside some arbitrarily large but finite domain. In the case of gravity, the field vanishes outside a finite spatial domain only if spacetime is flat. A partial integration performed in this case would leave the field equations unaffected but it would translate in a change for the definition of energy. Thus, from this and more rigorous considerations, the concept of energy in General Relativity is generally thought to have meaning only for asymptotically flat spacetimes.

In this canonical formulation, however, a problem quickly arose: the so-called "problem of constraints", which stems from the fact that general coordinate transformations constitute an invariance group for GR, just like the $U(1)$ abelian group of gauge transformations is a symmetry group for Electromagnetism. In the case of gravity, this translates, among other things, to the absence of a conjugate momentum for some field variables, and a dependence between the momenta of different variables.

The invariance under general coordinate transformations results in the redundancy of variables in the theory, so that not only the "physical" ones are left to work with. Thus, the goal for the canonical formulation was to separate the field into the parts carrying the true dynamical information and the redundant ones, that only characterize the coordinate system and appear to grant the correct transformation properties for the variables. The standard quantization techniques used for linear theories without constraints, such as those used for Electromagnetism, prove to be inappropriate in this context.

If the goal is to find canonical equations of motion, we have to make sure they are first-order in time. Therefore, a form of the GR Lagrangian which is linear in first derivatives should be employed, such as the Palatini Lagrangian, that regards the Christoffel symbols $\Gamma_{\mu\ \nu}^{\ \alpha}$ as independent quantities, so that variations with respect to them are performed separately. Starting from the usual Einstein-Hilbert action (in which the units are chosen so that $16\pi G_{\rm N}c^{-4} = 1$) $S = \int \mathrm{d}^4x\, \mathcal{L} = \int \mathrm{d}^4x\, \sqrt{-g}R$, we can rewrite it as

$$S = \int \mathrm{d}^4x\, \mathfrak{g}^{\mu\nu} R_{\mu\nu}(\Gamma), \tag{1.3.68}$$

where the Ricci tensor is given by (0.0.2) and $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$.

These covariant components of the Ricci tensor do not involve the metric but only the connection coefficients, so that by varying $g_{\mu\nu}$, the Einstein field equations are easily obtained. However, these equations no longer express the full content of the theory, as the relations between the affinity and the metric are not made explicit. This part of the physical content is obtained as a field equation by varying the affinity and gives the known relation (0.0.3).

Finally, the energy E of the gravitational field is defined to simply be the numerical value of the Hamiltonian, calculated for a particular solution to the field equations and in a way that is independent from the specific form of the Hamiltonian in terms of the canonical variables. ADM choose to express E as a surface integral:

$$P^0 \equiv E = -\oint \mathrm{d}S_i\, (g_{ij,j} - g_{jj,i}), \tag{1.3.69}$$

where $\mathrm{d}S_i$ is the two-dimensional surface element at spatial infinity. The total momentum is

$$P^i = -2\oint (\pi^i_{\ ,j} + \pi^j_{\ ,i}) dS_j = -2\oint \pi^{ij} dS_j, \tag{1.3.70}$$

where $\pi^{ij} \equiv \sqrt{-^4g}(^4\Gamma_{p\ q}^{\ 0} - g_{pq}\,^4\Gamma_{r\ s}^{\ 0} g^{rs}) g^{ip} g^{jq}$and the prefix ^{4} denotes a four-dimensional quantity, $g_{ij} \equiv {}^4g^{ij}$. The unmarked quantities are understood as being three-dimensional. P_0 and P_i correctly transform like a four-vector under Lorentz transformations at spatial infinity. The integrals are well-defined as long as asymptotic flatness is assumed. This comes down to the requirement that also $\Gamma_{\rho\ \sigma}^{\ \alpha}$ falls at least like $\sim 1/r$ at spatial infinity.

An important remark is now in order [22]: even if P_μ is proves to be invariant under a change of coordinates, this does not imply that different canonical forms of the theory in different frames will yield the same numerical result for the energy. In order for the generator of time translations to represent the physical energy of a system, the field has to be expressed in terms of the true and unconstrained degrees of freedom, but, additionally, the so-called Heisenberg representation ought to be employed, with the same meaning as in Quantum Mechanics. The Schrödinger or the interaction representations in this case describe the dynamics of the system but don't allow the identification of H with the energy [23].

Thus, the basic variables of the theory have to be in the Heisenberg representation: whereas in Classical Mechanics by basic variables we mean the directly measured position and momentum, in GR, every basic measurement refers directly to the metric $g_{\mu\nu}$. The condition that has to be met in order for $g_{\mu\nu}$ to be in the Heisenberg representation is that its full expression at any given time in terms of the canonical variables at that time does not have any explicit time dependence.

This condition also has a counterpart in the vanishing boundary terms that were mentioned earlier: transforming the metric from one canonical frame to another can yield, under specific circumstances, an oscillatory term that gives a finite, non-vanishing and ambiguous contribution to P_0. This contribution is not well-defined, since P_0 depends on the spacetime position of the surface at infinity, and can only disappear with the *ad hoc* prescription that an average is to be performed over oscillatory terms. Other authors have later showed that the term can disappear under more general assumptions, but we will not dive into the details.

Chapter 2

The corpuscular model of black holes

In this chapter, we want to provide the conceptual foundation of the thesis. We will firstly introduce the framework of classicalization elaborated in [15], and then present the corpuscular model of black holes described in [6], that provides the motivation for our work.

2.1 Classicalization of Einsteinian gravity

The idea of classicalization first came about in the context of gravity, but has been generalized to other field theories as well. However, gravity presents an uniqueness with respect to the other forces of Nature, as we will try to explain in the following.

The role of the Planck mass

The crucial role of the Planck mass in gravity is exemplified by the fact that it determines the strength of the interaction of the graviton with any arbitrary energy-momentum source. However, what is peculiar in gravity is that m_{p} also regulates the self-interactions of the gravitons to themselves, since gravity is a non-linear theory and the gravitons carry the charge to which they couple. This introduces a notorious complication in its treatment, just like the Yang-Mills non-linear theory is more complicated to deal with than the linear Maxwell theory for which photons do not couple to themselves. Moreover, the enormity of m_{p} compared to the scales of the other forces (for example that of the strong force, with the QCD scale being $\Lambda \sim 200MeV$) is the reason that accounts for gravity's feebleness. In the linearised theory, where gravitational interactions can be described by linear gravitons up to the strong-coupling Planck scale and the spacetime is asymptotically flat, we can find a dimensionless parameter to describe the strength of any interaction that involves a momentum transfer p,

$$\alpha_{\mathrm{GR}}(p^2) \equiv 16\pi G_{\mathrm{N}} p^2 = \frac{p^2}{m_{\mathrm{p}}^2}. \tag{2.1.1}$$

If an interaction is characterized by $p \ll m_{\mathrm{p}}$, we are in the safe weak-coupling regime where $\alpha_{\mathrm{GR}} \ll 1$, but since gravity couples universally to any energy-momentum source and the parameter α_{GR} depends itself on the momentum, in a perturbative expansion in powers of α_{GR}, one would encounter higher and higher powers of momenta at each order of the expansion. Hence, the higher the order, the higher the UV divergences. Eventually, we would need an infinite number of counterterms to cancel the infinities by making use of the standard renormalization procedures. Therefore, the non-renormalizable theory of gravity should be only interpreted as an effective theory, valid up to Planck scale.

2.1.1 The shortest length-scale of Nature

The Planck length is the shortest length-scale of nature and it acts like a boundary between the world of elementary particles governed by quantum mechanics and the one of black holes, governed by General Relativity. This claim will be substantiated both in this paragraph and in 2.1.2.

From what we have said earlier, we know that the Planck mass determines a strong-coupling scale for gravity. Switching to the length scale description, this means that the theory (as it stands) is inapplicable at lengths $L \ll \ell_{\mathrm{p}}$, because a theory of Quantum Gravity is needed (this is of course valid as long as we follow the standard lore, before taking the idea of classicalization into consideration).

This concept has powerful implications. In fact, it entails that it is impossible to probe shorter distances than ℓ_{p}, since beyond this scale no distance could be ever resolved and no new information derived from the system. This is valid *in principle* and regardless of the capabilities of an experimental apparatus. This principle goes under the name of Generalized Uncertainty Principle (GUP), that is commonly believed to hold in any picture of Quantum Gravity (see the following paragraph for more details).

What has been stated here represents one of the main supporting arguments of classicalization, presented in 1.2.2, since it posits that, even if we tried to introduce new propagating degrees of freedom above the energy scale that represents our boundary (as it is generally performed in the standard Wilsonian UV-completion) these degrees of freedom would be *physically meaningless*, because the corresponding distances could never be probed.

Therefore, if the physics beyond the Planck scale is inaccessible in principle, we are bounced back to the IR physics, which means the classical physics we are used to, endowed with the usual low-energy degrees of freedom. Thus, following this line of reasoning, the claim of classicalization seems to be supported: the physics that we can decode from trans-planckian distances and high energies is identical to the physics at macroscopic distances and low energies.

Generalized Uncertainty Principle

In the last decades, it has become clear that, when gravity is taken into account, basic principles of Quantum Mechanics such as Heisenberg's Uncertainty Principle need to be modified, because at scales comparable to the Planck scale, the very notion of spacetime needs to be revisited. A generalized version of Heisenberg's principle has been studied especially in the context of String Theory, where *Gedankenexperimente* have been conducted to analyze string collisions at the Planck scale, but also in a quantum theory of gravity of whatever nature, this problem cannot be ignored. We will briefly review some results related to the GUP that can be useful for our purposes but we have, of course, no goal of being exhaustive.

A relevant analysis has been that of [24], where a thought experiment to measure the area of the horizon of a black hole is performed, in a way that is independent by the employed model of Quantum Gravity and reproduces the results previously obtained in the context of String Theory.

The central idea is to obtain an "image" of a black hole by scattering photons off of it, thus building on the historically relevant idea of Heisenberg's microscope. The image can be obtained, in this context, because the black hole emits Hawking radiation, and if we measure the direction of propagation of photons emitted at different angles and trace them back, we could be, in principle, able to locate the center of the black hole and then measure the radius of its horizon. In the classical analysis by Heisenberg, the resolving power of the microscope has a minimum error

$$\Delta x_1 \sim \frac{\lambda}{sin\theta} \tag{2.1.2}$$

because the photon cannot resolve a length $L < \lambda$, where θ is the scattering angle. This represents the first source of uncertainty of the system. The second source of uncertainty comes from the fact that, in the emission process, the mass of the black hole changes from $M + \Delta M$ to M, where $\Delta M = \frac{\hbar}{c\lambda}$ and the radius varies accordingly. The corresponding error, intrinsic to the measuring process, is

$$\Delta x_2 \sim \frac{2G_\mathrm{N}}{c^2}\Delta M = \frac{2G_\mathrm{N}\hbar}{\lambda c^3}, \tag{2.1.3}$$

for a Schwarzschild black hole. Since

$$\ell_\mathrm{p} = \sqrt{\frac{\hbar G_\mathrm{N}}{c^3}}, \tag{2.1.4}$$

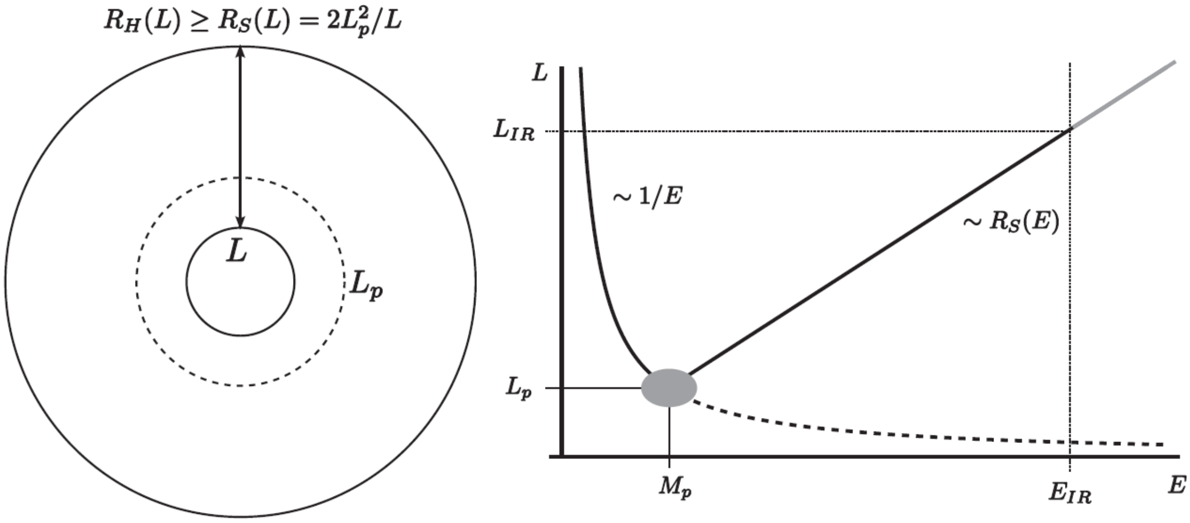


Figure 2.1: Trans-planckian distances are shielded by black hole barriers: to probe lengths $L^{-2} \ll m_{\rm p}^2$, energy of order L^{-1} has to be localized within L (on the left). The corresponding horizon of this energy, $R_{\rm H}(L) \geq R_{\rm S}(L) = 2\ell_{\rm p}^2/L$, shields the sub-Planckian region from being probed. The sub-Planckian distance L is mapped to the macroscopic $R_{\rm H}(L)$. On the right, a qualitative plot of the energy-distance relationship, where the grey blob indicates the lack of knowledge about the precise relationship between these quantities at the Planck scale itself. Source [26].

$$\Delta x_2 \sim \frac{2G_{\rm N}\hbar}{\lambda c^3} = \frac{2\ell_{\rm p}^2}{\lambda}. \tag{2.1.5}$$

Moreover, it is obvious that $\frac{\lambda}{sin\theta} \geq \lambda$, so the author chooses to linearly combine the two uncertainties in

$$\Lambda x \geq C\frac{\ell_{\rm p}^2}{\lambda}, \tag{2.1.6}$$

C being a constant that this model-independent approach cannot predict.

In [25], a different approach has been considered, that allows to obtain a GUP without relying on any specific physical phenomenon like the Hawking radiation, but only using the Heisenberg relation and the Schwarzschild radius.

In this article, the uncertainty principle is applied to a measuring process carried out in the presence of gravity, and in the end it is shown that the formation of a micro-black hole affects the process itself. This concept will be also dealt with in the following paragraphs.

For the moment, let us consider the principle in the form

$$\Delta E \Delta x \geq \frac{\hbar c}{2}, \tag{2.1.7}$$

because of $\Delta E \sim c\Delta p$. Observing a region of space of width Δx means observing quantum fluctuations of energy $\Delta E \sim \frac{\hbar c}{2\Delta x}$, confined in the region Δx. The Schwarzschild radius

$$r_g = \frac{2G_{\mathrm{N}}\Delta E}{c^4} \tag{2.1.8}$$

falls inside the same region, but if we want to observe shorter lengths than Δx, the energy ΔE and the corresponding r_g must grow accordingly. Of course, the radius can grow only until it reaches Δx and it can never exceed it. The critical length at which they become equal is the Planck length ℓ_{p}, corresponding to the Planck energy $\epsilon_{\mathrm{p}} = m_{\mathrm{p}}c^2$. At this critical point, a micro-black hole is created, that shields its interior according to the known properties of black holes. Any attempt to shrink the region Δx results in bigger quantum fluctuations that make the measurement itself impossible.

We then have two cases for the uncertainty in the position:

$$\Delta x \geq \begin{cases} \dfrac{\hbar c}{2\Delta E} & \text{for} \quad \Delta E \leq \epsilon_{\mathrm{p}} \\[2ex] \dfrac{2G_{\mathrm{N}}\Delta E}{c^4} & \text{for} \quad \Delta E > \epsilon_{\mathrm{p}}, \end{cases} \tag{2.1.9}$$

that can simply be linearly combined:

$$\Delta x \geq \frac{\hbar c}{2\Delta E} + \frac{2G_{\mathrm{N}}\Delta E}{c^4}. \tag{2.1.10}$$

In conclusion, we note that the introduction of a GUP has crucial consequences: firstly, the concept of the horizon in a black hole becomes uncertain because of quantum fluctuations as we get closer to the Planck scale, the horizon itself not being defined on scales smaller than ℓ_{p}. Secondly, this entails that it makes no sense to talk about black holes of $M < m_{\mathrm{p}}$, as well as of elementary particles with $M > m_{\mathrm{p}}$, as will be clarified in the following paragraphs.

We just briefly remark that several other more sophisticated formulations of the GUP exist in the literature, such as those that try to find a modified expression for the commutator $\left[\hat{X}, \hat{P}\right]$ with an additional term that is quadratic in the momentum. We also mention that the HQM formalism, developed in order to investigate the potential emergence of a horizon in the context of corspuscular black holes, also allows to recover an effective GUP [27].

Moreover, in the context of the GUP, some aspects of black holes have been recently recovered: for example, the Unruh effect and Unruh temperature in [28] and the corrections to the Schwarzschild metric necessary to reproduce the Hawking temperature in [29].

2.1.2 Black holes and elementary particles

The fundamental reason for the "censorship" that appears both in classicalization and in the GUP is, of course, the existence of black holes in GR.

Thorne's famous hoop conjecture states that whenever a hoop of radius $r_{\mathrm{S}}(M) = 2G_{\mathrm{N}}E$ can be placed around a region of space containing an energy E, a black hole is inevitably formed. The radius of course corresponds to the Schwarzschild radius of the object.

If we want to give a field-theoretical interpretation of the conjecture, let us perform a *Gedankenexperiment* very similar to the one proposed in the previous paragraph, but this time specifically in order to attempt to probe distances shorter than the Planck length. Any such measurement would have to be a scattering process that involves gravitons scattering off an object of energy $E < 1/L$ localized in a region of spacetime of size L. The Schwarzschild radius for this object is of course $R_{\mathrm{S}}(L) = \ell_{\mathrm{p}}^2/L$. It is straightforward to notice that if $L < \ell_{\mathrm{p}}$, the Schwarzschild radius exceeds both L and ℓ_{p}, so any attempt of probing such length scales will require an amount of energy corresponding to the probing length to be enclosed in a portion of spacetime of radius $R < R_S$, thus leading to the formation of a black hole, as was already noted in the context of the GUP. The black hole will be macroscopic and classical, and probing the length L turns out not to be possible, because of the way the BH shields the area inside its gravitational radius. No new information in the UV regime beyond the Planck scale could then be extracted, and it is here that the claim of classicalization finds justification, since the degrees of freedom of the theory would have to be the "old" IR ones. The more we increase the momentum transferred in our scattering *Gedankenexperiment*, the shorter distances and stronger gravitational couplings we would be able to probe. But when the Planck scale is reached and gravity becomes strongly coupled, the black hole formation takes over the process, impeding any further probing. Any attempt to further increase the momentum involved in the scattering experiment will just result in the creation of larger classical black holes.

Following the same line of reasoning, it has been also argued that, even if we were to modify gravity by the introduction of new degrees of freedom beyond the Planck scale, which would of course be represented by masses $m \gg m_{\mathrm{p}}$ and thus new poles in the graviton propagator, this modification could never be probed. The contribution of these poles must then be exponentially suppressed, with a suppression factor of $e^{-(\ell_{\mathrm{p}}/L)^2}$. For an alternative analysis of this problem, by means of the Horizon Wave Function formalism, see [30].

Thus, it can be said that in gravity and above the Planck scale there are no propagating degrees of freedom , but the latter just get mapped into those below the Planck scale. Considering the states beyond the Planck scale can be regarded as physically meaningless, since we cannot access them, *in principle*.

To understand the importance of the role played by the Planck mass, we can clarify the claim that elementary particles heavier than the Planck mass do not exist, because they are black holes [31]. To any particle of mass m we can associate two length scales, one quantum and one classical, namely its Compton wavelength L_{C} and its Schwarzschild radius R_{S}. The first represents the length at which the energy of quantum fluctuations equals the particle mass, and the second represents the radius for which an enclosed amount of energy becomes a black hole. These two length scales set the boundaries of two regimes, or even two different worlds, that of Quantum Mechanics and elementary particles, and that of General Relativity and black holes.

The two regimes are identified as follows:

- If we consider a particle of mass $m < m_{\mathrm{p}}$, then $L_{\mathrm{C}} > \ell_{\mathrm{p}} > r_{\mathrm{S}}$, so the dominant length scale is the quantum one, the Compton wavelength. The gravitational effects of such a particle are of course negligible, as we can see from approximately calculating its gravitational field in Newtonian terms:

$$\phi \sim \frac{m G_{\mathrm{N}}}{L_{\mathrm{C}}} = \frac{r_{\mathrm{S}}}{L_{\mathrm{C}}} \ll 1. \tag{2.1.11}$$

  Thus, the quantum effects dominate in the description of the particle, making it impossible to get close enough to it to probe the scale of the gravitational radius, that is shielded by Quantum Mechanics. This, of course, is the reason why gravitational interactions are neglected in the context of particle physics. We remark the striking property of gravity as the weakest of the fundamental forces, being $\sim 10^{40}$ times weaker than the electromagnetic force.

- In the opposite regime, that of General Relativity and black holes, for $m > m_{\mathrm{p}}$, we have $L_{\mathrm{C}} < \ell_{\mathrm{p}} < r_{\mathrm{S}}$. The gravitational radius dominates the physical description, so we are in presence of a black hole.

The only thing missing from this analysis is the contact point between the two worlds, where $L_{\mathrm{C}} = \ell_{\mathrm{p}} = r_{\mathrm{S}}$, which of course only happens for an object of $m = m_{\mathrm{p}}$.

It is striking to notice that the Planck mass represents an UV-cutoff parameter for the theory of gravitation, but also an upper bound on the mass of an elementary particle. Objects with $m \sim m_{\mathrm{p}}$ are defined as quantum black holes, since they have extremely strong quantum fluctuations as well as gravitational effects, for their Newtonian potential is of order one. It has been argued that such particles could physically exist as they represent the endpoint of the Hawking evaporation process. In [16], it is also clarified that trans-planckian distances cannot be probed by simply waiting longer. It is shown that no new poles in the graviton propagator arise in the strong-coupling regime even in the final stages of black hole evaporation.

Now that we have clarified the role of the Planck mass as a boundary between two regimes, we can argue, in the framework of classicalization, that gravity could be seen as a theory of the graviton at scales below m_p and a theory of objects called classicalons above m_p, which of course are black holes in this specific case. The previous description leads to the conclusion that a any degree of freedom with mass $M > m_\mathrm{p}$ is a classical object, it becomes obvious that there is no process, no matter how sophisticated, which can probe trans-Planckian physics (including black hole evaporation, primordial quantum fluctuations and scattering experiments).

If we were to see the black holes that gravity describes above the Planck scale as "new degrees of freedom" that gravity introduces, we could argue that classicalization acts just like a different kind of Wilsonian UV-completion. However, the fundamental difference between the two descriptions is that macroscopic black holes are of course not elementary particles, but composite particle states, in particular multi-gravitons states, as we will elaborate on in the following.

2.1.3 Estimates of the number of quanta

Classical states are characterized by the fact that they are not independent entities, but can be described by quantum degrees of freedom, such as the ones implied by quantum field theories. Moreover, classical states cannot be employed to probe distances that are shorter than the typical wavelength of the quantum elementary particles that constitute them. This wavelength is generally given by the size of the classical configuration.

The previous statements are equally valid for gravity and other field theories. However, for other field theories we can introduce new arbitrarily heavy degrees of freedom above the threshold energy, which will be able to probe shorter and shorter distances as they get heavier. Gravity, instead, presents a crucial difference: we cannot introduce arbitrarily heavy *independent* degrees of freedom, because above the Planck scales there are only classical *composite* states. The classical states can be composed of nothing else other than gravitons, the old IR degrees of freedom, now present in high-multiplicity states.

In [15], the author also argues that every state with a center of mass energy $\sqrt{s} \gg m_\mathrm{p}$ always consists of many soft (low-energy) quanta, whether this multiplicity is explicit or implicit. We now want to provide some estimates of the number of soft quanta that constitute such classical states.

The number of soft quanta can be seen as a measure of "classicality" of the physical state we are considering. In [32], it is argued that every mass M is surrounded by, on average,

$$N = \frac{M^2}{m_\mathrm{p}^2} \tag{2.1.12}$$

gravitons that make up the gravitational field generated by the mass. If the object is considered to be of a characteristic size R, a first approximate estimate of the number N can be drawn by identifying R with the de Broglie wavelength of the object, to yield:

$$\frac{N\hbar}{R} = \frac{M^2 G_{\mathrm{N}}}{R}, \tag{2.1.13}$$

where the definition of the Planck mass has been employed. If, instead, R is identified with the gravitational radius $r_{\mathrm{S}} = 2G_{\mathrm{N}}M$, the gravitons carry almost all the energy of the source contained in its mass, so that the size of the object cannot be reduced without the emission of some energy (compare with the "maximal packing" condition required in the following chapters for the corpuscular model of black holes). In [33], it is argued that this number of soft quanta can be explicitly calculated by bridging the classical and the quantum description by making use of coherent states. The straightforward analysis is carried out through a comparison between the gravitational and the electric field.

It is generally thought that an external source for the electromagnetic field, such as a moving charge, contains an infinite number of photons, which are soft but physical. The previously-cited two articles want to clarify this point, one in a more admittedly heuristic way and the other in a more formal standard fashion.

In [32], it is shown that the particle number is a good observable in a coherent state only if it is large, because the relative uncertainty on the particle number is found to be $(\Delta N)/N = 1/\sqrt{N}$. In order to interpret the classical Coulomb static field as a coherent state, dimensional analysis tells us that we need to introduce an auxiliary scale, identified with a very small photon mass μ that effectively acts as an infrared cut-off to cut off divergent integrals at large distances $r = \hbar/\mu$, instead of correcting the Coulomb potential with that of Yukawa and thus keeping the $1/r$ behaviour. We will heavily rely on this notion of a tiny effective mass in the following Chapters. Proceeding this way, a finite estimate of the number of photons can be recovered,

$$N = \frac{q^2}{4\pi\hbar}\left(1 - \frac{2\mu R}{3\hbar}\right) \approx \frac{q^2}{4\pi\hbar}, \tag{2.1.14}$$

where q is the charge of the electromagnetic field.

In this simple analysis, the fictitious photon mass has disappeared because of the large-distance cut-off.

Moving on to the gravitational case, a sort of "gravitational charge" $q_{\mathrm{M}} = \sqrt{4\pi GM}$ could be introduced, in order to make Newton's law and Coulomb's law identical. By using this charge in (2.1.14), we see that

$$N = \frac{4\pi GM^2}{4\pi\hbar} = \frac{M^2}{M_{\mathrm{P}}^2} \tag{2.1.15}$$

from [7] is recovered. Proceeding with the analogy of the electric field, it can be argued that an extremely small graviton mass must be introduced in the theory of gravity in order to cure the long-distance singularity. From the calculation of the average energy of photons and gravitons it comes out that this energy corresponds to a de Broglie wavelength which is approximately the size of the radius of the field source. If the corpuscular model of black holes (which will be presented in the following) is correct, than a coherent state of gravitons becomes the ground state of the system. This is one of the basic assumptions of our work, as will be clarified in the next Chapters.

The last remark we make on [33] is that the electromagnetic field with the choice of the Coulomb gauge fixing term and the presence of a small mass to regularize the IR behaviour can be quantized in the canonical formalism and the massless limit presents no pathologies.

2.2 The corpuscular model of black holes

In this section, we will present the quantum picture of black holes as Bose-Einstein condensates of gravitons, introduced by Dvali and Gomez, that we consider as a prerequisite for our work. The model will be explained in a qualitative way, in order to construct the needed conceptual framework.

The corpuscular model is a consistency test for non-Wilsonian self-completion of Einsteinian gravity as proposed in the context of classicalization, but also represents a novelty for its totally quantum nature, at variance with the semi-classical models that have been used so far to describe black holes. The latter are generally understood in geometric terms that allow for classical solutions, but the underlying quantum properties of Nature cannot be neglected at the most fundamental level. Even if these models have allowed physicists to gain extremely valuable insight into the realm of black holes (one crucial example being the discovery of the Hawking radiation from a semi-classical analysis) a deeper understanding of these extreme objects cries out for a fully quantum model. One central idea is that the semi-classical properties of black holes, such as the presence of a horizon, Bekenstein entropy and Hawking radiation should be recovered in the model as emergent phenomena from the quantum substructure.

The corpuscular model [6], [7] relies on the assumption that a black hole is a bound state of $N \gg 1$ weakly-interacting soft gravitons of wavelength $\lambda = \sqrt{N}\ell_{\mathrm{p}} \sim R_{\mathrm{S}}$. The non-propagating superposed gravitons interact with strength set by the coupling constant $\alpha_g = 1/N$. The condensate is energetically self-sustained and at the point of *maximal packing*, which means that any further increase in N will lead to an increase in its size R. The condensate is always on the verge of a quantum phase transition and is leaky, because of the quantum depletion effect that is understood as the emission of Hawking radiation.

The gravitons are weakly interacting and can be regarded as free for every practical purpose, because their wavelength is really long, but their collective behaviour and high occupation number cause them to produce the extreme gravitational potential of black holes. Their wavelength is

$$\lambda = \sqrt{N}\ell_{\mathrm{p}} \tag{2.2.16}$$

and each single graviton is subject to a collective binding potential of strength

$$V = \frac{\hbar}{\sqrt{N}\ell_{\mathrm{p}}}. \tag{2.2.17}$$

Just in the same way as a large number of photons gives rise to the emergent classical phenomenon of a laser beam, the classical geometric properties of black holes can be understood as an effective description of a quantum state with large graviton occupation number. When this state is a ground state, the black hole is *effectively* a Bose-Einstein condensate. The larger N becomes, the more the geometric picture becomes accurate.

The role of N All the characteristics of the black hole only depend on one parameter, the occupation number N, which is an intrinsically quantum parameter. In units of the fundamental length, the relevant properties of the condensate are:

- Occupation number: N
- Wavelength: $\lambda = \sqrt{N}\ell_{\mathrm{p}}$
- Coupling strength: $\alpha = 1/N$
- Mass: $M = \sqrt{N}\frac{\hbar}{\ell_{\mathrm{p}}}$

In the framework of classicalization and by making use of the large-N limit 1.2.2, any classical object is understood as a bound-state with high occupation number of quantum components. Among macroscopic classical objects, it is claimed that black holes are the simplest (and also the most classical) due to the the maximal packing condition. Since the gravitons are so overpacked that no more of them could be added without increasing the size of the condensate, N is maximized by the black hole state and effectively becomes the only relevant parameter. Thus, quantum black hole physics can be formulated in terms of only N, which is considered as a measure of classicalization. Furthermore, N is very large but finite, and it is infinite only in the classical limit. A black hole is thus a large-N system, in the sense intended by 't Hooft [18]. However, it must be kept in mind that N is not an input parameter which is chosen to be large for convenience, but rather it can assume arbitrarily large values: each value tunes all the other properties of the model to it, for example the wavelength of gravitons and their binding energy. Moreover, it is important to remember that the black hole state exists for arbitrarily large N.

To understand the field-theoretical meaning of classicality, let us consider a generic gravitating source of mass M. We can choose it to be of uniform density and radius

$R \gg R_\mathrm{S}$. Of course, for such a source, the approximation of linear gravity is well-suited and non-linearities are neglected, even though the potential can still be understood as a superposition of gravitons. The Newtonian component of the metric perturbation around flat space is $\Phi(r) = -\frac{R_\mathrm{S}}{r}$.

Considering the process of formation of a black hole, we can imagine trying to localize as many gravitons as possible in a region of size L. At the beginning, as long as $R \gg R_\mathrm{S}$, the gravitational interactions are very weak and an external source would be needed to maintain the condensate, such as in photon Bose-Einstein condensates, which require an external potential. Moreover, in this regime the energy of the system is simply the sum of the energies of the individual gravitons with wavelengths λ and occupation numbers N_λ,

$$E_\mathrm{grav} \sim \sum_\lambda N_\lambda \frac{\hbar}{\lambda}, \tag{2.2.18}$$

where $E_\mathrm{grav} \sim \frac{MR_\mathrm{S}}{R}$. The reason for writing the total gravitational energy as a simple sum is that their distribution is sharply peaked at $\lambda = R$ and the contribution from shorter wavelengths is exponentially suppressed. The interactions among the gravitons are very weak (both between the individual gravitons and the collective gravitational energy) and can be ignored in this regime, thus there is no gravitational self-sourcing. The occupation number of gravitons can be also found by dividing the total gravitational energy by the characteristic energy of a single quantum, which yields:

$$N \sim \frac{E_\mathrm{grav}}{\hbar R^{-1}} = \frac{MR_\mathrm{S}}{\hbar}. \tag{2.2.19}$$

As we keep adding gravitons and R becomes comparable with R_S, the interactions among the individual gravitons keep being negligible, but the interaction with the collective gravitational energy (the self-sourcing) becomes important, due to the fact that the gravitational energy is of the order of the energy of the source. At this point, the condensate is able to sustain itself, but the interactions between individual gravitons are still negligible and thus the number of gravitons can be still safely estimated by 2.2.19.

However, for a black hole we know that M can be estimated as the total gravitational energy of the source E. Since N can be rewritten as

$$N = \frac{M^2}{m_\mathrm{p}^2} = \frac{r_\mathrm{S}^2}{\ell_\mathrm{p}^2} \tag{2.2.20}$$

and considering the expression for λ, we obtain

$$N = \frac{\lambda^2}{\ell_\mathrm{p}^2} \equiv \frac{1}{\alpha_\mathrm{gr}}, \tag{2.2.21}$$

where $\alpha_{\rm gr}$ is the dimensionless self-coupling of gravitons $\alpha_{\rm gr} = \frac{\hbar G_{\rm N}}{\lambda^2}$. This means that the condensate satisfies the energy balance in [8] that defines the maximal packing condition. In the language of statistical physics, the condensate is at the critical point of a quantum phase transition. Furthermore, in this same language, established in [6], the degeneracy of the Bose-Einstein condensate at the critical point is connected to the Bekenstein entropy of a black hole and the quantum depletion of the condensate is interpreted as the emission of Hawking radiation. Before analyzing how these two correspondences arise, let us briefly review the basic concepts in the thermodynamics of black holes.

2.2.1 Bekenstein entropy and Hawking radiation

In this paragraph we want to provide a brief overview of two fundamental discoveries that revolutionized our understanding of black holes in the '70s, namely Bekenstein entropy and Hawking radiation. In the following, we will present how these aspects are recovered and simply explained in the corpuscular model of black holes.

One of the most remarkable developments in our understanding of black holes took place when a parallel between the laws that govern black hole dynamics and the ordinary laws of thermodynamics was established. Bekenstein was the first to notice a similarity between the Area Theorem of classical General Relativity and the second law of thermodynamics. The first, formulated by Hawking, states that the area A of a black hole can never decrease, while the second famously states that the total entropy of a closed system S can never decrease. The problem with the second law of thermodynamics in the context of black holes arises from the fact that such objects seem to violate it: according to classical arguments, when something falls into a black hole it disappears, in the sense that no information about it can ever be recovered, so its entropy seems to disappear with it and the second law appears to be violated. More precisely, it is said that black holes *transcend* the second law, because no external observer could ever verify by direct measurement that the total entropy of the Universe does not decrease. Bekenstein also solved this problem by formulating a law that accounts for this situation.

Let us first present a very straightforward argument in order to clarify the relationship between the area of a black hole and its entropy [34]. Imagine a small box of gas of size L, mass m and temperature T that falls into a black hole of mass M, Schwarzschild radius $R_{\rm S}$ and horizon area $A = 16\pi G_{\rm N}^2 M^2$. The box will start to merge with the black hole when its proper distance from the horizon is of order L. Its disappearance into the black hole leads to an entropy loss for the Universe of about $\Delta S \sim -m/T$.

If we consider a Schwarzschild black hole, the proper distance from the horizon is

$$\rho = \int_{2G_{\rm N}M}^{2G_{\rm N}M+\delta r} \frac{{\rm d}r}{\sqrt{1 - 2G_{\rm N}M/r}} \sim \sqrt{G_{\rm N}M\delta r} \tag{2.2.22}$$

and $\rho \sim L$ when $\delta r \sim L^2/G_{\rm N}M$. The initial mass m of the gas ends up being absorbed and leads to an increase ΔM in the black hole mass. This contribution, though, is m as seen from infinity, with the redshift evaluated in $r = 2G_{\rm N}M + \delta r$. Taking this effect into account, the black hole gains

$$\Delta M \sim m\sqrt{1 - \frac{2G_{\rm N}M}{2G_{\rm N}M + \delta r}} \sim \frac{mL}{G_{\rm N}M}. \tag{2.2.23}$$

If we want to calculate the variation in entropy, it is reasonable to consider the size of the box to be approximately the thermal wavelength of the gas, $L \sim \hbar/T$, so that:

$$\Delta S \sim -\frac{mL}{\hbar} \sim -\frac{G_{\rm N}M\Delta M}{\hbar} \sim -\frac{\Delta A}{\hbar G_{\rm N}}. \tag{2.2.24}$$

Thus, the analogy between entropy and the area of the black hole is justified.

In order to preserve the validity of the second law, the entropy of the object that falls in and disappears gets "converted" into an increase of the area of the black hole. The same result can also be recovered with more sophisticated approaches, such as with a generic object instead of a box of gas, or in the context of Kerr black holes and involving concepts regarding information theory [35]. Consequently, Bekenstein proposed that a black hole possesses a physical entropy of order:

$$S \sim \frac{A}{G_{\rm N}\hbar}. \tag{2.2.25}$$

But what does it mean to attribute entropy to a black hole? It is well-known from the No-hair Theorem that a stationary black hole can be parametrized by just a few quantities: its mass, electric charge and angular momentum. However, setting specific values for these parameters still leaves many possibilities for the black hole formation. There are many possible internal states corresponding to a black hole of given parameters, which means that many micro-states can correspond to one macro-state. Entropy quantifies such a degeneracy of micro-states, as well as accounting for the inaccessibility of information about the internal configuration; thus the attribution of entropy to a black holes can be justified. The concept of degeneracy is crucial, since it is because of it that the Bekenstein entropy result is found in the context of the corpuscular model of black holes.

Now that the parallel between the two quantities has been established, the Second Law of Thermodynamics needs to be reformulated in order to account for the presence of black holes:

$$\Delta S_{\rm BH} + \Delta S_{\rm th} \geq 0, \tag{2.2.26}$$

where $S_{\rm BH}$ is the entropy of the black hole and $S_{\rm th}$ is the entropy outside of it.

After these discoveries, Bardeen, Carter and Hawking formalized the relationship between the Laws of thermodynamics and the properties of black holes in the famous Four Laws of black hole dynamics [36]:

- Zeroth Law: The surface gravity κ of a stationary black hole is constant over the event horizon.
- First Law: $\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J$ in vacuum, where M, A, J and Ω are the mass, area, angular momentum, and angular velocity of the event horizon, respectively, of two nearby stationary black holes.
- Generalized Second Law: $\Delta S_{\mathrm{BH}} + \Delta S_{\mathrm{th}} \geq 0$
- Third Law: It is not possible to reduce the surface gravity κ of the horizon of a black hole in a finite number of steps.

The analogous quantities that belong, respectively, to the realms of thermodynamics and General Relativity are:

- $E \longleftrightarrow M$
- $T \longleftrightarrow \alpha\kappa$
- $S \longleftrightarrow \dfrac{A}{8\pi\alpha G_{\mathrm{N}}}$,

with α being a generic constant. The significant relationship between these quantities is supported by the fact that in General Relativity, the energy and the mass are physically equivalent. However, in a classical thermodynamical context the temperature of a black hole should be absolute zero, so the second analogy involved a conceptual leap that was only clarified later on.

The real confirmation of these analogies came from Hawking's discovery in 1974 that black holes radiate as a black body with a certain temperature, to be interpreted as a real physical temperature [5]. This crucial result justifies and completes the construction of a thermodynamics of black holes, so it is worth analyzing, at least in a qualitative way.

Intuitively, the Hawking effect can be described by taking two things into account. The first is that the vacuum is not empty from a quantum-mechanical point of view. It is instead filled with virtual particle-antiparticle pairs of opposite energy that fluctuate in and out of existence in such a short time that they don't violate the conservation of energy, since they are "shielded" by Heisenberg's uncertainty principle $\Delta E\Delta t \geq \hbar/2$. The existence of such particles gives rise to many experimentally testable effects, such as the polarization of vacuum observed in the Lamb shift. The second thing we must keep in mind is that energy in General Relativity is frame-dependent, so the energy of a particle can be positive in a frame and negative in another. Also, if we consider the well-known Schwarzschild metric, what happens is that inside the event horizon of the black hole, the role of the coordinates t and r is exchanged, with the first becoming a space coordinate and the second becoming a time coordinate. Hence, an ingoing virtual particle that has negative energy relative to an external observer might have positive energy to an observer inside the horizon. This suggests a way to "circumvent" the uncertainty principle, because if the negative-energy particle of the pair crosses the horizon, it does not need to vanish in a time dictated by the principle anymore, and its positive-energy partner can escape to infinity.

Making a slightly more quantitative estimate, if a virtual pair is at rest at a coordinate distance δr from the horizon, the time in which one of the members of the pair can reach the horizon is

$$\tau \sim \sqrt{G_{\mathrm{N}} M \delta r}. \tag{2.2.27}$$

If we set this equal to the lifetime of the pair $\hbar/E$, we get

$$|E| \sim \frac{\hbar}{\sqrt{G_{\mathrm{N}} M \delta r}}, \tag{2.2.28}$$

for both of the particles. The energy at infinity will be redshifted according to

$$E_{\infty} \sim \frac{\hbar}{\sqrt{G_{\mathrm{N}} M \delta r}} \sqrt{1 - \frac{2G_{\mathrm{N}} M}{2G_{\mathrm{N}} M + \delta r}} \sim \frac{\hbar}{G_{\mathrm{N}} M}. \tag{2.2.29}$$

This result leads to expecting a black hole to radiate with a characteristic temperature $kT \sim \hbar/G_{\mathrm{N}} M$. The precise computation, that we will not provide for the sake of brevity, yields the Hawking temperature

$$T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi k_{\mathrm{B}} G_{\mathrm{N}} M}, \tag{2.2.30}$$

in physical units and for a Schwarzschild black hole of mass M. The concept of a finite temperature for a black hole also stems from the fact that they possess a finite entropy. Furthermore, with a finite temperature, black holes should also to come into equilibrium with a neighboring system such as a gas of photons, thus they cannot only absorb radiation but also need to emit some. For a stellar mass black hole, the Hawking temperature is about eight orders of magnitude smaller that the cosmic microwave background, which means that its detection is at present outside of experimental capabilities. The emission of Hawking radiation leads to black hole evaporation, even if this process happens on enormous time scales: for example, a solar mass black hole would evaporate in a time which is several orders of magnitude greater than the age of the Universe.

If we associate such a temperature to a static black hole, the constant α in Bekenstein entropy can be set to $\alpha = \frac{\hbar}{2\pi k_{\mathrm{B}}}$ to yield the so-called Bekenstein-Hawking entropy:

$$S_{\mathrm{BH}} = \frac{A}{4\ell_{\mathrm{p}}^2} k_{\mathrm{B}} = \frac{c^3 A}{4 G_{\mathrm{N}} \hbar}. \tag{2.2.31}$$

It is important to notice that the Bekenstein entropy and the Hawking temperature depend explicitly on $\hbar$, so they are intrinsically quantum-mechanical. Moreover, the entropy also depends on G_{N}, thus it is, in a sense, quantum-gravitational.

2.2.2 Bekenstein entropy and Hawking radiation in corpuscular black holes

As we have already stated, in the corpuscular model, black holes are seen as Bose-Einstein condensates of gravitons at the critical point and thus of the verge of the quantum phase transition. The model claims that the peculiarity of black holes lies in fact that they keep being at the critical point, because of the self-similarity of their depletion. Let us clarify this point: firstly, that the condensate undergoes a quantum depletion, and secondly, that it does so in a self-similar way.

The cause of the leakage lies in the interactions, since some of the gravitons can gain enough energy to escape from the gravitational potential. Even if the interaction between a pair of individual gravitons is very weak because of their long wavelength, each graviton feels a collective binding potential created by the other ones, that for $r \gtrsim \lambda$, is $V(r)|_{r \gtrsim \lambda} = (\hbar \alpha_{\mathrm{gr}} N)/r$. For any λ, the potential reaches a maximal depth at $r = \lambda$ and determines an escape kinetic energy for a test particle that is $E_{\mathrm{esc}} \equiv \hbar/\lambda_{\mathrm{esc}}$. When the relevant quantities of the black hole are exactly

$$\lambda = \sqrt{N}\ell_{\mathrm{p}}, \qquad \alpha = \frac{1}{N}, \qquad M = \sqrt{N}\frac{\hbar}{\ell_{\mathrm{p}}}, \tag{2.2.32}$$

the escape wavelength exactly saturates the wavelength of the quanta and the condensate is at the critical point. Every time a graviton is emitted, the potential well created by all the other gravitons will become a bit more shallow as the gravity is a bit weaker. However, the occupation number N is so high and the condensate is so packed, that a state with $N - 1$ gravitons is extremely similar to the previous one. Since all the properties of the state depend on N alone, the state is said to be self-similar to the previous one and the condensate keeps itself on the verge of the phase transition.

The depletion of a Bose-Einstein condensate is well understood and studied in quantum physics, since, in this context, there are always some particles with energies above the ground state, even at zero temperature. For black holes, because of 2.2.32, it can be said that the escape energy is just above the energy of the quanta, thus a continuous production of quanta with energies above this threshold takes place and does not stop because of self-similarity. The interactions that have the biggest probability of driving a particle to escape are $2 \to 2$ graviton-graviton interactions. The momentum transfer is very small in such a process and its rate is

$$\Gamma = \frac{\hbar}{\sqrt{N}\ell_\mathrm{p}} + \frac{1}{\ell_\mathrm{p}}\mathcal{O}(N^{-3/2}), \tag{2.2.33}$$

where a factor of $1/N^2$ from the interaction strength and a factor of N^2 for combinatorics can be simplified, leaving only the factor that depends on the energy of the process. The emission rate is found to be

$$\dot{M} = -\frac{\hbar}{N\ell_\mathrm{p}^2} \tag{2.2.34}$$

and the leakage law is

$$\dot{N} = -\frac{1}{\sqrt{N}\ell_\mathrm{p}} + \ell_\mathrm{p}^{-1}\mathcal{O}(N^{-3/2}) + \frac{1}{\ell_\mathrm{p}}\mathcal{O}(N^{-3/2}), \tag{2.2.35}$$

where the dot is understood as a time derivative. The physical effect described by these laws is the emission of one quantum of the condensate in a time $\Delta t = \sqrt{N}\ell_\mathrm{p}$, which leads to a half-life for the black hole of $\tau = N^{3/2}\ell_\mathrm{p}$. If a temperature is defined as

$$T = \frac{\hbar}{\sqrt{N}\ell_\mathrm{p}}, \tag{2.2.36}$$

the expression for the emission rate and the half-life correctly reproduce the thermal evaporation of black holes described by Hawking:

$$\dot{M} = -\frac{T^2}{\hbar} \tag{2.2.37}$$

and

$$\tau = \frac{\hbar^2}{T^3 G_\mathrm{N}}. \tag{2.2.38}$$

The striking peculiarity of these results is that they have been recovered without relying on any geometrical notion such as the presence of a horizon. Instead, they emerge naturally in the corpuscular model of black holes from quantum-mechanical effects. The purely thermal spectrum of Hawking radiation is recovered only in the semi-classical limit, which is the limit in which Hawking's calculation has been performed in the first place:

$$N \to \infty, \qquad \ell_{\mathrm{p}} \to 0. \tag{2.2.39}$$

In the language of BECs, the quantum condensate in this limit becomes just a collection of infinitely soft and non-interacting bosons. The quantum model of black holes claims that all the seemingly mysterious properties of a black hole are consequences of this treatment, which is inadequate at the most fundamental level.

The negative heat capacity of black holes is recovered in this approach to Hawking radiation because N diminishes due to the depletion effect. The spectrum of Hawking radiation is thermal up to $1/N$ corrections, but we need to remark that the emergence of thermality has nothing to do with the temperature of the condensate; its origin lies in the dependence from N.

The universality of N also accounts for an explanation of the origin of black hole entropy, which is understood in terms of the occupation number. The concept of entropy itself, interpreted as the number of micro-states that are available to the system without requiring a change in its macro-state, can only emerge as the number of quantum states in which the N constituent gravitons can exist. Due to the fact that gravitons self-interact, the number of states is found to scale exponentially with N, as will be clarified in the following. Thus, since

$$S = k_{\mathrm{B}} log \Omega, \tag{2.2.40}$$

where k_{B} is Boltzmann's constant and Ω is the number of micro-states, we recover

$$S \propto N. \tag{2.2.41}$$

Let us calculate in how many states the gravitons could exist: without interactions, this number would be just N^{α}, where the exponent only depends on the possible states of a single graviton. Because of the interactions, however, the wavefunction of the condensate is interpreted as the product of wavefunctions of distinguishable flavours. The concept of flavour is clarified in [7]: if in first approximation we consider

$$N_{\text{states}} = \prod_j \xi_j, \tag{2.2.42}$$

writing the total number of states for the gravitons as the product of the number of states for the individual flavors, we can estimate N_{flavours} for which $j = 1, 2 ... N_{\text{flavours}}$.

To understand what flavours are, unions are defined first: a subset of N_j constituent gravitons can form an union of wavelength $\lambda = \sqrt{N_\alpha}\ell_{\mathrm{p}}$ and energy $M_\alpha = \sqrt{N}\hbar/\ell_{\mathrm{p}}$. A set of unions that form a bound state with mass $M = \sqrt{\sum_a N_a}$ equal to that of the black hole is defined as a flavour. At the leading order, the condition to be satisfied is $\Sigma_a N_a = N$. Of course, when the number of unions is of order one, $N_a \sim N$, so

$N_{\text{flavour}} \propto N$. The wavefunction of the black hole is thus, at least to the first order of the $1/N$ expansion, a product of one-flavour states that do not interact with each other and act like one-particle non-interacting states, characterized by a degeneracy ξ_j:

$$\Psi_{\text{BH}} = \prod_{j}^{N_{\text{flavours}}} \psi_j. \tag{2.2.43}$$

The number of states scales as

$$N_{\text{states}} \sim \prod_{j}^{N} \xi = \xi^N, \tag{2.2.44}$$

and therefore, the entropy scales as N because of (2.2.40).

In [6], another aspect of entropy is analyzed, namely its connection to the concept of information storage in black holes, while the identical physics underlying both corpuscular black holes and typical condensed matter BECs is uncovered. We cannot deal with this interesting topic in detail for the sake of brevity, but we will provide a brief explanation of how the corpuscular model and the universality of N attempt to provide a quantum foundation for holography.

In the semi-classical limit, in which $N \to \infty, \quad \ell_{\text{p}} \to 0, \quad L = \sqrt{N}\ell_{\text{p}} = \text{finite}, \quad \hbar =$ finite, the origin of the so-called holographic degrees of freedom that account for information storage in black holes is mysterious, but in the quantum picture they are explained as collective quantum excitations with a very low mass gap of the graviton condensate, which find an exact counterpart in the Bogolyubov excitations that occur in matter BECs (see Appendix A). The difference for the graviton BEC is that such excitations are almost degenerate, since they have a mass gap of $1/N$ with very large N, and are precisely the degrees of freedom called flavours in [7]. Because of their intrinsically quantum nature, such degrees of freedom are impossible to recover in the semi-classical limit: this is the reason why, in this limit, it appears as if black holes could store an infinite amount of information without ever releasing it. The Bekenstein entropy is an intrinsically quantum quantity because it is proportional to the Planck length 2.2.31. Technically, in the semi-classical limit the Planck length goes to zero, so the entropy would diverge: this behaviour is expected and consistent with the fact that we are not dealing with a classical quantity.

However, an infinite entropy translates into an infinite number of micro-states. If we interpret micro-states as excitations around the black hole vacuum, it would mean that an infinite number of perturbations could be generated at no energy cost whatsoever, but since such infinite perturbations are localized in a finite size box they cannot obviously cost zero energy. This is precisely the reason why the semi-classical limit is inappropriate for the understanding of the origin of information storage in black holes. Switching to the quantum picture, however, the storage of a huge (but finite) amount of information is explained with the (almost-)degeneracy of the holographic degrees of freedom, which translates in the very low amount of energy that is needed for exciting them. The very important property of black hole entropy is that it scales like the area, not the volume. 't Hooft and Susskind have connected this property to the information content of black holes: all the information about the objects that have fallen into them might be completely contained in surface fluctuations of the event horizon. This is the central idea of the so-called Holographic Principle that seems to resolve the black hole information paradox in the context of String Theory.

Let us now qualitatively explain the concept of Bogolyubov modes. The Bogolyubov approximation in the context of BECs relies on taking $N \gg 1$ while $\hbar \neq 0$, and is tantamount to replacing the quantum annihilation and creation operators with a c-number, which is of course only possible because $[a_k, a_k^\dagger] = 1 \sim 0$, if we confront it with the extremely large N. We remark that this approximation is not valid at the critical point, but it is reasonable to apply it around it. If we only consider terms that are quadratic in the operators, a simplified Hamiltonian can be found through a so-called Bogolyubov transformation, that leads to a diagonal hamiltonian, modulo a constant. The spectrum of this hamiltonian is quadratic and it is interpreted as the spectrum of quasi-particles called bogolons that emerge after the transformation. The ground state of the interacting system corresponds to the vacuum of quasi-particles. Exactly at the critical point, the energy vanishes and the system undergoes a phase transition, after which the gap between the ground state and the first excited state of the Bogolyubov modes collapses, because it goes like $1/N$ and $N \gg 1$. Therefore, it becomes energetically inexpensive to excite these modes. Such concepts are explained in a more quantitative way in Appendix A.

Chapter 3

Corpuscular gravitational potential

The goal of this chapter is to present the main results that constitute the starting point of this thesis, based on of [8] and [9]. These references attempt to formalize the corpuscular model of black holes, by further connecting it to features of General Relativity.

More specifically, the role of matter in the gravitational collapse was not addressed in [7] and [6], so that the black holes are considered to be in the final state of their collapse, in which the gravitational contribution has become dominant with respect to the matter that initially started the process. Such a picture seems in agreement with the result from Bekenstein that displays the enormous amount of gravitational entropy exhibited by astrophysical black holes [35].

However, the role of matter is certainly crucial since the gravitational collapse of a star in the end-phase of its evolution is the only known process that can lead to the formation of a black hole in the first place, if we exclude the possibility of formation of primordial black holes from a vacuum fluctuation in the very early Universe. This aspect is thoroughly considered in the references this thesis is based on, and its analysis eventually leads to an effective field theory for scalar toy gravitons, in the framework of the previously-mentioned weak-field limit of GR. In this context, the post-Newtonian corrections to the Newtonian potential (at the next-to-leading order in the expansion) can be computed. The same analysis is then replicated in a quantum uplifting of the model, in which the classical solution is reproduced by a coherent state in which the modes corresponding to different momenta k are superposed onto each other. The requirement that the source resides inside a finite volume naturally allows, in first approximation, only a single mode $k_c \sim R$. This result is undoubtedly in agreement with the key feature of the corpuscular model, that sees the black hole as made of a large number of bosons in the same mode, the defining property of BECs.

3.1 Scalar toy gravitons

We now want to briefly provide some justification and motivation for the scalar toy graviton model [37], starting from the Klein-Gordon equation for a real and massless scalar field $\phi(x)$ coupled to a scalar current $J(x)$

$$\Box\phi(x) = qJ(x), \tag{3.1.1}$$

where $\Box = \eta_{\mu\nu}\partial^\mu\partial^\nu$, the scalar field has the standard dimension of length^{-1} and the coupling q is dimensionless. The current is taken to be independent from time and spherical symmetry is assumed, so that $f(\mathbf{x}) = f(r)$ and $r = |\mathbf{x}|$.

In momentum space, this translates to

$$\tilde{f}(k) = 4\pi\int_0^\infty r^2 \mathrm{d}r\, j_0(kr) f(r), \tag{3.1.2}$$

where $j_0(kr) = \sin(kr)/kr$ is a spherical Bessel function of the first kind and $k = |\mathbf{k}|$. The formal classical solutions of the K-G equation in spherical symmetry can be formally expressed as

$$\phi(r) = q\Box^{-1}J(r), \tag{3.1.3}$$

and in momentum space they read

$$\tilde{\phi}(k) = -q\frac{\tilde{J}(k)}{k^2}. \tag{3.1.4}$$

In the simple case of a current with Gaussian support, we can show that the result asymptotically reproduces the Newtonian potential. Indeed, if we consider

$$J(r) = \frac{e^{-r^2/(2\sigma^2)}}{(2\pi\sigma^2)^{3/2}}, \tag{3.1.5}$$

so that

$$\tilde{J}(k) = e^{-k^2\sigma^2/2}, \tag{3.1.6}$$

the classical solution is

$$\phi(r) = -\frac{q}{2\pi^2}\int_0^\infty \mathrm{d}k\, j_0(kr)e^{-k^2\sigma^2/2} = -\frac{q}{2\pi^2}\,\mathrm{erf}\left(\frac{r}{\sqrt{2}\sigma}\right), \tag{3.1.7}$$

where erf is the error function. It is straightforward to see that, for $r \gg \sigma$, outside the source J, the Newtonian potential V_N is reproduced by the solution:

$$V_\mathrm{N} = \frac{4\pi}{q}G_\mathrm{N}M\phi \simeq -\frac{G_\mathrm{N}M}{r}, \tag{3.1.8}$$

where the coupling q now has the proper required dimensions.

Moving on to the quantum model, we now want to show in fairly general terms how the classical configurations can be successfully recovered though the use of coherent states.

Let us pause for a moment to consider the condensate that constitutes the black hole in the corpuscular model and view it from a different angle, that of matter physics. In this perspective, employing a coherent state is tantamount to considering a condensate in which the ground state is not the true vacuum $|0\rangle$, but a non-zero state $|g\rangle$. Without going into details, we simply mention that non-ground-state condensates are actively researched in the matter physics community [38].

Back to our general argument about coherent states, we can write the normal-ordered quantum Hamiltonian density for the scalar field as

$$\hat{\mathcal{H}} = k\hat{a}_k^{\prime\dagger}\hat{a}_k^{\prime} + \tilde{\mathcal{H}}_g, \tag{3.1.9}$$

where $\tilde{\mathcal{H}}_g$ is the ground-state energy density and it reads

$$\tilde{\mathcal{H}}_g = -q^2\frac{|\tilde{J}(k)|^2}{2k^2}. \tag{3.1.10}$$

and the standard creation and destruction operators are shifted through a Bogolyubov shift (see A)

$$\hat{a}_k^{\prime} = \hat{a}_k + q\frac{\tilde{J}(k)}{\sqrt{2k^3}}. \tag{3.1.11}$$

Coherent states are eigenstates of the annihilation operator, and in our case we find

$$\hat{a}_k\,|g\rangle = -q\frac{\tilde{J}(k)}{\sqrt{2k^3}}\,|g\rangle = g(k)\,|g\rangle \tag{3.1.12}$$

and

$$\hat{a}_k^{\prime}\,|g\rangle = 0, \tag{3.1.13}$$

$|g\rangle$ being the source-dependent coherent ground state. Therefore, $g(k)$ is an eigenvalue of the shifted annihilation operator, so that

$$|g\rangle = e^{-N/2}\exp\left\{\int\frac{k^2\mathrm{d}k}{2\pi^2}g(k)\hat{a}_k^\dagger\right\}|0\rangle\,, \tag{3.1.14}$$

and N represents the expectation value of the number of quanta in the coherent state:

$$N = \int\frac{k^2\,\mathrm{d}k}{2\pi^2}\,\langle g|\,\hat{a}_k^\dagger\hat{a}_k\,|g\rangle = \int\frac{k^2\,\mathrm{d}k}{2\pi^2}|g(k)|^2 = \frac{q^2}{(2\pi)^2}\int\frac{\mathrm{d}k}{k}|\tilde{J}(k)|^2. \tag{3.1.15}$$

This expression allows us to read the occupation number as

$$n_k = \left(\frac{q}{2\pi}\right)^2 \frac{|\tilde{J}(k)|^2}{k}. \tag{3.1.16}$$

The expectation value of the field in the coherent state recovers its classical value

$$\langle g| \hat{\phi}_k |g\rangle = \frac{1}{\sqrt{2k}} \langle g| (\hat{a}_k + \hat{a}_k^\dagger) |g\rangle = \frac{1}{\sqrt{2k}} \langle g| (\hat{a}'_k + \hat{a}'^\dagger_{-k}) |g\rangle - \frac{\tilde{J}(k)}{k^2} = \tilde{\phi}(k), \tag{3.1.17}$$

thus successfully showing the relevancy that the scalar toy graviton model that will be used throughout this work.

We also note, *en passant*, that another point remains unclear in the original references of the corpuscular model: the potential presence in the model of a horizon or trapping surface, which is of course one of the key features of black holes as we have come to know them. This aspect will not be addressed in this thesis, but has been analyzed by means of the Horizon Quantum Mechanics (HQM) formalism, for example in [37] and [39]. One of the results is the presence of an uncertainty in the horizon radius, that is shown to relate to the depletion effect of the condensate explained in terms of Hawking radiation.

3.2 Gravitational collapse

The aim of this section is to dig deeper into the role of matter in the gravitational collapse that leads to the formation of a black hole, which is effectively seen, in the corpuscular picture, as a gravitational well generated by a large number of superposed gravitons, whose depth is proportional to this number N.

Let us start by considering the Newtonian potential at distance r generated by a system of N gravitons, each of them possessing an effective mass m (the total mass is $M = Nm$ and in Planck units, if $c = 1$, $G_\mathrm{N} = \ell_\mathrm{p}/m_\mathrm{p}$)

$$V_\mathrm{N}(r) \simeq \frac{G_\mathrm{N} M}{r} = \frac{\ell_\mathrm{p} N m}{r m_\mathrm{p}}. \tag{3.2.18}$$

The gravitons that generate it are confined inside a finite spherical volume, up to the point that they are superposed onto each other, and it is precisely this confinement that generates an effective mass for the massless gravitons, due to the fact that the Compton wavelength of the particles is

$$\lambda \simeq \frac{\hbar}{m} = \ell_\mathrm{p} \frac{m_\mathrm{p}}{m} \tag{3.2.19}$$

and the confinement necessarily requires $\lambda < R$, where R is the radius of the spherical volume. We remark that confinement within a finite volume is also a crucial requirement for classicalization (see Chapter 2).

If we make the reasonable assumption that the gravitational interaction is constant within the "ball" that the gravitons constitute and negligible outside, taking $r = \lambda$ as the average interaction distance, the potential becomes

$$V_{\mathrm{N}}(r) \simeq -\frac{G_{\mathrm{N}} M}{\lambda} \Theta(\lambda - r) = V_{\mathrm{N}}(\lambda), \tag{3.2.20}$$

where Θ is the Heaviside step function, giving an average potential energy per graviton

$$U \simeq m V_{\mathrm{N}}(\lambda) = -N \alpha\, m, \tag{3.2.21}$$

where

$$\alpha = \frac{\ell_{\mathrm{p}}^2}{\lambda^2} = \frac{m^2}{m_{\mathrm{p}}^2} \tag{3.2.22}$$

is the gravitational coupling.

The formation of the black hole during the gravitational collapse can be thought of the simplified process in which the gravitons are superposed one by one and they progressively strengthen their reciprocal attraction, until they become confined inside a potential well deep enough that they cannot escape anymore.

If we now consider a lump of baryonic matter (that can be thought of a star) of density

$$\rho = M J, \tag{3.2.23}$$

where M is its total energy or mass, the Newtonian potential energy is $U_M = M V_{\mathrm{N}}$, and the scalar toy graviton ϕ in (3.1.1) is determined by (3.1.8) and the quantum state $\hat{\phi}$ by (3.1.17). We remark that the source $J(r)$ is considered as composed by gravitons only, under the assumption that gravity dominates the dynamics over matter. It is required that the energy density (3.2.23) that constitutes the source is the opposite of the total potential energy $N U_m / V$, where $V = 4\pi R^3/3$ is the volume in which the source is confined. This assumption is clearly translatable into the marginally bound condition $E_{\mathrm{K}} + U_m \simeq 0$. If we further take $N E_{\mathrm{K}} \sim J$, we obtain

$$J \simeq -\frac{3 N G_{\mathrm{N}} m}{q R^3} \phi, \tag{3.2.24}$$

in which m is the energy of each graviton. The substitution of this expression into (3.1.4) yields

$$\frac{3 N G_{\mathrm{N}} m}{R^3 k^2} = \frac{3 R_{\mathrm{S}}}{2 R^3 k^2} \simeq 1, \tag{3.2.25}$$

where $R_{\mathrm{S}} = 2 G_{\mathrm{N}} M$ is of course the Schwarzschild radius of the object. This results suggests that a self-sustained system of the type we are considering should only contain modes with wave numbers $k = k_c$, such that

$$Rk_c \simeq \sqrt{\frac{R_S}{R}}, \tag{3.2.26}$$

where a numerical coefficient has been neglected due to the qualitative nature of this description. The result that only one mode exists in the system might seem to be conflicting with the coherent state formulation presented previously, but it has to be kept in mind that, for $N \gg 1$, all scalars are considered to be in the same state $|k_c\rangle$, in agreement with the basic property of BECs, that the macroscopic number of bosons that condensate are all in the same quantum state. Further arguments in favour of the identification of this state with a BEC are given in e.g. [40].

Considering an ordinary star with typical radius $R \gg R_S$ and $k_c \ll R^{-1}$, only in the limit $R \sim R_S$ we obtain, due to $m = \hbar k$ and neglecting a numerical coefficient,

$$1 \simeq G_N M k_c = N \frac{m^2}{m_p^2}, \tag{3.2.27}$$

which leads to

$$m = \hbar k \simeq \frac{m_p}{\sqrt{N}} \qquad M \simeq Nm \simeq \sqrt{N} m_p \tag{3.2.28}$$

and $\lambda \simeq R_S$, that recover the central relations of the corpuscular model of black holes presented in Chapter 2.

In [8], the authors consider the effect of gravitons in the collapsing process of baryonic matter during the stages of formation of a black hole. The gravitational field is understood as being produced by gravitons that are soft (i.e. low-energy) and off-shell, coupled to the matter source by means of a coherent state. Moreover, the only gravitons considered are those with a wavelength comparable to the size of the collapsing body. This simple framework succeeds in recovering the Bekenstein area entropy law of black holes and provides a "post-Newtonian" correction to the total energy of the system, in a sense that will be clarified later on.

The starting point is a simple model of a spherically symmetric compact object of radius R, made out of N_B identical baryons with rest mass μ. The total energy of the system is the Arnowitt-Deser-Misner mass M (see 1.3) and its conservation is granted in GR by the Hamiltonian constraint, related to the invariance under time reparametrizations, that provides intrinsic, coordinate dependent dynamics of the gravitational field, as clarified in [41] and [21].

Considering an asymptotically flat space, the constraint is simply

$$H \equiv H_B + H_G = M, \tag{3.2.29}$$

where H_{B} and H_{G} are the Hamiltonians of matter and gravity respectively. During the gravitational collapse, R shrinks down all the way to the Schwarzschild radius R_{S}, starting from a configuration in which all the baryons are far apart from each other, so that no mutual influence is present:

$$H = E_{\mathrm{B}} \equiv \mu N_{\mathrm{B}} \simeq M. \tag{3.2.30}$$

It is important to point out that three more contributions to the energy emerge during the collapse. Firstly, a negative contribution to the interaction potential energy between baryons, $U_{\mathrm{BG}}(R)$, which is due to the gravitational attraction and is thus mediated by gravitons and grows as the radius shrinks. Secondly, a positive contribution to the kinetic energy K_{B} comes from the collapse and thirdly, a positive repulsive term U_{BB} accounts for the baryonic pressure. The energy is thus

$$E_{\mathrm{B}}(R) = M + K_{\mathrm{B}}(R) + U_{\mathrm{BG}}(R) + U_{\mathrm{BB}}(R). \tag{3.2.31}$$

In order to provide a consistent description, the quantum features of the gravitational interaction need to be taken into account, since baryons themselves are quantum. The bridge that connects the quantum model to the classical one is provided, of course, by the use of coherent states, that have minimum uncertainty: the classical gravitational field is described by a coherent state of virtual gravitons, as elaborated in [32]; likewise, the Coulomb field around a static charge can be reproduced by a coherent state of virtual photons [33].

The classical Newtonian field ϕ_{N} satisfies the Poisson equation, which in momentum space reads:

$$k^2 \phi_{\mathrm{N}}(k) = -\frac{M}{m_{\mathrm{p}}} j(k), \tag{3.2.32}$$

where $j(k)$ is the Fourier-transformed static source, that satisfies $\int_0^R r^2 \mathrm{d}r\, j(r) = 1$, and k is the dimensionless wave number. Performing the expansion of the graviton field operator in radial modes $\hat{\phi}_k \simeq (\hat{g}_k + \hat{g}^\dagger_{-k})/\sqrt{k}$, we can straightforwardly move on to the coherent state formulation.

Recalling the definition of a coherent state as an eigenstate of the quantum annihilation operator:

$$\hat{g}_k \left| g \right\rangle = g(k) \left| g \right\rangle , \tag{3.2.33}$$

we exploit the freedom to choose

$$g(k) \simeq -\frac{M j(k)}{m_{\mathrm{p}} k^{3/2}}, \tag{3.2.34}$$

in order to reproduce the classical field, since

$$\langle g|\, \hat{\phi}_k \,|g\rangle \simeq -\frac{M j(k)}{m_{\mathrm{p}} k^2} \simeq \phi_{\mathrm{N}}(k). \tag{3.2.35}$$

The expectation value of the graviton number scales just like Bekenstein's area law:

$$N_{\mathrm{G}} = \int k^2 \mathrm{d}k \,\langle g|\, (\hat{g}_k)^\dagger \hat{g}_k \,|g\rangle \simeq \frac{M^2}{m_{\mathrm{p}}^2} \int \mathrm{d}k\, \frac{j^2(k)}{k} \simeq \frac{M^2}{m_{\mathrm{p}}^2} \sim \frac{R_{\mathrm{S}}^2}{\ell_{\mathrm{p}}^2}, \tag{3.2.36}$$

where $R_{\mathrm{S}} = 2\ell_{\mathrm{p}} M/m_{\mathrm{p}}$ is the Schwarzschild radius in Planck units.

The typical graviton energy is

$$\epsilon_{\mathrm{G}} \simeq -\frac{\ell_{\mathrm{p}}}{R} m_{\mathrm{p}}, \tag{3.2.37}$$

which is determined by the typical length r_m of the coherent state, derived from the de Broglie relation

$$r_m \simeq \lambda_m \simeq \frac{\hbar}{m} = \ell_{\mathrm{p}} \frac{m_{\mathrm{p}}}{m}, \tag{3.2.38}$$

where m is the effective graviton mass. It is important to note the negative sign in (3.2.37): it is in accordance with the sign of gravitational interactions in (3.2.31) and with the negative non-relativistic Newtonian energy, but it also entails that the gravitons are off-shell. In addition, as it ought to, it grows in modulus as the radius diminishes. Moreover, since we are considering a constant M, the number of gravitons N_{G} ends up being conserved like the number of baryons N_{B} is. If instead we consider the interaction energy between gravitons,

$$U_{\mathrm{GG}}(R) \simeq N_{\mathrm{G}} \epsilon_{\mathrm{G}}(R) \phi_{\mathrm{N}}(R) \simeq N_{\mathrm{G}} \frac{M \ell_{\mathrm{p}}^2}{R^2}, \tag{3.2.39}$$

we can notice its positivity and its dependency $1/R^2$. These two properties allow the identification of U_{GG} with the standard post-Newtonian correction to the Newtonian potential. Its contribution, with respect to that of U_{BG}, is negligible for a small star because of

$$\left| \frac{U_{\mathrm{GG}}}{U_{\mathrm{BG}}} \right| \simeq \frac{R_{\mathrm{S}}}{R} \ll 1, \tag{3.2.40}$$

but starts to dominate when $R \simeq R_{\mathrm{S}}$, in the final stages of formation of the black hole. The condition (3.2.31) is thus updated to

$$M = E_{\mathrm{B}} + U_{\mathrm{GG}} = M + K_{\mathrm{B}}(R) + U_{\mathrm{BB}}(R) + U_{\mathrm{BG}}(R) + U_{\mathrm{GG}}(R). \tag{3.2.41}$$

If we consider the contraction of the system down to the Schwarzschild radius, the energy becomes

$$U_{\mathrm{GG}}(R_{\mathrm{S}}) \simeq -U_{\mathrm{BG}}(R_{\mathrm{S}}) \simeq M, \tag{3.2.42}$$

which can be interpreted exactly as one of the crucial features exhibited by the corpuscular model of black holes: the marginal bound condition found in [6] as

$$\alpha N_{\mathrm{G}} \simeq 1, \tag{3.2.43}$$

where $\alpha \simeq \epsilon_{\mathrm{G}}^2/m_{\mathrm{p}}^2$ is the coupling for the self-interaction among gravitons. In the limit $R \simeq R_{\mathrm{S}}$, the expression for the graviton mass found in the corpuscular model of black holes is recovered

$$m = -\epsilon_{\mathrm{G}} \simeq \frac{m_{\mathrm{p}}}{\sqrt{N_{\mathrm{G}}}} \simeq \frac{M}{N_{\mathrm{G}}}. \tag{3.2.44}$$

It is hypothesized, both in the corpuscular model of Dvali and Gomez and [8], that the effective number of soft gravitons in th black hole is much larger than the number of baryons, so that the gravitational contribution is starkly dominant and, in fact, the baryonic component is completely neglected in the original corpuscular model. This result is in agreement with [32], where the gravitational field around a mass M is

$$N = \frac{4\pi G_{\mathrm{N}} M^2}{4\pi\hbar} = \frac{M^2}{m_{\mathrm{p}}^2}. \tag{3.2.45}$$

Additionally, the depletion effect of the condensate, which is interpreted as Hawking radiation flux in the corpuscular model and is originated from graviton-graviton scatterings, is here recovered with an additional contribution given by baryon-graviton scatterings. The depletion law is

$$\dot{N}_{\mathrm{G}} \simeq -N_{\mathrm{G}}^2 \frac{1}{N_{\mathrm{G}}^2} \frac{1}{\ell_{\mathrm{p}}\sqrt{N_{\mathrm{G}}}} - N_{\mathrm{G}} N_{\mathrm{B}} \frac{1}{N_{\mathrm{G}}^2} \frac{1}{\ell_{\mathrm{p}}\sqrt{N_{\mathrm{G}}}} \simeq -\frac{1}{\ell_{\mathrm{p}}\sqrt{N_{\mathrm{G}}}} \left(1 + \frac{N_{\mathrm{B}}}{N_{\mathrm{G}}}\right), \tag{3.2.46}$$

where three factors are taken into account: the graviton and baryon multiplicity, the gravitational coupling α^2 and the typical energy of the process, seen as m. Of course, such a flux is negligible in objects of astrophysical size with $M \gg m_{\mathrm{p}}$ and thus $N_{\mathrm{G}} \gg N_{\mathrm{B}} \gg 1$.

3.3 Post-Newtonian corrections

The core ideas we have just presented were formalized and refined in [9], where a connection between the corpuscular model of black holes and post-Newtonian gravity is established.

The authors employ a "Newtonian-like" description of a simple static and isotropic gravitational compact source, that leads to an effective field theory with quantum corrections for the Newtonian potential. The corrections are found under the conditions of weak gravitational field and non-relativistic speeds for test particles, which justifies their identification as post-Newtonian. It is precisely the inclusion of matter terms, neglected in the original papers on the corpuscular model, that allows to draw the connection with the post-Newtonian approximation.

It is worth recalling from Section 1.1 that the term "post-Newtonian" represents the first non-linear order after the Newtonian zeroth-order, thus it accounts for the self-interaction of the gravitational field, and, in the quantum theory, for the self-interactions of gravitons.

The starting point is the derivation of an effective action for a static and spherically symmetric potential in the weak field and non-relativistic approximation, in which the higher order terms reproduce the post-Newtonian expansion of the Schwarzschild metric. The coherent state formalism is then used, as usual, to bridge the microscopic dynamics of gravity with the macroscopic description in curved spacetime. More precisely, the quantum state of the gravitational potential is identified with a coherent state of virtual soft gravitons.

One of the basic assumptions is that the "toy" gravitons that are studied in this context can be represented by a scalar field, with the goal of simplifying the treatment with respect to regular gravitons, represented by a spin-2 field. It is shown in the paper that a scalar field can also be used in order to describe the post-Newtonian correction that appears in the weak-field expansion of the Schwarzschild metric (the choice of a specific reference frame for static observers is assumed).

Starting from the action (1.1.3) coupled to matter and performing the weak-field limit (1.1.6), the well-known linearised Einstein field equations can be derived, as we have presented in Section 1.1. In order to adapt to the notation used in [9], we switch to the metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ and rewrite the Einstein equations for convenience

$$-\Box h_{\mu\nu} + \eta_{\mu\nu}\Box h + \partial_\mu \partial^\lambda h_{\lambda\nu} + \partial_\nu \partial^\lambda h_{\lambda\mu} - \eta_{\mu\nu}\partial^\lambda \partial^\rho h_{\lambda\rho} - \partial_\mu \partial_\nu h = 16\pi \frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}} T_{\mu\nu}. \tag{3.3.47}$$

Using the de Donder gauge,

$$2\partial^\mu h_{\mu\nu} = \partial_\nu h, \tag{3.3.48}$$

the equations become

$$-\Box h_{\mu\nu} = 16\pi \frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}} \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right), \tag{3.3.49}$$

with $T = \eta^{\mu\nu} T_{\mu\nu}$ being the trace of the energy-momentum tensor.

In addition to the weak-field limit, it is also assumed that the characteristic velocity of the system is $v/c \ll 1$, so that we are in the so-called slow-motion regime, which is a crucial requirement for the post-Newtonian approximation, since the latter is precisely a perturbative expansion that in the parameter v/c, as it is clarified in Section 1.1 This allows the stress-energy tensor to be determined by the energy density component alone,

$$T_{\mu\nu} = 2\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_{\mathrm{M}} \simeq u_\mu u_\nu \rho(r), \tag{3.3.50}$$

where $u^\mu = \delta_0^\mu$ is the four-velocity of the static source fluid and the expression above follows from the simple Lagrangian density

$$\mathcal{L}_{\mathrm{M}} \simeq -\rho(r). \tag{3.3.51}$$

Consequently, equations (3.3.49) take the form

$$\triangle h_{00} = -8\pi \frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}}\rho(r), \tag{3.3.52}$$

and we can identify the Newtonian potential

$$h_{00} = -2V_{\mathrm{N}}, \tag{3.3.53}$$

that satisfies

$$\triangle V_{\mathrm{N}} = 4\pi \frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}}\rho. \tag{3.3.54}$$

Additionally, we restrict the treatment to static spherically symmetric systems, which means that $\rho = \rho(r)$ and $V_{\mathrm{N}} = V_{\mathrm{N}}(r)$.

The last important modification is the replacement of the Einstein-Hilbert action with the massless Fierz-Pauli action. It is well-known that, beyond the linear order, the construction of an effective theory from the Einstein-Hilbert action coupled to matter suffers from inconsistencies [42], [43]. Therefore, in order to construct the NLO, where the first non-linearities begin to appear, an alternative action has to be employed. The basic reason is that, if we want to derive the Einstein equations not in a geometric way but from a field-theoretical point of view, we can note that the free massless spin-2 field equations, whose source is the matter stress-tensor $T_{\mu\nu}$, must actually be coupled to the *total* stress-tensor, including that of the gravitational field itself. This is of course due to the non-linear nature of gravity, that emerges whenever we include all the higher orders to take into account the self interaction of $h_{\mu\nu}$. The free-field equations are consistent as they stand, but this is no longer the case when the source is a dynamical system's energy-momentum tensor, since the latter would no longer be conserved.

With the aim of briefly sketching how the Fierz-Pauli action is presented in [9], we can perform the weak-field expansion in a slightly different form

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{3.3.55}$$

in order to keep track of the different orders of the expansion by means of ϵ. The effective Lagrangian for the classical Newtonian field that we are looking for is the sum of two terms

$$L[V_{\mathrm{N}}] = \epsilon^2 L_{\mathrm{FP}} + \epsilon L_{\mathrm{M}}, \tag{3.3.56}$$

in which the gravitational part is given by the massless Fierz-Pauli action

$$L_{\mathrm{FP}} = \frac{m_{\mathrm{p}}}{16\pi\ell_{\mathrm{p}}}\int \mathrm{d}^3x \left(\frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\mu h_{\nu\sigma}\partial^\mu h^{\nu\sigma} + \partial_\mu h_{\nu\sigma}\partial^\nu h^{\mu\sigma} - \partial_\mu h\partial_\sigma h^{\mu\sigma}\right) \tag{3.3.57}$$

that becomes

$$\begin{aligned} L_{\mathrm{FP}} &= \frac{m_{\mathrm{p}}}{16\pi\ell_{\mathrm{p}}}\int \mathrm{d}^3x \left(\partial_\mu h_{\nu\sigma}\partial^\nu h^{\mu\sigma} - \frac{1}{2}\partial_\mu h_{\nu\sigma}\partial^\mu h^{\nu\sigma}\right) \\ &\simeq -\frac{m_{\mathrm{p}}}{32\pi\ell_{\mathrm{p}}}\int \mathrm{d}^3x\, \partial_\mu h_{00}\partial^\mu h^{00} \\ &= -4\pi\int_0^\infty r^2\mathrm{d}r\frac{m_{\mathrm{p}}}{8\pi\ell_{\mathrm{p}}}(V_{\mathrm{N}}')^2, \end{aligned} \tag{3.3.58}$$

if we use $h_{00} = 2V_{\mathrm{N}}$

From the matter Lagrangian density (3.3.51), we obtain

$$L_{\mathrm{M}} \simeq 4\pi\int_0^\infty r^2\mathrm{d}r\,\frac{h_{00}}{2}\rho = -4\pi\int_0^\infty r^2\mathrm{d}r\, V_{\mathrm{N}}\rho \tag{3.3.59}$$

and summing the two contributions, we obtain

$$\begin{aligned} L[V_{\mathrm{N}}] &\simeq 4\pi\int_0^\infty r^2\mathrm{d}r\left(\frac{m_{\mathrm{p}}}{32\pi\ell_{\mathrm{p}}}h_{00}\triangle h_{00} + \frac{h_{00}}{2}\rho\right) \\ &= 4\pi\int_0^\infty r^2\mathrm{d}r\left(\frac{m_{\mathrm{p}}}{8\pi\ell_{\mathrm{p}}}V_{\mathrm{N}}\triangle V_{\mathrm{N}} - \rho V_{\mathrm{N}}\right) \\ &= -4\pi\int_0^\infty r^2\mathrm{d}r\left(\frac{m_{\mathrm{p}}}{8\pi\ell_{\mathrm{p}}}(V_{\mathrm{N}}')^2 + \rho V_{\mathrm{N}}\right), \end{aligned} \tag{3.3.60}$$

where an integration by parts has been performed.

We now want to move to the NLO order. The Hamiltonian can be easily obtained and it reads

$$H[V_{\rm N}] = -L[V_{\rm N}] = 4\pi \int_0^\infty r^2 {\rm d}r \left(-\frac{m_{\rm p}}{8\pi \ell_{\rm p}} V_{\rm N} \triangle V_{\rm N} + \rho V_{\rm N} \right), \tag{3.3.61}$$

which leads to the potential energy

$$U_{\rm N}(r) = -\frac{m_{\rm p}}{2\ell_{\rm p}} \int_0^r \tilde{r}^2 {\rm d}\tilde{r} [V_{\rm N}'(\tilde{r})]^2, \tag{3.3.62}$$

by using (3.3.54). Following [8], a self-gravitational source $J_{\rm V}$ given by the potential energy $U_{\rm N}$ per unit volume can be defined:

$$J_{\rm V}(r) = \frac{1}{4\pi r^2} \frac{\rm d}{{\rm d}r} U_{\rm N}(r) = -\frac{m_{\rm p}}{8\pi \ell_{\rm p}} [V_{\rm N}'(r)]^2. \tag{3.3.63}$$

Such a current can only be found at the next-to-leading order (NLO) in the expansion of the weak-field limit and comes from the geometric part of the Einstein-Hilbert action, while the term coming from the matter part is proportional to the leading order.

Including all the contributions, the total Lagrangian for a new field V is found:

$$L[V] = -4\pi \int_0^\infty r^2 {\rm d}r \left[\frac{m_{\rm p}}{8\pi \ell_{\rm p}} (1 - 4q_\Phi V)(V')^2 + q_{\rm B} V \rho (1 - 2q_\Phi V) \right], \tag{3.3.64}$$

where the parameters $q_{\rm B}$ and q_Φ keep up with the couplings of V with matter and with itself, respectively.

The Euler-Lagrange equations for V are

$$\frac{\delta L}{\delta V} - \frac{\rm d}{{\rm d}r} \left(\frac{\delta L}{\delta V'} \right) = 0, \tag{3.3.65}$$

since the only dependence is that on the radius. Their straightforward computation yields

$$(1 - 4q_\Phi V) \triangle V = 4\pi q_{\rm B} \frac{\ell_{\rm p}}{m_{\rm p}} \rho (1 - 4q_\Phi V) + 2q_\Phi (V')^2. \tag{3.3.66}$$

To allow for an analytic calculation, we can expand the field up to the first order in the self-coupling

$$V(r) = V_0(r) + q_\Phi V_1(r) \tag{3.3.67}$$

and solve each order separately. At the zeroth order, the solution of (3.3.66) is

$$\triangle V_0 = 4\pi \frac{\ell_{\rm p}}{m_{\rm p}}\rho, \tag{3.3.68}$$

which of course reproduces the Poisson equation (3.3.54) when $q_{\rm B} = 1$.

At the first order in the expansion, we obtain

$$\triangle V_1 = 2(V_0')^2. \tag{3.3.69}$$

The Hamiltonian for V reads

$$H[V] = -L[V] \simeq 2\pi \int_0^\infty r^2 {\rm d}r \left[q_{\rm B}\rho V(1-4q_\Phi V) - q_\Phi \frac{3m_{\rm p}}{2\pi\ell_{\rm p}} V(V')^2\right], \tag{3.3.70}$$

up to linear order in q_Φ. We remark that the graviton-graviton interaction energy $U_{\rm GG}$, with the same notation used in [8], now becomes

$$U_{\rm GG} = -3q_\Phi \frac{\ell_{\rm p}}{m_{\rm p}} \int_0^\infty r^2 {\rm d}r V_0 (V_0')^2 + \mathcal{O}(q_\Phi^2), \tag{3.3.71}$$

so that it is clear that the term $2q_\Phi(V')^2$ in the equations of motion (3.3.66) is the source of non-linearity.

The static and spherically symmetric classical solutions to (3.3.68) and (3.3.69), are then obtained in three specific cases: for a point-like source of mass M_0, in the case of a homogeneous mass distribution and for a Gaussian matter distribution. The classical solutions are found, under the assumption of static and spherically symmetric sources, by considering the eigenfunctions of the Laplace operator

$$\triangle j_0(kr) = -k^2 j_0(kr), \tag{3.3.72}$$

where the spherical Bessel functions of the first kind have been employed

$$j_0(kr) = \frac{\sin(kr)}{kr}. \tag{3.3.73}$$

The general solution at the zeroth order is

$$V_0(r) = -2q_{\rm B} \frac{\ell_{\rm p}}{m_{\rm p}} \int_0^\infty \frac{{\rm d}k}{\pi} j_0(kr)\tilde{\rho}(k), \tag{3.3.74}$$

where $\tilde{\rho}(k) = 4\pi \int_0^\infty r^2 {\rm d}r j_0(kr)\rho(r)$.

In the simplest case of a point-like source, the expected post-Newtonian correction is recovered, but this approximation obviously suffers from the divergence at small r.

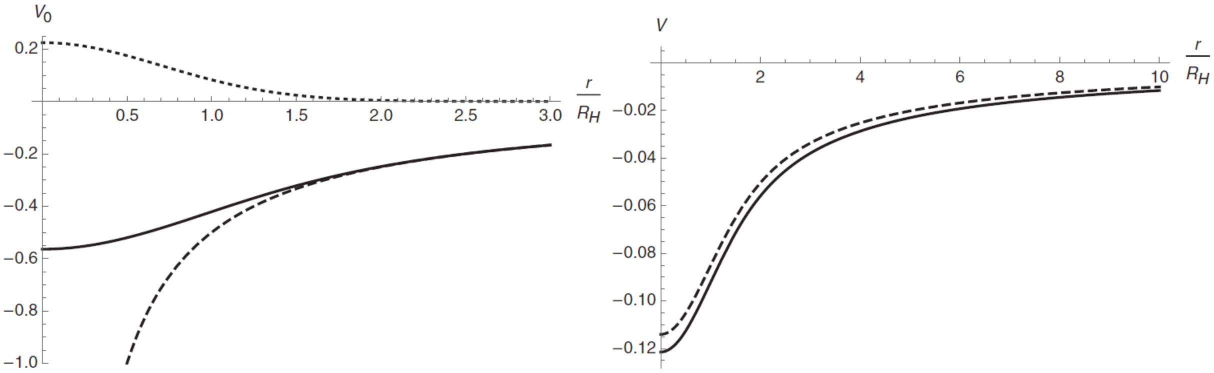


Figure 3.1: On the left, V_0 for a Gaussian matter distribution with $\sigma = 2\ell_\mathrm{p} M_0/m_\mathrm{p}$ (dotted line), shown against Newtonian potential (solid line) and V_0 for pointlike source of mass M_0 ($q_\mathrm{B} = 1$). On the right, full potential up to first order in q_Φ (solid line) vs Newtonian potential (dashed line) for Gaussian distribution with $\sigma = 2\ell_\mathrm{p} M_0/m_\mathrm{p} \equiv R_\mathrm{S}$ ($q_\mathrm{B} = q_\Phi = 1$).

Moreover, the maximal packing condition, a crucial feature of [6], [7], cannot be realized.

However, this feature is recovered in the case of a homogeneous matter distribution, in the limit $R \sim R_\mathrm{S}$, even though the precise value of the Schwarzschild radius falls outside the regime of validity of the approximations.

The best agreement among the three cases is unsurprisingly obtained for the most regular matter distribution considered, the Gaussian matter distribution of width σ,

$$\rho(r) = \frac{M_0 e^{-r^2/\sigma^2}}{\pi^{3/2}\sigma^3}, \tag{3.3.75}$$

where $M_0 = 4\pi \int_0^\infty r^2 \mathrm{d}r \rho(r)$.

Figure 3.1 shows how the potential generated from this distribution compares to the Newtonian potential and to the one generated by a point source, both for the zeroth order approximation V_0 and for the full potential V.

Quantum field and coherent ground state In order to perform a quantum uplifting of the model and reproduce the previous results in a quantum theory, the introduction of new, suitably rescaled quantities is necessary. The following notation is set now but will be relied upon throughout the next Chapter as well:

$$\Phi = \sqrt{\frac{m_\mathrm{p}}{\ell_\mathrm{p}}} V \qquad J_\mathrm{B} = 4\pi \sqrt{\frac{\ell_\mathrm{p}}{m_\mathrm{p}}} \rho. \tag{3.3.76}$$

After rescaling the whole Lagrangian (3.3.64) by a factor of 4π in order to have a canonically normalized term, we obtain the Lagrangian that will be our starting point in Chapter 4:

$$L[\Phi] = 4\pi \int_0^\infty r^2 \mathrm{d}r \left[\frac{1}{2}\Phi\Box\Phi - \mathrm{q_B J_B}\Phi \left(1 - 2\mathrm{q}_\Phi \sqrt{\frac{\mathrm{G}}{\mathrm{c}^3}}\Phi \right) + 2\mathrm{q}_\Phi \sqrt{\frac{\mathrm{G}}{\mathrm{c}^3}} (\partial_\mu \Phi)^2 \Phi \right] \quad (3.3.77)$$

The goal is now to define new quantum operators, starting from the free theory

$$\Box\Phi = 0, \quad (3.3.78)$$

where we still assume $\Phi = \Phi(t, r)$. Considering static and spherically symmetric states, we can again use the eigenfunctions of the Laplace operator (3.3.72) and define the modes $u_k(t, r) = j_0(kr)e^{i\omega t}$. The free dispersion relation is of course $\omega = k$ and the field can be rewritten in terms of the ladder operators

$$\hat{\Phi}(t, r) = \int_0^\infty \frac{k^2 \mathrm{d}k}{2\pi^2} \frac{\ell_\mathrm{p} m_\mathrm{p}}{2k} j_0(kr)(\hat{a}_k e^{ikt} + \hat{a}_k^\dagger e^{-ikt}). \quad (3.3.79)$$

If our aim is to find a quantum state $|g\rangle$ of Φ that will reproduce the classical solution, we shall first consider the Newtonian case, for $q_\Phi = 0$ and rewrite equation (3.3.68) in terms of Φ_c (where the subscript indicates that we are dealing with a classical quantity) and the new current $J_\mathrm{B} = J_\mathrm{B}(r)$:

$$\triangle\Phi_\mathrm{c}(r) = q_\mathrm{B} J_\mathrm{B}(r), \quad (3.3.80)$$

which straightforwardly leads to

$$\tilde{\Phi}_c(k) = -q_\mathrm{B} \frac{\tilde{J}_\mathrm{B}(k)}{k^2} \quad (3.3.81)$$

in momentum space.

Upon a coherent state of unit norm

$$\hat{a_k} |g\rangle = e^{i\gamma_k t} |g\rangle\,, \quad (3.3.82)$$

where we choose $\gamma_k = -kt$ for later convenience and where

$$g_k = \sqrt{\frac{k}{2\ell_\mathrm{p} m_\mathrm{p}}} \tilde{\Phi}_c(k) = -q_\mathrm{B} \frac{\tilde{J}_\mathrm{B}(k)}{\sqrt{2\ell_\mathrm{p} m_\mathrm{p} k^3}}, \quad (3.3.83)$$

the action of the field operator (3.3.79) yields the expectation value

$$
\begin{aligned}
\langle g|\,\hat{\Phi}(t,r)\,|g\rangle &= -q_{\rm B}\int_0^\infty \frac{k^2{\rm d}k}{2\pi^2}\,j_0(kr)\frac{\tilde{J}_{\rm B}(k)}{k^2} \\
&= \int_0^\infty \frac{k^2{\rm d}k}{2\pi^2}\,j_0(kr)\tilde{\Phi}_c(k) = \Phi_c(r),
\end{aligned}
\tag{3.3.84}
$$

precisely reproducing the classical solution of equation (3.3.80).

The coherent state can be written in terms of the true vacuum as

$$
|g\rangle = e^{-N_{\rm G}/2}\exp\left\{\int_0^\infty \frac{k^2{\rm d}k}{2\pi^2}\,g_k\hat{a}_k^\dagger\right\}|0\rangle \tag{3.3.85}
$$

where $N_{\rm G}$ is shown to reproduce the total occupation number of modes in the state $|g\rangle$

$$
N_{\rm G} = \int_0^\infty \frac{k^2{\rm d}k}{2\pi^2}\,g_k^2 = \langle g|\int_0^\infty \frac{k^2dk}{2\pi^2}\,\hat{a}_k{}^\dagger\hat{a}_k\,|g\rangle\,, \tag{3.3.86}
$$

which diverges for a point-like source, but is finite for an extended source of size R and mass M. Additionally, it has a slight dependence on the ratio R/R_∞ between the size of the source and the size of the region within which the gravitational potential is static, but a much stronger dependence on the mass M.

Post-Newtonian correction to the coherent state After having clarified that

$$
\sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\,\langle g|\,\hat{\Phi}(t,r)\,|g\rangle = V_{\rm N}(r) = V_0(r), \tag{3.3.87}
$$

we can move to the post-Newtonian NLO order through the definition of a modified coherent state $|g'\rangle$, such that

$$
\sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\,\langle g'|\,\hat{\Phi}\,|g'\rangle \simeq V_0 + q_\Phi V_1. \tag{3.3.88}
$$

This new coherent state can be written in a way that highlights the quantum corrections of the post-Newtonian order:

$$
|g'\rangle \simeq \mathcal{N}(|g\rangle + q_\Phi\,|\delta g\rangle), \tag{3.3.89}
$$

where $\mathcal{N}$ is a normalization constant and $\hat{a}_k\,|g'\rangle \simeq g_k\,|g\rangle + q_\Phi\delta g_k\,|\delta g\rangle$.

Finding an explicit expression for the perturbation of the state is quite hard in general, but it becomes easier if we make the reasonable assumption that most of the $N_{\rm G}$ gravitons are in one single mode of wavelength comparable to the size of the object $\lambda_{\rm G} \simeq R$, which is a crucial ingredient of the corpuscular model of black holes. By proceeding this way, the corrections obtained for the cases of the point-like source and the Gaussian distribution are, respectively:

$$\delta g_{k'} \sim \frac{R_{\mathrm{S}}}{r_0} g_{k'} \qquad \delta g_{k'} \sim \frac{R_{\mathrm{S}}}{\sigma} g_{k'} \tag{3.3.90}$$

where the characteristic size of the source is $R \sim r_0$, the ultraviolet cut-off needed to compute the diverging gravitational energy and $k' \simeq R^{-1}$. For $r_o \ll R_{\mathrm{S}}$, the result obviously falls out of the regime of validity of the approximation, but as $r_o \sim R_{\mathrm{S}}$, the correction becomes comparable to the Newtonian part and cannot be neglected. For the Gaussian source, instead, $k'^{-1} \simeq R \sim \sigma^{-1}$ is assumed, and it is easy to see that $\delta g_{k'} \ll g_{k'}$ when the source is much more extended than its gravitational radius, which is a quantum result perfectly consistent with the classical one.

Chapter 4

Dispersion relation

4.1 Recovering physical units

We want to rewrite the Lagrangian for the field Φ (3.3.77) in physical units instead of natural ones, thus reintroducing the fundamental constants $\hbar$, c and G. Of course, we have to identify the constants that appear from the beginning, in the theory for the Newtonian potential V_N. We start by introducing the relevant Planck units m_p, ℓ_p and t_p, which read

$$\ell_\mathrm{p} = \sqrt{\frac{\hbar G}{c^3}}\,, \qquad m_\mathrm{p} = \sqrt{\frac{\hbar c}{G}} \qquad \text{and} \qquad t_\mathrm{p} = \sqrt{\frac{\hbar G}{c^5}}\,. \tag{4.1.1}$$

These relations can be easily inverted, yielding

$$c = \frac{\ell_\mathrm{p}}{t_\mathrm{p}}\,, \qquad G = \frac{\ell_\mathrm{p}^3}{m_\mathrm{p} t_\mathrm{p}^2} \qquad \text{and} \qquad \hbar = \frac{\ell_\mathrm{p}^2 m_\mathrm{p}}{t_\mathrm{p}}\,. \tag{4.1.2}$$

4.1.1 The Newtonian potential V_N in physical units

In physical units, the Newtonian potential will, of course, not be adimensional anymore, as it was in [9], but has the dimensions of a squared velocity, $[V] = \frac{L^2}{T^2}$, as can be easily checked by considering e.g., the gravitational potential of a point mass. Making use of the Poisson equation,

$$\Delta V_\mathrm{N} = 4\pi G\,\rho = 4\pi \frac{\ell_\mathrm{p}^3}{m_\mathrm{p} t_\mathrm{p}^2}\,\rho, \tag{4.1.3}$$

we also find the expected dimensions of the classical mass density $[\rho] = L^{-2}[V]\,[G]^{-1} = M/L^3$. The Lagrangian and Hamiltonian for V_N (3.3.61) have been checked to have consistent dimensions.

We can now consider the dimensions of the current, representing the self-sourcing of gravitons, from its expression in natural units:

$$J_V = -\frac{1}{8\pi G}\left(V_{\mathrm{N}}'\right)^2 = -\frac{m_{\mathrm{p}}t_{\mathrm{p}}^2}{8\pi\ell_{\mathrm{p}}^3}\left(V_{\mathrm{N}}\right)^2 , \tag{4.1.4}$$

where $f' \equiv \mathrm{d}f/\mathrm{d}r$. Such dimensions should be $[J_V] = \frac{\text{Energy}}{\text{Volume}} = M/(LT^2)$, which is consistent with (4.1.4).

4.1.2 Up to NLO in physical units

The contribution provided by the current appears at the NLO in the expansion, as clarified in Section 3.3. We rewrite the Lagrangian for the full potential V (3.3.64) in a slightly different form for later convenience, in units of c as it was in the original reference

$$L[V] = 4\pi\int_0^\infty r^2\mathrm{d}r\left[\frac{m_{\mathrm{p}}}{8\pi\ell_{\mathrm{p}}}V\Delta V - q_{\mathrm{B}}\rho V\left(1-2q_\Phi V\right) + q_\Phi\frac{m_{\mathrm{p}}}{2\pi\ell_{\mathrm{p}}}V\left(V'\right)^2\right]. \tag{4.1.5}$$

In order to switch to physical units, we have to keep in mind is that the coupling q_Φ has to be rescaled by $t_{\mathrm{p}}^2/\ell_{\mathrm{p}}^2$ for dimensional reasons. Hence,

$$L[V] = 4\pi\int_0^\infty r^2\mathrm{d}r\left[\frac{m_{\mathrm{p}}t_{\mathrm{p}}^2}{8\pi\ell_{\mathrm{p}}^3}V\Delta V - q_{\mathrm{B}}\rho V\left(1-2q_\Phi\frac{t_{\mathrm{p}}^2}{\ell_{\mathrm{p}}^2}V\right) + q_\Phi\frac{m_{\mathrm{p}}t_{\mathrm{p}}^4}{2\pi\ell_{\mathrm{p}}^5}V\left(V'\right)^2\right], \tag{4.1.6}$$

or equivalently

$$L[V] = 4\pi\int_0^\infty r^2\mathrm{d}r\left[\frac{1}{8\pi G}V\Delta V - q_{\mathrm{B}}\rho V\left(1-\frac{2q_\Phi}{c^2}V\right) + \frac{q_\Phi}{2\pi Gc^2}V\left(V'\right)^2\right]. \tag{4.1.7}$$

In order to reproduce the classical results in a quantum framework, the scalar field $\Phi = \sqrt{\frac{m_{\mathrm{p}}}{\ell_{\mathrm{p}}}}V$ has been introduced, whose Lagrangian is (3.3.77)

$$L[\Phi] = 4\pi\int_0^\infty r^2\mathrm{d}r\left[\frac{1}{2}\Phi\Box\Phi - q_{\mathrm{B}}J_{\mathrm{B}}\Phi\left(1-2q_\Phi\sqrt{\frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}}}\Phi\right) + 2q_\Phi\sqrt{\frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}}}(\partial_\mu\Phi)^2\Phi\right]. \tag{4.1.8}$$

To understand how we need to suitably introduce physical units, we can compare the similar terms in the lagrangians with V and Φ, respectively. Taking into account the definition $S[\Phi] = \int \mathrm{d}t\, L[\Phi]$, we see that the action for V contains

$$4\pi\int\mathrm{d}t\int r^2\mathrm{d}r\left(\frac{m_{\mathrm{p}}t_{\mathrm{p}}^2}{8\pi\ell_{\mathrm{p}}^3}V\Delta V\right) ,$$

while the one for Φ incorporates

$$4\pi \int \frac{\ell_{\rm p}}{t_{\rm p}} {\rm d}t \int r^2 {\rm d}r \left(\frac{1}{2} \Phi \Box \Phi \right) \, .$$

The factor of $\frac{\ell_{\rm p}}{t_{\rm p}} = c$ needs to be explicit because it comes from the temporal part of the 4−vector x^μ, which is $x^0 = ct$. Therefore, ${\rm d}^4 x = c{\rm d}t \, {\rm d}\vec{x}$. Taking into account that the action has the dimensions of the Planck constant, $[S] = [\hbar] = ML^2T^{-1}$, these terms are both dimensionally consistent. Their comparison leads to the correct rescaling relation

$$\Phi = \frac{\sqrt{m_{\rm p} t_{\rm p}^3}}{\ell_{\rm p}^2} V \, . \tag{4.1.9}$$

Moreover, we shall bear in mind that there is an overall 4π factor, which is going to be rescaled away from $S[\Phi]$ itself, in order to have a canonically normalised field as in Lagrangian (4.1.8). Therefore, $[\Phi] = \frac{M^{1/2}T^{3/2}}{L^2}\frac{L^2}{T^2} = \sqrt{M}/T$ which is exactly $[\Phi] = {\rm erg}^{1/2}\,{\rm s}^{1/2}\,{\rm cm}^{-1}$ in cgs units.

In order to check the dimensions of the matter source $J_{\rm B}$ we need to consider the linearised equations of motion for Φ

$$\Box \Phi = q_{\rm B} J_{\rm B} , \tag{4.1.10}$$

to obtain $[J_{\rm B}] = L^{-2}[\Phi]$, or $[J_{\rm B}] = {\rm erg}^{1/2}\,{\rm s}^{1/2}\,{\rm cm}^{-3}$ in cgs units. We are therefore driven to define

$$J_{\rm B} = \frac{\Phi}{\ell_{\rm p}^2} = \frac{\sqrt{m_{\rm p} t_{\rm p}^3}}{\ell_{\rm p}^4} V = \frac{4\pi \ell_{\rm p}}{\sqrt{m_{\rm p} t_{\rm p}}} \rho \, , \tag{4.1.11}$$

where the Poisson equation has been used.

The action then becomes:

$$\begin{aligned} S[\Phi] = \int \frac{\ell_{\rm p}}{t_{\rm p}} {\rm d}t \int_0^\infty (4\pi r^2) {\rm d}r \Bigg[\frac{1}{2} \Phi \Box \Phi - q_{\rm B} J_{\rm B} \Phi \left(1 - 2 q_\Phi \sqrt{\frac{t_{\rm p}}{m_{\rm p}}} \Phi \right) \\ + 2 q_\Phi \sqrt{\frac{t_{\rm p}}{m_{\rm p}}} (\partial_\mu \Phi)^2 \Phi \Bigg] \, . \end{aligned} \tag{4.1.12}$$

We can easily verify that, in the limit for $c \to 1$, which means $\ell_{\rm p} \to t_{\rm p}$, we recover the correct expression in natural units. Using the fundamental constants instead of Planck units

$$S[\Phi] = \int c\mathrm{d}t \int_0^\infty (4\pi r^2)\,\mathrm{d}r \left[\frac{1}{2}\Phi\Box\Phi - q_\mathrm{B} J_\mathrm{B}\Phi\left(1 - 2q_\Phi\sqrt{\frac{G}{c^3}}\Phi\right)\right. \\ \left. + 2q_\Phi\sqrt{\frac{G}{c^3}}(\partial_\mu\Phi)^2\Phi\right] . \tag{4.1.13}$$

It is also useful to separate the temporal derivative from the spatial ones, in order to make it easier to find the conjugate momentum and the Hamiltonian

$$L[\Phi] \; = \; 4\pi \int_0^\infty r^2\mathrm{d}r \left[-\frac{1}{2c}\Phi\ddot{\Phi} + \frac{c}{2}\Phi\triangle\Phi - cq_\mathrm{B}J_\mathrm{B}\Phi\left(1 - 2q_\Phi\sqrt{\frac{G}{c^3}}\Phi\right)\right. \tag{4.1.14}$$

$$\left. -2q_\Phi\sqrt{\frac{G}{c}}\Phi\left(\frac{\dot{\Phi}^2}{c^2} - |\vec{\nabla}\Phi|^2\right)\right] , \tag{4.1.15}$$

where a dot stands for a time derivative. This expression translates to

$$L[\Phi] \; = \; 4\pi \int_0^\infty r^2\mathrm{d}r \left[-\frac{t_\mathrm{p}}{2\ell_\mathrm{p}}\Phi\ddot{\Phi} + \frac{\ell_\mathrm{p}}{2t_\mathrm{p}}\Phi\triangle\Phi - \frac{q_\mathrm{B}\ell_\mathrm{p}}{t_\mathrm{p}}J_\mathrm{B}\,\Phi\left(1 - 2q_\Phi\sqrt{\frac{t_\mathrm{p}}{m_\mathrm{p}}}\Phi\right)\right. \\ \left. -2q_\Phi\frac{\ell_\mathrm{p}}{\sqrt{m_\mathrm{p}t_\mathrm{p}}}\Phi\left(\frac{t_\mathrm{p}^2}{\ell_\mathrm{p}^2}\dot{\Phi}^2 - |\vec{\nabla}\Phi|^2\right)\right] \tag{4.1.16}$$

in Planck units.

4.2 Hamiltonian and equations of motion

In order to find the hamiltonian by means of a Legendre transformation, we need to first find the conjugate momentum of the scalar field Φ,

$$\Pi \; = \; \frac{\delta L[\Phi]}{\delta\partial_0\Phi} \; = \; \frac{c\delta L[\Phi]}{\delta\dot{\Phi}} \; = \; \dot{\Phi}\left(1 - 4q_\Phi\sqrt{\frac{G}{c^3}}\,\Phi\right) . \tag{4.2.17}$$

We can now compute the Hamiltonian, also taking into account that the first term of (4.1.15) can be rewritten using partial integration.

$$H[\Phi] = \int_0^\infty \mathrm{d}\vec{x} \left\{\dot{\Phi}\Pi - L[\Phi]\right\} \tag{4.2.18}$$

$$= \int_0^\infty \mathrm{d}\vec{x} \left[\frac{1}{2c}\dot{\Phi}^2 - \frac{c}{2}\Phi\triangle\Phi + cq_\mathrm{B}J_\mathrm{B}\Phi\left(1 - 2q_\Phi\sqrt{\frac{G}{c^3}}\Phi\right) - 2q_\Phi\sqrt{\frac{G}{c}}\Phi\left(\frac{\dot{\Phi}^2}{c^2} + |\vec{\nabla}\Phi|^2\right)\right] . \tag{4.2.19}$$

Writing the Hamiltonian using the Planck units, we have:

$$H[\Phi] = \int_0^\infty \mathrm{d}\vec{x} \left[\frac{t_\mathrm{p}^2}{2\ell_\mathrm{p}^2}\dot{\Phi}^2 - \frac{1}{2}\Phi\triangle\Phi + q_\mathrm{B}J_\mathrm{B}\Phi\left(1 - 2q_\Phi\sqrt{\frac{t_\mathrm{p}}{m_\mathrm{p}}}\Phi\right) - 2q_\Phi\sqrt{\frac{t_\mathrm{p}}{m_\mathrm{p}}}\Phi\left(\frac{t_\mathrm{p}^2}{\ell_\mathrm{p}^2}\dot{\Phi}^2 + |\vec{\nabla}\Phi|^2\right)\right] . \tag{4.2.20}$$

We find the equations of motion using the well-known Euler-Lagrange equations

$$\frac{\delta L[\Phi]}{\delta\Phi} - \partial_\mu\frac{\delta L[\Phi]}{\delta\partial_\mu\Phi} = 0 \tag{4.2.21}$$

to obtain

$$\frac{\delta L[\Phi]}{\delta\Phi} = c\left[\Box\Phi - q_\mathrm{B}J_\mathrm{B}\left(1 - 4q_\Phi\sqrt{\frac{G}{c^3}}\Phi\right) + 2q_\Phi\sqrt{\frac{G}{c^3}}(\partial_\mu\Phi)^2\right] \tag{4.2.22}$$

$$\partial_\mu\frac{\delta L[\Phi]}{\delta\partial_\mu\Phi} = 4q_\Phi\sqrt{\frac{G}{c}}\left(\partial_\mu\Phi\partial^\mu\Phi + \Phi\Box\Phi\right) , \tag{4.2.23}$$

thanks to Eq. (4.1.8). Putting all the contributions together and summing the similar terms, the E-L equations are

$$\left(1 - 4q_\Phi\sqrt{\frac{G}{c^3}}\Phi\right)\Box\Phi = q_\mathrm{B}J_\mathrm{B}\left(1 - 4q_\Phi\sqrt{\frac{G}{c^3}}\Phi\right) + 2q_\Phi\sqrt{\frac{G}{c^3}}(\partial_\mu\Phi)^2 \tag{4.2.24}$$

or

$$\left(1 - 4q_\Phi\sqrt{\frac{t_\mathrm{p}}{m_\mathrm{p}}}\Phi\right)\Box\Phi = q_\mathrm{B}J_\mathrm{B}\left(1 - 4q_\Phi\sqrt{\frac{t_\mathrm{p}}{m_\mathrm{p}}}\Phi\right) + 2q_\Phi\sqrt{\frac{t_\mathrm{p}}{m_\mathrm{p}}}(\partial_\mu\Phi)^2 \tag{4.2.25}$$

in Planck units.

Performing the non-relativistic limit, $c \to \infty$, we find the expected equation

$$\Box\Phi = q_{\mathrm{B}} J_{\mathrm{B}} \ . \tag{4.2.26}$$

In other words, this limit is tantamount to putting $q_\Phi = 0$ and considering the "linear" equation of motion.

4.3 Linearisation

Our goal is now to perform the splitting $\hat{\Phi} = \phi + \hat{\varphi}$, where $\langle\, g|\hat{\Phi}|g\,\rangle = \phi$, thus separating the scalar field into a classical background field and a quantum perturbation, considered to be small in modulus with respect to the background. The terms $\mathcal{O}(\varphi^2)$ are discarded; therefore, this procedure accounts for a linearisation of the expression of the Lagrangian.

The Lagrangian density thus becomes

$$\begin{aligned}\mathcal{L} \ = \ & \frac{1}{2}\phi\Box\phi + \frac{1}{2}\varphi\Box\varphi + \frac{1}{2}\varphi\Box\phi + \frac{1}{2}\phi\Box\varphi - q_{\mathrm{B}} J_{\mathrm{B}}(\phi+\varphi) \\ & +2q_{\mathrm{B}} J_{\mathrm{B}} q_\Phi \sqrt{\frac{G_{\mathrm{N}}}{c^3}}(\phi^2+\varphi^2+2\phi\varphi) + 2q_\Phi\sqrt{\frac{G_{\mathrm{N}}}{c^3}}\Big[\phi(\partial_\mu\phi)^2 \\ & +\varphi(\partial_\mu\varphi)^2 + \phi(\partial_\mu\varphi)^2 + \varphi(\partial_\mu\phi)^2 + 2\varphi\partial_\mu\phi\partial^\mu\varphi + 2\phi\partial_\mu\phi\partial^\mu\varphi\Big] \ .\end{aligned} \tag{4.3.27}$$

The terms $\mathcal{O}(\varphi)$ in the previous expression can be neglected by means of a double integration by parts and the equations of motion for ϕ. More specifically,

$$\begin{aligned}\mathcal{L}\Big|_{\mathcal{O}(\varphi)} \ = \ & \frac{1}{2}\left(\varphi\Box\phi + \phi\Box\varphi\right) - q_{\mathrm{B}} J_{\mathrm{B}}\varphi + 4q_{\mathrm{B}}\sqrt{\frac{G_{\mathrm{N}}}{c^3}} J_{\mathrm{B}}\phi\varphi \\ & +2\sqrt{\frac{G_{\mathrm{N}}}{c^3}}(\partial_\mu\phi)^2\varphi + 4\sqrt{\frac{G_{\mathrm{N}}}{c^3}}\phi\partial_\mu\phi\partial^\mu\varphi.\end{aligned} \tag{4.3.28}$$

Given the boundary of the spacetime $\partial\mathcal{M}$ and its normal unit vector n^μ, we perform the following integrations by parts

$$\int \mathrm{d}^4x\, \phi\Box\varphi \ = \ \left(\phi\, n^\mu\partial_\mu\varphi - \varphi\, n^\mu\partial_\mu\phi\right)\Big|_{\partial\mathcal{M}} + \int \mathrm{d}^4x\, \varphi\Box\phi \tag{4.3.29}$$

$$\equiv \int \mathrm{d}^4x\, \varphi\Box\phi, \tag{4.3.30}$$

$$\begin{aligned}\int \mathrm{d}^4x\, \phi\partial_\mu\phi\partial^\mu\varphi \ &= \ \phi\, n^\mu\partial_\mu\phi\, \varphi\Big|_{\partial\mathcal{M}} - \int \mathrm{d}^4x\, \varphi\left[\phi\Box\phi + (\partial_\mu\phi)^2\right] \\ &\equiv -\int \mathrm{d}^4x\, \varphi\left[\phi\Box\phi + (\partial_\mu\phi)^2\right] \ .\end{aligned} \tag{4.3.31}$$

Therefore, Eq. (4.3.28) yields

$$\begin{aligned}\mathcal{L}\Big|_{\mathcal{O}(\varphi)} &= \varphi(1-4\sqrt{\frac{G_{\mathrm{N}}}{c^3}}\phi)\Box\phi - q_{\mathrm{B}}J_{\mathrm{B}}\varphi(1-4\sqrt{\frac{G_{\mathrm{N}}}{c^3}}\phi) - 2\sqrt{\frac{G_{\mathrm{N}}}{c^3}}(\partial_\mu\phi)^2\varphi \qquad (4.3.32)\\ &= \varphi\left\{(1-4\sqrt{\frac{G_{\mathrm{N}}}{c^3}}\phi)\Box\phi - q_{\mathrm{B}}J_{\mathrm{B}}(1-4\sqrt{\frac{G_{\mathrm{N}}}{c^3}}\phi) - 2\sqrt{\frac{G_{\mathrm{N}}}{c^3}}(\partial_\mu\phi)^2\right\} \qquad (4.3.33)\\ &= \varphi\,\mathrm{EOM}(\phi)\;\equiv\;0\;. \qquad (4.3.34)\end{aligned}$$

This result avoids the mixing of kinetic terms kinetic mixing between the background ϕ and the fluctuation φ. Moreover, it can be shown to hold at the quantum level as well, since $\langle g|\,\hat{\varphi}\,|g\rangle = 0$.

The equations of motion for the fluctuations $\varphi(x)$ are

$$\Box\varphi\left(1-4q_\Phi\sqrt{\frac{G_{\mathrm{N}}}{c^3}}(\phi+\varphi)\right) - 4q_\Phi\sqrt{\frac{G_{\mathrm{N}}}{c^3}}\left[\frac{1}{2}\left(\partial_\mu\varphi\right)^2 + \partial_\mu\phi\partial^\mu\varphi\right] = 0, \qquad (4.3.35)$$

where a further simplification by means of the EOM of the background (4.1.10) has been performed.

We remark that the background is static and spherically symmetric, hence $\phi(x^\mu) = \phi(r = \sqrt{x_j x^j})$, whereas the fluctuations would technically not have such symmetry. However, for the sake of simplicity, we choose to restrict ourselves to $\varphi = \varphi(r)$ as well.

We now want to make the background field explicit, in order to better understand the physical significance of the equations of motion. The simplest possible choice for the background field is the Newtonian potential generated by a homogeneous spherical distribution of matter. For a density

$$\rho(r) = \frac{3M_0}{4\pi R^3}\Theta(R-r), \qquad (4.3.36)$$

where $M_0 = 4\pi\int_0^\infty r^2\mathrm{d}r\,\rho(r)$, the potential is

$$\phi(r) = \begin{cases} q_{\mathrm{B}}\dfrac{G_{\mathrm{N}}M_0}{2R^3}(r^2-3R^2), & \text{for } r<R\\[2ex] -q_{\mathrm{B}}\dfrac{G_{\mathrm{N}}M_0}{r}, & \text{for } r>R\end{cases} \qquad (4.3.37)$$

where we have used the same conventions of [9] in order to avoid unnecessary complications. We will need to use the expression for $r<R$, since we consider to be in the interior of the "ball" of gravitons that constitutes the black hole. Therefore, the derivative of $\phi(r)$

that also appears in the equations of motion (4.3.35) is

$$\partial_r \phi(r) = q_{\rm B} \frac{G_{\rm N} M_0 r}{R^3}, \tag{4.3.38}$$

since only the radial component of the derivative in spherical coordinate remains.

4.3.1 Dispersion relation

Starting from the equations of motion (4.3.35), we insert the ansatz of a spherical wave, where additionally we assume that ω is imaginary, so that we consider σ a real quantity

$$\varphi = \frac{e^{ikr}}{kr} e^{i\omega t + \sigma t}. \tag{4.3.39}$$

Moreover, in this context, we consider the WKB approximation to be applicable, since the waves are thought to oscillate on a much smaller scale than that on which the background potential varies. In order to calculate $\Box\varphi$ in spherical coordinates, we have to remark that the Laplacian is

$$\triangle\varphi = \nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right), \tag{4.3.40}$$

and the radial derivative of (4.3.39) is

$$\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{kr}\right) = \frac{(ikr-1)e^{ikr}}{kr^2}. \tag{4.3.41}$$

Therefore, let us calculate the terms that appear in the Lagrangian.

$$\begin{aligned}
\Box\varphi &= -\frac{\partial^2}{c^2\partial t^2}\varphi + \triangle\varphi \\
&= -\frac{\partial^2}{c^2\partial t^2}\frac{(i\omega+\sigma)}{kr} e^{ikr+i\omega t+\sigma t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{(ikr-1)e^{ikr}}{k}\right] e^{ikr+i\omega t+\sigma t} \\
&= \frac{(i\omega+\sigma)^2}{c^2 kr} e^{ikr+i\omega t+\sigma t} + \frac{1}{r^2}\left[ie^{ikr} - ie^{ikr} - kre^{ikr}\right] e^{ikr+i\omega t+\sigma t} \\
&= \left[\frac{(\omega^2-\sigma^2-2i\omega\sigma)}{c^2 kr} - \frac{k}{r}\right] e^{ikr+i\omega t+\sigma t}
\end{aligned} \tag{4.3.42}$$

$$\begin{aligned}\partial_\mu\varphi\partial_\mu\varphi = (\partial_\mu\varphi)^2 &= -\frac{1}{c^2}\partial_t\varphi\partial^t\varphi + \partial_r\varphi\partial^r\varphi \\ &= -\frac{1}{c^2}e^{ikr+i\omega t+\sigma t}\left\{\left[\frac{(i\omega+\sigma)}{kr}\right]^2 + \left[\frac{(ikr-1)}{kr^2}\right]^2\right\} \\ &= \left(\frac{\omega^2-\sigma^2-2i\omega\sigma}{c^2k^2r^2} - \frac{1}{r^2} - \frac{1}{k^2r^4} - \frac{2i}{kr^3}\right)e^{2(ikr+i\omega t+\sigma t)}\end{aligned} \tag{4.3.43}$$

$$\begin{aligned}\partial_\mu\phi\partial^\mu\varphi = \partial_r\phi\partial^r\varphi &= \frac{q_\mathrm{B}G_\mathrm{N}M_0 r}{R^3}\frac{(ikr-1)}{kr^2}e^{ikr+i\omega t+\sigma t} \\ &= \left(\frac{iq_\mathrm{B}G_\mathrm{N}M_0}{R^3} - \frac{q_\mathrm{B}G_\mathrm{N}M_0}{krR^3}\right)e^{ikr+i\omega t+\sigma t}\end{aligned} \tag{4.3.44}$$

where we used the Newtonian potential (4.3.37). The substitution of these expressions into (4.3.35) and the expansion $e^{i(kr+\omega t)} = \cos(kr+\omega t) + i\sin(kr+\omega t) \equiv \cos(x) + i\sin(x)$, yields

$$\begin{aligned}&\left(\frac{\omega^2-\sigma^2-2i\omega\sigma}{c^2kr} - \frac{k}{r}\right)\left[1 - 4q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}\left(q_\mathrm{B}\frac{G_\mathrm{N}M_0}{2R^3}(r^2-3R^2)+\varphi\right)\right] \\ &- 4q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}\left[\left(\frac{\omega^2-\sigma^2-2i\omega\sigma}{2c^2k^2r^2} - \frac{1}{2r^2} - \frac{1}{2k^2r^4} - \frac{i}{kr^3}\right)\cdot\right. \\ &\left.\cdot(\cos(x)+i\sin(x))e^{\sigma t} + \left(\frac{iq_\mathrm{B}G_\mathrm{N}M_0}{R^3} - \frac{q_\mathrm{B}G_\mathrm{N}M_0}{krR^3}\right)\right] = 0,\end{aligned} \tag{4.3.45}$$

discarding an overall exponential factor. It is now necessary to separate the real part $\Re$ of the equation from the imaginary one $\Im$, in order to find expressions for ω and σ.

We obtain

$$\begin{aligned}\Re = &\left(\frac{\omega^2-\sigma^2}{c^2kr} - \frac{k}{r}\right)\left[1 - 4q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}\left(q_\mathrm{B}\frac{G_\mathrm{N}M_0}{2R^3}(r^2-3R^2)+\varphi\right)\right] \\ &- 4q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}\left[\left(\frac{\omega^2-\sigma^2}{2c^2k^2r^2} - \frac{1}{2r^2} - \frac{1}{2k^2r^4}\right)\cos(x)e^{\sigma t}\right. \\ &\left.+\left(\frac{\omega\sigma}{c^2k^2r^2} + \frac{1}{kr^3}\right)\sin(x)e^{\sigma t} - \frac{q_\mathrm{B}G_\mathrm{N}M_0}{krR^3}\right] = 0,\end{aligned} \tag{4.3.46}$$

and

$$\begin{aligned}\mathfrak{I} = \frac{-2\omega\sigma}{c^2kr}&\left[1 - 4q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}\left(q_\mathrm{B}\frac{G_\mathrm{N}M_0}{2R^3}(r^2-3R^2)+\varphi\right)\right]\\ &- 4q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}\left[\left(\frac{\omega^2-\sigma^2}{2c^2k^2r^2} - \frac{1}{2r^2} - \frac{1}{2k^2r^4}\right)\sin(x)e^{\sigma t}\right.\\ &\left.-\left(\frac{\omega\sigma}{c^2k^2r^2} + \frac{1}{kr^3}\right)\cos(x)e^{\sigma t} + \frac{q_\mathrm{B}G_\mathrm{N}M_0}{R^3}\right] = 0.\end{aligned} \tag{4.3.47}$$

The solution for $\mathfrak{R}$ in the case of $q_\Phi = 0$, thus in the linear case, in which there is no self-interactions between gravitons, yields

$$\frac{\omega^2-\sigma^2}{c^2kr} - \frac{k}{r} = 0 \to \omega^2 = c^2k^2 + \sigma^2. \tag{4.3.48}$$

This solution clearly shows that the angular frequency ω is constituted by an "unperturbed" part, the usual c^2k^2 and a parte that must be proportional to the coupling q_Φ, which is σ^2. The claim that $\sigma \propto q_\Phi$ will be substantiated in the following. Therefore, we can write

$$\begin{cases}\omega = ck + q_\Phi\omega_1\\ \sigma = q_\Phi\sigma_1,\end{cases} \tag{4.3.49}$$

where k can be either positive or negative and the subscript 1 simply indicates the first (next-to-leading) order in the graviton self-interaction. These expression lead to the product $\omega\sigma = (ck + q_\Phi\omega_1)q_\Phi\sigma_1 = ckq_\Phi\sigma_1$, that we can use in (4.3.47) to find

$$\begin{aligned}\frac{2q_\Phi\sigma_1}{cr} =& - 4q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}\left(-\frac{1}{2k^2r^4}\sin(kr+ckt)\right.\\ &\left.-\frac{1}{kr^3}\cos(kr+ckt) + \frac{q_\mathrm{B}G_\mathrm{N}M_0}{R^3}\right),\end{aligned} \tag{4.3.50}$$

where we have used the expansions (4.3.49) and $e^{\sigma t} = e^{q_\Phi\sigma_1} \simeq 1 + q_\Phi\sigma_1$, with the aim of discarding terms of order $\mathcal{O}(q_\Phi^2)$.

Simplifying further, the expression for σ is found to be

$$\begin{aligned}q_\Phi\sigma_1 = \sigma =& - 2q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}c\left(-\frac{1}{2k^2r^3}\sin(kr+ckt)\right.\\ &\left.-\frac{1}{kr^2}\cos(kr+ckt) + \frac{q_\mathrm{B}G_\mathrm{N}M_0r}{R^3}\right),\end{aligned} \tag{4.3.51}$$

The second term in the parentheses disappears in the case of $\phi = 0$, thus in the absence of background field.

Discussion of the results In (4.3.51) we can read off the sign of σ, which is key to understanding the behaviour of φ with respect to the background field ϕ. That is, if $\sigma > 0$, the exponential in (4.3.39) grows and could technically end up dominating over the background, whereas if $\sigma < 0$, the perturbation φ decreases in time and gets reabsorbed in the background.

The WKB approximation, in which the wavelength $\lambda \sim 1/k$ is reasonably smaller than the scale r on which the potential ϕ varies, can be considered: we can discard the terms that contain $1/(kr^2) \ll 1$ and obtain a more straightforward expression for σ

$$q_\Phi \sigma_1 = \sigma = -2q_\Phi q_\mathrm{B}\sqrt{\frac{G_\mathrm{N}}{c^3}\frac{cG_\mathrm{N}M_0 r}{R^3}}. \tag{4.3.52}$$

Neglecting such terms, it is important to remark that the explicit dependence on k disappears. We can also consider the form that the expression above takes in the particular case $r = R_\mathrm{S}$:

$$\sigma = -4q_\Phi q_\mathrm{B}\sqrt{\frac{G_\mathrm{N}}{c^3}\frac{G_\mathrm{N}^2 M_0^2}{cR^3}}. \tag{4.3.53}$$

So far, we have considered $k > 0$ and we see that this leads to $\sigma < 0$. If we apply the WKB approximation, neglecting the oscillating terms with the trigonometric functions in (4.3.50), we can say that the perturbations decrease in time, as can be seen from the form (4.3.39). Therefore, the background ends up dominating over the perturbations, that get reabsorbed, thus granting the stability of the gravitational system described by (4.1.8). Of course, a more precise analysis of the behaviour of the oscillating terms is required if we wish to make more accurate predictions.

On the other hand, we can also consider (4.3.51) in the opposite limit to the one analyzed previously. More specifically, in the limit of long wavelengths, we can make the substitution $\lambda \sim R_\mathrm{S} \sim M$, in which M is the mass of the black hole, and, due to the coarse-grained nature of this approximation, we neglect the constants G_N and c in the Schwarzschild radius. We remark that such an approximation is reminiscent of the fact that gravitons, in their self-generated potential well, have a wavelength of the order of the well radius itself (see Chapter 2). With the same concept in mind, we can also consider r to be $\sim M$, and therefore $kr \sim 1$. Finally, in order to simplify the expression, we choose to restrict ourselves to $t = 0$. (4.3.51) can be rewritten in a slightly different form

$$\begin{aligned}\sigma = &-2q_\Phi\sqrt{\frac{G_\mathrm{N}}{c^3}}c\frac{1}{r}\left(-\frac{1}{2kr}\frac{\sin(kr)}{kr}\right.\\&\left.-\frac{\cos(kr)}{kr}+\frac{q_\mathrm{B}G_\mathrm{N}M_0 r}{R^3}\right),\end{aligned} \tag{4.3.54}$$

where, in the long-wavelength approximation, we can see how the last term of the expression, that was dominating in the WKB approximation, becomes negligible in this case, since, if $r \sim R_\mathrm{S} \sim M$,

$$\frac{q_\mathrm{B}G_\mathrm{N}M_0 r}{R^3} \sim \frac{M_0}{M^2}, \tag{4.3.55}$$

and $M_0/M \ll 1$. Therefore, the two oscillating terms with the trigonometric functions would end up dominating in this limit, and, due to their negative sign, would cause $\sigma > 0$. Such a peculiar behaviour would translate in an amplification of the perturbations, that can compromise the stability of the system of gravitons and might be linked to the emission of Hawking radiation. However, further studies on this scenario are required before a sound physical interpretation of the phenomenon can be provided.

Conclusions

In this thesis, we have presented an approach that describes fully quantum black holes, in a model without central singularity, therefore allowing considerations to be made on the internal black hole structure.

Firstly, we introduced the problem of the search for a quantum theory of gravity, presenting the issue of the non-renormalizability of General Relativity, and the concept of effective field theories. We have then shown how an effective quantum field theory of gravity at low energies was developed. With respect to the UV-completion of such a theory, an alternative path to the standard one has been introduced, that of classicalization, which might allow to avoid the need of a completion at all. The deep conceptual issues that emerge as we get close to the Planck scale have been highlighted as well.

In the context of classicalization, we have explored the corpuscular model of Dvali and Gomez, that describes black holes as BECs of gravitons at the critical point without central singularity, with emphasis on the role played by the occupation number of the condensate.

Subsequently, we have shown how the corpuscular model is the starting point for a more rigorous treatment of gravitational collapse and the construction of an effective theory for the Newtonian potential of spherically symmetric sources up to the post-Newtonian order, justifying the employment of a toy model of scalar gravitons.

Finally, starting from the Lagrangian of the gravitational field up to post-Newtonian corrections, we recovered physical units and found the equations of motion. Such equations have then been linearised, by considering the field as a Newtonian background plus a small perturbation, modelled as a spherical wave, for which we assumed the WKB approximation to be valid. We then found the equations of motion of the perturbation and the correspondent dispersion relation, up to first order in the graviton self-coupling. This procedure has shown that, in such a limit, the perturbations end up decreasing in time and disappear into the background. However, if the opposite, long-wavelength limit is considered, the terms that were negligible in the WKB approximation become dominant and cause an amplification of the perturbation that might signal an instability of the considered system.

Appendix A

Bose-Einstein Condensation

In this Appendix, we will provide a brief description of Bose-Einstein Condensation (BEC), highlighting some properties that particularly relate to the corpuscular model of black holes, relying on [44] and [45]. Therefore, the present treatment is by no means exhaustive or systematic, and the derivation of many results is not included for the sake of brevity.

Bose-Einstein condensation is a phase transition that occurs (in its simplest configuration) in a dilute gas of bosons at extremely low temperatures. It consists of the process in which a large fraction of bosons occupies the lowest quantum state, so that microscopic quantum phenomena start to appear at the macroscopic scale. The central idea of Bose-Einstein condensation has its roots in the 1925 work by A. Einstein, based on that of S. N. Bose one year earlier, in which a phase transition in a gas of non-interacting atoms was first explored. In 1947, N. N. Bogolyubov devised the first microscopic theory of superfluidity, based on the concept of Bose-Einstein condensation [46]. The first experimental observation of Bose-Einstein condensation, in a way that is not new to the history of science, was made after a long time, only in 1995, with the Nobel prize winning work by E. A. Cornell, W. Ketterle and E. Wieman. After this result, the study of BEC has become a vast and fertile area of research, both theoretical and experimental.

To understand the process of condensation, we will first introduce some basic concepts, such as the one-body density matrix $n^{(1)}(\mathbf{r}, \mathbf{r}') = \langle \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}') \rangle$, in which $\hat{\Psi}^\dagger(\mathbf{r})$ and $\hat{\Psi}(\mathbf{r})$ are bosonic field operators that create and annihilate, respectively, a particle at point $\mathbf{r}$. The angular brackets represent a thermodynamical average performed on the state of the system. Clearly, the diagonal density of the system is found when $\mathbf{r} = \mathbf{r}'$: $n(\mathbf{r}) = \langle \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \rangle = n^{(1)}(\mathbf{r}, \mathbf{r})$.

In order to model an ideal Bose-Einstein condensate, we first consider an isotropic and uniform system of N particles, that occupies a volume V and is not subject to external potentials. In the thermodynamic limit, $N, V \to \infty$, but $n = N/V$ is finite. If $\mathbf{t} = \mathbf{r} - \mathbf{r}'$, the one-body density should, in general, vanish at $t \to \infty$, but in the presence of a BEC, a very peculiar behaviour is displayed: the ground state (the single-particle state with momentum $\mathbf{p} = 0$) becomes populated by a macroscopic number of particles. Therefore, the resulting momentum distribution includes a delta function

$$n(\mathbf{p}) = N_0\delta(\mathbf{p}) + \tilde{n}(\mathbf{p}), \tag{A.0.1}$$

where N_0 is proportional to the number of particles and defines the condensate fraction $N_0/N \leq 1$. Therefore, in the presence of BEC, the one-body density matrix approaches a finite value at large distances, $n^{(1)}(t)_{t\to\infty} \to n_0$, where $n_0 = N_0/V$.

This behaviour is called off-diagonal long-range order and for a generic system, it means that its constituents have a strongly correlated behaviour that does not disappear when the distance between them becomes large. We can also rewrite

$$\int \mathrm{d}\mathbf{r}'\, n^{(1)}(\mathbf{r}, \mathbf{r}')\varphi_i(\mathbf{r}') = n_i\varphi_i(\mathbf{r}). \tag{A.0.2}$$

The eigenvalues n_i are normalized by $\sum_i n_i = N$ and represent the single-particle occupation numbers for the corresponding particle states φ_i. Moreover, they allow the density matrix to be rewritten in diagonal form as $n^{(1)}(\mathbf{r}, \mathbf{r}') = \sum_i n_i\varphi_i^*\varphi_i(\mathbf{r}')$.

When Bose-Einstein condensation occurs, the single-particle state corresponding to $i = 0$ becomes macroscopically occupied, with $n_0 \equiv N_0 \sim N$, while all the other states have occupation of order 1. After the diagonalization, the single-particle wavefunctions can be used to provide an expression for the field operator we have previously seen:

$$\Psi(\mathbf{r}) = \sum_i a_i\varphi_i(\mathbf{r}), \tag{A.0.3}$$

(the hats on the operators are going to be used only when necessary to avoid ambiguities), where the a_i and $a_i^\dagger$ are the annihilation and creation operators of a particle in the state φ_i, obeying the usual bosonic commutation relations. The field operator can be separated into a term representing the condensate (i.e. the macroscopically occupied single-particle state) and another representing the non-condensate part

$$\Psi(\mathbf{r}) = \varphi_0(\mathbf{r})a_0 + \sum_{i\neq 0}\varphi_i(\mathbf{r})a_i. \tag{A.0.4}$$

The field operator can also be rewritten as $\Psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \psi_1(\mathbf{r})$. The normalization condition relative to the number of particles is basically the quantum-number conservation condition $\langle a_i^\dagger a_j\rangle = \delta_{ij}\langle a_i^\dagger a_j\rangle$, that is $\langle\psi_0^\dagger(\mathbf{r})\psi_1(\mathbf{r})\rangle = 0$. Thus, we can define two number operators $N_0 \equiv a_0^\dagger a_0$ and $N_1 \equiv \sum_{i\neq 0} a_i a_i^\dagger$, for the condensate and non-condensate parts, respectively, giving rise to the total number operator $\hat{N} = \hat{N}_0 + \hat{N}_1$. The number of condensed particles is then the statistical average $N_0 = \langle\hat{N}_0\rangle = \langle a_0^\dagger a_0\rangle$.

Several criteria can be used to identify the onset of the process of condensation in the thermodynamic limit, such as the Einstein criterion: it states that a BEC occurs when

$$\lim_{N\to\infty} \frac{N_0}{N} > 0, \tag{A.0.5}$$

an expression that captures the macroscopic occupation of a single state.

However, these definitions do not highlight that, in order for the condensation (and thus the phase transition) to occur, a symmetry breaking needs to take place. If we consider a Hamiltonian $H[\psi]$ endowed with a generic $U(1)$ symmetry, it means that the Hamiltonian is invariant under the transformation $\psi(\mathbf{r}) \to \psi(\mathbf{r})e^{i\alpha}$, with $\alpha \in \mathbb{R}$. Such a symmetry can be broken by performing a Bogolyubov shift. This method was first devised by by N. N. Bogolyubov and relies on the inclusion of infinitesimal sources as symmetry-breaking terms into the Hamiltonian: $H'[\psi] \equiv H[\psi] + \epsilon\Gamma[\psi]$, where the statistical average of $\Gamma[\psi]$ is proportional to N and ϵ is a small real quantity. This result allows to show, via a series of straightforward theorems, that spontaneous symmetry breaking is the *necessary and sufficient* condition for Bose-Einstein condensation.

Bogolyubov shift The Bogolyubov shift is one of the easiest ways to implement the gauge symmetry breaking for a system of bosons. It is tantamount to replacing the field operator $\hat{\psi}$ of a non-condensed system with

$$\hat{\psi}(\mathbf{r}) = \eta(\mathbf{r}) + \hat{\delta}\psi_1(\mathbf{r}), \tag{A.0.6}$$

where the first term is called the condensate wavefunction and the second is the field operator of non-condensed particles. The first term behaves like a classical quantity, not an operator; thus, it can be stated that the shift rewrites $\hat{\psi}(\mathbf{r})$ as a classical quantity plus quantum corrections. If the non-condensed component can be neglected, the system as a whole behaves like a classical object. The validity of this approximation holds as long as the macroscopic occupation of a single-particle state ($N_0 \gg 1$) occurs.

The shift is used in the context of the Bogolyubov approximation, which consists of replacing the creation and annihilation operators with the c-number $\sqrt{N_0}$. It is equivalent to neglecting the non-commutativity of the operators, because $[a^\dagger_{\mathbf{p}}, a_{\mathbf{p}}] = 1 \ll N_0$. The approximation is justified for systems in which $N_0 \gg 1$ such as BECs, as we have mentioned in Chapter 2.

Of course, ψ_1 satisfies the commutation relations for a Bose operator and is orthogonal to the operator representing non-condensed particles. The total number of particles is again $N = \langle \hat{N} \rangle = N_0 + N_1$, where $\hat{N} \equiv \int \mathrm{d}\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})$. Moreover, the quantum number conservation condition takes the form $\langle \psi_1(\mathbf{r}) \rangle = 0$, while from (A.0.6), we see that $\langle \hat{\psi}(\mathbf{r}) \rangle = \eta(\mathbf{r})$. This implies that the condensate function plays the role of an order parameter, since it assumes a non-zero value when the symmetry is broken, after performing the Bogolyubov shift. The condensate function is a complex quantity $\eta(\mathbf{r}) = |\eta(\mathbf{r})|e^{iS(\mathbf{r})}$ and is always defined up to a phase factor; this property of course reflects the gauge invariance of the system.

Another property of this system is reminiscent of the large-N limit described in Chapter 2: due to the great number of particles N_0 in the condensate, adding a single particle does not change the physical properties of the system, at least up to corrections of order $1/N$. Therefore, the state $|N\rangle$ and the state $|N+1\rangle$ are physically equivalent, allowing us to employ the notation $\eta = \langle\hat{\psi}\rangle$. The time-evolution of this quantity is not given by $e^{-iEt/\hbar}$ as usual, but by

$$\eta(\mathbf{r},t) = \eta(\mathbf{r})e^{-i\mu t/\hbar}, \tag{A.0.7}$$

where $\mu = E(N) - E(N-1) \approx \partial E/\partial N$ is the chemical potential.

The condensate wavefunction is orthogonal to the field operator, as are their Fock spaces. The condensate function can also be named coherent field, because the related coherent state $|\eta\rangle$ is the vacuum state in $\mathcal{F}(\psi_1)$ [47].

Bogolyubov approximation and Bogolyubov transformations The Bose gas described in basic Statistical Mechanics textbooks is just a pedagogical idealized system, in which the interactions are neglected with the aim of simplifying the calculations. In more realistic situations, however, in order to calculate the quantities of interest, an approximation is needed: the most powerful one was devised by Bogolyubov and is valid for low temperature and weak interactions. Such systems allow a description of the interactions that only takes into account the effect of pairs of interacting particles, neglecting those involving three or more. Such an approximation is also justified in the context of black hole BECs, since the effects of graviton-graviton scattering are dominant with respect to all the other possible scattering configurations.

Let us first employ this approximation at the lowest possible order. In the context of the Bogolyubov approximation, the s-wave scattering length a (the one at the first order of the partial wave expansion, describing the lowest-energy interactions) is considered crucial to characterize all the scattering effects. The holding condition is $|a| \ll d$, where d is the inter-particle distance. The Hamiltonian of the system is

$$\hat{H} = \int \mathrm{d}\mathbf{r} \left(\frac{\hbar^2}{2m}\nabla\hat{\Psi}^\dagger(\mathbf{r})\nabla\hat{\Psi}(\mathbf{r})\right) + \frac{1}{2}\int \mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}\, \hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}^\dagger(\mathbf{r})V(\mathbf{r}'-\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}^\dagger(\mathbf{r}), \tag{A.0.8}$$

where $V(\mathbf{r})$ is the two-body potential and no external fields are considered. Under the assumption of a uniform gas that occupies a volume V, the field operator can take the form $\hat{\Psi}(\mathbf{r}) = \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}_i}\frac{1}{\sqrt{V}}e^{i\mathbf{p}\cdot\mathbf{r}}/\hbar$ and the Hamiltonian can be rewritten with the ladder operators and simplified: the microscopic potential V can be replaced by an effective, soft potential V_{eff} that yields the same s-wave scattering length.

At this point, the Bogolyubov approximation can be employed, which replaces operators with c-numbers: $a_0 \equiv \sqrt{N_0}$, as justified by the largeness of N_0. It is important to remark that such an approximation is valid in the case of a soft potential which has small perturbations even at short distances; a realistic potential that becomes strong at short distances will therefore undermine the validity of this approach. At the first order, all the terms in the Hamiltonian that contain the creation and annihilation operators relative to momenta $\mathbf{p} \neq 0$ can be safely neglected. Using $a_0 \sim \sqrt{N}$ because $N_0 \sim N$, we get a simple expression for the ground state energy of the considered system, $E_0 = (N^2 V_0)/2V$, where $V_0 = \int \mathrm{d}\mathbf{r}\, V_{\text{eff}}(r)$.

Elementary excitations If we now consider the next-to-leading order in the Bogolyubov approximation, that is, the contribution of creation and annihilation operators with $\mathbf{p} \neq 0$, we can see how the elementary excitations of the system emerge [48].

By employing the approximation, the Hamiltonian becomes quadratic in the ladder operators, and as any quadratic form, can be diagonalized by means of a linear transformation, a Bogolyubov transformation. Let us first diagonalize a generic quadratic Hamiltonian for a system of bosons

$$\hat{H} = \sum_{\mathbf{p}} T_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}} (\Phi_{\mathbf{p}} a^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} + \Phi^*_{\mathbf{p}} a_{-\mathbf{p}} a_{\mathbf{p}}), \tag{A.0.9}$$

in which $T_{\mathbf{p}}$=$T^*_{\mathbf{p}}$, $\Phi_{\mathbf{p}}$=$\Phi_{-\mathbf{p}}$ and $\Phi^*_{\mathbf{p}}$=$\Phi^*_{-\mathbf{p}}$ are functions of the momentum $\mathbf{p}$. The Bogolyubov transformations are the following

$$b_{\mathbf{p}} = u_{\mathbf{p}} a_{\mathbf{p}} - v_{\mathbf{p}} a^\dagger_{-\mathbf{p}} \qquad b^\dagger_{\mathbf{p}} = u^*_{\mathbf{p}} a^\dagger_{\mathbf{p}} - v^*_{\mathbf{p}} a_{-\mathbf{p}}, \tag{A.0.10}$$

in which the parameters $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ satisfy the conditions

$$|u_{\mathbf{p}}|^2 - |v_{\mathbf{p}}|^2 = 1, \qquad u_{\mathbf{p}} = u_{-\mathbf{p}}, \qquad v_{\mathbf{p}} = v_{-\mathbf{p}}, \tag{A.0.11}$$

which ensure that the transformation presented above is canonical, so that the new operators $b_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ still satisfy the bosonic commutation relations. Expressing the usual creation and annihilation operators in terms of the new ones, we find:

$$a_{\mathbf{p}} = u^*_{\mathbf{p}} b_{\mathbf{p}} - v_{\mathbf{p}} b^\dagger_{-\mathbf{p}}, \qquad a^\dagger_{\mathbf{p}} = u_{\mathbf{p}} b^\dagger_{\mathbf{p}} - v^*_{\mathbf{p}} b_{-\mathbf{p}}. \tag{A.0.12}$$

By making a suitable choice for the parameters $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$, the Hamiltonian (A.0.9) can be recast in a diagonal form:

$$\hat{H} = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + E_0 \mathbb{1}, \tag{A.0.13}$$

where E_0 is a constant that needs to be determined. By computing the commutator $[b_{\mathbf{p}}, H]$, we can derive the set of equations that determine the quantities $\epsilon_{\mathbf{p}}$ and $u_{\mathbf{p}}$. From the conditions of solubility of this set, a quadratic spectrum is obtained $\epsilon^2_{\mathbf{p}} = T_{\mathbf{p}} - |\Phi_{\mathbf{p}}|^2$, which yields the dispersion relation for the elementary excitations of the system, the quasi-particles or collective excitations called bogolons, created and annihilated by the new bosonic operators $b_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$. Therefore, $\langle 0| \, b^\dagger_{\mathbf{p}} b_{\mathbf{p}} \, |0\rangle = |v_{\mathbf{p}}|^2$, which leads to

$$\langle 0| \, H \, |0\rangle = \frac{1}{2}\sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \left(\frac{T_{\mathbf{p}}}{\epsilon_{\mathbf{p}}} - 1\right) + E_0; \qquad E_0 = -\frac{1}{2}\sum_{\mathbf{p}} (T_{\mathbf{p}} - \epsilon_{\mathbf{p}}). \tag{A.0.14}$$

In the case of a weakly interacting Bose gas, after employing the Bogolyubov transformations, we find

$$\epsilon(p) = \left[\frac{gn}{m}p^2 + \left(\frac{p^2}{2m}\right)^2\right]^{1/2} \tag{A.0.15}$$

which is the celebrated Bogolyubov dispersion law, where m is the mass of the bosons, p their momentum, g their coupling constant and $n = N/V$ the density of the gas. It should be remarked that the description in terms of the bogolons is independent from that of the gas particles and provides and alternative picture. The ground state of the interacting system is thus the vacuum of quasi-particles, $b_{\mathbf{p}} \, |vac\rangle = 0$, for $\mathbf{p} \neq 0$. The occupation number of quasi-particles is $N_{\mathbf{p}} = \langle b^\dagger_{\mathbf{p}} b_{\mathbf{p}} \rangle$, but it is important to notice that it is not conserved, since quasi-particles can appear and disappear. Similarly to how it happens for photons in a cavity, their number is only determined by the condition of thermodynamic equilibrium. This causes the chemical potential of the quasi-particles to be zero by definition. It is an interesting consequence of quantum fluctuations caused by the interactions that, even at absolute zero, when $N_{\mathbf{p}} = 0$, the number of actual gas particles with $\mathbf{p} \neq 0$ is not zero. This can be seen if we express the number of particles in terms of the number of quasi-particles

$$n_{\mathbf{p}} = \langle a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \rangle = |v_{\mathbf{p}}|^2 + |u_{\mathbf{p}}|^2 \langle b^\dagger_{\mathbf{p}} b_{\mathbf{p}} \rangle + |v_{-\mathbf{p}}|^2 \langle b^\dagger_{-\mathbf{p}} b_{-\mathbf{p}} \rangle. \tag{A.0.16}$$

We can shed light on another crucial aspect of bogolons if we look at the behaviour of their spectrum in two different limits. For momenta $p \ll mc$, their dispersion law becomes that of phonons $\epsilon(p) = v_{\mathrm{s}} p$, where $v_{\mathrm{s}} = \sqrt{gn/m}$ is the sound velocity. This means that the low-energy excitations of the Bose gas are sound waves. In the other limit, $p \gg mc$, the behaviour of quasi-particles tends to that of free particles $\epsilon(p) \approx p^2/(2m) + gn$, where the coefficients are $|v_{-\mathbf{p}}| \ll u_{\mathbf{p}} \sim 1$ and $a^\dagger_{\mathbf{p}} \sim b_{\mathbf{p}}$. The middle ground between these two regimes is found when $p \sim mc$, or $p^2/2m \sim gn = mc^2$.

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