Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE

Corso di Laurea Magistrale in Matematica

GOOD $\mathbb{Z}$-GRADINGS OF SIMPLE LIE ALGEBRAS

Tesi di Laurea in Algebra

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Introduzione

Questa tesi intende studiare le buone graduazioni delle algebre di Lie semplici di dimensione finita e classifica tutte le buone $\mathbb{Z}$-graduazioni di $\mathfrak{gl}_n$: l'algebra di Lie di tutte le matrici $n \times n$.
Nel loro articolo [3], su cui si è basato il nostro lavoro, Elashvili e Kac studiano e danno una classificazione completa di tali $\mathbb{Z}$-graduazioni; più precisamente, per ogni elemento nilpotente e di un'algebra di Lie semplice $\mathfrak{g}$, determinano tutte le $\mathbb{Z}$-graduazioni buone di $\mathfrak{g}$ per cui e è un buon elemento. Questa analisi si basa profondamente sulle proprietà delle classi di coniugio degli elementi nilpotenti in un'algebra di Lie semisemplice.

Una $\mathbb{Z}$-graduazione $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ si dice *buona* se ammette un elemento $e \in \mathfrak{g}_2$, detto buon elemento, tale che

a) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ è iniettiva per $j \leq -1$;

b) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ è suriettiva per $j \geq -1$.

La condizione a) è equivalente al fatto che il centralizzatore $C_{\mathfrak{g}}(e)$ di e è contenuto nella parte non negativa della $\mathbb{Z}$-graduazione. Questa osservazione sarà rilevante nel seguito per classificare le graduazioni buone di $\mathfrak{gl}_n$.
L'esempio più importante di $\mathbb{Z}$-graduazione buona di $\mathfrak{g}$ è il cosiddetto *Dynkin grading*. Infatti, se $\mathfrak{g}$ è un'algebra di Lie semisemplice su un campo algebricamente chiuso di caratteristica 0 ed e è un elemento nilpotente non nullo di $\mathfrak{g}$ allora, per il Teorema di Jacobson-Morozov (si veda il Teorema 2.12), esiste in $\mathfrak{g}$ una $\mathfrak{sl}_2$-tripla $\{e, h, f\}$ contentente e. Il Dynkin grading è la $\mathbb{Z}$-

graduazione di $\mathfrak{g}$ indotta dalla decomposizione in autospazi di ad_h; questa risulta essere una graduazione buona con buon elemento e.

Più in generale, se $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ è una $\mathbb{Z}$-graduazione buona con buon elemento $e \in \mathfrak{g}_2$, $H \in \mathfrak{g}$ è l'elemento che definisce la $\mathbb{Z}$-graduazione, $\mathfrak{s} = \{e, h, f\}$ è una $\mathfrak{sl}_2$-tripla contenente e, allora $z := H - h$ appartiene al centro di $C_\mathfrak{g}(\mathfrak{s})$, che è una sottoalgebra riduttiva in $C_\mathfrak{g}(e)$. Questa osservazione spiega perché la struttura del centro di $C_\mathfrak{g}(\mathfrak{s})$ gioca un ruolo cruciale nella descrizione delle graduazioni buone. In particolare, se questo centro è banale, allora l'unica graduazione buona per cui e è un buon elemento è quella di Dynkin.

Allo scopo di classificare le graduazioni buone dell'algebra di Lie $\mathfrak{gl}_n$, si è introdotto il concetto combinatorio di piramide, che è definita come una collezione finita di scatole di taglia 1×1 nel piano, centrate in (i, j), con $i, j \in \mathbb{Z}$, tale che valgono le seguenti condizioni:

- la seconda coordinata dei centri delle scatole della riga più bassa è 1;
- la prima coordinata della j-esima riga forma una progressione aritmetica $f_j, f_j + 2, \ldots, l_j$ di differenza 2 e $f_1 = -l_1$;
- $f_j \le f_{j+1}, l_j \ge l_{j+1}$ for all j.

Data questa definizione, si è calcolata la funzione generatrice per il numero Pyr_n delle piramidi di taglia n e si sono associati a una piramide P data di taglia n due endomorfismi dello spazio vettoriale $\mathbb{F}^n$:

- uno nilpotente $e(P)$ definito facendolo agire lungo le righe della piramide (si veda la Definizione 4.10);
- uno diagonale $h(P)$ imponedo che il j-esimo elemento diagonale della matrice ad esso associata sia uguale alla prima coordinata del centro della j-esima scatola della piramide.

Grazie a un'esplicita descrizione combinatoria della base di $C_{\mathfrak{gl}_n}(e(P))$, si sono trovate le condizioni sotto le quali il centralizzatore di $e(P)$ è contenuto nella parte non negativa della $\mathbb{Z}$-graduazione data dalla decomposizione in autospazi di $\mathrm{ad}(k)$, dove $k \in \mathfrak{gl}_n$ è una matrice diagonale, e quindi si sono ottenute le condizioni necessarie e sufficienti affinchè una coppia $(k, e(P))$ sia buona:

Theorem 0.1. *Sia e(p) l'endomorfismo nilpotente definito dalla partizione $p = (p_1^{m_1}, p_2^{m_2}, \ldots, p_d^{m_d})$ del numero n. Definiamo $t_i := \sum_{j=1}^{i} m_j p_j$ per $1 \leq i \leq d$, tali che $t_1 < t_2 < \ldots < t_d = n$. Sia $P(p)$ la piramide simmetrica associata determinata dalla partizione p, sia h(p):=h(P(p)) la corrispondente matrice diagonale in $\mathfrak{gl}_n$ e sia $h = (h_1, h_2, \ldots, h_n)$ una matrice diagonale. Allora, la coppia $(h(p)+h, e(p))$ è buona se e solo se le coordinate h_i soddisfano le seguenti condizioni:*

1. *$h_i - h_j$ sono interi;*
2. *$h_1 = h_2 = \ldots = h_{t_1}, h_{t_1+1} = \ldots = h_{t_2}, \ldots, h_{t_{d-1}+1} = \ldots = h_{t_d}$;*
3. *$|h_{t_1} - h_{t_2}| \leq p_1 - p_2, \ldots, |h_{t_{d-1}} - h_{t_d}| \leq p_{d-1} - p_d$;*
4. *$\sum_{i=1}^{n} h_i = 0$.*

Inoltre, si è data una descrizione combinatoria della caratteristica di una $\mathbb{Z}$-graduazione di $\mathfrak{gl}_n$ (si veda Osservazione 4.18).

La tesi è organizzata come segue:
nel Capitolo 1 richiamiamo alcuni risultati di base di teoria di Lie su un campo algebricamente chiuso di caratteristica 0. In questo capitolo le dimostrazioni sono omesse, ma possono essere reperite in [5] dove non specificato diversamente. Tutti gli altri risultati presenti nella tesi, invece, sono provati.
Il secondo capitolo è dedicato alle algebre di Lie riduttive e alle loro principali caratterizzazioni. In particolare si sono provate le seguenti proprietà:

1. $\mathfrak{g}$ è un'algebra di Lie riduttiva se e solo se $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, con $[\mathfrak{g}, \mathfrak{g}]$ semisemplice.

2. Sia $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ un'algebra di Lie non nulla che agisce irriducibilmente su V (attraverso l'azione naturale). Allora $\mathfrak{g}$ è riduttiva, con dim$Z(\mathfrak{g}) \leq 1$. Se, inoltre, $\mathfrak{g} \subseteq \mathfrak{sl}(V)$, allora $\mathfrak{g}$ è semisemplice.

3. Sia $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ una rappresentazione di $\mathfrak{g}$ su V (con dim$V < \infty$), e sia β la forma di traccia associata su $\mathfrak{g}$. Se β è non degenere, allora $\mathfrak{g}$ è riduttiva.

Inoltre, si è provato il teorema di Jacobson-Morozov, che, come già spiegato, gioca un ruolo essenziale nella trattazione delle graduazioni buone.
Le $\mathbb{Z}$-graduazioni buone sono definite e studiate nel Capitolo 3. Abbiamo, quindi, introdotto i Dynkin grading e descritto esplicitamente quelli di $\mathfrak{sl}_n$.
Infine, nel Capitolo 4 abbiamo classificato tutte le graduazioni buone di $\mathfrak{gl}_n$.

Nell'ultimo capitolo, seguendo Hoyt [4], abbiamo esteso la definizione di graduazioni buone alle superalgebre di Lie e abbiamo iniziato a studiare le graduazioni buone della superalgebra di Cartan $W(n)$. Abbiamo, perciò provato, i seguenti risultati:

- l'unica graduazione buona per $W(2)$ è, a meno di isomorfismi, la graduazione di tipo $(2, 0)$ con buon elemento $e = \xi_1 \frac{\partial}{\partial \xi_2}$;

- non esistono buoni elementi di $W(3)$ che appartengono alla copia di $\mathfrak{gl}_3$ contenuta in $W(3)_{\bar{0}}$.

Questo è solo il punto di partenza di un'analisi che intendiamo proseguire in futuro.

Introduction

This thesis aims to study the good gradings of simple finite-dimensional Lie algebras and classifies all good $\mathbb{Z}$-gradings of $\mathfrak{gl}_n$: the Lie algebra of all $n \times n$ matrices.
In their paper [3], which we based our work on, Elashvili and Kac study and give a complete classification of such $\mathbb{Z}$-gradings; more precisely, for each nilpotent element e of a simple Lie algebra $\mathfrak{g}$, they find all good $\mathbb{Z}$-gradings of $\mathfrak{g}$ for which e is a good element. This analysis deeply relies on the properties of the conjugacy classes of the nilpotent elements in a semisimple Lie algebra.

A $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is *good* if it admits an element $e \in \mathfrak{g}_2$, called good element, such that

a) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$;

b) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$.

Condition a) is equivalent to the fact that the centralizer $C_{\mathfrak{g}}(e)$ of e lies in the non-negative part of the $\mathbb{Z}$-grading. This observation will be relevant in order to classify the good gradings of $\mathfrak{gl}_n$.
The most important example of a good $\mathbb{Z}$-grading of $\mathfrak{g}$ is the so-called *Dynkin grading*. Namely, if $\mathfrak{g}$ is a semisimple Lie algebra over an algebraically closed field of characteristic 0 and e is a nonzero nilpotent element of $\mathfrak{g}$ then, by the Jacobson-Morozov Theorem (see Theorem 2.12), there exists in $\mathfrak{g}$ a standard $\mathfrak{sl}_2$-triple $\{e, h, f\}$ containing e. The Dynkin grading is the $\mathbb{Z}$-grading of $\mathfrak{g}$

induced by the eigenspace decomposition of ad_h; this turns out to be a good grading with good element e.

More in general, if $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is a good $\mathbb{Z}$-grading with good element $e \in \mathfrak{g}_2$, $H \in \mathfrak{g}$ is the element defining the $\mathbb{Z}$-grading, $\mathfrak{s} = \{e, h, f\}$ is an $\mathfrak{sl}_2$-triple containing e, then $z := H - h$ lies in the center of $C_{\mathfrak{g}}(\mathfrak{s})$, which is a reductive subalgebra in $C_{\mathfrak{g}}(e)$. This observation explains why the structure of the center of $C_{\mathfrak{g}}(\mathfrak{s})$ plays a crucial role in the description of the good gradings. In particular, if this center is trivial, then the only good grading for which e is a good element is the Dynkin one.

In order to classify the good gradings of the Lie algebra $\mathfrak{gl}_n$, we introduce the combinatorial concept of pyramid that is a finite collection of boxes of size 1×1 on the plane, centered at (i, j), where $i, j \in \mathbb{Z}$, such that the following conditions hold:

- the second coordinates of the centers of the boxes of the lowest row equal 1;
- the first coordinates of the j^{th} row form an arithmetic progression $f_j, f_j + 2, \ldots, l_j$ with difference 2 and $f_1 = -l_1$;
- $f_j \leq f_{j+1}, l_j \geq l_{j+1}$ for all j.

Given this definition, we calculate the generating function for the number Pyr_n of pyramids of size n and associate to a given pyramid P of size n two endomorphisms of the vector space $\mathbb{F}^n$:

- a nilpotent one $e(P)$ by letting it act along the rows of the pyramid (see Definition 4.10);
- a diagonal one $h(P)$ by letting the j^{th} diagonal entry of its associated matrix equal to the first coordinate of the center of the j^{th} box of the pyramid.

Thanks to an explicit combinatorial description of the basis of $C_{\mathfrak{gl}_n}(e(P))$, we can find the conditions under which the centralizer of $e(P)$ is contained in the non-negative part of the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(k)$ where $k \in \mathfrak{gl}_n$ is a diagonal matrix, and thus obtain the necessary and sufficient conditions such that a pair $(k, e(P))$ is good:

Theorem 0.2. *Let e(p) be the nilpotent element defined by a partition* $p = (p_1^{m_1}, p_2^{m_2}, \ldots, p_d^{m_d})$ *of the number n. Define* $t_i := \sum_{j=1}^{i} m_j p_j$ *for* $1 \leq i \leq d$ *so that* $t_1 < t_2 < \ldots < t_d = n$*. Let P(p) be the symmetric pyramid determined by the partition p, let h(p):=h(P(p)) be the corresponding diagonal matrix in* $\mathfrak{gl}_n$ *and let* $h = (h_1, h_2, \ldots, h_n)$ *be a diagonal matrix. Then, the pair* $(h(p) + h, e(p))$ *is good if and only if the coordinates* h_i *satisfy the following conditions:*

1. $h_i - h_j$ *are integers;*

2. $h_1 = h_2 = \ldots = h_{t_1}, h_{t_1+1} = \ldots = h_{t_2}, \ldots, h_{t_{d-1}+1} = \ldots = h_{t_d}$*;*

3. $|h_{t_1} - h_{t_2}| \leq p_1 - p_2, \ldots, |h_{t_{d-1}} - h_{t_d}| \leq p_{d-1} - p_d$*;*

4. $\sum\limits_{i=1}^{n} h_i = 0.$

Furthermore, we are able to give a combinatorial description of the characteristic of a $\mathbb{Z}$-grading of $\mathfrak{gl}_n$ (see Remark 4.18).

The thesis is organized as follows:
in Chapter 1 we recall some basic results of Lie theory on an algebraically closed field of characteristic 0. In this chapter proofs are omitted and can be found in [5] where not else specified. All other results in the thesis are proved.
The second chapter is dedicated to reductive Lie algebras and their main characterizations. In particular we prove the following:

1. $\mathfrak{g}$ is a reductive Lie algebra if and only if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.

2. Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a nonzero Lie algebra acting irreducibly on V (via the natural action). Then $\mathfrak{g}$ is reductive, with dim $Z(\mathfrak{g}) \leq 1$. If in addition $\mathfrak{g} \subseteq \mathfrak{sl}(V)$, then $\mathfrak{g}$ is semisimple.

3. Let $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V (with $\dim V < \infty$), and let β be the associated trace form on $\mathfrak{g}$. If β is non-degenerate, then $\mathfrak{g}$ is reductive.

Moreover, we prove the Jacboson-Morozov theorem which, as we already explained, plays an essential role in the discussion on good gradings.
The good $\mathbb{Z}$-gradings are defined and studied in Chapter 3. We introduce the Dynkin gradings and explicitly describe those of $\mathfrak{sl}_n$.
Finally, in Chapter 4 we classify all good gradings of $\mathfrak{gl}_n$.

In the last chapter, following Hoyt [4], we extend the definition of good gradings to Lie superalgebras and start studying the good gradings of the Cartan superalgebra $W(n)$. We prove the following:

- the unique good grading for $W(2)$ is, up to isomorphisms, the grading of type $(2,0)$ with good element $e = \xi_1 \frac{\partial}{\partial \xi_2}$;

- there are no good elements of $W(3)$ belonging to the copy of $\mathfrak{gl}_3$ contained in $W(3)_{\bar{0}}$.

This is just the starting point of an analysis which we intend to carry out in the future.

Contents

Chapter 1

Preliminaries

In this chapter we recall some basic fundamental results on semisimple Lie algebras. All the proofs can be found in [5].

1.1 Basic results on Lie algebras

Definition 1.1. A vector space $\mathfrak{g}$ over a field $\mathbb{F}$, with a bilinear operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ and called the *bracket* or *commutator* of x and y, is called a *Lie algebra* over $\mathbb{F}$ if the following axioms are satisfied:

(L1) $[x, x] = 0$ for all x in $\mathfrak{g}$.

(L2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for $x, y, z \in \mathfrak{g}$ (*Jacobi identity*).

There is a standard way to associate a Lie algebra to an associative algebra. Indeed, if $(A, \cdot)$ is an associative algebra, defining

$$[x, y] := x \cdot y - y \cdot x \text{ for } x, y \in A, \tag{1.1}$$

$(A, [,])$ becomes a Lie algebra.

Example 1.2.
- Let $M_n(\mathbb{F})$ be the set of matrices of order n over $\mathbb{F}$ and let $\cdot$ be the usual product of matrices. Then the Lie algebra associated to the algebra $(M_n(\mathbb{F}), \cdot)$ is denoted by $\mathfrak{gl}(n, \mathbb{F}) \equiv \mathfrak{gl}_n$.

We write down the multiplication table for $\mathfrak{gl}_n$ relative to the standard basis consisting of the matrices E_{ij} (having 1 in the (i,j) position and 0 elsewhere). Since $E_{ij}E_{kl} = \delta_{jk}E_{il}$, we have:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}.$$

- If V is a finite dimensional vector space over $\mathbb{F}$, denote by $\mathrm{End}(V)$ the set of linear transformations $V \to V$. Define the bracket as in (1.1), so that $\mathrm{End}(V)$ becomes a Lie algebra over $\mathbb{F}$. In order to distinguish this new algebra structure from the old associative one, we write $\mathfrak{gl}(V)$ for $\mathrm{End}(V)$ viewed as a Lie algebra and call it the *general linear algebra.*

Definition 1.3. Let $\mathfrak{g}, \mathfrak{g}'$ be two Lie algebras. A linear map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ such that $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$ is called a *Lie algebra homomorphism.* If φ is also bijective, it is called *Lie algebra isomorphism.*

Remark 1.4. Let V be an n-dimensional vector space over $\mathbb{F}$. The Lie algebras $\mathfrak{gl}(V)$ and $\mathfrak{gl}_n$ are isomorphic, hence in the following we will identify them.

Definition 1.5. Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if $[x,y] \in \mathfrak{h}$ whenever $x, y \in \mathfrak{h}$.

Example 1.6. Let $\mathfrak{sl}_n \equiv \mathfrak{sl}_n(\mathbb{F}) := \{X \in \mathfrak{gl}_n \mid \mathrm{tr}X = 0\}$. Then $\mathfrak{sl}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$ since, if $X, Y \in \mathfrak{sl}_n$, then also $[X,Y] \in \mathfrak{sl}_n$.

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra. A subset $I \subseteq \mathfrak{g}$ is called *ideal* if $[x,y] \in I$ for every $x \in I, y \in \mathfrak{g}$.

Definition 1.8. Let $\mathfrak{g}$ be a Lie algebra. We define the following ideals of $\mathfrak{g}$:

- the *derived algebra* of $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}] := \langle [x,y] \mid x, y \in \mathfrak{g} \rangle$
- the *center* of $\mathfrak{g}$ is $Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x,z] = 0 \text{ for all } z \in \mathfrak{g}\}$

Definition 1.9. A Lie algebra $\mathfrak{g}$ is called *simple* if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and it has no proper ideals.

Example 1.10.

1. Let us consider the following basis of $\mathfrak{sl}_2(\mathbb{F})$, that is usually called the *standard basis* of $\mathfrak{sl}_2$:

$$\{e,h,f\} := \{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}.$$

   Then:

   - $[h,e] = 2e$;
   - $[e,f] = h$;
   - $[h,f] = -2f$.

   If $\text{char}\mathbb{F} \neq 2$, then $\mathfrak{sl}_2(\mathbb{F})$ is simple.

2. $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \langle I_n \rangle$, where $\langle I_n \rangle = Z(\mathfrak{gl}_n)$. In partivular $\mathfrak{gl}_n$ is not simple.

Definition 1.11. Let $(A, \cdot)$ be an algebra over the field $\mathbb{F}$. A *derivation* of A is a linear map $\delta : A \to A$ satisfying the familiar product rule $\delta(ab) = a\delta(b) + \delta(a)b$, called the *Leibniz rule*.

It is easily checked that the collection $\text{Der}(A)$ of all derivations of A is a vector subspace of $\text{End}(A)$ and also that the commutator $[\delta, \delta']$ of two derivations is again a derivation. So $\text{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$. Note that, on the contrary, $\text{Der}(A)$ is not a subalgebra of the associative algebra $\text{End}(A)$.
Since a Lie algebra $\mathfrak{g}$ is an $\mathbb{F}$-algebra, $\text{Der}(\mathfrak{g})$ is defined. Certain derivations arise quite naturally, as follows.

Definition 1.12. Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. We denote by $\text{ad}_x : \mathfrak{g} \to \mathfrak{g}$ the endomorphism of $\mathfrak{g}$ defined as $y \mapsto [x,y]$.

Remark 1.13. $\text{ad}_x \in \text{Der}(\mathfrak{g})$, because, due to the Jacobi identity, ad_x satisfies the Leibniz rule with respect to the bracket.

Derivations of the form ad_x, $x \in \mathfrak{g}$, are called *inner* and all others *outer*.

Definition 1.14. A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V over $\mathbb{F}$ ($\dim V < \infty$) is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$.

An important example to keep in mind is the *adjoint representation*

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

which sends x to ad_x.

Definition 1.15. A representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called *irreducible* if there does not exist a non-zero subspace $W \subsetneq V$ such that $\varphi(\mathfrak{g})(W) \subseteq W$.

Example 1.16. Consider the standard representation of $\mathfrak{gl}_n$ on $\mathbb{F}^n$, i.e.,

$$\begin{aligned} \varphi : \mathfrak{gl}_n &\to \mathfrak{gl}(\mathbb{F}^n) \\ X &\mapsto X \end{aligned}$$

where $X : \mathbb{F}^n \to \mathbb{F}^n$ is such that $v \mapsto Xv$. Then φ is irreducible since $\mathfrak{gl}_n$ acts on $\mathbb{F}^n$ transitively.

It is often convenient to use the language of modules along with the language of representations. As in other algebraic theories, there is a natural definition.

Definition 1.17. Let $\mathfrak{g}$ be a Lie algebra. A vector space V, endowed with a bilinear operation $\mathfrak{g} \times V \to V$ (denoted $(x, v) \mapsto x.v$) is called a $\mathfrak{g}$-*module* if the following condition is satisfied:

$$[x, y].v = x.y.v - y.x.v \text{ for all } x, y \in \mathfrak{g} \text{ and } v \in V.$$

Definition 1.18. Let V, W be $\mathfrak{g}$-modules. A $\mathfrak{g}$-*module homomorphism* between V and W is a homomorphism of vector spaces $\varphi : V \to W$ such that $\varphi(x.v) = x.\varphi(v)$ for every $x \in \mathfrak{g}, v \in V$.

Remark 1.19. The concept of $\mathfrak{g}$-module is equivalent to the one of representation. Indeed, if $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, then V may be viewed as a $\mathfrak{g}$-module via the action $x.v = \varphi(x)(v)$. Conversely, given a $\mathfrak{g}$-module V, this equation defines a representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Definition 1.20. A $\mathfrak{g}$-module V is called *irreducible* if it has precisely two $\mathfrak{g}$-submodules (itself and 0), i.e., if it does not exist a non-zero $\mathfrak{g}$-submodule $W \subsetneq V$ such that $\mathfrak{g}.W \subseteq W$.

Definition 1.21. A $\mathfrak{g}$-module V is called *completely reducible* if V is a direct sum of irreducible $\mathfrak{g}$-submodules.

An important role in the theory of Lie algebras (and in the following) is the one played by the finite irreducible representations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of characteristic zero. Here we recall the principal properties of these representations.

Theorem 1.22. *Let V be an $\mathfrak{sl}_2$-module of dimension $s+1$ for some $s \in \mathbb{Z}_{\geq 0}$. Then:*

1. *there exists a unique (up to scalar multiples) nonzero element $w \in V$ such that $e.w = 0$ and $V = \langle w, f.w, \ldots, f^s.w \rangle$, called the highest weight vector of V;*

2. *$V = \bigoplus_{k=0}^{s} V_{s-2k}$, where $V_{s-2k} = \{v \in V \mid h.v = (s-2k)v\} = \langle f^k.w \rangle$ is called weight space of weight $s-2k$;*

3. *either 0 or 1 appears exactly once as a weight and the weights are all even or all odd;*

4. *$e.f^k.w = k(s-k+1)f^{k-1}.w$ for all $k \in \mathbb{Z}^+$.*

Now we give the definitions and the main results concerning nilpotent and solvable Lie algebras.

Definition 1.23. Let $\mathfrak{g}$ be a Lie algebra. The sequence of ideals of $\mathfrak{g}$ defined by

$$\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1], \ldots, \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$$

is called the *descending central sequence* of $\mathfrak{g}$.

Definition 1.24. $\mathfrak{g}$ is called *nilpotent* if there exists $n \in \mathbb{Z}_+$ such that $\mathfrak{g}^n = 0$.

Example 1.25.
- Let $\mathfrak{n}(n, \mathbb{F})$ be the Lie algebra of all strictly upper triangular matrices. Then $\mathfrak{n}(n, \mathbb{F})$ is nilpotent.
- Let $\mathfrak{t}(n, \mathbb{F})$ be the Lie algebra of all upper triangular matrices. Then $\mathfrak{t}(n, \mathbb{F})$ is not nilpotent.

Remark 1.26. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then ad_x is a nilpotent endomorphism for every $x \in \mathfrak{g}$. We will say that x is *ad-nilpotent.*

Theorem 1.27. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent endomorphisms. Then there exists a nonzero $v \in V$ such that $\mathfrak{g}.v = 0$, i.e., such that v is an eigenvector common to all endomorphisms in $\mathfrak{g}$, relative to the eigenvalue 0.*

Theorem 1.28 (Engel)**.** *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if x is ad-nilpotent for every $x \in \mathfrak{g}$.*

Definition 1.29. Let $\mathfrak{g}$ be a Lie algebra. The sequence of ideals of $\mathfrak{g}$ defined by

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \ldots, \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$$

is called the *derived series* of $\mathfrak{g}$.

Definition 1.30. $\mathfrak{g}$ is called *solvable* if there exists $n \in \mathbb{Z}^+$ such that $\mathfrak{g}^{(n)} = 0$.

Example 1.31.
- If a Lie algebra $\mathfrak{g}$ is nilpotent, then it is solvable (since $\mathfrak{g}^{(k)} \subset \mathfrak{g}^k$ for every k).
- $\mathfrak{t}(n, \mathbb{F})$ is solvable (but not nilpotent).
- If a Lie algebra $\mathfrak{g}$ is simple, then it is not solvable, since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Next we assemble a few simple observations about solvability.

Proposition 1.32. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *If $\mathfrak{g}$ is solvable, then so are all subalgebras and homomorphic images of $\mathfrak{g}$.*

2. *If I is a solvable ideal of $\mathfrak{g}$ such that $\mathfrak{g}/I$ is solvable, then $\mathfrak{g}$ itself is solvable.*

3. *If I, J are solvable ideals of $\mathfrak{g}$, then so is $I + J$.*

1.2 Semisimple Lie algebras and Cartan decomposition

In this section we recall the definition of semisimple Lie algebras, their main properties and how the Cartan decomposition arises.

Remark 1.33. Let $\mathfrak{g}$ be an arbitrary Lie algebra and let S be a maximal (with respect to inclusion) solvable ideal of $\mathfrak{g}$. If I is any other solvable ideal of $\mathfrak{g}$ then, by point 3. of Proposition 1.32, we know that $S + I = S$, i.e., $I \subseteq S$. This proves the existence of a unique maximal solvable ideal of $\mathfrak{g}$, called the *radical* of $\mathfrak{g}$ and denoted by $\mathrm{Rad}(\mathfrak{g})$.

Definition 1.34. $\mathfrak{g}$ is called *semisimple* if $\mathrm{Rad}(\mathfrak{g}) = 0$.

Example 1.35. A simple Lie algebra is also semisimple.

Theorem 1.36 (Lie). *Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$ ($\mathrm{dim}V < \infty$) over an algebraically closed field $\mathbb{F}$ of characteristic 0. If $V \neq 0$, then V contains a common eigenvector for all the endomorphisms of $\mathfrak{g}$.*

We recall the following standard result of linear algebra, which will be very helpful in the following.

Proposition 1.37. 1. *Let $x \in \mathrm{End}(V)$ be diagonalizable. Let $W \subsetneq V$ be a vector subspace such that $x(W) \subseteq W$. Then $x_{|W}$ is diagonalizable.*

2. *Let $x, y \in \mathrm{End}(V)$ be diagonalizable such that $[x, y] = 0$. Then x, y are simultaneously diagonalizable, i.e. they have the same eigenvectors.*

Definition 1.38. An element $x \in \mathfrak{gl}(V)$ is called *semisimple* if it is diagonalizable.

Proposition 1.39. *Let $x \in \mathfrak{gl}(V)$. Then:*

1. *there exist unique $x_s, x_n \in \mathfrak{gl}(V)$ with x_s semisimple and x_n nilpotent such that $[x_s, x_n] = 0$ and $x = x_s + x_n$.*

2. *There exist two polynomials $p(\lambda), q(\lambda) \in \mathbb{F}(\lambda)$ such that $p(0) = 0 = q(0)$ and $p(x) = x_s$ and $q(x) = x_n$. In particular x_s and x_n commute with all the endomorphisms commuting with x.*

3. *If $A \subset B \subset V$ are linear subspaces such that $x(B) \subseteq A$, then $x_s(B) \subseteq A$ and $x_n(B) \subseteq A$.*

The decomposition $x = x_s + x_n$ is called the Jordan-Chevalley decomposition of x.

Proposition 1.40. *If $x \in \mathfrak{gl}(V)$ is semisimple, then so is ad_x. Furthermore, if $v_1, \ldots, v_n$ is a basis of eigenvectors of V relative to the eigenvalues $\lambda_1, \ldots, \lambda_n$, then the eigenvectors of ad_x are the standard basis of $\mathfrak{gl}(V)$ relative to $\{v_1, \ldots, v_n\}$ with eigenvalues $\lambda_i - \lambda_j$.*

Remark 1.41. Let $x = x_s + x_n$ be the Jordan-Chevalley decomposition of $x \in \mathfrak{gl}(V)$. Then the Jordan-Chevalley decomposition of ad_x is $\mathrm{ad}_x = \mathrm{ad}_{x_s} + \mathrm{ad}_{x_n}$.

Definition 1.42. Let $\mathfrak{g}$ be any Lie algebra. The bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ defined by $K(x, y) = \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y)$ for $x, y \in \mathfrak{g}$ is called the *Killing form* of $\mathfrak{g}$.

Remark 1.43. The Killing form K of $\mathfrak{g}$ is:

1. symmetric;

2. associative, i.e., $K([x, y], z) = K(x, [y, z])$.

Remark 1.44. We recall that the *radical* rad_K of the Killing form K is defined as follows:

$$\mathrm{rad}_K = \{x \in \mathfrak{g} \mid K(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}.$$

Then K is non-degenerate if and only if $\mathrm{rad}_K = 0$. Furthermore rad_K is an ideal of $\mathfrak{g}$.

Theorem 1.45. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate.*

Theorem 1.46. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exist ideals $I_1, \ldots, I_k$ of $\mathfrak{g}$ which are simple (as Lie algebras) such that $\mathfrak{g} = I_1 \oplus \ldots \oplus I_k$. Moreover, if I is a simple ideal of $\mathfrak{g}$, then $I = I_j$ for some j.*

Corollary 1.47. *If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

Theorem 1.48. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then* $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

Definition 1.49. Let $\mathfrak{g}$ be semisimple and let $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful finite dimensional representation of $\mathfrak{g}$. The bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ such that $\beta(x, y) = \mathrm{tr}(\Phi(x)\Phi(y))$ for $x, y \in \mathfrak{g}$ is called the *trace form* of $\mathfrak{g}$ associated to Φ.

Example 1.50. If $\mathfrak{g}$ is a Lie algebra, then the trace form associated to the adjoint representation is the Killing form.

Remark 1.51. As for the Killing form, one can show that β is symmetric and associative (i.e. $\beta([x, y], z) = \beta(x, [y, z])$ for every $x, y, z \in \mathfrak{g}$). Furthermore, if $\mathfrak{g}$ is semisimple then β is non-degenerate.

Theorem 1.52 (Weyl). *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ a finite dimensional representation of $\mathfrak{g}$. Then Φ is completely reducible.*

Definition 1.53. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We define the *normalizer* of $\mathfrak{h}$ in $\mathfrak{g}$ as the following subalgebra:

$$N_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] \subseteq \mathfrak{g}\}.$$

Definition 1.54. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. $\mathfrak{h}$ is called a *Cartan subalgebra* of $\mathfrak{g}$ if it is nilpotent and $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Definition 1.55. A *toral subalgebra* of a Lie algebra $\mathfrak{g}$ is a non-zero subalgebra of $\mathfrak{g}$ consisting of semisimple elements.

Proposition 1.56. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if it is a maximal toral subalgebra of $\mathfrak{g}$.*

Proposition 1.56 holds more in general for reductive Lie algebras which will be introduced in Chapter 2.

From now on, $\mathfrak{g}$ will denote a (non-zero) semisimple Lie algebra.

Remark 1.57. Every semisimple Lie algebra $\mathfrak{g}$ contains at least one semisimple element. Hence $\mathfrak{g}$ always contains a non-zero toral subalgebra.

Lemma 1.58. *A toral subalgebra T of $\mathfrak{g}$ is abelian.*

Now fix a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Since $\mathfrak{h}$ is abelian by Lemma 1.58, $\{\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g} \mid h \in \mathfrak{h}\}$ is a commuting family of semisimple endomorphisms. Then, according to point 2. of Proposition 1.37, the family above is simultaneously diagonalizable.
Hence we can write

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$.
Notice that $\mathfrak{g}_0 = C_\mathfrak{g}(\mathfrak{h}) \supseteq \mathfrak{h} \neq 0$, i.e. $\mathfrak{g}_0 \neq 0$. So we can decompose $\mathfrak{g}$ as follows.

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Phi = \{\alpha \in \mathfrak{h}^*, \alpha \neq 0 \mid \mathfrak{g}_\alpha \neq 0\}$. This decomposition is known as the *Cartan decomposition* of $\mathfrak{g}$ and Φ is called the *root system of* $\mathfrak{g}$ relative to $\mathfrak{h}$.

Now we give some simple observations about the root space decomposition of a semisimple Lie algebra.

Remark 1.59. $\#\Phi < \infty$.

Proposition 1.60. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be the Cartan decomposition of $\mathfrak{g}$. Then, for all $\alpha, \beta \in \mathfrak{h}^*$:*

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$;
2. *if* $\alpha, \beta \in \mathfrak{h}^*$ *such that* $\alpha + \beta \neq 0$, *then* $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

Corollary 1.61. *The restriction of the Killing form of* $\mathfrak{g}$ *to* $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$ *is non-degenerate.*

Proposition 1.62. *Let* $\mathfrak{h}$ *be a maximal toral subalgebra of* $\mathfrak{g}$. *Then* $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$.

Thus we can rewrite the Cartan decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Remark 1.63. Corollary 1.61 combined with Proposition 1.62 allows us to say that the restriction of the Killing form to $\mathfrak{h}$ is non-degenerate. Hence we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via the isomorphism given by:

$$\begin{array}{rclll} h & \mapsto & \varphi_h(t) = K(h,t) & & \\ t_\alpha & \leftarrow\!\shortmid & \alpha & \text{if } \alpha(h) = K(t_\alpha, h) & (*) \end{array}$$

Theorem 1.64. *Let* Φ *be the root system of* $\mathfrak{g}$.

1. Φ *spans* $\mathfrak{h}^*$.
2. *If* $\alpha \in \Phi$, *then* $-\alpha \in \Phi$.
3. *Let* $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$. *Then* $[x, y] = K(x, y)t_\alpha$ *(*t_α *as in* $(*)$*).*
4. *If* $\alpha \in \Phi$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle t_\alpha \rangle$.
5. *If* $\alpha \in \Phi$, *then* $K(t_\alpha, t_\alpha) \neq 0$.
6. *If* $\alpha \in \Phi$ *and* x_α *is a nonzero element of* $\mathfrak{g}_\alpha$, *then there exists* $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ *such that* $\langle x_\alpha, x_{-\alpha}, h_\alpha := [x_\alpha, x_{-\alpha}] \rangle \cong \mathfrak{sl}_2$.
7. $h_\alpha = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$ *and* $h_{-\alpha} = -h_\alpha$.

Proposition 1.65. 1. $\alpha \in \Phi$ *implies dim* $\mathfrak{g}_\alpha = 1$.

2. *If $\alpha \in \Phi$, the only scalar multiples of α which are roots are α and $-\alpha$.*

3. *If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$, where h_α is the element introduced in Theorem 1.64. The numbers $\beta(h_\alpha)$ are called Cartan integers.*

4. *If $\alpha, \beta, \alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

5. *Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Let r, q be (respectively) the largest integers for which $\beta - r\alpha, \beta + q\alpha$ are roots. Then $\beta + k\alpha \in \Phi$ for all $k = -r, \dots, q$, and $\beta(h_\alpha) = r - q$.*

Remark 1.66. Since the restriction to $\mathfrak{h}$ of the Killing form is non-degenerate by Remark 1.63, we may transfer the form to $\mathfrak{h}^*$, letting $(\alpha, \beta) := K(t_\alpha, t_\beta)$ for all $\alpha, \beta \in \mathfrak{h}^*$. This is a non-degenerate bilinear form on $\mathfrak{h}^*$, with $(\alpha, \beta) \in \mathbb{Q}$ for every $\alpha, \beta \in \mathfrak{h}^*$. We know that Φ spans $\mathfrak{h}^*$ by point 1. of Theorem 1.64, so we can choose a basis $\{\alpha_1, \dots, \alpha_l\}$ of $\mathfrak{h}^*$ consisting of roots. If we set $E_\mathbb{Q} := \text{span}_\mathbb{Q}\{\alpha_1, \dots, \alpha_l\}$, we can prove that $(.,.)$ is a positive definite form on $E_\mathbb{Q}$. Now, let $E_\mathbb{R}$ be the real vector space obtained by extending the base field from $\mathbb{Q}$ to $\mathbb{R}$ (i.e., $E_\mathbb{R} := \mathbb{R} \otimes E_\mathbb{Q}$); the form extends canonically to $E_\mathbb{R}$ and is positive definite.

In the following, E will denote a Euclidean space, i.e., a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric bilinear form $(.,.)$. Moreover, α will denote a nonzero element in E.

Definition 1.67. Define the *reflecting hyperplane* of α as follows:

$$P_\alpha = \{v \in E \mid (v, \alpha) = 0\}.$$

Definition 1.68. The invertible linear transformation $\sigma_\alpha : E \to E$ such that $\sigma_\alpha(v) = v$ for all $v \in P_\alpha$ and $\sigma_\alpha(\alpha) = -\alpha$ is called the *reflecion* with respect to P_α.

Remark 1.69. It is easy to write down an explicit formula for σ_α:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

Definition 1.70. Let $\alpha, \beta \in E$. We define

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Definition 1.71. A subset Φ of the Euclidean space E is called an *abstract root system* in E if the following axioms are satisfied:

(R1) Φ is finite, spans E and does not contain 0.

(R2) If $\alpha \in \Phi$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$.

(R3) For all $\alpha, \beta \in \Phi$, $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in \Phi$.

(R4) For all $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

Example 1.72. The root system of a semisimple Lie algebra is an abstract root system in $E_\mathbb{R}$.

Definition 1.73. Call $\ell := \dim E$ the *rank* of the root system Φ.

Proposition 1.74. *Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Then:*

1. *if $(\alpha, \beta) > 0$, $\alpha - \beta \in \Phi$;*

2. *if $(\alpha, \beta) < 0$, $\alpha + \beta \in \Phi$.*

Definition 1.75. A subset Δ of Φ is called a *base* if:

(B1) Δ is a basis of E;

(B2) each root $\beta \in \Phi$ can be written as $\beta = \sum_{\gamma \in \Delta} n_\gamma \gamma$, with integral coefficients n_γ all non-negative or all non-positive.

The roots in Δ are called *simple*. We call $\beta \in \Phi$ *positive* (resp. *negative*) if $n_\gamma \geq 0$ (resp. $n_\gamma \leq 0$) for every $\gamma \in \Delta$.

Theorem 1.76. *Every root system Φ has a base.*

Definition 1.77. Let $(\Phi, E_{\mathbb{R}})$ be a root system and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a base of Φ. The *Coxeter graph* associated to (Φ, Δ) is defined as a graph having l vertices in which the i^{th} and the j^{th} vertix $(i \neq j)$ are linked by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges.

Definition 1.78. In the same setting as before, we call *Dynkin diagram* of Φ the Coxeter graph of Φ in which we add an arrow from the i^{th} to the j^{th} vertix if $(\alpha_i, \alpha_i) > (\alpha_j, \alpha_j)$.

Chapter 2

On the Jacobson-Morozov Theorem

2.1 Basic results on reductive Lie algebras

Definition 2.1. A Lie algebra $\mathfrak{g}$ for which $\mathrm{Rad}(\mathfrak{g}) = Z(\mathfrak{g})$ is called *reductive*.

Example 2.2.

1. A commutative Lie algebra $\mathfrak{g}$ is reductive, since $\mathrm{Rad}(\mathfrak{g}) = \mathfrak{g} = Z(\mathfrak{g})$.

2. If $\mathfrak{g}$ is a semisimple Lie algebra, then it is reductive since $\mathrm{Rad}(\mathfrak{g}) = \{0\} = Z(\mathfrak{g})$.

3. $\mathfrak{gl}_n$ is reductive, since $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \langle I_n \rangle$, and $Z(\mathfrak{gl}_n) = \langle I_n \rangle = \mathrm{Rad}(\mathfrak{gl}_n)$.

Proposition 2.3.

1. $\mathfrak{g}$ is a reductive Lie algebra if and only if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.

2. Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a nonzero Lie algebra acting irreducibly on V (via the natural action). Then $\mathfrak{g}$ is reductive, with $\dim Z(\mathfrak{g}) \leq 1$. If in addition $\mathfrak{g} \subseteq \mathfrak{sl}(V)$, then $\mathfrak{g}$ is semisimple.

Proof. 1. By definition of reductive Lie algebra we have $\mathrm{Rad}(\mathfrak{g}) = Z(\mathfrak{g})$; hence $\mathfrak{g}' := \mathfrak{g}/Z(\mathfrak{g})$ is semisimple.

The adjoint action induces an action of $\mathfrak{g}'$ on $\mathfrak{g}$: for $\bar{x} := x + Z(\mathfrak{g}) \in \mathfrak{g}'$, we consider $\mathrm{ad}_{\bar{x}} : \mathfrak{g} \to \mathfrak{g}$ such that $y \mapsto [x, y]$ (it is easy to see that it is well defined).
Since $\mathfrak{g}'$ is semisimple, $\mathfrak{g}$ is a completely reducible $\mathfrak{g}'$-module by Weyl's theorem (Theorem 1.52). Hence we can write $\mathfrak{g} = Z(\mathfrak{g}) \oplus M$, where $Z(\mathfrak{g})$ is a trivial $\mathfrak{g}'$-submodule of $\mathfrak{g}$ because it is an ideal of $\mathfrak{g}$, and M is an ideal of $\mathfrak{g}$ by definition of $\mathrm{ad}_{\bar{x}}$. Moreover, $[\mathfrak{g}, \mathfrak{g}] = [M, M] = M$ by Corollary 1.47 because $M \cong \mathfrak{g}'$, that is semisimple.
The converse is true since, if we consider any solvable ideal $I \subset [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, then it must be $I \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$ since otherwise $I \cap [\mathfrak{g}, \mathfrak{g}]$ would be a nonzero solvable ideal in $[\mathfrak{g}, \mathfrak{g}]$, that is semisimple. So $I \subseteq Z(\mathfrak{g})$ and hence $\mathrm{Rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$.

2. Let $S = \mathrm{Rad}\mathfrak{g}$. By Lie's theorem (Theorem 1.36), there exists an eigenvector $v \in V$ common to all the elements of S, i.e., such that $s.v = \lambda(s)v$ for all $s \in S$. Now, if $x \in \mathfrak{g}$ then $[x, s] \in S$; thus

$$s.(x.v) = x.(s.v) - [s, x].v = \lambda(s)x.v - \lambda([s, x])v. \tag{2.1}$$

Since $\mathfrak{g}$ acts irreducibly on V, all vectors in V are obtainable by repeated applications of elements of $\mathfrak{g}$ to v and formation of linear combinations. It therefore follows from (2.1) that the matrices of all $s \in S$ (relative to a suitable basis of V) are upper triangular, with $\lambda(s)$ the only diagonal entry. However, the commutators $[s, x] \in S$ ($s \in S$, $x \in \mathfrak{g}$) have trace 0, so this condition forces λ to vanish on $[S, \mathfrak{g}]$. Referring back to (2.1), we now conclude that $s \in S$ acts diagonally on V as the scalars $\lambda(s)$. In particular, $S = Z(\mathfrak{g})$; so $\mathfrak{g}$ is reductive and $\dim S \leq 1$.
Finally, if $\mathfrak{g} \subseteq \mathfrak{sl}(V)$, since $\mathfrak{sl}(V)$ contains no scalars except 0, $S = 0$ and thus $\mathfrak{g}$ is semisimple.

□

Proposition 2.4. *Let* $n \in \mathbb{N}$. *Then:*

1. *$\mathfrak{sl}_n$ is semisimple;*

2. *$\mathfrak{sp}_{2n}$ is semisimple.*

Proof. 1. Let V be an n-dimensional vector space. Since $\mathfrak{gl}(V) = \mathfrak{sl}(V) \oplus \langle I_V \rangle$ and since $\mathfrak{gl}(V)$ acts irreducibly on V (see Example 1.16), then it is clear that $\mathfrak{sl}(V)$ acts irreducibly as well. Thus, by point 2. of Proposition 2.3, $\mathfrak{sl}_n$ is semisimple.

2. Let V be a $2n$-dimensional vector space. Notice that any subspace W of V which is invariant under the action of a subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is also invariant under the action of the (associative) subalgebra of End$V$ generated by I_V and $\mathfrak{g}$. Indeed, if $w \in W$ then $(\alpha I_V + \sum_i \beta_i x_i)(w) = \alpha w + \sum_i \beta_i x_i(w) \in W$ for all $x_i \in \mathfrak{g}$, $\alpha, \beta_i \in \mathbb{F}$. We now want to prove that all the endomorphisms in V are obtainable from I_V and $\mathfrak{sp}_{2n}$ using addiction, scalar multiplication and ordinary multiplication. From I_V we get all scalars. Now, $E_{ii} = \frac{1}{2}((E_{ii} - E_{i+n,i+n}) + I_{2n})(E_{ii} - E_{i+n,i+n})$ for all $i = 1, \dots, n$ and similarly for $i = n+1, \dots, 2n$. Therefore we get all possible diagonal matrices. Now, multiplying various other basis elements (such as $E_{ij} - E_{ji}$) by suitable E_{ii} yields all the possible off-diagonal matrices E_{ij}.
Thus, using Example 1.16 combined with point 2. of Proposition 2.3, we get that $\mathfrak{sp}_{2n}$ is semisimple.

□

Proposition 2.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $x \in \mathfrak{g}$ a semisimple element. Then $C_{\mathfrak{g}}(x)$ is reductive. Furthermore, if $\mathfrak{h} \subset \mathfrak{g}$ is a maximal toral subalgebra containing x, $C_{\mathfrak{g}}(x) = \mathfrak{h} \oplus \oplus_{\alpha \in \Phi_x} \mathfrak{g}_\alpha$, where $\Phi_x = \{\alpha \in \Phi \mid \alpha(x) = 0\}$ and Φ is the root system of $\mathfrak{g}$.*

Proof. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. On one hand, if $w \in C_{\mathfrak{g}}(x)$ lies in $\mathfrak{g}_\alpha$ for some $\alpha \in \Phi$ then, by definition, $[h, w] = \alpha(h)w$ for all $h \in \mathfrak{h}$. But the element $x \in \mathfrak{h}$ must centralize w, and so we get $\alpha(x)w = 0$.

Let $\Phi_x = \{\alpha \in \Phi \mid \alpha(x) = 0\}$. These remarks show that

$$C_{\mathfrak{g}}(x) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_x} \mathfrak{g}_\alpha.$$

One can easily show that Φ_x satisfies the axioms of a root system; hence we can consider a base of simple roots Δ_x for Φ_x.
In order to show that $C_{\mathfrak{g}}(x)$ is reductive, we may show (using point 1. of Proposition 2.3) the decomposition $[C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)] \oplus Z(C_{\mathfrak{g}}(x))$. Begin by defining

$$\mathfrak{h}_1 := \bigcap_{\alpha \in \Phi_x} \ker\alpha.$$

Since the span of the roots in Φ_x has dimension equal to the rank of the root system Φ_x by Definition 1.73 and Remark 1.66, we get that $\dim(\mathfrak{h}_1) = \dim(\mathfrak{h}) - \text{rank}(\Phi_x)$. For each $\alpha \in \Phi$, pick elements $h_\alpha \in \mathfrak{h}$ as in point 6. of Theorem 1.64. Now define

$$\mathfrak{h}_2 := \text{span}_{\mathbb{F}}\{h_\alpha \mid \alpha \in \Delta_x\}.$$

Clearly $\dim(\mathfrak{h}_2) = \text{rank}(\Phi_x)$. We have now the following refined decomposition of $C_{\mathfrak{g}}(x)$:

$$C_{\mathfrak{g}}(x) = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \bigoplus_{\alpha \in \Phi_x} \mathfrak{g}_\alpha.$$

Finally we can say that:

- $Z(C_{\mathfrak{g}}(x)) = \mathfrak{h}_1$. Indeed, if we take $h_1 \in \mathfrak{h}_1$ then $[h_1, \mathfrak{h}_1] = [h_1, \mathfrak{h}_2] = 0$ since toral subalgebras are commutative by Lemma 1.58. Furthermore, if $z_\alpha \in \mathfrak{g}_\alpha$, then $[h_1, z_\alpha] = \alpha(h_1) z_\alpha = 0$. Hence $\mathfrak{h}_1 \subseteq Z(C_{\mathfrak{g}}(x))$. The converse follows using standard properties of root systems.

- $[C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)] = \mathfrak{h}_2 \oplus \oplus_{\alpha \in \Phi_x} \mathfrak{g}_\alpha$. Indeed we know that $[\mathfrak{h}_1, C_{\mathfrak{g}}(x)] = 0$. Furthermore $[\mathfrak{h}_2, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle h_\alpha \rangle$ for all $\alpha \in \Delta_x$.

Notice that $K([C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)], \mathfrak{h}_1) = K(C_{\mathfrak{g}}(x), [C_{\mathfrak{g}}(x), \mathfrak{h}_1]) = 0$. Hence the Killing form of $[C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)]$ is non-degenerate because otherwise it would result $K([C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)], \mathfrak{g}) = 0$, against the semisemplicity of $\mathfrak{g}$ (see Theorem 1.45). Thus, by point 1. of Proposition 2.3, $C_{\mathfrak{g}}(x)$ is reductive. □

Lemma 2.6. *Let* $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ *be an irreducible representation of* $\mathfrak{g}$ *on* V *(with* $\dim V < \infty$*). Then* $\Phi([\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]) = 0$.

Proof. We know that $\mathrm{Rad}(\mathfrak{g})$ is solvable; then $\Phi(\mathrm{Rad}(\mathfrak{g})) \subset \mathfrak{gl}(V)$ is solvable by Proposition 1.32. Hence, by Lie's Theorem, there exists a nonzero $v \in V$ such that $\Phi(h)(v) = \lambda(h)v$ for every $h \in \mathrm{Rad}(\mathfrak{g})$, where $\lambda \in \mathrm{Rad}(\mathfrak{g})^*$.
Now, let $V_\lambda = \{w \in V : \Phi(h)(w) = h.w = \lambda(h)w \text{ for all } h \in \mathrm{Rad}(\mathfrak{g})\} \neq 0$. We want to show that $\mathfrak{g}.V_\lambda \subset V_\lambda$. If $x \in \mathfrak{g}$, $h \in \mathrm{Rad}(\mathfrak{g})$, $w \in V_\lambda$, then:

$$h.x.w = [h, x].w + x.h.w = \lambda([h, x])w + \lambda(h)x.w \tag{2.2}$$

Now we want to show that $\lambda([h, x]) = 0$. Let $W_n = \mathrm{span}_{\mathbb{F}}\{v, x.v, \dots, x^{n-1}.v\}$, where $n \in \mathbb{N}$ is the minimum such that $\{v, x.v, \dots, x^n.v\}$ are linearly dependent. Denote $W_i = \mathrm{span}_{\mathbb{F}}\{v, x.v, \dots, x^{i-1}.v\}$, for $i = 1, \dots, n$. We claim that $h.x^r.v = \lambda(h)x^r.v + \omega_r$, where $\omega_r \in W_r$. Indeed, by induction on r:

- if $r = 1$ we are in the case of (2.2);
- if $r > 1$,

$$\begin{aligned} h.(x^{r+1}.v) &= h.(x.(x^r.v)) = [h, x].(x^r.v) + x.(h.(x^r.v)) \\ &= \lambda([h, x])x^r.v + \omega_r + \lambda(h)x^{r+1}.v + x.\omega'_r, \text{ where } \omega_r, \omega'_r \in W_r. \end{aligned}$$

  Hence $h.(x^{r+1}.v) = \lambda(h)x^{r+1}.v + \omega_{r+1}$, where $\omega_{r+1} \in W_{r+1}$.

Now, let $h \in \mathrm{Rad}(\mathfrak{g})$. Consider $h_{|W_n} : W_n \to W_n$; its matrix, with respect to the basis $\{v, x.v, \dots, x^{n-1}.v\}$ is:

$$\begin{pmatrix} \lambda(h) & * & \dots & * \\ 0 & \lambda(h) & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda(h) \end{pmatrix}$$

Hence the trace of $h_{|W_n}$ is $n\lambda(h)$. In particular, every element of $\mathfrak{g}$ of the form $[h, x] \in \mathfrak{g}$ has trace $n\lambda([h, x])$; but the trace of a commutator is 0, and so $\lambda([h, x]) = 0$.

Hence, by (2.2) we obtain that $h.x.v = \lambda(h)x.v$. So V_λ is a submodule of V and, since $V_\lambda \neq 0$ and V is irreducible, it holds $V = V_\lambda$, that implies $\Phi(h) = \lambda(h)\mathrm{id}_V$ for every $h \in \mathrm{Rad}(\mathfrak{g})$.
Now, if $h = [x, h'] \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$, then $\Phi(h)v = \Phi(x)\Phi(h')v - \Phi(h')\Phi(x)v = \lambda(h')\Phi(x)v - \lambda(h')\Phi(x)v = 0$ for every $v \in V$. Hence $\Phi([\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]) = 0$. □

Theorem 2.7. *Let $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V (with* $\mathrm{dim}V < \infty$*), and let β_Φ be the associated trace form on $\mathfrak{g}$. If β_Φ is non-degenerate, then $\mathfrak{g}$ is reductive.*

Proof. We can construct a sequence of $\mathfrak{g}$-submodules of V:

$$\{0\} = V_0 \subset V_1 \subset \dots V_t = V$$

such that V_i/V_{i-1} is irreducible. Indeed:

- if V is irreducible, we take $V_0 = \{0\}$, $V_1 = V$;
- if V is not irreducible, we take W to be a maximal $\mathfrak{g}$-submodule of V. Then V/W is irreducible and we can construct the sequence of $\mathfrak{g}$-submodules of V above iterating this procedure.

By Lemma 2.6, $[\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ acts trivially on V_i/V_{i-1}, i.e., for every $x \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ we have $x.V_i \subseteq V_{i-1}$. But the V_i's are $\mathfrak{g}$-modules, and hence we can say that $y.x.V_i \subseteq V_{i-1}$ for every $y \in \mathfrak{g}$. So, if we take a basis of V obtained by completing a basis of V_{i-1} to a basis of V_i for every $i = 1, \dots, t$, we have that the matrix associated to $\Phi(y)\Phi(x)$ is strictly upper triangular, and hence $\beta_\Phi(y, x) = \mathrm{tr}(\Phi(y)\Phi(x)) = 0$ for every $y \in \mathfrak{g}$, $x \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$. But β_Φ is non-degenerate by assumption, hence $[\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})] = 0$. So $\mathrm{Rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$ and, since the converse is always true, $\mathfrak{g}$ is reductive. □

2.2 The Jacobson Morozov theorem

Remark 2.8. Let $A \in \mathfrak{gl}(V)$, where V is a finite dimensional vector space. Then A is nilpotent if and only if $\mathrm{tr}(A^k) = 0$ for all $k \in \mathbb{Z}^+$.

Proof. If A is nilpotent, then $\operatorname{tr}(A^k) = 0$ for all $k \in \mathbb{Z}^+$ because all eigenvalues of A are 0, and hence so are all eigenvalues of A^k.
Conversely, suppose that $\operatorname{tr}(A^k) = 0$ for all $k \in \mathbb{Z}^+$. By contradiction, suppose that A is not nilpotent, with nonzero eigenvalues $\lambda_1, \dots, \lambda_r$ and corresponding multiplicities $m_1, \dots, m_r$. Then $\operatorname{tr}(A^k) = m_1\lambda_1^k + \ldots + m_r\lambda_r^k$ for every k. Hence, we have:

$$\begin{cases} m_1\lambda_1 + \ldots + m_r\lambda_r = 0 \\ \qquad\vdots \\ m_1\lambda_1^r + \ldots + m_r\lambda_r^r = 0 \end{cases} \tag{2.3}$$

i.e.,

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

But

$$\det \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{pmatrix} = \lambda_1 \cdot \ldots \cdot \lambda_r \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \cdots & \lambda_r^{r-1} \end{pmatrix}.$$

This is the determinant of the Vandermonde matrix, that is nonzero. Hence the unique solution of (2.3) is $m_1 = \ldots = m_r = 0$, that is absurd. □

Lemma 2.9. *Let $C \in \mathfrak{gl}(V)$, where V is a finite-dimensional vector space. Suppose that $C = \sum_{i=1}^{r}[A_i, B_i]$ (with $A_i, B_i \in \mathfrak{gl}(V)$) and that $[C, A_i] = 0$ for $i = 1, 2, \dots r$. Then C is nilpotent.*

Proof. For $i \in \{1, \dots r\}$, we have $[C^{k-1}, A_i] = 0$ for $k \geq 1$ where $C^0 = Id_V$, indeed:

$$[C^{k-1}, A_i] = [\underbrace{C \cdot \ldots \cdot C}_{k-1}, A_i] = \underbrace{C \cdot \ldots \cdot C}_{k-1} A_i - A_i \underbrace{C \cdot \ldots \cdot C}_{k-1} =$$
$$= \underbrace{C \cdot \ldots \cdot C}_{k-2} A_i C - A_i \underbrace{C \cdot \ldots \cdot C}_{k-1} =$$
$$= \ldots = A_i \underbrace{C \cdot \ldots \cdot C}_{k-1} - A_i \underbrace{C \cdot \ldots \cdot C}_{k-1} = 0$$

Hence
$C^k = C^{k-1}C = \sum_{i=1}^{r} C^{k-1}[A_i, B_i] = \sum_{i=1}^{r} C^{k-1}(A_iB_i - B_iA_i) = \sum_{i=1}^{r}(A_i(C^{k-1}B_i) - (C^{k-1}B_i)A_i) = \sum_{i=1}^{r}[A_i, C^{k-1}B_i]$.
Since the trace of any commutator is 0, this gives $\mathrm{tr}(C^k) = 0$ for $k \geq 1$. Hence C is nilpotent by Remark 2.8. □

Lemma 2.10 (Morozov)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0. Suppose that there exist $e, h \in \mathfrak{g}$ such that $[h, e] = 2e$ and $h \in [e, \mathfrak{g}]$. Then there exists $f \in \mathfrak{g}$ such that $[h, f] = -2f$, $[e, f] = h$ (and $[h, e] = 2e$).*

Proof. By assumption $h \in [e, \mathfrak{g}]$, then there exists an element $z \in \mathfrak{g}$ such that $h = [e, z]$. Moreover, since ad is a homomorphism of Lie algebras, we have that:

- $[\mathrm{ad}_h, \mathrm{ad}_e] = \mathrm{ad}_{[h,e]} = \mathrm{ad}_{2e} = 2\mathrm{ad}_e$;
- $[\mathrm{ad}_e, \mathrm{ad}_z] = \mathrm{ad}_{[e,z]} = \mathrm{ad}_h$.

The first of these relations together with Lemma 2.9 implies that ad_e is nilpotent. Besides,

$$[e, [h, z] + 2z] = [e, [h, z]] + 2[e, z] = [[e, h], z] - [h, [e, z]] + 2h$$
$$= [-2e, z] - [h, h] + 2h = -2h + 2h = 0$$

Hence $[h, z] = -2z + x_1$, where $x_1 \in C_{\mathfrak{g}}(e)$, the centralizer of e in $\mathfrak{g}$.
Since $[\mathrm{ad}_h, \mathrm{ad}_e] = 2\mathrm{ad}_e$, if $b \in C_{\mathfrak{g}}(e)$, then:

$$\mathrm{ad}_e\mathrm{ad}_h(b) = \mathrm{ad}_h\mathrm{ad}_e(b) - 2\mathrm{ad}_e(b) = 0 \tag{2.4}$$

Hence $\mathrm{ad}_h(b) \in C_{\mathfrak{g}}(e)$; therefore, $\mathrm{ad}_h(C_{\mathfrak{g}}(e)) \subseteq C_{\mathfrak{g}}(e)$.
Moreover we notice that the following relation holds:

$$\begin{aligned}[\mathrm{ad}_e^i, \mathrm{ad}_z] &\underset{(*)}{=} \mathrm{ad}_e^{i-1}[\mathrm{ad}_e, \mathrm{ad}_z] + \mathrm{ad}_e^{i-2}[\mathrm{ad}_e, \mathrm{ad}_z]\mathrm{ad}_e + \cdots + [\mathrm{ad}_e, \mathrm{ad}_z]\mathrm{ad}_e^{i-1} \\ &\underset{(**)}{=} \mathrm{ad}_e^{i-1}\mathrm{ad}_h + \mathrm{ad}_e^{i-2}\mathrm{ad}_h\mathrm{ad}_e + \cdots + \mathrm{ad}_h\mathrm{ad}_e^{i-1}\end{aligned}$$

(the right hand side of $(*)$ is $(\mathrm{ad}_e^i\mathrm{ad}_z - \mathrm{ad}_e^{i-1}\mathrm{ad}_z\mathrm{ad}_e) + (\mathrm{ad}_e^{i-1}\mathrm{ad}_z\mathrm{ad}_e - \mathrm{ad}_e^{i-2}\mathrm{ad}_z\mathrm{ad}_e^2) + (\mathrm{ad}_e^{i-2}\mathrm{ad}_z\mathrm{ad}_e^2 - \mathrm{ad}_e^{i-3}\mathrm{ad}_z\mathrm{ad}_e^3) + \cdots + (\mathrm{ad}_e\mathrm{ad}_z\mathrm{ad}_e^{i-1} - \mathrm{ad}_z\mathrm{ad}_e^i)$; thus only the first and the last term of the sum survive, that is exactly $[\mathrm{ad}_e^i, \mathrm{ad}_z]$).

By induction on $k \in \mathbb{N}$, we can also prove that:

$$\mathrm{ad}_e^k\mathrm{ad}_h = \mathrm{ad}_h\mathrm{ad}_e^k - 2k\mathrm{ad}_e^k. \tag{2.5}$$

Indeed:

- if $k = 1$
  $\mathrm{ad}_e\mathrm{ad}_h - \mathrm{ad}_h\mathrm{ad}_e = [\mathrm{ad}_e, \mathrm{ad}_h] = -2\mathrm{ad}_e$.
- if $k > 1$

$$\begin{aligned}\mathrm{ad}_e^k\mathrm{ad}_h &= \mathrm{ad}_e\mathrm{ad}_e^{k-1}\mathrm{ad}_h = \mathrm{ad}_e(\mathrm{ad}_h\mathrm{ad}_e^{k-1} - 2(k-1)\mathrm{ad}_e^{k-1}) \\ &= (\mathrm{ad}_e\mathrm{ad}_h)\mathrm{ad}_e^{k-1} - 2(k-1)\mathrm{ad}_e^k \\ &= (\mathrm{ad}_h\mathrm{ad}_e - 2\mathrm{ad}_e)\mathrm{ad}_e^{k-1} - 2(k-1)\mathrm{ad}_e^k \\ &= \mathrm{ad}_h\mathrm{ad}_e^k - 2\mathrm{ad}_e - 2(k-1)\mathrm{ad}_e^k \\ &= \mathrm{ad}_h\mathrm{ad}_e^k - 2k\mathrm{ad}_e^k.\end{aligned}$$

Now, applying relation (2.5) in the equality $(**)$, we get:

$$[\mathrm{ad}_e^i, \mathrm{ad}_z] = i(\mathrm{ad}_h - (i-1))\mathrm{ad}_e^{i-1}.$$

Let $b \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^{i-1})$. Then there exists $a \in \mathfrak{g}$ such that $b = \mathrm{ad}_e^{i-1}(a)$ and $\mathrm{ad}_e(b) = \mathrm{ad}_e(\mathrm{ad}_e^{i-1}(a)) = \mathrm{ad}_e^i(a) = 0$. Hence:

$$i(\mathrm{ad}_h - (i-1))\mathrm{ad}_e^{i-1}(a) = [\mathrm{ad}_e^i, \mathrm{ad}_z](a) = \mathrm{ad}_e^i\mathrm{ad}_z(a) - \mathrm{ad}_z\mathrm{ad}_e^i(a) = \mathrm{ad}_e^i(\mathrm{ad}_z(a)),$$

meaning that $i(\mathrm{ad}_h - (i-1))\mathrm{ad}_e^{i-1}(a) \in \mathrm{Im}(\mathrm{ad}_e^i)$.
Thus, by this and since $\mathrm{ad}_h(C_{\mathfrak{g}}(e)) \subseteq C_{\mathfrak{g}}(e)$, we have

$$i(\mathrm{ad}_h - (i-1))(b) \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^i). \tag{2.6}$$

It follows from this relation and the nilpotency of ad_e that, if b is any element of $C_{\mathfrak{g}}(e)$, there exists a positive integer m such that:

$$(\mathrm{ad}_h - m)(\mathrm{ad}_h - (m-1)) \cdot \ldots \cdot (\mathrm{ad}_h - 1)\mathrm{ad}_h(b) = 0.$$

In fact, by (2.6) we have that:

$$\begin{aligned}
&b \in C_{\mathfrak{g}}(e) = C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^0)\\
&\Rightarrow \mathrm{ad}_h(b) \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^1)\\
&\Rightarrow (\mathrm{ad}_h - 1)\mathrm{ad}_h(b) \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^2)\\
&\vdots\\
&\Rightarrow (\mathrm{ad}_h - m)(\mathrm{ad}_h - (m-1)) \cdot \ldots \cdot (\mathrm{ad}_h - 1)\mathrm{ad}_h(b) \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^{m+1}) = \{0\}.
\end{aligned}$$

This tells us that the characteristic roots of $\mathrm{ad}_{h|C_{\mathfrak{g}}(e)} : C_{\mathfrak{g}}(e) \to C_{\mathfrak{g}}(e)$ are non-negative integers. Hence $\mathrm{ad}_h + 2$ induces a non-singular linear transformation in $C_{\mathfrak{g}}(e)$ and consequently there exists $y_1 \in C_{\mathfrak{g}}(e)$ such that $(\mathrm{ad}_h + 2)(y_1) = x_1$, where $x_1 \in C_{\mathfrak{g}}(e)$ is the element such that $[h, z] = 2z + x_1$. Then $[h, y_1] = -2y_1 + x_1$. Hence, if we set $f = z - y_1$, we have $[h, f] = [h, z] - [h, y_1] = -2z + x_1 + 2y_1 - x_1 = -2(z - y_1) = -2f$. Also, thanks to the fact that $y_1 \in C_{\mathfrak{g}}(e)$, we have $[e, f] = [e, z] - [e, y_1] = [e, z] = h$. Hence the thesis holds. □

Lemma 2.11. *Let $e \in \mathfrak{g}$ be a nilpotent element and K be the Killing form on $\mathfrak{g}$. Then $K(e, C_{\mathfrak{g}}(e)) = 0$.*

Proof. Take $y \in C_{\mathfrak{g}}(e)$. Then $\mathrm{ad}_{[e,y]} = 0$, i.e., $[\mathrm{ad}_e, \mathrm{ad}_y] = 0$. Therefore $\mathrm{ad}_e\mathrm{ad}_y = \mathrm{ad}_y\mathrm{ad}_e$. This means that, for arbitrary $k \in \mathbb{Z}^+$, $(\mathrm{ad}_e\mathrm{ad}_y)^k = \mathrm{ad}_e^k\mathrm{ad}_y^k$. By the nilpotency of e we can take $k >> 1$ such that $\mathrm{ad}_e^k = 0$. Then $(\mathrm{ad}_e\mathrm{ad}_y)^k = 0$, i.e., $\mathrm{ad}_e\mathrm{ad}_y$ is nilpotent and hence its trace is zero. Therefore $K(e, y) = \mathrm{tr}(\mathrm{ad}_e\mathrm{ad}_y) = 0$. □

Now we prove a strengthened version of point 6. of Theorem 1.64, the so called Jacobson-Morozov Theorem.

Theorem 2.12 (Jacobson-Morozov). *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field of charactesristic 0. If e is a nonzero nilpotent element of $\mathfrak{g}$, then there exists a standard triple $\{e, h, f\}$ for $\mathfrak{g}$.*

Proof. We will argue by induction on the dimension of $\mathfrak{g}$.
If this is 3 (the smallest dimension for a semisimple Lie algebra), then $\mathfrak{g}$ must be isomorphic to $\mathfrak{sl}_2$. Indeed, if $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is the Cartan decomposition of $\mathfrak{g}$ then, by point 1. of Theorem 1.64, $\#\Phi > 0$. Thus by point 6. of Theorem 1.64 we know that $\mathfrak{g}$ contains an $\mathfrak{sl}_2$-triple. But, since $\dim\mathfrak{g} = 3$, then $\mathfrak{g} \cong \mathfrak{sl}_2$. Now, take $z \in \mathfrak{sl}_2 \cong \mathfrak{g}$ nilpotent and denote by $\{e, h, f\}$ the standard basis of $\mathfrak{sl}_2$. Then $z = ae + bf$ (h can not appear because it is semisimple); but $\det(z - \lambda I_2) = \det \begin{pmatrix} -\lambda & a \\ b & -\lambda \end{pmatrix} = \lambda^2 - ab$. So z is nilpotent if and only if either $a = 0$ or $b = 0$. Therefore,

- if $z = ae$, then $\{z, h, \frac{1}{a} f\}$ is an $\mathfrak{sl}_2$-triple containing z;
- if $z = bf$, then $\{z, -h, \frac{1}{b} e\}$ is an $\mathfrak{sl}_2$-triple containing z.

Assume $\dim(\mathfrak{g}) > 3$. If e lies in a proper semisimple Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, then by induction we can find an $\mathfrak{sl}_2$-triple in $\mathfrak{a}$, that is an $\mathfrak{sl}_2$-triple also in $\mathfrak{g}$.
Thus we may assume for the remainder of the proof that e does not lie in any proper semisimple Lie subalgebra of $\mathfrak{g}$.
Let K be the Killing form on $\mathfrak{g}$. First of all, notice that $(C_\mathfrak{g}(e))^\perp = [\mathfrak{g}, e]$ where the orthogonal complement is take relative to the Killing form. Indeed:

- $[\mathfrak{g}, e] \subseteq (C_\mathfrak{g}(e))^\perp$ because, if $x = [z, e] \in [\mathfrak{g}, e]$, then $K(x, C_\mathfrak{g}(e)) = K([z, e], C_\mathfrak{g}(e)) = K(z, [e, C_\mathfrak{g}(e)]) = 0$;
- $\dim(C_\mathfrak{g}(e))^\perp = \dim\mathfrak{g} - \dim(C_\mathfrak{g}(e))$ and, considering $\mathrm{ad}_e : \mathfrak{g} \to \mathfrak{g}$, we can say that $\dim\mathfrak{g} = \dim[e, \mathfrak{g}] + \dim C_\mathfrak{g}(e)$.

By Lemma 2.11 we can say that $K(e, C_{\mathfrak{g}}(e)) = 0$ and so $e \in (C_{\mathfrak{g}}(e))^{\perp} = [\mathfrak{g}, e]$. Thus there exists $h' \in \mathfrak{g}$ such that $[h', e] = 2e$.
Claim 1. There exists a semisimple element h such that $[h, e] = 2e$.
To see this, let $h' = h'_s + h'_n$ be the Jordan-Chevalley decomposition of h' in $\mathfrak{g}$. By point 3. of Proposition 1.39 we know that h'_s and h'_n stabilize every subspace that h' stabilizes. So h'_s acts semisimply and h'_n act nilpotently on the subspace $\langle e \rangle$; hence $[h'_s, e] = 2e$, $[h'_n, e] = 0$. Thus we may take $h = h'_s$.
Claim 2. If h is as in Claim 1, then $h \in [\mathfrak{g}, e]$
By contradiction, suppose that $h \notin [\mathfrak{g}, e]$. Then, as $[\mathfrak{g}, e] = (C_{\mathfrak{g}}(e))^{\perp}$, we must have

$$K(h, C_{\mathfrak{g}}(e)) \neq 0. \tag{2.7}$$

By an easy calculation with the Jacobi identity we see that ad_h leaves $C_{\mathfrak{g}}(e)$ invariant. Hence ad_h must act semisimply on $C_{\mathfrak{g}}(e)$, so we may decompose $C_{\mathfrak{g}}(e)$ into ad_h eigenspaces:

$$C_{\mathfrak{g}}(e) = \bigoplus_{\tau_i \in \mathbb{F}} C_{\mathfrak{g}}(e)_{\tau_i}.$$

Note that $C_{\mathfrak{g}}(e)_0 = \{z \in C_{\mathfrak{g}}(e) \mid [h, z] = 0\} = C_{C_{\mathfrak{g}}(e)}(h)$. So we have:

$$C_{\mathfrak{g}}(e) = C_{C_{\mathfrak{g}}(e)}(h) \oplus \bigoplus_{\tau_i \neq 0} C_{\mathfrak{g}}(e)_{\tau_i}. \tag{2.8}$$

By the invariance of the Killing form $K(h, [h, C_{\mathfrak{g}}(e)]) = K([h, h], C_{\mathfrak{g}}(e)) = 0$. Thus, if z is a nonzero element of $C_{\mathfrak{g}}(e)_{\tau_i}$ with $\tau_i \neq 0$, then $0 = K(h, [h, z]) = K(h, \tau_i z) = \tau_i K(h, z)$. This shows that

$$h \in (C_{\mathfrak{g}}(e)_{\tau_i})^{\perp} \text{ for all } \tau_i \neq 0. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we can say that there exists $z \in C_{C_{\mathfrak{g}}(e)}(h)$ such that $K(h, z) \neq 0$. If z is nilpotent then, by Lemma 2.11 we can say that $K(h, z) = 0$, a contradiction. Hence $z_s \neq 0$. By point 3. of Proposition 1.39, we can say that z_s is a nonzero semisimple element in $C_{C_{\mathfrak{g}}(e)}(f)$. By Proposition 2.5 we know that $C_{\mathfrak{g}}(z_s)$ is reductive, whence $[C_{\mathfrak{g}}(z_s), C_{\mathfrak{g}}(z_s)]$ is a semisimple Lie subalgebra of $\mathfrak{g}$. It is a proper subalgebra, since $C_{\mathfrak{g}}(z_s) = \mathfrak{g}$

only if $z_s = 0$. We have now shown that $h \in C_{\mathfrak{g}}(z_s)$ and $e \in C_{\mathfrak{g}}(z_s)$. Hence $2e = [h, e] \in [C_{\mathfrak{g}}(z_s), C_{\mathfrak{g}}(z_s)]$. Thus our nilpotent element e belongs to a proper semisimple subalgebra of $\mathfrak{g}$, in contradiction to our assumption.

Hence, by Lemma 2.10, we conclude that there exists $f \in \mathfrak{g}$ such that $\{e, h, f\}$ in an $\mathfrak{sl}_2$-triple. □

Chapter 3

Good $\mathbb{Z}$-gradings

3.1 Basic definitions

From now on, we shall assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0.

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra. A $\mathbb{Z}$-*grading* of $\mathfrak{g}$ is a decomposition:

$$\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$$

where the $\mathfrak{g}_j$'s are vector subspaces of $\mathfrak{g}$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

Remark 3.2. If $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ is semisimple, then there exists an element $H \in \mathfrak{g}_0$ defining the $\mathbb{Z}$-grading, i.e., such that $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [H, x] = kx\}$ for all $k \in \mathbb{Z}$.

Proof. Define $\phi : \mathfrak{g} \to \mathfrak{g}$ such that, for $x \in \mathfrak{g}_k$, $\phi(x) = kx$, and extend it on $\mathfrak{g}$ by linearity. This endomorphism is a derivation. Indeed, if $x \in \mathfrak{g}_k$ and $y \in \mathfrak{g}_j$, then

$$\phi([x, y]) = (k + j)[x, y] \text{ since } [x, y] \in \mathfrak{g}_{k+j}$$
$$[\phi(x), y] + [x, \phi(y)] = k[x, y] + j[x, y] = (k + j)[x, y].$$

Since all derivations of $\mathfrak{g}$ are inner by Theorem 1.48, i.e., $\mathrm{Der}\mathfrak{g} = \mathrm{ad}\mathfrak{g}$, there exists $H \in \mathfrak{g}$ such that $\phi = \mathrm{ad}_H$. So if $x \in \mathfrak{g}_k$, we have that $\phi(x) = \mathrm{ad}_H(x) \Leftrightarrow kx = [H, x]$. Hence $\mathfrak{g}_k = \{x \in \mathfrak{g} : [H, x] = kx\}$. $\square$

Definition 3.3. A $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is called *even* if $\dim\mathfrak{g}_j = 0$ for all j odd.

Proposition 3.4. *Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a semisimple Lie algebra. Then $K(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i + j \neq 0$.*

Proof. Take $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$ $(i + j \neq 0)$ and $H \in \mathfrak{g}_0$ defining the $\mathbb{Z}$-grading. Then:

$$-iK(x, y) = K([x, H], y) = K(x, [H, y]) = jK(x, y).$$

Hence $(i + j)K(x, y) = 0$ and, as $i + j \neq 0$, $K(x, y) = 0$. □

Proposition 3.5. *Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a semisimple Lie algebra. Then $\mathfrak{g}_0$ is reductive.*

Proof. By Proposition 3.4, we have $K(\mathfrak{g}_0, \mathfrak{g}_i) = 0$ for every $i \neq 0$. Hence $K_{|\mathfrak{g}_0\times\mathfrak{g}_0}$ is non-degenerate. Indeed, if we take $z \in \mathfrak{g}_0$ such that $K(z, \mathfrak{g}_0) = 0$ then, since $K(z, \mathfrak{g}_i) = 0$ for every $i \neq 0$, $K(z, \mathfrak{g}) = 0$. But K is non-degenerate on $\mathfrak{g}$ because $\mathfrak{g}$ is semisimple, and so $z = 0$. Hence, by Theorem 2.7, $\mathfrak{g}_0$ is reductive. □

Definition 3.6. Let $\mathfrak{g}$ be a Lie algebra and $S \subset \mathfrak{g}$. The *centralizer* of S in $\mathfrak{g}$ is defined as follows:

$$C_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid [x, S] = 0\}.$$

Definition 3.7. An element $e \in \mathfrak{g}_2$ is called *good* if the following properties hold:

a) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$;

b) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$.

Remark 3.8. Given the definition of good element, we can immediately observe that:

1. e is a nonzero ad-nilpotent element of $\mathfrak{g}$;

2. Point a) of Definition 3.7 is equivalent to the fact that the centralizer $C_{\mathfrak{g}}(e)$ of e lies in $\oplus_{j\geq 0}\mathfrak{g}_j$;

3. $\mathrm{ad}_e : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is bijective;

4. $[\mathfrak{g}_0, \mathfrak{g}_2] = \mathfrak{g}_2$.

Proof. 1. If $e = 0$, then $\mathrm{ad}_e(x) = 0$ for all $x \in \mathfrak{g}_j$. But this contradicts point a) of Definition 3.7.
Moreover $\mathrm{ad}_e^k \in \mathfrak{g}_{2k} = 0$ for $k >> 1$ since $\mathfrak{g}$ is finite-dimensional.

2. 3.7a) $\Rightarrow C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$.
Suppose by contradiction that $x \in C_{\mathfrak{g}}(e)$, $x \neq 0$ such that $x \in \bigoplus_{j\leq -1} \mathfrak{g}_j$.
Write $x = \sum_{j\leq -1} x_j$, with $x_j \in \mathfrak{g}_j$. Then $0 = \mathrm{ad}_e(x) = \sum_{j\leq -1} \mathrm{ad}_e(x_j)$.
Since every summand lies in a different homogeneous component of the $\mathbb{Z}$-grading, then $\mathrm{ad}_e(x_j) = 0$ for all $j \leq -1$. But ad_e is injective for $j \leq -1$, i.e., $x_j = 0$ for all j. Hence $x = 0$.
$C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j \Rightarrow$ 3.7a).
Fix $j \leq -1$ and let $x, y \in \mathfrak{g}_j$ with $x \neq y$, such that $\mathrm{ad}_e(x) = \mathrm{ad}_e(y)$, then
$$[e, x] = [e, y] \Leftrightarrow [e, x - y] = 0 \Leftrightarrow \mathrm{ad}_e(x - y) = 0.$$
Since $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$, $x - y \in \oplus_{j\geq 0}\mathfrak{g}_j$. This is a contradiction because, by assumption, $0 \neq x - y \in \mathfrak{g}_j$, with $j \leq -1$.

3. It follows from a) and b) for $j = -1$.

4. Obvious by property b) of Definition 3.7 for $j = 0$.

□

Definition 3.9. A $\mathbb{Z}$-grading of $\mathfrak{g}$ is called *good* if it admits a good element.

3.2 Dynkin $\mathbb{Z}$-gradings

The most important examples of good $\mathbb{Z}$-gradings of $\mathfrak{g}$ correspond to $\mathfrak{sl}_2$-triples $\{e, h, f\}$, where $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. We call the good

$\mathbb{Z}$-gradings thus obtained the *Dynkin $\mathbb{Z}$-gradings.* In this section we show more precisely what a Dynkin $\mathbb{Z}$-grading is and why it is good.

Let $e \in \mathfrak{g}$ be a nonzero nilpotent element. By the Jacobson-Morozov Theorem (Theorem 2.12), e embeds into a $\mathfrak{sl}_2$-triple $\{e, h, f\}$, i.e., $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Since ad_h acts semisimply on $\mathfrak{g}$, we can decompose $\mathfrak{g}$ into the direct sum of its eigenspaces:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_\lambda,$$

where $\mathfrak{g}_\lambda = \{z \in \mathfrak{g} \mid \mathrm{ad}_h(z) = \lambda z\}$.
Let $\mathfrak{a} := \langle e, h, f \rangle$ and consider the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ such that $x \mapsto \mathrm{ad}_x$). Then, by Weyl's Theorem (Theorem 1.52), since $\mathfrak{a}$ is semisimple, $\mathfrak{g}$ decomposes as a direct sum of irreducible finite-dimensional ($\mathfrak{a} \cong \mathfrak{sl}_2$)-modules $\mathfrak{g}_{s_k}$:

$$\mathfrak{g} = \bigoplus_{k=1}^{r} \mathfrak{g}_{s_k} \tag{3.1}$$

where $\mathfrak{g}_{s_k} = \langle w_k, f.w_k, \ldots, f^{s_k}.w_k \rangle$ with $h.w_k = s_k w_k$ and $e.w_k = 0$ (by $x.z$ we denote $\mathrm{ad}_x(z)$).
Now, since the weights of h on $\mathfrak{g}_{s_k}$ are integers for every k, we can write:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \tag{3.2}$$

where $\mathfrak{g}_i = \{z \in \mathfrak{g} \mid [h, z] = ix\}$.
This decomposition of $\mathfrak{g}$ is called *Dynkin $\mathbb{Z}$-grading* associated to the nilpotent element e. Notice that it does not depend on the choice of the $\mathfrak{sl}_2$-triple containing e (see [9]).

Remark 3.10. The decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ introduced in (3.2) is a $\mathbb{Z}$-grading.

Proof. Let $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, then $[h, [x, y]] = [[h, x], y] + [x, [h, y]] = i[x, y] + j[x, y] = (i + j)[x, y]$, i.e., $[x, y] \in \mathfrak{g}_{i+j}$. $\square$

Proposition 3.11. *The $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ introduced in* (3.2) *is good with good element e.*

Proof.

- $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$.
  By point 2. of Remark 3.8, it is enough to show that $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$. Thanks to the decomposition (3.1) of $\mathfrak{g}$ as sum of irreducible $\mathfrak{sl}_2$-modules, we can say that $C_{\mathfrak{g}}(e) = \langle w_1, \dots, w_r\rangle$, with $h.w_i = s_i w_i$ and $s_i \geq 0$ for all $i = 1, \dots, r$. Hence $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$.

- $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$.
  Fix $j \geq -1$. Thanks to the decomposition (3.1), we can find a basis of $\mathfrak{g}_{j+2}$ consisting of elements of the form $f^k.w_i$, where $h.f^k.w_i = (j+2)f^k.w_i$ for some $k \in \{0, \dots, s_i\}$ and $i \in \{1, \dots, r\}$. By the representation theory of $\mathfrak{sl}_2$ (in particular point 4. of Theorem 1.22), we know that:

  $$e.f^l.w_i = l(s_i - l + 1)f^{l-1}.w_i \text{ for all } l \in \mathbb{Z}^+.$$

  Then, for $l = k+1$, we have $e.f^{k+1}.w_i = (k+1)(s_i - k)f^k.w_i$. Hence ad_e is surjective on $\mathfrak{g}_j$ because:
  · $f^{k+1}.w_i \in \mathfrak{g}_j$ since $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is a $\mathbb{Z}$-grading;
  · $s_i \neq k$ because if not, we would have $h.f^{s_i}.w_i = (j+2)f^{s_i}.w_i$. But, by representation theory of $\mathfrak{sl}_2$ we know that $h.f^{s_i}.w_i = -s_i f^{s_i}.w_i$. So we would get $j + 2 = -s_i$, which can not happen because $j + 2 \geq 1$ and $-s_i \leq 0$.
  · $f^{k+1}.w_i \neq 0$ because we have seen that $k < s_i$ in the previous point.

□

Example 3.12 ($\mathfrak{sl}_2$)**.** Consider the Cartan decomposition of $\mathfrak{sl}_2$ that corresponds to the eigenspace decomposition of ad_h, i.e.:

$$\mathfrak{sl}_2 = \langle f\rangle \oplus \langle h\rangle \oplus \langle e\rangle = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2 \tag{3.3}$$

where we denote by $\mathfrak{g}_j$ the eigenspace of ad_h with eigenvalue j.
This decomposition (3.3) is a good $\mathbb{Z}$-grading of $\mathfrak{sl}_2$. Indeed, we have:

1. (3.3) is a $\mathbb{Z}$-grading of $\mathfrak{sl}_2$, since:
   - $[h, f] = -2f \in \mathfrak{g}_{-2} \Rightarrow [\mathfrak{g}_0, \mathfrak{g}_{-2}] \subseteq \mathfrak{g}_{-2}$
   - $[e, f] = h \in \mathfrak{g}_0 \Rightarrow [\mathfrak{g}_2, \mathfrak{g}_{-2}] \subseteq \mathfrak{g}_0$
   - $[h, e] = 2e \in \mathfrak{g}_2 \Rightarrow [\mathfrak{g}_0, \mathfrak{g}_2] \subseteq \mathfrak{g}_2$
2. (3.3) is a good $\mathbb{Z}$-grading since it admits a good element that is $e \in \mathfrak{g}_2$, indeed:
   - $\mathrm{ad}_e : \mathfrak{g}_{-2} \to \mathfrak{g}_0$ is injective because $\mathrm{ad}_e(\lambda f) = [e, \lambda f] = \lambda h = 0 \Leftrightarrow \lambda = 0$,
   - $\mathrm{ad}_e : \mathfrak{g}_0 \to \mathfrak{g}_2$ is surjective because $\mathrm{ad}_e(h) = [e, h] = -2e \neq 0$

Example 3.13 ($\mathfrak{sl}_3$)**.** Consider the Cartan decomposition of $\mathfrak{sl}_3$:

$$\mathfrak{sl}_3 = H \oplus L_{\pm\alpha} \oplus L_{\pm\beta} \oplus L_{\pm(\alpha+\beta)}$$

Taken $x_\alpha \in L_\alpha$, we know by point 6. of Proposition 1.64 that there exists $x_{-\alpha} \in L_{-\alpha}$ such that $\langle x_\alpha, x_{-\alpha}, h_\alpha := [x_\alpha, x_{-\alpha}] \rangle \cong \mathfrak{sl}_2$.
Now we want to construct a $\mathbb{Z}$-grading of $\mathfrak{sl}_3$ given by the eigenspace decomposition of ad_{h_α}, so:

$$\begin{aligned}
&[h_\alpha, H] = 0 \text{ (since } h_\alpha \in H) && \Rightarrow H \subseteq \mathfrak{g}_0,\\
&[h_\alpha, x_\beta] = \beta(h_\alpha)x_\beta = \langle \beta, \alpha \rangle x_\beta = -x_\beta && \Rightarrow x_\beta \in \mathfrak{g}_{-1},\\
&[h_\alpha, x_{-\beta}] = x_{-\beta} && \Rightarrow x_{-\beta} \in \mathfrak{g}_1,\\
&[h_\alpha, x_\alpha] = 2x_\alpha && \Rightarrow x_\alpha \in \mathfrak{g}_2,\\
&[h_\alpha, x_{-\alpha}] = -2x_{-\alpha} && \Rightarrow x_{-\alpha} \in \mathfrak{g}_{-2},\\
&[h_\alpha, x_{\alpha+\beta}] = (\alpha+\beta)(h_\alpha)x_{\alpha+\beta} = (2-1)x_{\alpha+\beta} = x_{\alpha+\beta} && \Rightarrow x_{\alpha+\beta} \in \mathfrak{g}_1,\\
&[h_\alpha, x_{-\alpha-\beta}] = -x_{-\alpha-\beta} && \Rightarrow x_{-\alpha-\beta} \in \mathfrak{g}_{-1}.
\end{aligned}$$

Hence

$$\mathfrak{sl}_3 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{3.4}$$

is the eigenspace decomposition of ad_{h_α} of $\mathfrak{sl}_3$ where

- $\mathfrak{g}_0 = H$,

- $\mathfrak{g}_1 = \langle x_{-\beta}, x_{\alpha+\beta} \rangle$ and $\mathfrak{g}_{-1} = \langle x_\beta, x_{-\alpha-\beta} \rangle$,

- $\mathfrak{g}_2 = \langle x_\alpha \rangle$ and $\mathfrak{g}_{-2} = \langle x_{-\alpha} \rangle$.

This decomposition (3.4) is a good $\mathbb{Z}$-grading of $\mathfrak{sl}_3$. Indeed:

1. (3.4) is a $\mathbb{Z}$-grading of $\mathfrak{sl}_3$ by Remark 3.10.

2. (3.4) is a good $\mathbb{Z}$-grading since it admits a good element that is $x_\alpha \in \mathfrak{g}_2$, in fact:

   - $\mathrm{ad}_{x_\alpha} : \mathfrak{g}_{-2} \to \mathfrak{g}_0$ such that $x_\alpha \mapsto h_\alpha$ is injective because $\dim \mathfrak{g}_{-2} = 1$ and $\mathrm{ad}_{x_\alpha}(x_{-\alpha}) = h_\alpha \neq 0$,

   - $\mathrm{ad}_{x_\alpha} : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ such that $x_\beta \mapsto x_{\alpha+\beta}$, $x_{-\alpha-\beta} \mapsto x_{-\beta}$, up to scalars, is bijective because it maps a basis into a basis and $\mathrm{ad}_{x_\alpha}(x_\beta) \neq 0$ since $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ and $\dim L_{\alpha+\beta} = 1$ by points 4. and 1. of Proposition 1.65,

   - $\mathrm{ad}_{x_\alpha} : \mathfrak{g}_0 \to \mathfrak{g}_2$ such that $h_\alpha \mapsto -2x_\alpha$ is surjective because $\dim \mathfrak{g}_2 = 1$.

Now, we want to see what happens if we start with an element in another root space. So, instead of choosing $x_\alpha \in L_\alpha$, we take $x_{\alpha+\beta} \in L_{\alpha+\beta}$.
We know by point 6. of Proposition 1.64 that there exist $x_{-\alpha-\beta} \in L_{-\alpha-\beta}$ such that $\langle x_{\alpha+\beta}, x_{-\alpha-\beta}, h_{\alpha+\beta} := [x_{\alpha+\beta}, x_{-\alpha-\beta}] \rangle \cong \mathfrak{sl}_2$.
Now we want to construct a $\mathbb{Z}$-grading of $\mathfrak{sl}_3$ given by the eigenspace decom-

position of ad_{h_α}, so:

$$
\begin{aligned}
&[h_{\alpha+\beta}, H] = 0 \text{ (since } h_{\alpha+\beta} \in H) && \Rightarrow H \subseteq \mathfrak{g}_0,\\
&[h_{\alpha+\beta}, x_\alpha] = \alpha(h_{\alpha+\beta})x_\alpha = \langle \alpha, \alpha+\beta \rangle x_\alpha = x_\alpha && \Rightarrow x_\alpha \in \mathfrak{g}_1,\\
&[h_{\alpha+\beta}, x_{-\alpha}] = -x_{-\alpha} && \Rightarrow x_{-\alpha} \in \mathfrak{g}_{-1},\\
&[h_{\alpha+\beta}, x_\beta] = \beta(h_{\alpha+\beta})x_\beta = \langle \beta, \alpha+\beta \rangle x_\beta \underset{(*)}{=} x_\beta && \Rightarrow x_\beta \in \mathfrak{g}_1,\\
&[h_{\alpha+\beta}, x_{-\beta}] = -x_{-\beta} && \Rightarrow x_{-\beta} \in \mathfrak{g}_{-1},\\
&[h_{\alpha+\beta}, x_{\alpha+\beta}] = 2x_{\alpha+\beta} && \Rightarrow x_{\alpha+\beta} \in \mathfrak{g}_2,\\
&[h_{\alpha+\beta}, x_{-\alpha-\beta}] = -2x_{-\alpha-\beta} && \Rightarrow x_{-\alpha-\beta} \in \mathfrak{g}_{-2}.
\end{aligned}
$$

Hence

$$\mathfrak{sl}_3 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{3.5}$$

is the eigenspace decomposition of ad_{h_α} of $\mathfrak{sl}_3$ where

- $\mathfrak{g}_0 = H$,
- $\mathfrak{g}_1 = \langle x_\alpha, x_\beta \rangle$ and $\mathfrak{g}_{-1} = \langle x_{-\alpha}, x_{-\beta} \rangle$,
- $\mathfrak{g}_2 = \langle x_{\alpha+\beta} \rangle$ and $\mathfrak{g}_{-2} = \langle x_{-\alpha-\beta} \rangle$.

This decomposition (3.5) is a good $\mathbb{Z}$-grading of $\mathfrak{sl}_3$. In fact:

1. (3.5) is a $\mathbb{Z}$-grading of $\mathfrak{sl}_3$ by Remark 3.10.
2. (3.5) is a good $\mathbb{Z}$-grading since it admits a good element that is $x_{\alpha+\beta} \in \mathfrak{g}_2$, in fact:
    - $\mathrm{ad}_{x_{\alpha+\beta}} : \mathfrak{g}_{-2} \to \mathfrak{g}_0$ such that $x_{-\alpha-\beta} \mapsto h_{\alpha+\beta}$ is injective because $\dim \mathfrak{g}_{-2} = 1$ and $\mathrm{ad}_{x_{\alpha+\beta}}(x_{-\alpha-\beta}) = h_{\alpha+\beta} \neq 0$,
    - $\mathrm{ad}_{x_{\alpha+\beta}} : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ such that $x_{-\alpha} \mapsto x_\beta$, $x_{-\beta} \mapsto x_\alpha$, up to scalars, is bijective because it maps a basis into a basis and $\mathrm{ad}_{x_{\alpha+\beta}}(x_{-\alpha}) \neq 0$ since $[L_{\alpha+\beta}, L_{-\alpha}] = L_\beta$ and $\dim L_\beta = 1$ by points 4. and 1. of Proposition 1.65,

- $\mathrm{ad}_{x_{\alpha+\beta}} \colon \mathfrak{g}_0 \to \mathfrak{g}_2$ such that $h_{\alpha+\beta} \mapsto -2x_{\alpha+\beta}$ is surjective because $\dim \mathfrak{g}_2 = 1$.

Example 3.14 ($\mathfrak{sl}_n$)**.** Consider the Cartan decomposition of $\mathfrak{sl}_n$

$$\mathfrak{sl}_n = H \oplus L_{\pm\alpha_1} \oplus L_{\pm\alpha_2} \oplus \ldots \oplus L_{\pm\alpha_{n-1}} \oplus L_{\pm(\alpha_1+\alpha_2)} \oplus \ldots \oplus L_{\pm(\alpha_{n-2}+\alpha_{n-1})} \oplus \ldots \oplus L_{\pm(\alpha_1+\ldots+\alpha_{n-1})}.$$

Taken $x_{\alpha_1} \in \mathfrak{g}_{\alpha_1}$, we know by point 6. of Proposition 1.64 that there exist $x_{-\alpha_1} \in \mathfrak{g}_{-\alpha_1}$ such that $\langle x_{\alpha_1}, x_{-\alpha_1}, h_{\alpha_1} := [x_{\alpha_1}, x_{-\alpha_1}] \rangle \cong \mathfrak{sl}_2$.
Now we construct a $\mathbb{Z}$-grading of $\mathfrak{sl}_n$ given by the eigenspace decomposition of $\mathrm{ad}_{h_{\alpha_1}}$, so:

$$[h_{\alpha_1}, H] = 0 \text{ (since } h_{\alpha_1} \in H) \qquad \Rightarrow H \subseteq \mathfrak{g}_0,$$

$$[h_{\alpha_1}, x_{\alpha_1}] = 2x_{\alpha_1} \qquad \Rightarrow x_{\alpha_1} \in \mathfrak{g}_2,$$

$$[h_{\alpha_1}, x_{\alpha_2}] = \alpha_2(h_{\alpha_1})x_{\alpha_2} = \langle \alpha_2, \alpha_1 \rangle x_{\alpha_2} = -x_{\alpha_2} \qquad \Rightarrow x_{\alpha_2} \in \mathfrak{g}_{-1},$$

$$[h_{\alpha_1}, x_{\alpha_j}] = \langle \alpha_j, \alpha_1 \rangle x_{\alpha_j} = 0 \text{ for all } j \neq 1, 2 \qquad \Rightarrow x_{\alpha_j} \in \mathfrak{g}_0,$$

$$[h_{\alpha_1}, x_{\alpha_1+\alpha_2}] = (\alpha_1+\alpha_2)(h_{\alpha_1})x_{\alpha_1+\alpha_2} = (2-1)x_{\alpha_1+\alpha_2} = x_{\alpha_1+\alpha_2} \qquad \Rightarrow x_{\alpha_1+\alpha_2} \in \mathfrak{g}_1,$$

$$[h_{\alpha_1}, x_{\alpha_2+\alpha_3}] = (\alpha_2+\alpha_3)(h_{\alpha_1})x_{\alpha_2+\alpha_3} = (-1+0)x_{\alpha_2+\alpha_3} = -x_{\alpha_2+\alpha_3} \qquad \Rightarrow x_{\alpha_2+\alpha_3} \in \mathfrak{g}_{-1},$$

$$[h_{\alpha_1}, x_{\alpha_j+\alpha_{j+1}}] = 0 \text{ for all } j \neq 1, 2 \qquad \Rightarrow x_{\alpha_j+\alpha_{j+1}} \in \mathfrak{g}_0.$$

In general, it holds:

$$\begin{aligned}
[h_{\alpha_1}, x_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_{j+k}}] &= (\alpha_j + \alpha_{j+1} + \ldots + \alpha_{j+k})(h_{\alpha_1})x_{\alpha_j+\ldots+\alpha_{j+k}} \\
&= (\langle \alpha_j, \alpha_1 \rangle + \langle \alpha_{j+1}, \alpha_1 \rangle + \underbrace{\langle \alpha_{j+2}, \alpha_1 \rangle}_{=0} + \ldots + \underbrace{\langle \alpha_{j+k}, \alpha_1 \rangle}_{=0})x_{\alpha_j+\ldots+\alpha_{j+k}} \\
&= (\langle \alpha_j, \alpha_1 \rangle + \langle \alpha_{j+1}, \alpha_1 \rangle)x_{\alpha_j+\ldots+\alpha_{j+k}} \\
&= \begin{cases} (\langle \alpha_1, \alpha_1 \rangle + \langle \alpha_2, \alpha_1 \rangle)x_{\alpha_j+\ldots+\alpha_{j+k}} = x_{\alpha_j+\ldots+\alpha_{j+k}} & \text{if } j = 1 \\ \langle \alpha_2, \alpha_1 \rangle x_{\alpha_j+\ldots+\alpha_{j+k}} = -x_{\alpha_j+\ldots+\alpha_{j+k}} & \text{if } j = 2 \\ 0 & \text{if } j > 2 \end{cases}
\end{aligned}$$

Hence, for $j = 1$ (resp. $j = 2$ and $j > 2$) $x_{\alpha_j+\ldots+\alpha_{j+k}} \in \mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$).
So we can write

$$\mathfrak{sl}_n = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{3.6}$$

where:

- $\mathfrak{g}_{-2} = \langle x_{-\alpha_1} \rangle$;

- $\mathfrak{g}_{-1} = \langle x_{\alpha_2+\ldots+\alpha_{2+k}}, x_{-(\alpha_1+\ldots\alpha_{1+j})}, k = 0, \ldots, n-3, j = 0, \ldots, n-2\rangle$;
- $\mathfrak{g}_0 = \langle H, x_{\pm(\alpha_j+\ldots+\alpha_{j+k})}, j > 2, k = 0, \ldots, n-j-1\rangle$;
- $\mathfrak{g}_1 = \langle x_{-(\alpha_2+\ldots+\alpha_{2+k})}, x_{\alpha_1+\ldots\alpha_{1+j}}, k = 0, \ldots, n-3, j = 0, \ldots, n-2\rangle$;
- $\mathfrak{g}_2 = \langle x_{\alpha_1}\rangle$;

The decomposition (3.6) is a good $\mathbb{Z}$-grading of $\mathfrak{sl}_n$. In fact:

1. (3.6) is a $\mathbb{Z}$-grading of $\mathfrak{sl}_n$ by Remark 3.10.
2. (3.6) is a good $\mathbb{Z}$-grading since it admits a good element that is $x_{\alpha_1} \in \mathfrak{g}_2$, in fact:
   - $\mathrm{ad}_{x_{\alpha_1}} : \mathfrak{g}_{-2} \to \mathfrak{g}_0$ such that $x_{-\alpha_1} \mapsto h_{\alpha_1}$ is injective because $\dim\mathfrak{g}_{-2} = 1$ and $\mathrm{ad}_{x_{\alpha_1}}(x_{-\alpha_1}) = h_{\alpha_1} \neq 0$,
   - $$\begin{array}{cccc} \mathrm{ad}_{x_{\alpha_1}} : & \mathfrak{g}_{-1} & \to & \mathfrak{g}_1 \\ & x_{\alpha_2+\ldots+\alpha_k} & \mapsto & x_{\alpha_1+\ldots+\alpha_k} \\ & x_{-(\alpha_1+\ldots+\alpha_k)} & \mapsto & x_{-(\alpha_2+\ldots+\alpha_k)} \end{array}$$
     is defined in this way, up to nonzero scalars, by Proposition 1.65. Hence $\mathrm{ad}_{x_{\alpha_1}}$ is bijective on $\mathfrak{g}_{-1}$.
   - $\mathrm{ad}_{x_{\alpha_1}} : \mathfrak{g}_0 \to \mathfrak{g}_2$ such that $h_{\alpha_1} \mapsto -2x_{\alpha_1}$ is surjective because $\dim\mathfrak{g}_2 = 1$.

3.3 Properties of good gradings

From now on, we shall assume that $\mathfrak{g}$ is a semisimple Lie algebra. Fix a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j \tag{3.7}$$

Lemma 3.15. *Let $e \in \mathfrak{g}_2$, $e \neq 0$. Then there exists $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-2}$ such that $\{e, h, f\}$ forms an $\mathfrak{sl}_2$-triple, i.e., $[h, e] = 2e$, $[e, f] = h$, $[h, f] = -2f$.*

Proof. By the Jacobson Morozov Theorem (Theorem 2.12), there exist $h, f \in \mathfrak{g}$ such that $\{e,\ h,\ f\}$ is an $\mathfrak{sl}_2$-triple. We write $h = \sum_{j\in\mathbb{Z}} h_j$, $f = \sum_{j\in\mathbb{Z}} f_j$ according to the given $\mathbb{Z}$-grading of $\mathfrak{g}$. Then

- $[h_0, e] = 2e$ because $2e = [h, e] = [\sum_{j\in\mathbb{Z}} h_j, e] = \sum_{j\in\mathbb{Z}}[h_j, e]$. But $e \in \mathfrak{g}_2$ and $[h_j, e] \in \mathfrak{g}_{j+2}$, so $[h_j, e] = 0$ for $j \neq 0$ and $[h_0, e] = 2e$.
- $[e, \mathfrak{g}] \ni h_0$ since $[e, f_{-2}] = h_0$; indeed: $\sum_{j\in\mathbb{Z}} h_j = h = [e, f] = [e, \sum_{j\in\mathbb{Z}} f_j] = \sum_{j\in\mathbb{Z}} [e, f_j] \Rightarrow h_{j+2} = [e, f_j]$ for $j \in \mathbb{Z}$.

Therefore, by Morozov's lemma (Lemma 2.10), there exists f' such that $\{e,\ h_0,\ f'\}$ is an $\mathfrak{sl}_2$-triple. But then $\{e,\ h_0,\ f'_{-2}\}$ is an $\mathfrak{sl}_2$-triple, in fact:

- $[h_0, e] = 2e$,
- $[e, f'] = h_0$, then $\mathfrak{g}_0 \ni h_0 = [e, f'] = [e, \sum_{j\in\mathbb{Z}} f'_j] = \sum_{j\in\mathbb{Z}} [e, f'_j]$ so, as above, $[e, f'_{-2}] = h_0$,
- $[h_0, f'] = -2f'$, but $-2f' = -2\sum_{j\in\mathbb{Z}} f'_j = \sum_{j\in\mathbb{Z}} -2f'_j]$ and $[h_0, f'] = [h_0, \sum_{j\in\mathbb{Z}} f'_j] = \sum_{j\in\mathbb{Z}} [h_0, f'_j]$, hence $-2f'_j = [h_0, f'_j]$ for $j \in \mathbb{Z}$.

□

Lemma 3.16. *Let $e \in \mathfrak{g}$ be a nonzero nilpotent element, $\mathfrak{s} = \{e, h, f\}$ an $\mathfrak{sl}_2$-triple and $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ the Dynkin grading introduced in* (3.2)*. Set $C_{\mathfrak{g}}(e)_i = C_{\mathfrak{g}}(e) \cap \mathfrak{g}_i$. Then:*

1. $C_{\mathfrak{g}}(e) = \bigoplus_{i\geq 0} C_{\mathfrak{g}}(e)_i$;
2. $C_{\mathfrak{g}}(e) \cap [\mathfrak{g}, e] = \bigoplus_{i>0} C_{\mathfrak{g}}(e)_i$;
3. $C_{\mathfrak{g}}(e)_0 = C_{\mathfrak{g}}(\mathfrak{s})$.

Proof. Thanks to the decomposition of $\mathfrak{g} = \oplus_{j=1}^{k}\mathfrak{g}_{s_k}$ introduced in (3.1), we can say that $C_{\mathfrak{g}}(e) = \langle w_1, \ldots, w_r\rangle$, with $h.w_i = s_i w_i$ and $s_i \geq 0$ for all $i = 1, \ldots, r$.

1. This implies that $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$. Thus $C_{\mathfrak{g}}(e) = \oplus_{j\geq 0}C_{\mathfrak{g}}(e)_j$.

2. In order to prove the second point, we want to show that $w_i \in [\mathfrak{g}, e]$ if and only if $s_i > 0$. Indeed $s_i > 0$ is equivalent to $\dim\mathfrak{g}_{s_i} > 1$. This means that $f.w_i \neq 0$. Thus, since $e.f^k.w_i = k(s_i - k + 1)f^{k-1}.w_i$ for all k (see point 4. in Theorem 1.22), we have that $e.f.w_i = s_i w_i$, i.e., $w_i \in [\mathfrak{g}, e]$.

3. $C_{\mathfrak{g}}(e)_0 = \{w_{i_1}, \dots, w_{i_k}\}$, with $h.w_{i_j} = 0$ (i.e., $s_{i_j} = 0$).
$C_{\mathfrak{g}}(\mathfrak{s}) = \{z \in \mathfrak{g} \mid e.z = f.z = h.z = 0\}$; this means that $z \in C_{\mathfrak{g}}(\mathfrak{s})$ if and only if $z = w_i$ (because $e.z = 0$) with $s_i = 0$ (because $h.z = 0$). Thus $C_{\mathfrak{g}}(\mathfrak{s}) = \{w_{i_1}, \dots, w_{i_k}\}$.

□

Proposition 3.17. *Let e be a non-zero nilpotent element of $\mathfrak{g}$ and let $\mathfrak{s} = \{e, h, f\}$ be an $\mathfrak{sl}_2$-triple. Then $C_{\mathfrak{g}}(\mathfrak{s})$ is a reductive subalgebra of $C_{\mathfrak{g}}(e)$, called the reductive part of $C_{\mathfrak{g}}(e)$.*

Proof. By Theorem 2.7, it is enough to show that the Killing form K is non-degenerate when restricted to $C_{\mathfrak{g}}(\mathfrak{s})$. For the remainder of the proof, we will use the notation $\perp$ to denote the K-orthogonal. By the proof of the Jacobson-Morozov Theorem (see Theorem 2.12), we know that $C_{\mathfrak{g}}(e)^{\perp} = [\mathfrak{g}, e]$. Then K restricts to a non-degenerate form on

$$C_{\mathfrak{g}}(e)/(C_{\mathfrak{g}}(e) \cap (C_{\mathfrak{g}}(e))^{\perp}) = C_{\mathfrak{g}}(e)/(C_{\mathfrak{g}}(e) \cap [\mathfrak{g}, e]) \cong C_{\mathfrak{g}}(e)_0,$$

where the last isomorphism is due to points 1. and 2. of Lemma 3.16.
Moreover, by the third point of Lemma 3.16, we can say that $C_{\mathfrak{g}}(e)_0 = C_{\mathfrak{g}}(\mathfrak{s})$; hence K is non-degenerate when restricted to $C_{\mathfrak{g}}(\mathfrak{s})$. □

Theorem 3.18. *Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a good $\mathbb{Z}$-grading and $e \in \mathfrak{g}_2$ a good element. Let $H \in \mathfrak{g}$ be the element defining the $\mathbb{Z}$-grading, and let $\mathfrak{s} = \{e, h, f\}$ be an $\mathfrak{sl}_2$-triple given by Lemma 3.15. Then $z := H - h$ lies in the center of $C_{\mathfrak{g}}(\mathfrak{s})$.*

Proof. The existence of H is guaranteed by Remark 3.2. The eigenvalues of ad_H on $C_{\mathfrak{g}}(e)$ are non-negative since, if $a \in C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$ is an eigenvector of ad_H, by point 2. of Remark 3.8 we have $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$. Thus there exists $\bar{j} \geq 0$ such that $a \in \mathfrak{g}_{\bar{j}}$, so $\mathrm{ad}_H(a) = [H, a] = \bar{j}a$ with $\bar{j} \geq 0$.
Hence the eigenvalues of ad_H on $C_{\mathfrak{g}}(\mathfrak{s})$ are non-negative because $C_{\mathfrak{g}}(\mathfrak{s}) = \{a \in \mathfrak{g} : [a, e] = 0, [a, h] = 0, [a, f] = 0\} \subseteq C_{\mathfrak{g}}(e)$. So we can write $C_{\mathfrak{g}}(\mathfrak{s}) = \oplus_{i\geq 0} C_{\mathfrak{g}}(\mathfrak{s})_i$. By Proposition 3.17, $C_{\mathfrak{g}}(\mathfrak{s})$ is reductive; thus we can say that $C_{\mathfrak{g}}(\mathfrak{s}) = [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})] \oplus Z(C_{\mathfrak{g}}(\mathfrak{s}))$, where $[C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]$ is semisimple thanks to Proposition 2.3. Notice the following facts.

1. $[H, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] = 0$. Indeed, since $[C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]$ is semisimple, the Killing form restricted to it is non-degenerate by Theorem 1.45. Since $K([H, [C_{\mathfrak{g}}(\mathfrak{s}), \oplus_{j>0} C_{\mathfrak{g}}(\mathfrak{s})_j]], [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]) = 0$ by Proposition 3.4 and $[H, [C_{\mathfrak{g}}(\mathfrak{s})_0, C_{\mathfrak{g}}(\mathfrak{s})_0]] = 0$ because H is the element defining the $\mathbb{Z}$-grading, then $[H, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] = 0$.

2. $[h, C_{\mathfrak{g}}(\mathfrak{s})] = 0$ by the definition of $C_{\mathfrak{g}}(\mathfrak{s})$.

3. $[H - h, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] = [H, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] - [h, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] = 0$ by points 1., 2. and the Jacobi identity.

4. $H - h \in C_{\mathfrak{g}}(\mathfrak{s})$ because H is the element defining the $\mathbb{Z}$-grading and $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple. Thus $[H - h, Z(C_{\mathfrak{g}}(\mathfrak{s}))] = 0$.

Therefore $[z, C_{\mathfrak{g}}(\mathfrak{s})] = 0$, i.e., $z \in Z(C_{\mathfrak{g}}(\mathfrak{s}))$. □

Corollary 3.19. *If $\mathfrak{s} = \{e, h, f\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and the center of $C_{\mathfrak{g}}(\mathfrak{s})$ is trivial, then the only good grading for which e is a good element is the Dynkin grading.*

Proof. If $Z(C_{\mathfrak{g}}(\mathfrak{s})) = 0$, then $z := H - h = 0$ because by Theorem 3.18 we have $z \in Z(C_{\mathfrak{g}}(\mathfrak{s}))$. Hence $H = h$. But $H \in \mathfrak{g}$ is the element defining the $\mathbb{Z}$-grading, i.e., $\mathfrak{g}_j = \{a \in \mathfrak{g} : [H, a] = ja\} = \{a \in \mathfrak{g} : [h, a] = ja\}$. Since $H = h$, the good $\mathbb{Z}$-grading of $\mathfrak{g}$ with good element e is the one obtained by the eigenspace decomposition of ad_h in $\mathfrak{g}$, that means the Dynkin $\mathbb{Z}$-grading. □

Example 3.20 ($\mathfrak{sl}_2$).

Consider $\mathfrak{g} = \mathfrak{sl}_2 = \langle \mathfrak{s} \rangle = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$.

Notice that $C_{\mathfrak{g}}(\mathfrak{s}) = Z(\mathfrak{sl}_2) = 0$ because $\mathfrak{sl}_2$ is semisimple. Hence $Z(C_{\mathfrak{g}}(\mathfrak{s})) = 0$ and, by Corollary 3.19, the Dyinkin grading is the only $\mathbb{Z}$-grading for which $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a good element.

Example 3.21 ($\mathfrak{sl}_3$).

Consider $\mathfrak{g} = \mathfrak{sl}_3$. Up to conjugation, the only nilpotent elements of $\mathfrak{g}$ are

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We start analyzing the case of an $\mathfrak{sl}_2$-triple containing e_1.

With a simple calculation one can check that $\mathfrak{s}_1 = \{e_1, h_1, f_1\}$ is an $\mathfrak{sl}_2$-triple, with $h_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ and $f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. Now we want to compute $C_{\mathfrak{g}}(\mathfrak{s}_1)$. Consider $x = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \in C_{\mathfrak{g}}(\mathfrak{s}_1)$. Then:

$$\begin{aligned} 0 = [x, h_1] &= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2b & -4c \\ 2d & 0 & -2f \\ 4g & 2h & 0 \end{pmatrix} \end{aligned}$$

if and only if $b = c = d = f = g = h = 0$. Thus $x = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix}$.

Moreover,

$$0 = [x, e_1] = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix}$$
$$= \begin{pmatrix} 0 & a+e & 0 \\ 0 & 0 & a+2e \\ 0 & 0 & 0 \end{pmatrix}$$

if and only if $a = e = 0$.
Therefore $C_{\mathfrak{g}}(\mathfrak{s}_1) = 0$; hence $Z(C_{\mathfrak{g}}(\mathfrak{s}_1)) = 0$ and, by Corollary 3.19, the Dyinkin grading is the only $\mathbb{Z}$-grading for which e_1 is a good element.

Now we analyze the second case, in which we consider an $\mathfrak{sl}_2$-triple containing e_2.

It easy to see that $\mathfrak{s}_2 = \{e_2, h_2, f_2\}$ is an $\mathfrak{sl}_2$-triple, with $h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Now we want to compute $C_{\mathfrak{g}}(\mathfrak{s}_2)$.

Consider $x = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \in C_{\mathfrak{g}}(\mathfrak{s}_2)$. Then:

$$0 = [x, h_2] = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -2b & -c \\ 2d & 0 & f \\ g & -h & 0 \end{pmatrix}$$

if and only if $b = c = d = f = g = h = 0$. Thus $x = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix}$.

Moreover,

$$0 = [x, e_2] = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix}$$
$$= \begin{pmatrix} 0 & a-e & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

if and only if $a = e$. Thus $x = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \in C_{\mathfrak{g}}(\mathfrak{s}_2)$ (it is easy to see that such x commutes with f_2).

Therefore $C_{\mathfrak{g}}(\mathfrak{s}_2) = \langle \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \rangle$. So $Z(C_{\mathfrak{g}}(\mathfrak{s}_2)) = C_{\mathfrak{g}}(\mathfrak{s}_2)$ because $C_{\mathfrak{g}}(\mathfrak{s}_2)$ is one-dimensional and hence commutative.

Hence, in this case we can not establish if the Dynkin grading is the only good $\mathbb{Z}$-grading with good element e_2.

By Proposition 3.5, we know that $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$. Furthermore, it can be proven that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}$ (see [8]).

Let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_\alpha \mathfrak{g}_\alpha)$ be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Δ_0^+ be a system of positive roots of the subalgebra $\mathfrak{g}_0$. It is well known that $\Delta^+ = \Delta_0^+ \cup (\alpha \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_s,\ s > 0)$ is a set of positive roots of $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta^+$ be the set of the simple roots. Setting $\Pi_s = (\alpha \in \Pi | \mathfrak{g}_\alpha \subset \mathfrak{g}_s)$ we obtain a decomposition of Π into a disjoint union of subsets $\Pi = \cup_{s \geq 0} \Pi_s$. This decomposition is called the *characteristic* of the $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. So we obtain a bijection between all $\mathbb{Z}$-gradings up to conjugation and all characteristics.

Theorem 3.22. *If the $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is good, then* $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2$.

Proof. Let $e \in \mathfrak{g}_2$ be a good element. From the construction above, we can write $e = \sum_{\rho_j \in \Phi^+} e_{\rho_j}$, with $\rho_j = \alpha_{j_1} + \ldots + \alpha_{j_{k_j}}$ for some non-negative simple roots α_{j_i}. Suppose, by contradiction, that there exists a simple root $\alpha_j \notin \Pi_0 \cup \Pi_1 \cup \Pi_2$. Then e lies in the Lie subalgebra generated by e_{α_i}, $i \neq j$. Indeed, if not, we could find an addend e_{ρ_r} of e such that $\alpha_j \in \{\alpha_{r_1}, \dots, \alpha_{r_{k_r}}\}$. But, since $\deg e_{\rho_r} = \sum_k \deg e_{\alpha_{r_k}}$, then e could not belong to $\mathfrak{g}_2$. Therefore $[e_{\rho_i}, e_{-\alpha_j}] \in \mathfrak{g}_{\rho_i - \alpha_j} = \{0\}$ for all i and hence $[e, e_{-\alpha_j}] = 0$. This contradicts property a) of Definition 3.7.

□

Corollary 3.23. *All good $\mathbb{Z}$-gradings are among those defined by* $\deg e_{\alpha_i} = -\deg e_{-\alpha_i} = 0, 1$ *or* 2, $i = 1, \dots, r$.

Lemma 3.24. *Let $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ be a $\mathbb{Z}$-grading, $e \in \mathfrak{g}_2$ and K the Killing form on $\mathfrak{g}$. Then $[e, \mathfrak{g}_j] \neq \mathfrak{g}_{j+2}$ if and only if there exists a non-zero $a \in \mathfrak{g}_{-j-2}$ such that $K([e, \mathfrak{g}_j], a) = 0$.*

Proof. Suppose that $K([e, \mathfrak{g}_j], a) = 0$ for some non-zero $a \in \mathfrak{g}_{-j-2}$. Suppose by contradiction that $[e, \mathfrak{g}_j] = \mathfrak{g}_{j+2}$ for some $j \geq -1$. Then $K(\mathfrak{g}_{j+2}, a) = 0$. Now, take $H \in \mathfrak{g}_0$ defining the grading. $K(\mathfrak{g}_k, a) = 0$ for all $k \in \mathbb{Z}$, $k \neq j+2$ by Proposition 3.4. Hence $(\mathfrak{g}, a) = 0$. This is a contradiction because K is non-degenerate and $a \neq 0$.
Conversely, suppose that $[e, \mathfrak{g}_j] \subsetneq \mathfrak{g}_{j+2}$. Notice that $\mathfrak{g}_{j+2}^{\perp} = \oplus_{k \neq -j-2} \mathfrak{g}_k$ by the non-degeneracy of K (see Proposition 3.4). Then $[e, \mathfrak{g}_j]^{\perp} \supsetneq \mathfrak{g}_{j+2}^{\perp} = \oplus_{k \neq -j-2} \mathfrak{g}_k$. This implies that $[e, \mathfrak{g}_j]^{\perp} \cap \mathfrak{g}_{-j-2} \neq 0$, i.e., there exists a nonzero element $a \in \mathfrak{g}_{-j-2}$ such that $K([e, \mathfrak{g}_j], a) = 0$. □

Theorem 3.25. *Properties a) and b) of the definition of good element (Definition 3.7) of a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ are equivalent.*

Proof. By Lemma 3.24 we know that the property $[e, \mathfrak{g}_j] \neq \mathfrak{g}_{j+2}$ for $j \geq -1$ is equivalent to the existence of a non-zero element $a \in \mathfrak{g}_{-j-2}$ such that $K([e, \mathfrak{g}_j], a) = 0$. But the latter equality is equivalent to $K([e, a], \mathfrak{g}_j) = 0$ by the invariance of K and this is equivalent to $[e, a] = 0$ by the non-degeneracy of K. Then $\mathrm{ad}_e : \mathfrak{g}_{-j-2} \to \mathfrak{g}_{-j}$ is not injective. □

Theorem 3.26. *Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a good $\mathbb{Z}$-grading with good element e. Then $C_{\mathfrak{g}}(e) \cong \mathfrak{g}_0 + \mathfrak{g}_{-1}$ as $C_{\mathfrak{g}_0}(e)$-modules.*

Proof. Due to properties a) and b) of Definition 3.7 we have the following exact sequence of $C_{\mathfrak{g}_0}(e)$-modules:

$$0 \to C_{\mathfrak{g}}(e) \overset{\text{id}}{\to} \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \overset{\text{ad}_e}{\to} \mathfrak{g}_+ \to 0.$$

Indeed,

- $\text{id} : C_{\mathfrak{g}}(e) \to \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+$ is injective because, by point 2. of Remark 3.8, $C_{\mathfrak{g}}(e) \subset \mathfrak{g}_{\geq}$;
- $\text{ad}_e : \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \to \mathfrak{g}_+$ is surjective by property b) of Definition 3.7;
- $\ker(\text{ad}_e) = \{x \in \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \mid [e, x] = 0\} = C_{\mathfrak{g}}(e) = \text{Im}(\text{id})$

Moreover, we can note that $C_{\mathfrak{g}}(e)$, $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$, and $\mathfrak{g}_+$ are $C_{\mathfrak{g}_0}(e)$-modules. We show that only for $C_{\mathfrak{g}}(e)$ (it will be analogue in the other cases).
Since the concept of $\mathfrak{g}$-module is equivalent to the concept of representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, it is enough to consider the adjoint representation:

$$\begin{array}{rccc} \text{ad}: & C_{\mathfrak{g}_0}(e) & \to & \mathfrak{gl}(C_{\mathfrak{g}}(e)) \\ & x & \mapsto & \text{ad}_x : C_{\mathfrak{g}}(e) \to C_{\mathfrak{g}}(e) \end{array}$$

and verify that ad_x is well defined as $C_{\mathfrak{g}}(e)$-endomorphism, i.e., $\text{ad}_x(C_{\mathfrak{g}}(e)) \subseteq C_{\mathfrak{g}}(e)$. Thus, let $x \in C_{\mathfrak{g}_0}(e)$ and $y \in C_{\mathfrak{g}}(e)$; then $[e, [x, y]] = [[e, x], y] + [x, [e, y]] = [0, y] + [x, 0] = 0$, so $[x, y] \in C_{\mathfrak{g}}(e)$. Hence, $C_{\mathfrak{g}}(e) \cong \mathfrak{g}_0 + \mathfrak{g}_{-1}$ as $C_{\mathfrak{g}_0}(e)$-modules because for all $x \in C_{\mathfrak{g}_0}(e)$ and $y \in \mathfrak{g}_0 + \mathfrak{g}_{-1}$ we have that $\text{ad}_e(\text{ad}_x(y)) = \text{ad}_x(\text{ad}_e(y))$ since $[e, x] = 0$. □

Corollary 3.27. *Let $\mathfrak{g} = \oplus_j\mathfrak{g}_j$ be a $\mathbb{Z}$-grading and let $e \in \mathfrak{g}_2$. Then* $\dim C_{\mathfrak{g}}(e) \geq \dim\mathfrak{g}_{-1} + \dim\mathfrak{g}_0$*, and equality holds if and only if e is a good element.*

Proof. We have an exact sequence of vector spaces (the proof of its exactness is analogue to the one in Theorem 3.26):

$$0 \to C_{\mathfrak{g}}(e) \cap (\mathfrak{g}_{-1} + \mathfrak{g}_{\geq}) \overset{\text{id}}{\to} \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \overset{\text{ad}_e}{\to} [e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq}] \to 0.$$

Hence $\dim C_{\mathfrak{g}}(e) + \dim[e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq}] \geq \dim(C_{\mathfrak{g}}(e) \cap (\mathfrak{g}_{-1} + \mathfrak{g}_{\geq})) = \dim\mathfrak{g}_{-1} + \dim\mathfrak{g}_0 + \dim\mathfrak{g}_+$. But, since $[e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq}] \subseteq \mathfrak{g}_+$ (and equality holds if and only if e is good), one has $\dim C_{\mathfrak{g}}(e) + \dim\mathfrak{g}_+ \geq \dim\mathfrak{g}_{-1} + \dim\mathfrak{g}_0 + \dim\mathfrak{g}_+$, i.e. $\dim C_{\mathfrak{g}}(e) \geq \dim\mathfrak{g}_{-1} + \dim\mathfrak{g}_0$, and hence the Corollary follows.

□

Definition 3.28. Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module. Then V is called *self-dual* if it is isomorphic to V^* as $\mathfrak{g}$-module.

Lemma 3.29. *Let $\mathfrak{g}$ be a Lie algebra, V be a $\mathfrak{g}$-module via the adjoint action. If there exists a non-degenerate $\mathfrak{g}$-invariant bilinear form $(.,.) : V \times V \to \mathbb{F}$, then V is self-dual.*

Proof. Set

$$\begin{array}{rccc} \varphi : & V & \to & V^* \\ & v & \mapsto & (w \mapsto (v, w)) \end{array}$$

Then:

- φ is bijective since $(.,.)$ is non-degenerate;
- $\varphi(x.v) = x.\varphi(v)$ for every $x \in \mathfrak{g}$ and $v \in V$. Indeed, if $w \in V$:
  $(\varphi(x.v))(w) = (x.v, w) = ([x, v], w)$
  $(x.\varphi(v))(w) = -\varphi(v)(x.w) = -(v, x.w) = -(v, [x, w]) = -([v, x], w) = ([x, v], w)$
  where we used the invariancy of $(.,.)$.

□

Corollary 3.30. *Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a good $\mathbb{Z}$-grading with good element e. Then the representation of $C_{\mathfrak{g}_0}(e)$ on $C_{\mathfrak{g}}(e)$ is self-dual.*

Proof. Consider the bilinear form on $\mathfrak{g}_{-1}$ given by $\langle a, b\rangle := K(e, [a, b])$. Note that $\langle .,.\rangle$ has the following properties:

1. It is $C_{\mathfrak{g}_0}(e)$-invariant. Indeed, if we take $c \in C_{\mathfrak{g}_0}(e)$, $a, b \in \mathfrak{g}_{-1}$, one has $[a, c], [b, c] \in \mathfrak{g}_{-1}$. Furthermore $\langle [a, c], b\rangle = K(e, [[a, c], b]) = K(e, [a, [c, b]]) - K(e, [c, [a, b]]) = \langle a, [c, b]\rangle - K([e, c], [a, b]) = \langle a, [c, b]\rangle$ because $c \in C_{\mathfrak{g}_0}(e)$.

2. It is non-degenerate. Indeed, if we take $a \in \mathfrak{g}_{-1}$ such that $\langle \mathfrak{g}_{-1}, a \rangle = 0$, then $K(e, [\mathfrak{g}_{-1}, a]) = 0$, i.e. $K([e, \mathfrak{g}_{-1}], a) = 0$. Using point 3. of Remark 3.8 we can say that the latter is equivalent to $K(\mathfrak{g}_1, a) = 0$. Moreover, by Proposition 3.4, $K(\mathfrak{g}_k, a) = 0$ for all $k \neq 1$. So $K(\mathfrak{g}, a) = 0$ and, by non-degeneracy of K, $a = 0$.

Hence the $C_{\mathfrak{g}_0}(e)$-module $\mathfrak{g}_{-1}$ is self-dual by Lemma 3.29.
Similarly, the $C_{\mathfrak{g}_0}(e)$-module $\mathfrak{g}_0$ is self-dual since the bilinear form K is non-degenerate on $\mathfrak{g}_0$. So we can conclude using Theorem 3.26. □

Chapter 4

Good gradings of $\mathfrak{gl}_n$

Definition 4.1. A *partition* of n is a tuple $p = (p_1, \ldots, p_s)$ with $p_i \in \mathbb{N}$, $p_i \geq p_{i+1}$ and $p_1 + \ldots + p_s = n$. We denote by $Par(n)$ the set of all the partitions of n.

It will be convenient to assume that in fact p has an arbitrary number of further components on the right, all equal to 0, i.e., that $p_{s+1} = p_{s+2} = \ldots = 0$.

Definition 4.2. We denote by $\text{mult}_p(j)$ the *multiplicity* of the number j in the partition p, i.e.,

$$\text{mult}_p(j) := \#\{i : p_i = j\}.$$

Remark 4.3. A partition $p = (p_1, \ldots, p_s)$ can be also written as

$$\sum_{i \geq 1} i\text{mult}_p(i) = n. \tag{4.1}$$

Definition 4.4. Let $p = (p_1, \ldots, p_s) \in \text{Par}(n)$. Then $p^* = (p_1^*, p_2^*, \ldots)$, where $p_j^* := \#\{i : p_i \geq j\}$, $j = 1, 2, \ldots$, is called the *dual partition* of p.

Remark 4.5. Note in particular that

$$\text{mult}_p(j) = p_j^* - p_{j+1}^*. \tag{4.2}$$

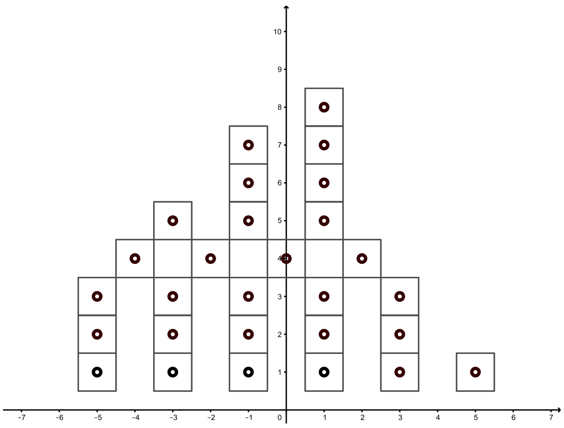
Theorem 4.6. *[10] Let $e \in \mathfrak{gl}_n$ be a nilpotent element, and $p = (p_1, \dots, p_s)$ the partition of the Jordan canonical form of e. Let $(p_1^*, \dots, p_r^*)$ be the dual partition of p. Then the dimension of the centralizer of e in $\mathfrak{gl}_n$ is:*

$$\sum_{i=1}^{r} (p_i^*)^2.$$

Definition 4.7. A *pyramid* P is a finite collection of boxes of size 1×1 on the plane, centered at (i, j), where $i, j \in \mathbb{Z}$, such that the following conditions hold:

1. the second coordinates of the centers of the boxes of the lowest row equal 1, i.e., there are no $j \leq 0$, $i \in \mathbb{Z}$ such that $(i, j) \in P$;
2. the first coordinates of the j^{th} row form an arithmetic progression $f_j, f_j + 2, \dots, l_j$ with difference 2, and $f_1 = -l_1$;
3. $f_j \leq f_{j+1}$, $l_j \geq l_{j+1}$ for all j.

Example 4.8. The following figure is an example of a pyramid where the circles represent the centers of the boxes.



Definition 4.9. The *size* of a pyramid is the number of boxes in it.

To a given pyramid P of size n we associate a nilpotent endomorphism of the vector space $\mathbb{F}^n$ in the following manner.

Definition 4.10. Enumerate the squares of P in some linear order, label the standard basis of $\mathbb{F}^n$ with the corresponding number, and define an endomorphism $e(P)$ of $\mathbb{F}^n$ by letting it act "along the rows of the pyramid", i.e., by sending the basis vector labeled by a box to the basis vector labeled by its neighbour on the right, if it belongs to P, and to 0 otherwise.

Remark 4.11. The nilpotent endomorphism $e(P)$ can be graphically represented by a collection of arrows.

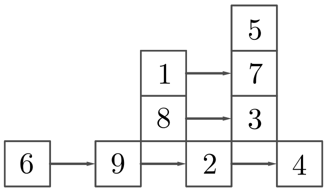


Figure 4.1: Endomorphism $e(P)$

For example, the endomorphism $e(P)$ represented in Figure 4.1 corresponds to the following endomorphism:

$e_6 \mapsto e_9 \mapsto e_2 \mapsto e_4 \mapsto 0$

$e_8 \mapsto e_3 \mapsto 0$

$e_1 \mapsto e_7 \mapsto 0$

$e_5 \mapsto 0$

where e_i is the i^{th} basis vector of the standard basis of $\mathbb{F}^9$.

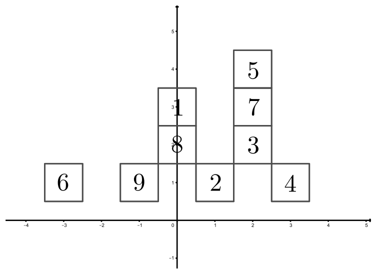
Remark 4.12. Denote by p_j the number of the squares in the j^{th} row of P. Then the endomorphism $e(P)$ is a nilpotent one corresponding to the partition (i.e., with sizes of Jordan blocks given by) $p = (p_1, \dots, p_k)$ and the endomorphisms corresponding to all pyramids with n boxes and fixed lengths of rows belong to the conjugacy class of the nilpotent $e(P)$.

Proof. It is evident that $e(P)$ is nilpotent and corresponds to the partition p by its definition (see Figure 4.1).

Furthermore, the endomorphism *e(Q)* associated to another pyramid Q with the same lengths of rows and to any another linear order for the elements of Q has the same Jordan form of *e(P)*, and hence lies in the same conjugacy class. □

Definition 4.13. In the same setting as before, we define the diagonal matrix $h(P) \in \mathfrak{gl}_n$ by letting its j^{th} diagonal entry equal to the first coordinate of the center of the j^{th} box.

Example 4.14. Consider the same pyramid P as in Figure 4.1:



Then the diagonal matrix $h(P)$ associated to P is:

$$h(P) = \operatorname{diag}(0, 1, 2, 3, 2, -3, 2, 0, -1).$$

Remark 4.15. The eigenspace decomposition of $\operatorname{ad}(h(P))$ is a $\mathbb{Z}$-grading of $\mathfrak{gl}_n$.

Proof. By Theorem 1.40, we know that $\operatorname{ad}(h(P))$ is semisimple. Furthermore, if $h(P) = \operatorname{diag}(h_1, \ldots, h_n)$, then the eigenvalues of $\operatorname{ad}(h(P))$ are equal to $h_i - h_j \in \mathbb{Z}$. Hence we can write the eigenspace decomposition of $\mathfrak{gl}_n$ as $\mathfrak{gl}_n = \bigoplus_{i,j=1}^n \mathfrak{gl}(\mathbb{F}^n)_{i-j}$, where $\mathfrak{gl}(\mathbb{F}^n)_{i-j} = \{g \in \mathfrak{gl}_n : [h(P), g] = (h_i - h_j)g\}$. We also know by Theorem 1.40 that a basis of $\mathfrak{gl}(\mathbb{F}^n)_{i-j}$ is given by the elementary matrices E_{st}, such that $h_s - h_t = h_i - h_j$. So, if we take $E_{st} \in \mathfrak{gl}(\mathbb{F}^n)_{i-j}$, $E_{uv} \in \mathfrak{gl}(\mathbb{F}^n)_{l-m}$, we have that $[E_{st}, E_{uv}] = \delta_{tu}E_{sv} - \delta_{sv}E_{ut}$; if $t = u$ and $s = v$, then $[E_{st}, E_{uv}] = E_{ss} - E_{tt} \in \mathfrak{gl}(\mathbb{F}^n)_{i-j+l-m}$ since $[h(P), E_{ss} - E_{tt}] = 0$. If $t = u$, $s \neq v$, then $[E_{st}, E_{uv}] = E_{sv} \in \mathfrak{gl}(\mathbb{F}^n)_{i-j+l-m}$ because $h_s - h_v = h_s - h_t + h_u - h_v = h_i - h_j + h_l - h_m$ and similarly in the other cases. So the eigenspace decomposition of $\operatorname{ad}(h(P))$ is a $\mathbb{Z}$-grading of $\mathfrak{gl}_n$. □

Remark 4.16. Let $\mathfrak{gl}_n = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be the $\mathbb{Z}$-grading defined by $\mathrm{ad}(h(P))$. Then $e(P) \in \mathfrak{g}_2$.

Proof. By definition, $e(P)$ is a sum of elementary endomorphisms E_{st} which connect boxes with centers (h_s, j) with boxes with centers $(h_s + 2, j)$ (see Figure 4.1). Thus, by the proof of Remark 4.15, $[h(P), E_{st}] = 2E_{st}$. So $E_{st} \in \mathfrak{g}_2$ and also $e(P) \in \mathfrak{g}_2$. □

Remark 4.17. The nilpotent endomorphism $e(P)$ can be embedded into an $\mathfrak{sl}_2$-triple $\{e(P), h(P), f(P)\}$ containing $h(P)$.

Proof. By Remark 4.16, $[h(P), e(P)] = 2e(P)$. Thus, by Claim 2. of the proof of the Jacobson-Morozov Theorem (Theorem 2.12), $h(P) \in [\mathfrak{gl}_n, e(P)]$. So, by Lemma 2.10, there exists an $\mathfrak{sl}_2$-triple as requested. □

Remark 4.18. The characteristic of this $\mathbb{Z}$-grading can be described as follows. First, denote by b_j the number of squares in the j^{th} column of P, for $j = 1, \ldots, 2p_1 - 1$.

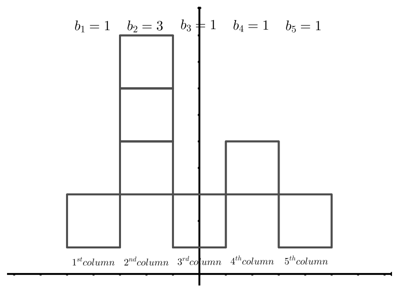


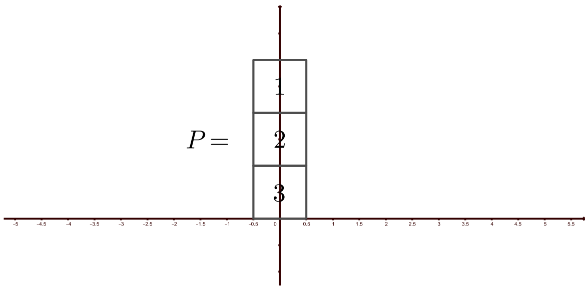
Figure 4.2: Numeration of the columns

Note that for j odd, necessarily $b_j > 0$, since the box $(j, 1)$ surely belongs to the pyramid (with this new enumeration of the columns). Next, for each j, construct a sequence which begins with $b_j - 1$ zeros and is followed by a:

- 2 if the right neighbour of b_j (i.e. b_{j+1}) is zero;
- 1 if the right neighbour of b_j (i.e. b_{j+1}) is nonzero;
- nothing at all if b_j doesn't have any right neighbour, i.e., $j = 2p_1 - 1$.

Now concatenate the sequence obtained, to form the sequence of $n-1$ integers equal to 0, 1 or 2, which defines the characteristic in question by assigning these integers to the corresponding simple roots.
Notice also that, from what we saw in the proof of Remark 4.15, an elementary matrix E_{ij} has a non-negative degree if and only if the label i is not located strictly to the left of the label j in the pyramid P.

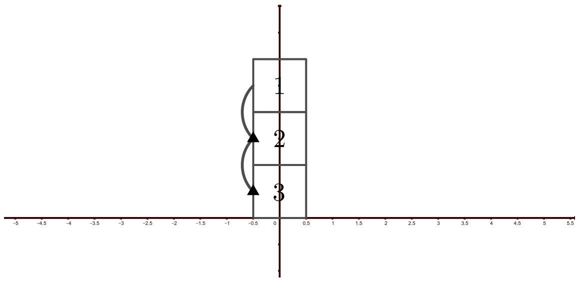
Example 4.19. Consider the following pyramid P with boxes labeled by the standard basis vectors of $\mathbb{F}^3$:

$P=$ 

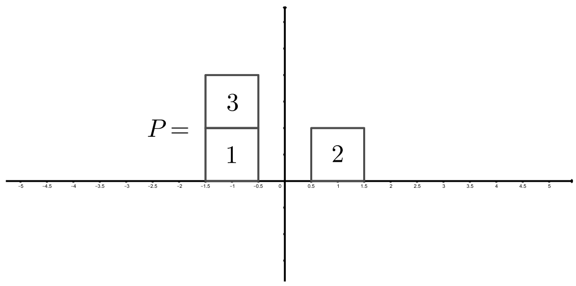
The sequence of integers equal to 0, 1 or 2 obtained by P is $(0,0)$. This sequence defines the characteristic of the $\mathbb{Z}$-grading of $\mathfrak{gl}_3$ given by the eigenspace decomposition of $\mathrm{ad}(h(P))$ (where $h(P) = \mathrm{diag}(0,0,0)$) by assigning these integers to the corresponding simple roots, i.e., by finding two simple roots α, β such that $\mathfrak{g}_\alpha \subset (\mathfrak{gl}_3)_0$ and $\mathfrak{g}_\beta \subset (\mathfrak{gl}_3)_0$.
In this case, we can consider the simple roots α, β such that $\mathfrak{g}_\alpha = \langle E_{21} \rangle$ and $\mathfrak{g}_\beta = \langle E_{32} \rangle$ because:

- $E_{21} \in (\mathfrak{gl}_3)_0$ since $[h(P), E_{21}] = 0$;
- $E_{32} \in (\mathfrak{gl}_3)_0$ since $[h(P), E_{32}] = 0$;
- $E_{31} = [E_{32}, E_{21}] \in [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, then $\alpha + \beta \in \Phi$.

Hence, also $-\alpha, -\beta, -\alpha-\beta \in \Phi$ by Proposition 1.64. This shows that $\{\alpha, \beta\}$ is a base of Φ because every root of Φ can be written as sum of α and β with integer coefficients all positive or all negative. This choice of α and β can be represented in the following way, where the arrows linking the i^{th} and the j^{th} boxes denote the matrix E_{ji} of $\mathfrak{gl}_3$.



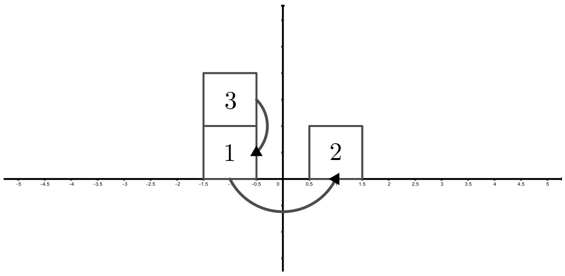
Example 4.20. Consider the following pyramid P with boxes labeled by the standard basis vectors of $\mathbb{F}^3$:



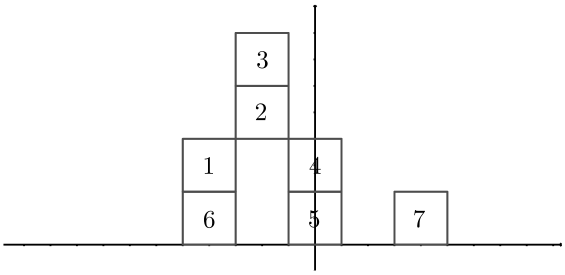
The sequence of integers equal to 0, 1 or 2 obtained by P is $(0,2)$. This sequence defines the characteristic of the $\mathbb{Z}$-grading of $\mathfrak{gl}_3$ given by the eigenspace decomposition of $\mathrm{ad}(h(P))$, where $h(P) = \mathrm{diag}(-1,1,-1)$.
As in the previous example, we want to assign these integers to the corresponding simple roots, i.e., we want to find two simple roots α, β such that $\mathfrak{g}_\alpha \subset (\mathfrak{gl}_3)_0$ and $\mathfrak{g}_\beta \subset (\mathfrak{gl}_3)_2$.
In this case, we can consider the simple roots α, β such that $\mathfrak{g}_\alpha = \langle E_{13} \rangle$ and $\mathfrak{g}_\beta = \langle E_{21} \rangle$ because:

- $E_{13} \in (\mathfrak{gl}_3)_0$ since $[h(P), E_{13}] = 0$;
- $E_{21} \in (\mathfrak{gl}_3)_2$ since $[h(P), E_{21}] = 2E_{21}$;
- $E_{23} = [E_{21}, E_{13}] \in [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, then $\alpha + \beta \in \Phi$.

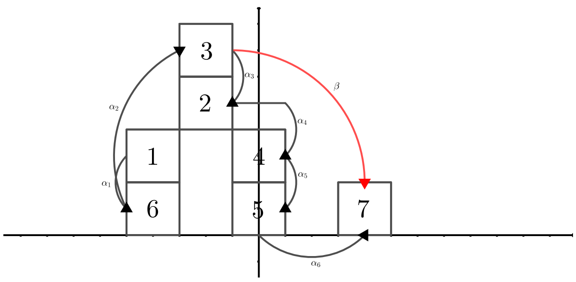
Hence, also $-\alpha, -\beta, -\alpha - \beta \in \Phi$ by point 2. of Proposition 1.64. This shows that $\{\alpha, \beta\}$ is a base of Φ because every root of Φ can be written as sum of α and β with integer coefficients all positive or all negative. With the same conventions as before, this choice of α and β can be visualized in the following way:



Example 4.21. Consider the following pyramid P with boxes labeled by the standard basis vectors of $\mathbb{F}^7$:



The sequence of integers equal to 0, 1 or 2 obtained by P is $(0,1,0,1,0,2)$, and it defines the characteristic of the $\mathbb{Z}$-grading of $\mathfrak{gl}_7$ given by the eigenspace decomposition of $\text{ad}(h(P))$, where $h(P) = \text{diag}(-2,-1,-1,0,0,-2,2)$.
As in the previous examples, we want to find six simple roots $\alpha_1,\ldots,\alpha_6$ such that $\mathfrak{g}_{\alpha_1},\mathfrak{g}_{\alpha_3},\mathfrak{g}_{\alpha_5} \subset (\mathfrak{gl}_7)_0$, $\mathfrak{g}_{\alpha_2},\mathfrak{g}_{\alpha_4} \subset (\mathfrak{gl}_7)_1$ and $\mathfrak{g}_{\alpha_6} \subset (\mathfrak{gl}_7)_2$.
Now consider the following figure, where α_j denotes the simple root correspondent to the root space spanned by the endomorphism represented by the arrow, with the same notation as before. For example α_1 denotes the root corresponding to the root space $\langle E_{61}\rangle$.

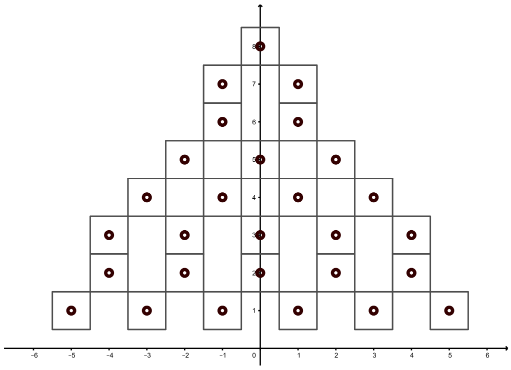


Note that α_1, α_3 and α_5 have degree 0, α_2 and α_4 have degree 1, and α_6 has degree 2 (meaning that the relative root spaces have such degrees in the $\mathbb{Z}$-grading we are considering). Furthermore $\Delta := \{\alpha_1, \ldots, \alpha_6\}$ is a base of the root system Φ. Indeed, if we take any non-negative endomorphism E_{ij} (i.e., such that $b_j - b_i \geq 0$), then E_{ij} belongs to some root space $\mathfrak{g}_\beta$, where β is a positive root (with respect to Δ). For example, if we take the elementary endomorphism $E_{73} \in (\mathfrak{gl}_7)_3$ (indicated in picture by the red arrow β), then $\beta = \alpha_6 + \alpha_5 + \alpha_4 + \alpha_3$, meaning that $E_{73} = [E_{75}, [E_{54}, [E_{42}, E_{23}]]] \in \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$.
Moreover, if $b_j - b_i \leq 0$ and $\langle E_{ji} \rangle = \mathfrak{g}_\gamma$ for some $\gamma \in \Phi^+$, then $\langle E_{ij} \rangle = \mathfrak{g}_{-\gamma}$.

Remark 4.22. In general, consider a pyramid P with some labeling of its boxes. Then, starting from the highest box of the first column, connect every box with the box immediately below until the bottom. Then connect this box with the highest box of the nonzero nearest right column. Continue this procedure until arriving to a box with no lower boxes or right neighbours.
Hence, the roots defining the characteristic of the $\mathbb{Z}$-grading induced by $\mathrm{ad}(h(P))$ are the ones whose root spaces are spanned by the elementary matrices represented by the arrows described above. Indeed, if we take any non-negative elementary endomorphism and represent it in the pyramid by an arrow, then this arrow can be written as composition of arrows representing simple roots, as shown in Examples 4.19 - 4.21.
Moreover, notice that by construction this choice of the simple roots agrees with Remark 4.18, i.e., the degrees of these roots correspond to the integers in the sequence constructed in Remark 4.18.

Definition 4.23. Given $p = (p_1, \ldots, p_k) \in Par(n)$, denote by $P(p)$ the symmetric pyramid corresponding to p, i.e., the pyramid with k rows such that the j^{th} row contains p_j boxes centered at (i, j), where i runs over the arithmetic progression with difference 2 and $f_j = -p_j + 1 = -l_j$.

Example 4.24. The following figure represents the symmetric pyramid associated to the partition $p = (6, 5, 5, 4, 3, 2, 2, 1)$



Denote by $Pyr(p)$ the set of all pyramids attached to the partition p, i.e. the pyramids containing p_j boxes in the j^{th} row, for $j = 1, \ldots, k$.

Lemma 4.25.

$$\#Pyr(p) = \prod_{j=1}^{k-1}(2(p_j - p_{j+1}) + 1)$$

In case $k = 1$, the empty product is understood to be 1.

Proof. Obviously, all the pyramids from $Pyr(p)$ are obtained from the symmetric one, $P(p)$, by a horizontal shift for each $j \geq 1$ of the boxes of the $(j+1)^{\text{th}}$ row (regarded as a whole) in a way such that the 3^{rd} condition in Definition 4.7 is satisfied (we can do this operation in $2(p_j - p_{j+1}) + 1$ ways for each $j = 1, \ldots, k-1$). □

Proposition 4.26. *The generating function for the number Pyr_n of pyramids of size n is given by*

$$F(q) = \sum_{n\geq 1} \left(\prod_{k=1}^{n-1} \frac{1+q^k}{(1-q^k)^2} \right) \frac{q^n}{1-q^n}.$$

Proof. Using Lemma 4.25, we can calculate the generating function

$$F(q) = \sum_n Pyr_n q^n$$

for the number Pyr_n. Indeed, we have

$$Pyr_n = \sum_{p \in Par(n)} \#Pyr(p).$$

Thus, according to Lemma 4.25, we can write

$$F(q) = \sum_p \left(\prod_{i:p_{i+1}>0} (2(p_i - p_{i+1}) + 1) \right) q^{\sum_i p_i}$$

with the sum ranging over all partitions of all natural numbers.
Notice now that, since partitions are in one-to-one correspondence with duals of partitions, we obviously have

$$F(q) = \sum_p \left(\prod_{i:p^*_{i+1}>0} (2(p^*_i - p^*_{i+1}) + 1) \right) q^{\sum_i p_i}$$

Then, using (4.1) and (4.2), we can write

$$F(q) = \sum_p \left(\prod_{i:p^*_{i+1}>0} (2\text{mult}_p(i) + 1) \right) q^{\sum_{i\geq 1} i\text{mult}_p(i)}$$

Now observe thar for any i, the condition $p^*_{i+1} > 0$ (i.e. $\#\{j : p_j \geq i+1\} > 0$) is equivalent to $i < p_1$. Indeed, if $i < p_1$, then $\{j; p_j \geq i+1\} \supseteq \{1\}$, and hence $p^*_{i+1} > 0$ (similarly the other way around).
Thus, since $\sum_{i\geq 1} i\text{mult}_p(i) = \sum_{i=1}^{p_1} i\text{mult}_p(i)$, the above can be rewritten as

$$F(q) = \sum_p \left(\prod_{i<p_1} (2\text{mult}_p(i) + 1) q^{i\text{mult}_p(i)} \right) q^{p_1 \text{mult}_p(p_1)}$$

or as well

$$F(q) = \sum_n \sum_{p:p_1=n} \left(\prod_{i<n} (2\text{mult}_p(i) + 1) q^{i\text{mult}_p(i)} \right) q^{n\text{mult}_p(n)}$$

The last expression can be rewritten as

$$\sum_n \sum_{m_1,m_2,\ldots,m_{n-1}\geq 0, m_n>0} \left((2m_1+1)q^{m_1} \right) \cdots (2m_{n-1}+1) q^{(n-1)m_{n-1}} \Big) q^{nm_n}$$

$$= \sum_n \left(\sum_{m_1\geq 0} (2m_1+1) q^{m_1} \cdots \sum_{m_{n-1}\geq 0} (2m_{n-1}+1)(q^{n-1})^{m_{n-1}} \right) \sum_{m_n>0} (q^n)^{m_n}$$

Since for $|t| < 1$,

$$\sum_{m\geq 0}(2m+1)t^m = \frac{1+t}{(1-t)^2}, \qquad \sum_{m>0} t^m = \frac{t}{1-t}$$

we obtain the thesis. □

Lemma 4.27. *Let $p = (p_1, \dots, p_k)$ be a partition on the number n, with dual partition $p^* = (p_1^*, \dots, p_r^*)$. Then*

$$n + 2\sum_{i=1}^{k}(i-1)p_i = \sum_{i=1}^{r}(p_i^*)^2.$$

Proof. First of all notice that $n + 2\sum(i-1)p_i = \sum p_i + 2\sum ip_i - 2\sum p_i = 2\sum ip_i - \sum p_i = \sum(2i-1)p_i$. Now, enumerate the boxes of the partition p by placing 1's in the boxes of the first row, 3's in the boxes of the second row,..., $(2k-1)$'s in the boxes of the k^{th} row, as in Figure 4.3.

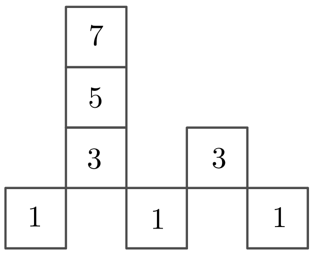
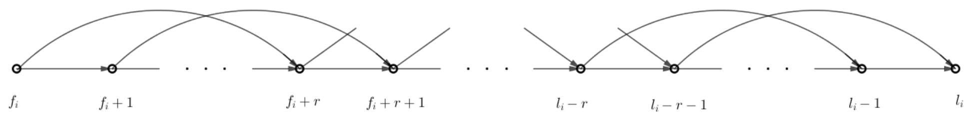


Figure 4.3: Enumeration of the boxes

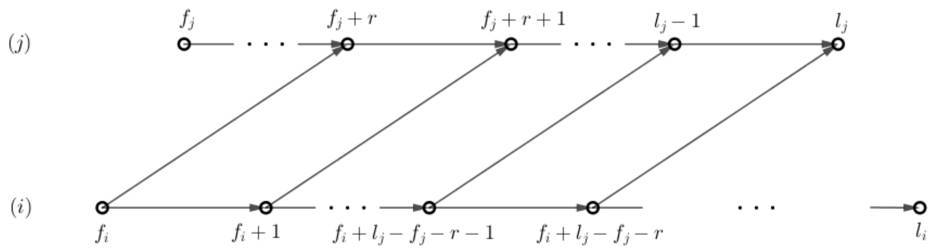
Then $\sum_{i=1}^{k}(2i-1)p_i$ is the sum over all boxes. If we do the sum instead by columns, using the standard identity $\sum_{i=1}^{m}(2i-1) = m^2$ for $m = (p_i^*)^2$, $i = 1, \dots, r$, we get $\sum_{i=1}^{r}(p_i^*)^2$, and hence the identity. □

Remark 4.28. Let p be a partition of the number n and $P \in \mathrm{Pyr}(p)$. Recall that we defined $e(P)$ as the endomorphism which acts "along the rows of the pyramid" and for this reason it is natural to depict it via horizontal arrows which connect centers of the boxes with their right neighbours on the same row (see Figure 4.1).

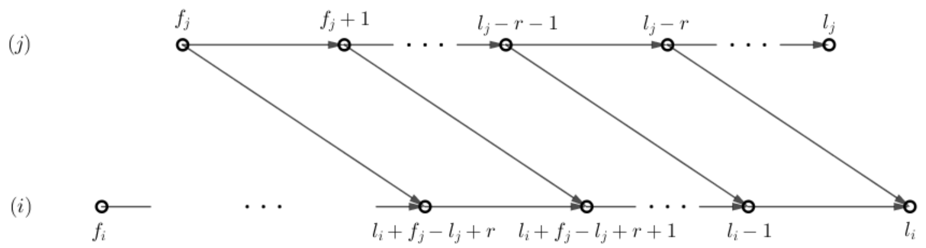
Furthermore, the elementary endomorphisms E_{ij} map the j^{th} basis vector of $\mathbb{F}^n$ to the i^{th} basis vector, and we represent E_{ij} by the arrow connecting the corresponding centers of the boxes of the pyramid P. Then the endomorphisms commuting with *e(P)* are precisely those represented by collections of arrows, which fit with arrows of *e(P)* into commutative diagrams. The figures of Type 1, 2, 3 represent examples of such commutative diagrams.



Type 1



Type 2

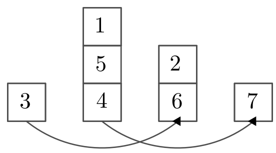


Type 3

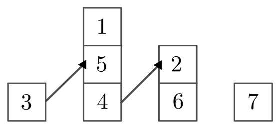
The loops in these figures mean identity mappings.

Now we give some examples of the endomorphisms defined in Remark 4.28.

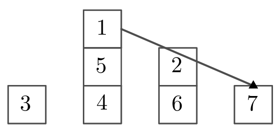
Example 4.29.



This figure represents the endomorphism (of Type 1, with $i = 1$, $r = 2$) described by $E_{63} + E_{74}$.



This figure represents the endomorphism (of Type 2, with $i = 1$, $j = 2$ and $r = 0$) described by $E_{53} + E_{24}$.



This figure represents the endomorphism (of Type 3, with $i = 1$, $j = 3$ and $r = 0$) described by E_{71}.

Proposition 4.30. *Let p be a partition of the number n, $P \in Pyr(p)$ and $e(P)$ the endomorphism acting "along the rows of the pyramid". Then the endomorphisms of Type 1, 2, 3 form a basis of $C_{\mathfrak{gl}_n}(e(P))$.*

Proof. It is easy to see that when i runs through the set $\{1,\ldots,k\}$, the corresponding endomorphisms of Type 1, 2, 3 are linearly independent. Indeed, every such endomorphism f is a sum of elementary ones, which appear as summands only in f, and in no other endomorphism of Type 1, 2, 3.
Furthermore, the number of diagrams of first type is $p_1 + p_2 + \ldots + p_k = n$. Indeed, for every fixed $i \in \{1,\ldots,k\}$, every choice of $r \in \{0,\ldots,p_i - 1\}$ uniquely determines the endomorphism. Hence, for each i we have p_i endomorphisms. Summing in i we obtain the number of all endomorphisms of Type 1.
Notice now that counting the number of diagrams of second and third Type is equivalent, so we count those of the second type. For every fixed $i \in \{1,\ldots,k\}$ and $j \in \{i+1,\ldots,k\}$, we have p_j endomorphisms of Type 2 (indeed we have one such endomorphism of this type for every choice of $r \in \{0,\ldots,p_j - 1\}$). So the number of endomorphisms of this type for fixed i is $p_{i+1} + \ldots + p_k$. Hence the total number is $\sum_{i=1}^{k}(p_{i+1} + \ldots + p_k) = (p_2 + \ldots + p_k) + (p_3 + \ldots + p_k) + \ldots + (p_{k-1} + p_k) + p_k = \sum_{i=1}^{k}(i-1)p_i$.
So the number of endomorphisms of Type 1, 2, 3 is $n + 2\sum_{i=1}^{k}(i-1)p_i$. But, by Lemma 4.27 $n + 2\sum_{i=1}^{k}(i-1)p_i = (p_1^*)^2 + \cdots + (p_k^*)^2$.
All this means that diagrams of Type 1, 2, 3 form a basis of the centralizer of the nilpotent $e(p)$ because by Theorem 4.6 $\dim C_{\mathfrak{gl}_n}(e(p)) = (p_1^*)^2 + \cdots + (p_k^*)^2$. □

Corollary 4.31. *Let $p = (p_1^{m_1},\ldots,p_s^{m_s})$ be a partition of the number n, $P \in Pyr(p)$. Choose a labeling of P in a way such that the endomorphism $e(P)$ is in its canonical Jordan form. Then an element $x \in \mathfrak{gl}_n$ belongs to $C_{\mathfrak{gl}_n}(e(P))$ if and only if*

$$x = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix},$$

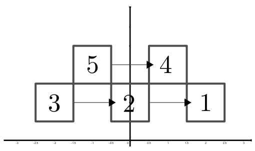
where $A_{ij} \in M_{m_ip_i \times m_jp_j}(\mathbb{F})$ consists of m_im_j independent $p_i \times p_j$ matrices of the form:

- $$\begin{pmatrix} a_1 & a_2 & \cdots & a_{p_j} \\ & a_1 & \ddots & \vdots \\ & & \ddots & a_2 \\ & & & a_1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ if } i \geq j;$$

- $$\begin{pmatrix} 0 & \cdots & 0 & a_1 & a_2 & \cdots & a_{p_i} \\ 0 & \cdots & 0 & & a_1 & \cdots & a_{p_{i-1}} \\ \vdots & & \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & & & & a_1 \end{pmatrix} \text{ if } i < j.$$

Proof. This follows from Theorem 4.30 and in particular from matrix representation of the endomorphism of Type 1, 2, 3. □

Example 4.32. Consider the partition $p = (3, 2)$ of the number 5. Consider the following pyramid in $Pyr(p)$:



and choose the labeling in a way such that $e(P) = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

By Corollary 4.31, an element of the centralizer of $e(P)$ in $\mathfrak{gl}_5$ is of the form

$$x = \left(\begin{array}{ccc|cc} a & b & c & f & g \\ 0 & a & b & 0 & f \\ 0 & 0 & a & 0 & 0 \\ \hline 0 & h & i & d & e \\ 0 & 0 & h & 0 & d \end{array}\right).$$

This description of the centralizer of $e(P)$ agrees with the one given in Proposition 4.30. Indeed each parameter corresponds to an endomorphism of Type 1, 2 or 3 in the following way:

- the endomorphism of Type 1 relative to the first row of P with step $r = 0$ (resp. $r = 1, 2$) is represented by the parameter a (resp. b, c);
- the endomorphism of Type 1 relative to the second row of P with step $r = 0$ (resp. $r = 1$) is represented by the parameter d (resp. e);
- the endomorphism of Type 2 connecting the first and the second row of P with step $r = 0$ (resp. $r = 1$) is represented by the parameter f (resp. g);
- the endomorphism of Type 3 connecting the second and the first row of P with step $r = 0$ (resp. $r = 1$) is represented by the parameter h (resp. i).

Definition 4.33. Let $k \in \mathfrak{gl}_n$ be a diagonal matrix, e a nilpotent endomorphism. Then we say that the pair (k, e) is good if e is a good element in the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(k)$.

Proposition 4.34. *The pair $(h(P), e(P))$, where $P \in Pyr(p)$ and p is a partition of the number n, is good.*

Proof. By Remark 4.17, we know that there exists an $\mathfrak{sl}_2$-triple containing $h(P)$ and $e(P)$; thus the grading induced by $\mathrm{ad}(h(P))$ is the Dynkin grading, which is good with good element $e(P)$ by Proposition 3.11. □

However, we want to show this in a more explicit way, following the strategy explained in Remark 4.35.

Remark 4.35. In the following two theorems we will use Proposition 4.30 in order to show that some pair $(k, e(P))$ is good. Namely, we will show that the endomorphisms of Type 1, 2, 3 have non-negative degree with respect to the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(k)$, and conclude that the centralizer of $e(P)$ is contained in the non-negative part of this $\mathbb{Z}$-grading. This is equivalent to property a) of Definition 3.7. But by Theorem 3.25, we have that Properties a) and b) of the definition of good element (Definition 3.7) are equivalent; hence the pair $(k, e(P))$ is good.

Theorem 4.36. *Let p be a partition of the number n, $P \in Pyr(p)$ and $h(P)$ the corresponding diagonal matrix in $\mathfrak{gl}_n$. Then the pair $(h(P), e(P))$ is good.*

Proof. Since the ending of no arrow in the diagrams described in Remark 4.28 is located strictly to the left of its source, all the corresponding endomorphisms have non-negative degree with respect to $h(P)$, meaning that every such endomorphism is sum of elementary ones with non-negative degree with respect to the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(h(P))$. Indeed, we already noticed that an elementary endomorphism E_{ij} (connecting the j^{th} basis vector of $\mathbb{F}^n$ with the i^{th}) has degree $h_i - h_j$, where h_i and h_j are the first coordinates of the i^{th} and of the j^{th} box, respectively. Hence, since the ending of no arrow in these diagrams is located strictly to the left of its source, $h_i - h_j \geq 0$.
By Proposition 4.30, the endomorphisms of Type 1, 2, 3 form a basis of the centralizer of $e(P)$ and all the elements of this centralizer have non-negative degree with respect to the $\mathbb{Z}$-grading determined by $h(P)$ (i.e., the centralizer of $e(P)$ in $\mathfrak{gl}_n$ lies in the non-negative part of the $\mathbb{Z}$-grading determined by $h(P)$). Thus, using Remark 4.35, we can conclude that the pair $(h(P), e(P))$ is good. □

In the next theorem we want to characterize all the good $\mathbb{Z}$-gradings with good element $e(p)$. In order to do this we need an explicit description of the center of the reductive part of the centralizer of $e(p)$.

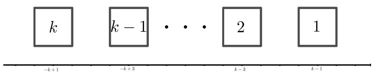
Proposition 4.37. *Consider the partition $p = (k)$ of the number k and $P \in Pyr(p)$. Choose a labeling of P such that $e(p) \in \mathfrak{gl}_k$ is a Jordan block.*

Then $e(p)$ embeds into an $\mathfrak{sl}_2$-triple $\{e(p), h(p), f(p)\}$, where

$$f(p) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & a_{k-1} & 0 \end{pmatrix}$$

with $a_j = a_{k-j} = j(k-j)$ for $j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$.

Proof. With the choice of the labeling of P such that $e(p)$ is a Jordan block, the pyramid P is:



It follows that $h(p) = \mathrm{diag}(k-1, k-3, \ldots, -k+3, -k+1)$. By Lemma 2.10, since $[h(p), e(p)] = 2e(p)$, we only need to find $f(p)$ such that $[e(p), f(p)] = h(p)$. Let $f(p)$ be of the form:

$$f(p) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & a_{k-1} & 0 \end{pmatrix},$$

with $a_1, \ldots, a_{k-1} \in \mathbb{F}$. Then $h(p) = [e(p), f(p)]$ if and only if

$$\begin{pmatrix} k-1 & & & \\ & k-3 & & \\ & & \ddots & \\ & & & -k+1 \end{pmatrix} = \begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & a_{k-1} & \\ & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_{k-1} \end{pmatrix}.$$

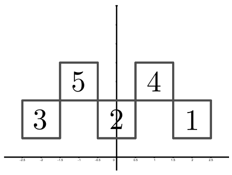
So we obtain the system:

$$\begin{cases} a_1 = k-1 \\ a_2 - a_1 = k-3 \\ \quad \vdots \\ a_{k-1} - a_{k-2} = -k+3 \\ -a_{k-1} = -k+1 \end{cases}$$

whose solutions are $a_j = a_{k-j} = j(k-j)$ for $j = 1, \dots, \left\lfloor \dfrac{k}{2} \right\rfloor$. □

Corollary 4.38. *Let $p = (p_1^{m_1}, p_2^{m_2}, \dots, p_d^{m_d})$ be a partition of the number n and $P(p)$ be the symmetric pyramid determined by the partition p. Choose a labeling of P such that $e(p) \in \mathfrak{gl}_n$ is in its canonical Jordan form. Then $e(p)$ embeds into an $\mathfrak{sl}_2$-triple $\{e(p), h(p), f(p)\}$, where $f(p)$ is a block matrix with each block of the form described in Proposition 4.37.*

Example 4.39. Consider the partition $p = (3, 2)$ of the number 5. Consider the following labeling of the symmetric pyrimid $P(p)$:



Then $e(p) = \left(\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ \hline & & & 0 & 1 \\ & & & & 0 \end{array}\right)$ and, by Corollary 4.38, $f(p) = \left(\begin{array}{ccc|cc} 0 & & & & \\ 2 & 0 & & & \\ & 2 & 0 & & \\ \hline & & & 0 & \\ & & & 1 & 0 \end{array}\right)$.

Let $\mathfrak{s} = \{e(p), h(p), f(p)\}$. In order to describe $C_{\mathfrak{gl}_5}(\mathfrak{s})$, it is enough to see which conditions an element $x \in C_{\mathfrak{gl}_5}(e(p))$ has to satisfy so that it also belongs to $C_{\mathfrak{gl}_5}(f(p))$.

By Corollary 4.31, an element $x \in C_{\mathfrak{gl}_5}(e(p))$ if it is of the form $\left(\begin{array}{ccc|cc} a & b & c & f & g \\ & a & b & & f \\ & & a & & \\ \hline & h & i & d & e \\ & & h & & d \end{array}\right)$.

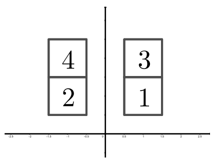
So $x \in C_{\mathfrak{gl}_5}(f(p))$ if and only if

$$0 = [f(p), x] = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 2a & 2b & 2c & 2f & 2g \\ 0 & 2a & 2b & 0 & 2f \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & h & i & d & e \end{array}\right) - \left(\begin{array}{ccc|cc} 2b & 2c & 0 & g & 0 \\ 2a & 2b & 0 & f & 0 \\ 0 & 2a & 0 & 0 & 0 \\ \hline 2h & 2i & 0 & e & 0 \\ 0 & 2h & 0 & d & 0 \end{array}\right),$$

from which we get the conditions $b = c = f = g = h = i = 0$. So an element x belongs to $C_{\mathfrak{gl}_5}(\mathfrak{s})$ if and only if it is of the form

$$\left(\begin{array}{ccc|cc} a & & & & \\ & a & & & \\ & & a & & \\ \hline & & & d & \\ & & & & d \end{array}\right).$$

Example 4.40. Consider the partition $p = (2, 2)$ of the number 4. Consider the following labeling of the symmetric pyrimid $P(p)$:



Then $e(p) = \left(\begin{array}{cc|cc} 0 & 1 & & \\ & 0 & & \\ \hline & & 0 & 1 \\ & & & 0 \end{array}\right)$ and, by Corollary 4.38, $f(p) = \left(\begin{array}{cc|cc} 0 & & & \\ 1 & 0 & & \\ \hline & & 0 & \\ & & 1 & 0 \end{array}\right)$.

Let $\mathfrak{s} = \{e(p), h(p), f(p)\}$. By Corollary 4.31, an element $x \in C_{\mathfrak{gl}_4}(e(p))$ if it is of the form $\left(\begin{array}{cc|cc} a & b & e & f \\ & a & & e \\ \hline g & h & c & d \\ & g & & c \end{array}\right)$. So $x \in C_{\mathfrak{gl}_4}(f(p))$ if and only if

$$0 = [f(p), x] = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ a & b & e & f \\ \hline 0 & 0 & 0 & 0 \\ g & h & c & d \end{array}\right) - \left(\begin{array}{cc|cc} b & 0 & f & 0 \\ a & 0 & e & 0 \\ \hline h & 0 & d & 0 \\ g & 0 & c & 0 \end{array}\right),$$

from which we get the conditions $b = d = f = h = 0$. So an element x

belongs to $C_{\mathfrak{gl}_4}(\mathfrak{s})$ if and only if it is of the form

$$\left(\begin{array}{cc|cc} a & & e & \\ & a & & e \\ \hline g & & c & \\ & g & & c \end{array}\right).$$

One can argue as in Examples 4.39 and 4.40 in order to describe the reductive part of $C_{\mathfrak{gl}_n}(e(p))$.

Proposition 4.41. *Let $p = (p_1^{m_1}, \dots, p_d^{m_d})$ be a partition of the number n, $P \in Pyr(p)$. Choose a labeling of P in a way such that the endomorphism $e(p)$ is in its canonical Jordan form. Let $\mathfrak{s} = \{e(p), h(p), f(p)\}$ be the $\mathfrak{sl}_2$-triple given by Corollary 4.38. Then an element $x \in \mathfrak{gl}_n$ belongs to $C_{\mathfrak{gl}_n}(\mathfrak{s})$ if and only if*

$$x = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_d \end{pmatrix},$$

where $A_i \in M_{m_i p_i \times m_i p_i}(\mathbb{F})$ consists of m_i^2 independent $p_i \times p_i$ diagonal matrices that are scalar multiples of I_{p_i}.

Example 4.42. In the same setting as in Example 4.39, we can notice that an element $x = \left(\begin{array}{ccc|cc} a & & & & \\ & a & & & \\ & & a & & \\ \hline & & & d & \\ & & & & d \end{array}\right) \in C_{\mathfrak{gl}_5}(\mathfrak{s})$ also belongs to $Z(C_{\mathfrak{gl}_5}(\mathfrak{s}))$ because it is diagonal.

Example 4.43. In the same setting as in Example 4.40, consider an element $x = \left(\begin{array}{cc|cc} a & & e & \\ & a & & e \\ \hline g & & c & \\ & g & & c \end{array}\right) \in C_{\mathfrak{gl}_4}(\mathfrak{s})$. In order to understand when $x \in Z(C_{\mathfrak{gl}_4}(\mathfrak{s}))$, we need to impose that x commutes with the elements of the basis of $C_{\mathfrak{gl}_4}(\mathfrak{s})$. So it must hold:

- $0 = [x, \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right)] = \left(\begin{array}{c|c} aI_2 & 0 \\ \hline cI_2 & 0 \end{array}\right) - \left(\begin{array}{c|c} aI_2 & bI_2 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & -bI_2 \\ \hline cI_2 & 0 \end{array}\right).$
  So $b = c = 0$. Hence $x = \left(\begin{array}{c|c} aI_2 & 0 \\ \hline 0 & dI_2 \end{array}\right).$
- $0 = [x, \left(\begin{array}{c|c} 0 & I_2 \\ \hline 0 & 0 \end{array}\right)] = \left(\begin{array}{c|c} 0 & aI_2 \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} 0 & dI_2 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & (a-d)I_2 \\ \hline 0 & 0 \end{array}\right).$
  So $a = d$.

Hence $x = \left(\begin{array}{c|c} aI_2 & 0 \\ \hline 0 & aI_2 \end{array}\right).$

One can argue as in Examples 4.42 and 4.43 in order to generalize the description of the center of the reductive part of $C_{\mathfrak{gl}_n}(e(p))$, that is given by the following Proposition.

Proposition 4.44. *Let $p = (p_1^{m_1}, \ldots, p_d^{m_d})$ be a partition of the number n, $P \in Pyr(p)$. Choose a labeling of P in a way such that the endomorphism $e(p)$ is in its canonical Jordan form. Let $\mathfrak{s} = \{e(p), h(p), f(p)\}$ be the $\mathfrak{sl}_2$-triple given by Corollary 4.38. Then an element $x \in \mathfrak{gl}_n$ belongs to $Z(C_{\mathfrak{gl}_n}(\mathfrak{s}))$ if and only if*

$$x = \begin{pmatrix} a_1 I_{m_1 p_1} & & & \\ & a_2 I_{m_2 p_2} & & \\ & & \ddots & \\ & & & a_d I_{m_d p_d} \end{pmatrix}.$$

Theorem 4.45. *Let e(p) be the nilpotent element defined by a partition*

$$p = (p_1^{m_1}, p_2^{m_2}, \ldots, p_d^{m_d}) = (\underbrace{p_1, \ldots, p_1}_{m_1}, \underbrace{p_2, \ldots, p_2}_{m_2}, \ldots, \underbrace{p_d, \ldots, p_d}_{m_d})$$

of the number n. Define $t_i := \sum_{j=1}^{i} m_j p_j$ for $1 \leq i \leq d$ so that $t_1 < t_2 < \ldots < t_d = n$. Let $P(p)$ be the symmetric pyramid determined by the partition p, let $h(p) := h(P(p))$ be the corresponding diagonal matrix in $\mathfrak{gl}_n$ and let $h = (h_1, h_2, \ldots, h_n)$ be a diagonal matrix. Then, the pair $(h(p) + h, e(p))$ is good if and only if the coordinates h_i satisfy the following conditions:

1. $h_i - h_j$ are integers;

2. $h_1 = h_2 = \ldots = h_{t_1}, h_{t_1+1} = \ldots = h_{t_2}, \ldots, h_{t_{d-1}+1} = \ldots = h_{t_d}$;

3. $|h_{t_1} - h_{t_2}| \leq p_1 - p_2, \ldots, |h_{t_{d-1}} - h_{t_d}| \leq p_{d-1} - p_d$;

4. $\sum_{i=1}^{n} h_i = 0$.

Moreover, if for each $i \in \{2, \ldots, d\}$ we set $h_{t_{i-1}} - h_{t_i} = a_i$, where $a_i \in \{-p_{i-1} + p_i, \ldots, p_{i-1} - p_i\}$, then the system of linear equations given by conditions 2. and 4. has a unique solution.

Proof. Suppose that the pair $(h(p)+h, e(p))$ is good. First of all, if we denote by $\tilde{h}_i$ the i^{th} diagonal entry of $h(p)+h$, then $\tilde{h}_i - \tilde{h}_j \in \mathbb{Z}$ because $\mathrm{ad}(h(p)+h)$ defines a $\mathbb{Z}$-grading. Hence $h_i - h_j = (\tilde{h}_i - h(p)_i) - (\tilde{h}_j - h(p)_j) \in \mathbb{Z}$, i.e., condition 1. is satisfied.
Now, consider a labeling of $P(p)$ such that the matrix associated to $e(p)$ coincides with its canonical Jordan form. By Remark 4.17, we know that there exists an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e(p), h(p), f(p)\}$ containing $e(p)$ and $h(p)$. Hence, if the pair $(h(p) + h, e(p))$ is good, then $h \in Z(C_{\mathfrak{gl}_n}(\mathfrak{s}))$ by Theorem 3.18. Hence, by Proposition 4.44, we have condition 2.
Notice that $(h(p)+h, e(p))$ is a good pair if and only if $C_{\mathfrak{gl}_n}(e(p))$ is contained in the non-negative part of the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(h(p) + h)$. We want to see when the latter happens to be true. Since we know the description of a basis of $C_{\mathfrak{gl}_n}(e(p))$ (that is given by the endomorphisms of Type 1, 2, 3), we need to exhibit conditions that tell us when such elements of the basis lie in the non-negative part of the $\mathbb{Z}$-grading induced by $h(p) + h$. First of all notice the following facts.

1. By definition of $h(p)$, if we have an elementary endomorphism E_{ij} connecting a box centered in (j_1, j_2) with a box centered in (i_1, i_2), then $[h(p), E_{ij}] = (i_1 - j_1)E_{ij}$.

2. Due to condition 2. we can say that, if we consider the endomorphism E_{ij} as before, where we suppose that (i_1, i_2) and (j_1, j_2) are centers

of boxes belonging to rows of length p_i and p_j (respectively), then $[h, E_{ij}] = (h_{t_i} - h_{t_j})E_{ij}$.

Notice that endomorphisms of Type 1 belong to the non-negative part of the $\mathbb{Z}$-grading induced by $h(p) + h$. Indeed if φ is such an endomorphism, then $[h(p)+h, \varphi] = [h(p), \varphi]+[h, \varphi] = r\varphi$, with $r \geq 0$ the step defining φ, because $[h, \varphi] = (h_{t_i} - h_{t_i})\varphi = 0$.

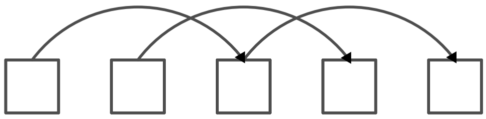


Figure 4.4

As it concerns the endomorphisms of Type 2 and 3, we may limit ourselves to examine the extreme case when the step r is equal to 0. Indeed, if such endomorphisms belong to the non-negative part of the $\mathbb{Z}$-grading induced by $(h(p) + h)$, also the endomorphisms of Type 2 or 3 with step $r > 0$ belong to the latter. This is clear if we look at Figure 5.5; indeed here the endomorphism with step $r = 1$ is given by $E_{61} + E_{72}$. But the latter can be also written as $[E_{65}, E_{51}] + [E_{76}, E_{62}]$ and, since $E_{65}, E_{51}, E_{76}, E_{62}$ belong to the non-negative part of the $\mathbb{Z}$-grading induced by $h(p) + h$, also $E_{61} + E_{72}$ does.

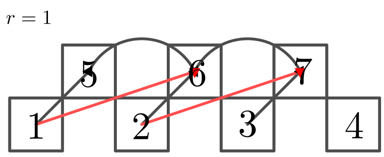


Figure 4.5

Also, we may just consider the extreme endomorphisms connecting boxes in rows of length p_i with boxes in rows of length p_{i+1} for endomorphisms of Type 2, and the other way around for endomorphisms of Type 3. Indeed, we can clearly see from Figure 4.6 that, if we know that E_{51} and E_{85} belong to the non-negative part of the $\mathbb{Z}$-grading induced by $h(p)+h$, also $E_{81} = [E_{85}, E_{51}]$ does.

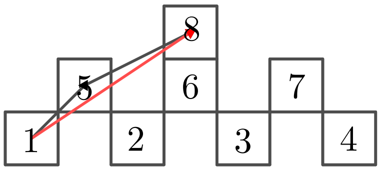
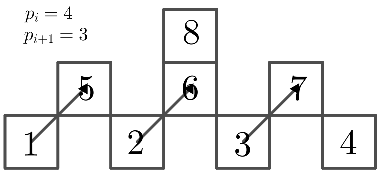
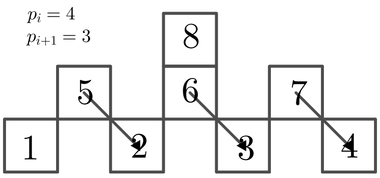


Figure 4.6

So, examine the case of an extreme endomorphism φ of Type 2 connecting boxes in a row of length p_i with boxes in a row of length p_{i+1}. Then $[h(p) + h, \varphi] = [h(p), \varphi] + [h, \varphi] = (p_i - p_{i+1})\varphi + (h_{t_{i+1}} - h_{t_i})\varphi = (p_i - p_{i+1} + h_{t_{i+1}} - h_{t_i})\varphi$. So we require that $p_i - p_{i+1} + h_{t_{i+1}} - h_{t_i} \geq 0$, i.e., that $h_{t_i} - h_{t_{i+1}} \leq p_i - p_{i+1}$, for φ to be contained in the non-negative part of the $\mathbb{Z}$-grading induced by $h(p) + h$.



Now consider an extreme endomorphism $\bar{\varphi}$ of Type 3 connecting boxes in a row of length p_{i+1} with boxes in a row of length p_i. Then, as above, we can say that $[h(p) + h, \bar{\varphi}] = (p_i - p_{i+1} + h_{t_i} - h_{t_{i+1}})\bar{\varphi}$. Thus $\bar{\varphi}$ belongs to the non-negative part of the $\mathbb{Z}$-grading induced by $h(p) + h$ if and only if $p_i - p_{i+1} + h_{t_i} - h_{t_{i+1}} \geq 0$, i.e., if $h_{t_i} - h_{t_{i+1}} \geq -(p_i - p_{i+1})$.



So we have shown that the endomorphisms of Type 1, 2, 3 belong to the non-negative part of the $\mathbb{Z}$-grading induced by $h(p)+h$ if and only if $|h_{t_i} - h_{t_{i+1}}| \leq$

$p_i - p_{i+1}$ for all i, i.e., condition 3. holds.
Now, notice that $h(p)+h \in [e(p), \mathfrak{gl}_n]$. Indeed, since $(h(p)+h, e(p))$ is a good $\mathbb{Z}$-grading, we know that $[h(p)+h, e(p)] = 2e(p)$. Furthermore, by Claim 2. of the proof of Theorem 2.12, we have $h(p)+h \in [e(p), \mathfrak{gl}_n]$. So, suppose by contradiction that condition 4. doesn't hold, i.e., that $\sum h_i \neq 0$. Then $h \notin \mathfrak{sl}_n$; thus, since $h(p) \in \mathfrak{sl}_n$, $h(p)+h \notin \mathfrak{sl}_n$. But $\mathfrak{gl}_n = \underbrace{[\mathfrak{gl}_n, \mathfrak{gl}_n]}_{=\mathfrak{sl}_n} \oplus Z(\mathfrak{gl}_n)$; so $h(p)+h \notin [\mathfrak{gl}_n, \mathfrak{gl}_n]$ and hence $h(p)+h \notin [e(p), \mathfrak{gl}_n]$.
Conversely, suppose that conditions 1., 2., 3., 4. hold. Then condition 1. implies that the eigenspace decomposition of $\mathrm{ad}(h(p)+h)$ is a $\mathbb{Z}$-grading. Furthermore, due to condition 2., condition 3. is equivalent to the fact that the centralizer of $e(p)$ in $\mathfrak{gl}_n$ is contained in the non-negative part of the $\mathbb{Z}$-grading induced by $h(p)+h$ (as previously shown). But this is equivalent to the injectivity of $\mathrm{ad}_{e(p)}$ on the negative part of the $\mathbb{Z}$-grading (see point 2.) of Remark 3.8). Hence, by Theorem 3.25, the pair $(h(p)+h, e(p))$ is good.
The last statement is clear because the matrix of coefficients associated to the system given by conditions 2. and 4. is:

$$\begin{pmatrix} 1 & & & 1 \\ -1 & 1 & & 1 \\ & \ddots & \ddots & \vdots \\ & & -1 & 1 \end{pmatrix}.$$

This matrix has maximum rank, and hence the system has a unique solution.
□

Remark 4.46. In the notation of Theorem 4.45, each possible choice of $a_i \in \{-p_{i-1}+p_i, \dots, p_{i-1}-p_i\}$ for $i = 2, \dots, d$ determines:

1. a good $\mathbb{Z}$-grading with good element $e(p)$ given by the pair $(h(p)+h, e(p))$ by Theorem 4.45;

2. a pyramid $\tilde{P} \in \mathrm{Pyr}(p)$ which can be obtained from the symmetric pyramid $P(p)$ by shifting by $-a_i$ the i^{th} block with respect to the

$(i-1)^{\text{th}}$ block for $i = 2, \ldots, d$, where we call i^{th} block all the rows with p_i boxes.

Corollary 4.47. *There is a one-to-one correspondence between the pyramids in Pyr(p) and the good gradings with good element $e(p)$. Furthermore, the number of such good gradings is:*

$$\prod_{i=2}^{d} (2(p_{i-1} - p_i) + 1) \, .$$

Proof. First of all, notice that every pyramid in Pyr(p) can be obtained as described in the construction of Remark 4.46. Hence we get the requested correspondence. This correspondence, using Lemma 4.25, gives us the number of good pairs of the form $(h + h(p), e(p))$.
Thus we only need to show that the $\mathbb{Z}$-grading induced by $h(\tilde{P})$ is the same as the one induced by $h(p) + h$. In the notation of Theorem 4.45 fix $h_{t_{i-1}} - h_{t_i} = a_i \in \{-p_{i-1} + p_i, \ldots, p_{i-1} - p_i\}$ for every $i = 2, \ldots, d$. Then, solving the corresponding system of linear equations, we obtain:

- $h_1 = \ldots = h_{t_1}$;
- $h_{t_1+1} = \ldots = h_{t_2} = h_{t_1} - a_2 = h_1 - a_2$;
- $h_{t_2+1} = \ldots = h_{t_3} = h_{t_2} - a_3 = h_1 - (a_2 + a_3)$;

$\vdots$

- $h_{t_{d-1}+1} = \ldots = h_{t_d} = h_{t_{d-1}} - a_d = h_1 - (a_2 + \ldots + a_d)$.

The grading on $\mathfrak{gl}_n$ determined by $h(p) + h$ will not change if we subtract from $h + h(p)$ the scalar matrix $h_1 I_n$. The semisimple element

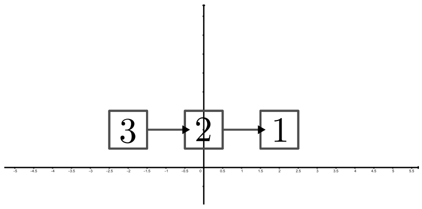
$$h(p)+h-h_1 I_n = h(p)-\text{diag}(\underbrace{0,\ldots,0}_{t_1},\underbrace{a_2,\ldots,a_2}_{t_2-t_1},\ldots,\underbrace{a_2+\ldots+a_d,\ldots,a_2+\ldots+a_d}_{t_d-t_{d-1}})$$

corresponds to the grading determined by the pyramid $\tilde{P}$ obtained from the symmetric pyramid *P(p)* by shifting to the left by a_i the i^{th} block with respect to the $(i-1)^{\text{th}}$ block for $i = 2, \dots, d$. Therefore we have obtained the correspondence:

$$\begin{array}{ccc} \mathrm{Pyr}(p) & \longleftrightarrow & \{\text{good gradings with good element } e(p)\} \\ \tilde{P} & \leftrightarrow & (h(\tilde{P}), e(\tilde{P})) = (h(p)+h, e(p)) \end{array}$$

□

Example 4.48. Consider the partition $p = (3)$ of $n = 3$. Consider the following labeling of the symmetric pyramid $P(p)$:



With this choice of the basis, we have:

$$e(p) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h(p) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Consider now a diagonal matrix $h = \mathrm{diag}(h_1, h_2, h_3)$. We want to see when the pair $(h(p)+h, e(p))$ is good, i.e., by Theorem 4.45, when conditions 1., 2., 3., 4. are satisfied. We necessarily have:

1. $h_1 - h_2, h_2 - h_3 \in \mathbb{Z}$;

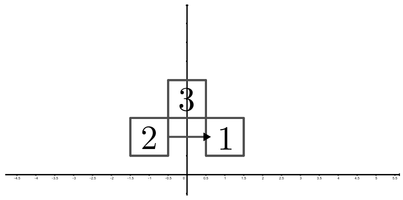
2. $h_1 = h_2 = h_3$;

3. this condition is empty as the partition p consists of only one block;

4. $h_1 + h_2 + h_3 = 0$.

Conditions 2. and 4. imply that $h_1 = h_2 = h_3 = 0$. Thus $(h(p)+h, e(p))$ is good if and only if $h = 0$. This agrees with what we obtained in Example 3.21.

Example 4.49. Consider the partition $p = (2, 1)$ of $n = 3$. Consider the following labeling of the symmetric pyramid $P(p)$:



With this choice of the basis, we have:

$$e(p) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider now a diagonal matrix $h = \mathrm{diag}(h_1, h_2, h_3)$. We want to see when tha pair $(h(p) + h, e(p))$ is good, i.e., by Theorem 4.45 when conditions 1., 2., 3., 4. are satisfied. We have:

1. $h_1 - h_2, h_2 - h_3 \in \mathbb{Z}$;

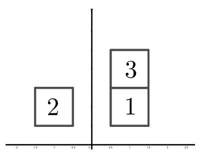
2. $h_1 = h_2$;

3. $|h_2 - h_3| \leq p_1 - p_2 = 2 - 1 = 1$;

4. $h_1 + h_2 + h_3 = 0$.

Moreover by Theorem 4.45, for any choice of $a_2 \in \{-1, 0, 1\}$, if we set $a_2 = h_2 - h_3$, then the system of linear equations given by conditions 2. and 4. has a unique solution. Thus:

- if $a_2 = -1$, the pyramid described in Corollary 4.47 is:

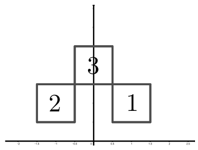


and the system we get is:

$$\begin{cases} h_1 = h_2 \\ h_2 - h_3 = -1 \\ h_1 + h_2 + h_3 = 0 \end{cases}$$

that has the unique solution $(h_1, h_2, h_3) = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$. So $(h(p) + h, e(p))$ is good;

- if $a_2 = 0$, the pyramid described in Corollary 4.47 is:

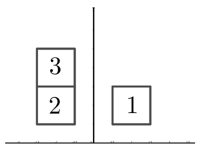


and the system we get is:

$$\begin{cases} h_1 = h_2 \\ h_2 - h_3 = 0 \\ h_1 + h_2 + h_3 = 0 \end{cases}$$

that has the unique solution $(h_1, h_2, h_3) = (0, 0, 0)$. Thus, in this case we get the pair $(h(p), e(p))$;

- if $a_2 = 1$, the pyramid described in Corollary 4.47 is:



and the system we get is:

$$\begin{cases} h_1 = h_2 \\ h_2 - h_3 = 1 \\ h_1 + h_2 + h_3 = 0 \end{cases}$$

that has the unique solution given by $(h_1, h_2, h_3) = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$. So $(h(p) + h, e(p))$ is good.

The fact that there is more than one good $\mathbb{Z}$-grading with good element $e(p)$ does not contradict Example 3.21 because the latter didn't exclude the existence of other good $\mathbb{Z}$-gradings with good element $e(p)$ except from the Dynkin grading.

Remark 4.50. By definition of even $\mathbb{Z}$-grading (see Definition 3.3), in the $\mathfrak{sl}_n$ case a partition p determines an even grading $h(p)$ if and only if all parts p_i of p have the same parity. In terms of the pyramids $P \in Pyr(p)$ this means that the first coordinates of the centers of all boxes constituting the pyramid P have the same parity. Indeed, let us recall that in the $\mathbb{Z}$-grading induced by $h(p)$, the component of degree k is non-zero if and only if there exist i, j such that $i - j = k$ and $(i, i_2), (j, j_2) \in P(p)$ for some i_2, j_2. Now, since $h(p)$ corresponds to the symmetric pyramid $P(p)$, then $i - j$ is always even (i.e., the $\mathbb{Z}$-grading induced by $h(p)$ is even) if and only $p_i \equiv p_j \pmod 2$ for all i, j.

Proposition 4.51. *Let $e(p)$ be a nilpotent element of $\mathfrak{sl}_n$ determined by a partition p. Then there exists a semisimple element h^e such that the pair $(h^e, e(p))$ is good and h^e determines an even grading.*

Proof. If all the parts p_i of p have the same parity, then one can take $h^e = h(p)$, the semisimple element determining the Dynkin grading.
Otherwise, let $i_1, \dots i_k \in \{1, \dots, d\}$ be all those natural numbers i for which $p_{i-1} - p_i \equiv 1 \pmod 2$. Put $a_{i_1} = a_{i_2} = \ldots = a_{i_k} = 1$, $a_j = 0$ for $j \neq i_1, \dots, i_k$ (see the notation of Theorem 4.45) and denote by $h(a_{i_1}, \dots, a_{ik})$ the solution of the corresponding system of equations. Then $h^e := h(a_{i_1}, \dots, a_{i_k}) + h(p)$ will be the required semisimple element. We explain this with an easier case, then the general proof will follow similarly.
Suppose that there exists only one i such that $p_{i-1} - p_i \equiv 1 \pmod 2$. Then, by Theorem 4.45, we know that there exists a unique $h = h(a_i)$ satisfying the conditions

- $h_1 = \ldots = h_{t_{i-1}}, \; h_{t_{i-1}+1} = \ldots = h_{t_d} = h_1 - 1$;

- $h_1 + \ldots + h_{td} = 0$.

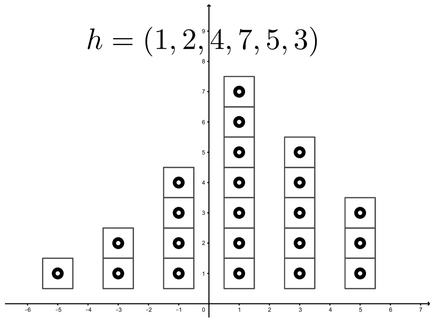
Thus, if the matrix associated to $h(p)$ has diagonal $(b_1, \ldots, b_{t_{i-1}}, b_{t_{i-1}+1}, \ldots, b_{t_d})$ (with $b_{t_{i-1}+1} - b_{t_i} \equiv 1 \pmod 2$ and $b_{j+1} - b_j \equiv 0 \pmod 2$ for all $j \neq t_i$), then $h(p)+h(a_i)$ has diagonal $(b_1+h_1, \ldots, b_{t_{i-1}}+h_1, b_{t_{i-1}+1}+h_1-1, \ldots, b_{t_d}+h_1-1)$. Hence all the elements in the diagonal of $h(p) + h(a_i)$ have the same parity, and consequently determine an even grading (good by Theorem 4.45). □

Definition 4.52. A *unimodal sequence* of size n is a sequence of natural numbers $h_1 \leq h_2 \leq \ldots \leq h_i \geq h_{i+1} \geq \ldots \geq h_k$ satisfying $\sum_{j=1}^{k} h_j = n$.

Proposition 4.53. *There exists a one-to-one correspondence between even good gradings of the simple Lie algebra $\mathfrak{sl}_n$ and unimodal sequences of size n.*

Proof. Given a unimodal sequence $h = (h_1, \ldots, h_k)$ of size n, we construct a pyramid in the following way: the first row of the pyramid will consist of k boxes, with the first coordinates constituting an arithmetic progression with difference 2 and first entry $-k+1$. The pyramid will consist of $2k-1$ columns, with columns at even places consisting of 0 boxes and the column at the $(2i-1)^{\text{th}}$ place consisting of h_i boxes, $i = 1, 2, \ldots, k$. This pyramid P belongs to the set $\mathrm{Pyr}(h^*)$ (where $h_i^* = \#\{j;\ h_j \geq i\}$) and determines an even good grading via $h(P)$(even because all the first coordinates in P have the same parity, and good by Theorem 4.36).
Conversely, the sequence of nonzero column heights of a pyramid P determined by an even grading will be an unimodal sequence of size n, just by definition of pyramid. □

Example 4.54. Here is an example of the pyramid associated to a unimodal sequence of size 22.



Corollary 4.55. *The generating function for the numbers of even good $\mathbb{Z}$-gradings of Lie algebras $\mathfrak{sl}_n$, $n = 1, 2, \dots,$ is*

$$U(q) = \sum_{n\geq 1}(-1)^{n+1}q^{\binom{n+1}{2}}\prod_{k\geq 1}\frac{1}{(1-q^k)^2}. \tag{4.3}$$

Proof. This follows directly from Proposition 4.53 since, according to Corollary A.6 in Appendix A, the generating function for unimodal sequences is *U(q)*. □

Remark 4.56. If one looks at the proof of the aforementioned Corollary A.6 in Appendix A, there the generating function is obtained by transforming in a clever way the series

$$\sum_{n\geq 1}\left(\prod_{k=1}^{n-1}\frac{1}{(1-q^k)^2}\right)\frac{q^n}{1-q^n},$$

which resembles our generating function *F(q)* of Proposition 4.26. This makes one wonder whether the latter can be similarly transformed to a more satisfactory form. Now, the analogue of the second factor of (4.3) for the series *F(q)* is more or less obviously the product

$$\prod_{k\geq 1}\frac{1+q^k}{(1-q^k)^2},$$

so a natural thing to do is to look at the result of dividing *F(q)* by this product. This result (see [3]) gives the equality

$$F(q) = \sum_{n\geq 1}\left(\prod_{k=1}^{n-1}\frac{1+q^k}{(1-q^k)^2}\right)\frac{q^n}{1-q^n} = \sum_{n\geq 1}\left(q^{\frac{3n^2-n}{2}} - q^{\frac{3n^2+n}{2}}\right)\prod_{k\geq 1}\frac{1+q^k}{(1-q^k)^2}.$$

Chapter 5

Good $\mathbb{Z}$-gradings of superalgebras

5.1 Basic definitions

Definition 5.1. A *superalgebra* is a $\mathbb{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, i.e., such that if $a \in A_\alpha, b \in A_\beta$ with $\alpha, \beta \in \mathbb{Z}_2$, then $ab \in A_{\alpha+\beta}$. The elements of $A_{\bar{0}}$ are called *even*, those of $A_{\bar{1}}$ *odd*.

In what follows, if $a \in A_\alpha$ we will say that α is the parity of a and denote it by $p(a)$. Moreover, if $p(a)$ occurs in an expression, then it is assumed that a is homogeneous and that the expression extends to the other elements by linearity.

Associativity of superalgebras is defined as for algebras.

Example 5.2. Let $\Lambda(n)$ be the Grassmann algebra in n variables $\xi_1, \ldots, \xi_n$. Then we can define a $\mathbb{Z}_2$-grading on $\Lambda(n)$ by setting deg$\xi_i = \bar{1}$, $i = 1, \ldots, n$. The result is called a *Grassmann superalgebra*. It is associative and commutative in the sense that

$$ab = (-1)^{p(a)p(b)} ba.$$

Example 5.3. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. Then the associative algebra $\mathrm{End}(V)$ is equipped with the induced $\mathbb{Z}_2$-grading $\mathrm{End}(V) = \mathrm{End}_{\bar{0}}(V) \oplus \mathrm{End}_{\bar{1}}(V)$, where

$$\mathrm{End}_\alpha(V) = \{a \in \mathrm{End}(V) \mid a(V_\beta) \subseteq V_{\alpha+\beta} \text{ for every } \beta \in \mathbb{Z}_2\}$$

Definition 5.4. A *Lie superalgebra* is a superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with an operation $[.,.]$ satisfying the following axioms:

(LS1) $[a,b] = -(-1)^{p(a)p(b))}[b,a]$ for $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_\beta$;

(LS2) $[a,[b,c]] = [[a,b],c] + (-1)^{(p(a)p(b)}[b,[a,c]]$ for $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_\beta$.

Remark 5.5. Observe that $\mathfrak{g}_{\bar{0}}$ is an ordinary Lie algebra and that multiplication on the left by elements of $\mathfrak{g}_{\bar{0}}$ determines a structure of a $\mathfrak{g}_{\bar{0}}$-module on $\mathfrak{g}_{\bar{1}}$.

Remark 5.6. If A is an associative superalgebra, then there is a natural way of defining a *bracket* $[.,.]$ in A, by the equality:

$$[a,b] = ab - (-1)^{p(a)p(b)}ba \qquad \text{for all } a,b \in A. \tag{5.1}$$

Example 5.7. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra, and $\mathrm{End}(A)$ the superalgebra of endomorphisms of A. Then, $\mathrm{End}(A)$ with bracket (5.1) is a Lie superalgebra, denoted by $\mathfrak{gl}(A)$.

Definition 5.8. A *derivation of degree* s, $s \in \mathbb{Z}_2$, of a superalgebra A is an endomorphism $D \in \mathrm{End}(A)$ with the property

$$D(ab) = D(a)b + (-1)^{s\,p(a)}aD(b).$$

We denote by $\mathrm{der}_s A \subset \mathrm{End}(A)$ the space of all derivations of degree s, and we set $\mathrm{der}A = \mathrm{der}_{\bar{0}}A \oplus \mathrm{der}_{\bar{1}}A$.

Remark 5.9. The space $\mathrm{der}A \subset \mathrm{End}(A)$ is easily seen to be closed under the bracket (5.1), in other words, it is a subalgebra of the Lie superalgebra $\mathfrak{gl}(A)$.

Definition 5.10. der$A$ is called the *superalgebra of derivations of* A and every element of it is called a *derivation of* A.

Definition 5.11. A $\mathbb{Z}$-*grading* of a Lie superalgebra $\mathfrak{g}$ is a decomposition of $\mathfrak{g}$ into a direct sum of $\mathbb{Z}_2$-graded subspaces

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

Definition 5.12. Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a $\mathbb{Z}$-graded Lie superalgebra. An element $e \in (\mathfrak{g}_{\bar{0}})_2$ is called *good* if the following properties hold:

a) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$;

b) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$.

A $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$ is called *good* if it admits a good element.

Remark 5.13. If $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is a good grading of the Lie superalgebra $\mathfrak{g}$ with good element $e \in (\mathfrak{g}_{\bar{0}})_2$, then e is good in the Lie algebra $\mathfrak{g}_{\bar{0}} = \oplus_{j\in\mathbb{Z}}(\mathfrak{g}_j \cap \mathfrak{g}_{\bar{0}})$.

5.2 $W(n)$

Let $\Lambda(n)$ be the Grassmann superalgebra. Then we can write

$$\Lambda(n) = \bigoplus_{k=0}^{n} \Lambda_k(n), \tag{5.2}$$

where $\Lambda_k(n) = \langle \xi_{i_1} \ldots \xi_{i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n\rangle$.
Recall that $\xi_i\xi_j = -\xi_j\xi_i$ for all $i, j = 1, \ldots, n$.

Definition 5.14. Define $W(n)$ as the superalgebra of derivations of the Grassmann superalgebra $\Lambda(n)$, i.e.,

$$W(n) := \mathrm{der}\Lambda(n).$$

Remark 5.15. Notice that $W(n) = \{\sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i} \mid P_i \in \Lambda(n)\}$.

Proof. For this purpose it is convenient to represent $\Lambda(n)$ in the form $\tilde{\Lambda}(n)/I$, where $\tilde{\Lambda}(n)$ is the free associative superalgebra with generators $\xi_1, \ldots, \xi_n$, whose $\mathbb{Z}_2$-grading is given by $p(\xi_i) = \bar{1}$, $i = 1, \ldots, n$, and I is the ideal generated by all the elements $\xi_i\xi_j + \xi_j\xi_i$. Note that if P and Q are homogeneous elements of $\tilde{\Lambda}(n)$, then $PQ - (-1)^{p(P)p(Q)}QP \in I$.
Let D be a derivation of degree s of $\tilde{\Lambda}(n)$. Then

$$\begin{aligned} D(\xi_i\xi_j + \xi_j\xi_i) &= D(\xi_i)\xi_j + (-1)^s\xi_i D(\xi_j) + D(\xi_j)\xi_i + (-1)^s\xi_j D(\xi_i) \\ &= (D(\xi_i)\xi_j + (-1)^s\xi_j D(\xi_i)) + (D(\xi_j)\xi_i + (-1)^s\xi_i D(\xi_j) \in I \end{aligned}$$

from which it follows that I is invariant under D. Since, obviously, there is one and only one derivation of $\tilde{\Lambda}(n)$ with prescribed values $D(\xi_i) \in \tilde{\Lambda}(n)$, we see that for any $P_1, \ldots, P_n \in \Lambda(n)$ there is one and only one derivation $D \in \mathrm{der}\Lambda(n)$ for which $D(\xi_i) = P_i \in \Lambda(n)$.
In particular, the relations $\frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij}$ define the derivation $\frac{\partial}{\partial \xi_i}$, $i = 1, \ldots, n$. The derivation $D \in \mathrm{der}\Lambda(n)$ for which $D(\xi_i) = P_i$ can now be written as a linear differential operator:

$$D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i}, \qquad P_i \in \Lambda(n).$$

□

By Remark 5.15, we can write:

$$W(n) = \bigoplus_{i=1}^{n} \bigoplus_{k=0}^{n} W_i^k(n), \tag{5.3}$$

where $W_i^k(n) = \langle \xi_{j_1} \ldots \xi_{j_k} \frac{\partial}{\partial \xi_i} \mid 1 \leq j_1 < j_2 < \ldots < j_k \leq n \rangle$.

Remark 5.16. $\mathrm{dim}W(n) = n2^n$.

Proof. First of all, notice that $\mathrm{dim}\Lambda_k(n) = \binom{n}{k}$. Thus, by (5.2), $\mathrm{dim}\Lambda(n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n$. So, since $\mathrm{dim}W_i^k(n) = \mathrm{dim}\Lambda_k(n) = \binom{n}{k}$, from the decomposition (5.3) we get that $\mathrm{dim}W(n) = \sum_{i=1}^{n} 2^n = n2^n$. □

A $\mathbb{Z}$-grading can be defined on $W(n)$ by setting $\deg\xi_i = -\deg\frac{\partial}{\partial\xi_i} = a_i \in \mathbb{Z}_{\geq 0}$ [7]. Such a grading is called *of type* $(a_1, \ldots, a_n)$. Then an element $\xi = \xi_{i_1} \ldots \xi_{i_r}\frac{\partial}{\partial\xi_j}$ of the basis of $W(n)$ has degree $\deg\xi = \deg\xi_{i_1} + \ldots + \deg\xi_{i_r} + \deg\frac{\partial}{\partial\xi_j} = a_{i_1} + \ldots + a_{i_r} - a_j$. Then

$$W(n) = \bigoplus_{k\in\mathbb{Z}} W(n)_k, \tag{5.4}$$

where $W(n)_k = \langle \xi_{i_1} \ldots \xi_{i_r}\frac{\partial}{\partial\xi_j} \mid a_{i_1} + \ldots + a_{i_r} - a_j = k\rangle$.
Notice that (5.4) is a $\mathbb{Z}$-grading of $W(n)$. Indeed, if we take $P = X\frac{\partial}{\partial\xi_i} \in W(n)_l$ and $Q = Y\frac{\partial}{\partial\xi_j} \in W(n)_m$, then

$$\left[X\frac{\partial}{\partial\xi_i}, Y\frac{\partial}{\partial\xi_j}\right] = X\frac{\partial}{\partial\xi_i}Y\frac{\partial}{\partial\xi_j} - (-1)^{p(P)p(Q)}Y\frac{\partial}{\partial\xi_j}X\frac{\partial}{\partial\xi_i}.$$

Now, $\deg\Big(X\frac{\partial}{\partial\xi_i}Y\frac{\partial}{\partial\xi_j}\Big) = m + l$ and $\deg\Big(Y\frac{\partial}{\partial\xi_j}X\frac{\partial}{\partial\xi_i}\Big) = m + l$, so we can conclude that $\deg\Big[X\frac{\partial}{\partial\xi_i}, Y\frac{\partial}{\partial\xi_j}\Big] = l + m$, i.e., (5.4) is a $\mathbb{Z}$-grading.

We now want to describe the good gradings of $W(2)$.

Example 5.17. Consider $W(2)$ with a grading of type (a, b). Then

$$W(2) = \langle \overbrace{\frac{\partial}{\partial\xi_1}}^{-a}, \overbrace{\frac{\partial}{\partial\xi_2}}^{-b}, \overbrace{\xi_1\frac{\partial}{\partial\xi_1}}^{0}, \overbrace{\xi_2\frac{\partial}{\partial\xi_1}}^{b-a}, \overbrace{\xi_1\frac{\partial}{\partial\xi_2}}^{a-b}, \overbrace{\xi_2\frac{\partial}{\partial\xi_2}}^{0}, \overbrace{\xi_1\xi_2\frac{\partial}{\partial\xi_1}}^{b}, \overbrace{\xi_1\xi_2\frac{\partial}{\partial\xi_2}}^{a}\rangle, \tag{5.5}$$

where we have written above every element of the chosen basis of $W(2)$ its degree with respect to the $\mathbb{Z}$-grading of type (a, b).
Without loss of generality, we can suppose that $a \geq b$; indeed, the $\mathbb{Z}$-grading of type (a, b) with $a < b$ is obtained by the grading of type (b, a) exchanging the variables ξ_1 and ξ_2.
We want to exhibit the good $\mathbb{Z}$-gradings of $W(2)$.
First of all notice that, under our assumptions, the only possible good element $e \in (W(2)_{\bar{0}})_2$ is:

$$e = \xi_1\frac{\partial}{\partial\xi_2}$$

because all the other basis elements of (5.5) have degree ≤ 0 or they are odd and thus cannot belong to $(W(2)_{\bar{0}})_2$.

Now we want to get conditions on a, b such that e can be good element. Suppose that e is good. Then $a - b = 2$. Furthermore ad_e must be injective on the negative part of the grading (5.4). But, since $\mathrm{ad}_e\left(\frac{\partial}{\partial\xi_2}\right) = 0$ and $\frac{\partial}{\partial\xi_2} \in W(2)_{-b}$, $b = 0$.
So the only possible $\mathbb{Z}$-grading of $W(2)$ for which e can be good is the one of type $(2, 0)$. Now we want to see whether this is effectively good, and which are the good elements.

The $\mathbb{Z}$-grading of type $(2, 0)$ is the following:

$$W(2) = \underbrace{\langle\frac{\partial}{\partial\xi_1}, \xi_2\frac{\partial}{\partial\xi_1}\rangle}_{=W(2)_{-2}} \oplus \underbrace{\langle\frac{\partial}{\partial\xi_2}, \xi_1\frac{\partial}{\partial\xi_1}, \xi_2\frac{\partial}{\partial\xi_2}, \xi_1\xi_2\frac{\partial}{\partial\xi_1}\rangle}_{=W(2)_0} \oplus \underbrace{\langle\xi_1\frac{\partial}{\partial\xi_2}, \xi_1\xi_2\frac{\partial}{\partial\xi_2}\rangle}_{=W(2)_2}.$$

First of all, we want to see if e is a good element.
$\left[e, \frac{\partial}{\partial\xi_1}\right] = \xi_1\frac{\partial}{\partial\xi_2}\frac{\partial}{\partial\xi_1} - \frac{\partial}{\partial\xi_2} - \xi_1\frac{\partial}{\partial\xi_1}\frac{\partial}{\partial\xi_2} = -\frac{\partial}{\partial\xi_2}$;
$\left[e, \xi_2\frac{\partial}{\partial\xi_1}\right] = \xi_1\frac{\partial}{\partial\xi_1} + \xi_2\xi_1\frac{\partial}{\partial\xi_2}\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2} - \xi_1\xi_2\frac{\partial}{\partial\xi_1}\frac{\partial}{\partial\xi_2} = \xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}$.
So $\mathrm{ad}_e : W(2)_{-2} \to W(2)_0$ is injective.
$\left[e, \xi_1\frac{\partial}{\partial\xi_1}\right] = -\xi_1\frac{\partial}{\partial\xi_2}$;
$\left[e, \xi_1\xi_2\frac{\partial}{\partial\xi_1}\right] = -\xi_1\xi_2\frac{\partial}{\partial\xi_2}$.
So $\mathrm{ad}_e : W(2)_0 \to W(2)_2$ is surjective. Hence e is a good element.

We have thus proved the following.

Proposition 5.18. *The unique good grading for $W(2)$ is, up to isomorphisms, the grading of type $(2, 0)$ with good element:*

$$e = \xi_1\frac{\partial}{\partial\xi_2}.$$

Remark 5.19. Notice that $W(2)_{\bar{0}} \cong \mathfrak{gl}_3$ under the Lie algebra isomorphism

$$\begin{aligned}\varphi: \quad \langle\xi_i\frac{\partial}{\partial\xi_j}; i, j \in \{1, 2\}\rangle = W(2)_{\bar{0}} &\to \mathfrak{gl}_2 \\ \xi_i\frac{\partial}{\partial\xi_j} &\mapsto E_{ij}\end{aligned}$$

We point out that the good element $\xi_1\frac{\partial}{\partial\xi_2}$, we found in $W(2)$, is exactly the one corresponding to E_{12}, the only good element of $\mathfrak{gl}_2$ as follows by Theorem 4.45.

Example 5.20. Consider

$$W(3) = \langle\frac{\partial}{\partial\xi_i}, \xi_i\frac{\partial}{\partial\xi_j}, \xi_i\xi_j\frac{\partial}{\partial\xi_k}, \xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_k};\ i,j,k \in \{1,2,3\}\rangle \tag{5.6}$$

with a grading of type (a,b,c). Without loss of generality, we can suppose that $a \geq b \geq c$; indeed, the $\mathbb{Z}$-grading of type (a,b,c) with a, b, c not in decreasing order is obtained by the grading of type $(\sigma(a),\sigma(b),\sigma(c))$ with $\sigma \in \mathbb{S}_3$ and $\sigma(a) \geq \sigma(b) \geq \sigma(c)$ and replacing the variable ξ_i with $\xi_{\sigma(i)}$.
First of all notice that, under our assumptions, the only possible basis good elements $e \in (W(3)_{\bar{0}})_2$ are:

1. $\xi_1\frac{\partial}{\partial\xi_2}$;
2. $\xi_1\frac{\partial}{\partial\xi_3}$;
3. $\xi_2\frac{\partial}{\partial\xi_3}$;
4. $\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}$;
5. $\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}$;
6. $\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}$.

because all the other basis elements of (5.6) either have degree ≤ 0 or they are odd and thus cannot belong to $(W(3)_{\bar{0}})_2$.
Since there exists a Lie algebra isomorphism $\varphi : \mathfrak{gl}_3 \to \langle\xi_i\frac{\partial}{\partial\xi_j};\ i,j \in \{1,2,3\}\rangle$ given by $E_{ij} \mapsto \xi_i\frac{\partial}{\partial\xi_j}$, the images of good elements of $\mathfrak{gl}_3$ will be good also for $W(3)_{\bar{0}}$. Thus, we start our analysis of good elements of $W(3)$ verifying

if $\varphi(E_{12}) = \xi_1 \dfrac{\partial}{\partial \xi_2} =: e_1$ and $\varphi(E_{12}+E_{23}) = \xi_1 \dfrac{\partial}{\partial \xi_2} + \xi_2 \dfrac{\partial}{\partial \xi_3} =: e_2$ are good elements of $W(3)$.

· Suppose that e_1 is good. Then $a - b = 2$. Furthermore ad_{e_1} must be injective on the negative part of the grading (5.4). But,

- $\mathrm{ad}_{e_1}\left(\frac{\partial}{\partial \xi_2}\right) = 0$ and $\frac{\partial}{\partial \xi_2} \in W(3)_{-b}$, then $b = 0$;

- $\mathrm{ad}_{e_1}\left(\frac{\partial}{\partial \xi_3}\right) = 0$ and $\frac{\partial}{\partial \xi_3} \in W(3)_{-c}$, then $c = 0$.

So the only possible $\mathbb{Z}$-grading of $W(3)$ for which e_1 can be good is the one of type $(2,0,0)$. The $\mathbb{Z}$-grading of type $(2,0,0)$ is the following:

$$W(3) = W(3)_{-2} \oplus W(3)_0 \oplus W(3)_2$$

with

$W(3)_{-2} = \langle \xi_1 \frac{\partial}{\partial \xi_2}, \xi_1 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2 \frac{\partial}{\partial \xi_2}, \xi_1\xi_3 \frac{\partial}{\partial \xi_2}, \xi_1\xi_2 \frac{\partial}{\partial \xi_3}, \xi_1\xi_3 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_2}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_3} \rangle$;

$W(3)_0 = \langle \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_3}, \xi_i \frac{\partial}{\partial \xi_i}, \xi_3 \frac{\partial}{\partial \xi_2}, \xi_2 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2 \frac{\partial}{\partial \xi_1}, \xi_1\xi_3 \frac{\partial}{\partial \xi_1}, \xi_2\xi_3 \frac{\partial}{\partial \xi_2}, \xi_2\xi_3 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_1} \rangle$;

$W(3)_2 = \langle \frac{\partial}{\partial \xi_1}, \xi_2 \frac{\partial}{\partial \xi_1}, \xi_3 \frac{\partial}{\partial \xi_1}, \xi_2\xi_3 \frac{\partial}{\partial \xi_1} \rangle$.

Now we want to see whether e_1 is effectively good.

Notice that $\mathrm{ad}_{e_1} : W(3)_0 \to W(3)_2$ is not surjective because $\xi_1\xi_2 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_3} \notin \mathrm{Im}(\mathrm{ad}_{e_1})$. Hence e_1 is not a good element.

· Suppose that e_2 is good. Then $a - b = 2 = b - c$. Furthermore ad_{e_2} must be injective on the negative part of the grading (5.4). But $\mathrm{ad}_{e_2}\left(\frac{\partial}{\partial \xi_3}\right) = 0$ and $\frac{\partial}{\partial \xi_3} \in W(3)_{-c}$, then $c = 0$. Hence $b = 2$ and $a = 4$.

So the only possible $\mathbb{Z}$-grading of $W(3)$ for which e_2 can be good is the one of type $(4,2,0)$. The $\mathbb{Z}$-grading of type $(4,2,0)$ is the following:

$$W(3) = W(3)_{-4} \oplus W(3)_{-2} \oplus W(3)_0 \oplus W(3)_2 \oplus W(3)_4 \oplus W(3)_6$$

with

$W(3)_{-4} = \langle \frac{\partial}{\partial \xi_1}, \xi_3 \frac{\partial}{\partial \xi_1} \rangle$

$W(3)_{-2} = \langle \frac{\partial}{\partial \xi_2}, \xi_2 \frac{\partial}{\partial \xi_1}, \xi_3 \frac{\partial}{\partial \xi_2}, \xi_2\xi_3 \frac{\partial}{\partial \xi_1} \rangle$;

$W(3)_0 = \langle \frac{\partial}{\partial \xi_3}, \xi_i \frac{\partial}{\partial \xi_i}, \xi_1\xi_3 \frac{\partial}{\partial \xi_1}, \xi_2\xi_3 \frac{\partial}{\partial \xi_2} \rangle;$
$W(3)_2 = \langle \xi_1 \frac{\partial}{\partial \xi_2}, \xi_2 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2 \frac{\partial}{\partial \xi_1}, \xi_1\xi_3 \frac{\partial}{\partial \xi_2}, \xi_2\xi_3 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_1} \rangle;$
$W(3)_4 = \langle \xi_1 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2 \frac{\partial}{\partial \xi_2}, \xi_1\xi_3 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_2} \rangle;$
$W(3)_6 = \langle \xi_1\xi_2 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_3} \rangle.$

Now we want to see whether e_2 is effectively good.
Notice that $\mathrm{ad}_{e_2} : W(3)_0 \to W(3)_2$ is not surjective because
$\xi_1\xi_2 \frac{\partial}{\partial \xi_1}, \xi_1\xi_3 \frac{\partial}{\partial \xi_2}, \xi_2\xi_3 \frac{\partial}{\partial \xi_3}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_1} \notin \mathrm{Im}(\mathrm{ad}_{e_2})$. Hence e_2 is not a good element.

Remark 5.21. We have thus proved that the image of good elements of $\mathfrak{gl}_3$ are not good elements of $W(3)$, instead of what happened in $W(2)$.

Remark 5.22. Notice also that the even part of $W(3)$ is

$$\langle \xi_i \frac{\partial}{\partial \xi_j}, \xi_1\xi_2\xi_3 \frac{\partial}{\partial \xi_k} \mid i,j,k \in \{1,2,3\} \rangle.$$

So $W(3)_{\bar{0}}$ does not contain only a copy of $\mathfrak{gl}_3$ (i.e., $\langle \xi_i \dfrac{\partial}{\partial \xi_j} \mid i,j \in \{1,2,3\} \rangle$), but also the subspace $\langle \xi_1\xi_2\xi_3 \dfrac{\partial}{\partial \xi_k} \mid i,j,k \in \{1,2,3\} \rangle$; this fact makes the study of such gradings more complicated. We aim to complete this study in the future.

Appendix A

Unimodal sequence

Lemma A.1. *Let $u(n)$ be the number of unimodal sequences of size n and set $U(q) = \sum_{n \geq 0} u(n)q^n$. Then*

$$U(q) = \sum_{k \geq 1} \frac{q^k}{[k-1]![k]!} \tag{A.1}$$

where $[j]! := (1-q)(1-q^2)\cdots(1-q^j)$.

Proof. We want to show that the number of unimodal sequences of size n with largest term k is the coefficient of q^n in $\frac{q^k}{[k-1]![k]!}$. We can rewrite it as

$$q^k \left(\sum_{r_1 \geq 0} q^{r_1}\right)\left(\sum_{r_2 \geq 0} q^{2r_2}\right)\cdots\left(\sum_{r_{k-1} \geq 0} q^{(k-1)r_{k-1}}\right)\left(\sum_{s_1 \geq 0} q^{s_1}\right)\cdots\left(\sum_{s_k \geq 0} q^{ks_k}\right) \tag{$*$}$$

Now, if we take a unimodal sequence h of size n with largest term k, we can write it as $h = (1^{i_1}, 2^{i_2}, \ldots, (k-1)^{i_{k-1}}, k, k^{j_k}, (k-1)^{j_{k-1}}, \ldots, 1^{j_1})$. Hence, if we set $V_k(n)$ to be the set of all unimodal sequence of size n with largest term k, we can establish a bijective correspondence between $V_k(n)$ and the terms in $(*)$ contributing to q^n, by sending h in the addend

$$q^k q^{i_1} q^{2i_2} \cdots q^{(k-1)i_{k-1}} q^{j_1} \cdots q^{kj_k} = q^n.$$

□

However, we would like to write (A.1) in a way which is more similar to the expression of the generating function of the number of partitions of natural number which states that

$$\sum_{n\geq 0} p(n)q^n = \prod_{i\geq 1} \frac{1}{1-q^i}$$

It turns out to be easier to work with objects slightly different from unimodal sequences, and then relate them to unimodal sequences at the end.

Definition A.2. A *V-partition* of n is an $\mathbb{N}$-array

$$\begin{bmatrix} c & \begin{matrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{matrix} \end{bmatrix} \tag{A.2}$$

such that $c + \sum a_i + \sum b_i = n$, $c \geq a_1 \geq a_2 \geq \ldots$ and $c \geq b_1 \geq b_2 \geq \ldots$.

Hence a V-partition may be regarded as a unimodal sequence "rooted" at one of its largest parts.

Lemma A.3. *Let $v(n)$ be the number of V-partitions of n, with $v(0) := 1$. Set $V(q) = \sum_{n\geq 0} v(n)q^n$. Then we have*

$$V(q) = \sum_{k\geq 0} \frac{q^k}{[k]!^2}$$

Proof. The proof is analogue to the one of (A.1). □

As before, anyway, we want a product formula for $V(q)$. Let V_n be the set of all V-partitions of n, and let D_n be the set of all *double partitions* of n, that is, the $\mathbb{N}$-arrays

$$\begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} \tag{A.3}$$

such that $\sum a_i + \sum b_i = n$, $a_1 \geq a_2 \geq \ldots$ and $b_1 \geq b_2 \geq \ldots$. If $d(n) = \#D_n$, then clearly

$$\sum_{n\geq 0} d(n)q^n = \prod_{i\geq 1} \frac{1}{(1-q^i)^2}. \tag{A.4}$$

Now define $\Gamma_1 : D_n \to V_n$ by

$$\Gamma_1 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} a_1 & \begin{matrix} a_2 & a_3 & \cdots \\ b_1 & b_2 & \cdots \end{matrix} \end{bmatrix}, & \text{if } a_1 \geq b_1 \\ \begin{bmatrix} b_1 & \begin{matrix} a_1 & a_2 & \cdots \\ b_2 & b_3 & \cdots \end{matrix} \end{bmatrix}, & \text{if } b_1 > a_1 \end{cases}$$

Clearly Γ_1 is surjective, but not injective. Indeed, every V-partition in the set

$$V_n^1 = \{ \begin{bmatrix} c & \begin{matrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{matrix} \end{bmatrix} \in V_n \,:\, c > a_1 \}$$

appears twice as a value of Γ_1, so

$$\#V_n = \#D_n - \#V_n^1.$$

Next define $\Gamma_2 : D_{n-1} \to V_n^1$ by

$$\Gamma_2 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} a_1+1 & \begin{matrix} a_2 & a_3 & \cdots \\ b_1 & b_2 & \cdots \end{matrix} \end{bmatrix}, & \text{if } a_1+1 \geq b_1 \\ \begin{bmatrix} b_1 & \begin{matrix} a_1+1 & a_2 & \cdots \\ b_2 & b_3 & \cdots \end{matrix} \end{bmatrix}, & \text{if } b_1 > a_1+1 \end{cases}$$

Again, Γ_2 is surjective, but every V-partition in the set

$$V_n^2 = \{ \begin{bmatrix} c & \begin{matrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{matrix} \end{bmatrix} \in V_n \,:\, c > a_1 > a_2 \}$$

appears twice as a value of Γ_2. Hence $\#V_n^1 = \#D_{n-1} - \#V_n^2$, so we have

$$\#V_n = \#D_n - \#D_{n-1} + \#V_n^2.$$

Next define $\Gamma_3 : D_{n-3} \to V_n^2$ by

$$\Gamma_2 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} a_1+2 & \begin{matrix} a_2+1 & a_3 & \cdots \\ b_1 & b_2 & \cdots \end{matrix} \end{bmatrix}, & \text{if } a_1+2 \geq b_1 \\ \begin{bmatrix} b_1 & \begin{matrix} a_1+2 & a_2+1 & \cdots \\ b_2 & b_3 & \cdots \end{matrix} \end{bmatrix}, & \text{if } b_1 > a_1+2 \end{cases}$$

We obtain

$$\#V_n = \#D_n - \#D_{n-1} + \#D_{n-3} - \#V_n^3,$$

where

$$V_n^3 = \{\begin{bmatrix} c & \begin{matrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{matrix} \end{bmatrix} \in V_n : c > a_1 > a_2 > a_3\}.$$

Continuing this process, we obtain maps $\Gamma_i : D_{n-\binom{i}{2}} \to V_n^{i-1}$. The process stops when $\binom{i}{2} > n$, so we obtain the formula

$$v(n) = d(n) - d(n-1) + d(n-3) - d(n-6) + \dots, \tag{A.5}$$

where we set $d(m) := 0$ for $m < 0$. Thus, using equation (A.4) we obtain the following result.

Proposition A.4. *We have*

$$V(q) = \left(\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}}\right)\prod_{i\geq 1}\frac{1}{(1-q^i)^2}.$$

Proof. By equation (A.5), it holds

$$\begin{aligned} V(q) &= \sum_{n\geq 0}(d(n) - d(n-1) + d(n-3) - d(n-6) + \dots)q^n \\ &= \sum_{n\geq 0} d(n)q^n - \sum_{n\geq 0} d(n-1)q^n + \sum_{n\geq 0} d(n-3)q^n - \sum_{n\geq 0} d(n-6)q^n + \dots \\ &= \sum_{n\geq 0} d(n)q^n - \sum_{n\geq -1} d(n)q^{n+1} + \sum_{n\geq -3} d(n)q^{n+3} - \sum_{n\geq -6} d(n)q^{n+6} + \dots . \end{aligned}$$

But $d(k) = 0$ for any $k < 0$; thus

$$\begin{aligned} V(q) &= \prod_{i\geq 1}(1-q^i)^{-2} - q\prod_{i\geq 1}(1-q^i)^{-2} + q^3\prod_{i\geq 1}(1-q^i)^{-2} - q^6\prod_{i\geq 1}(1-q^i)^{-2} + \dots \\ &= \left(\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}}\right)\prod_{i\geq 1}(1-q^i)^{-2}. \end{aligned}$$

□

We can now obtain an expression for $U(q)$ using the following result.

Proposition A.5. *We have*

$$U(q) + V(q) = \prod_{i \geq 1} \frac{1}{(1-q^i)^2}.$$

Proof. Let U_n be the set of all unimodal sequences of weight n. We need to find a bijection $D_n \to U_n \cup V_n$. Such a bijection is given by

$$\begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} \mapsto \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, & \text{if } a_1 \geq b_1 \\ \cdots \ a_2 \ a_1 \ b_1 \ b_2 \ \cdots, & \text{if } b_1 > a_1 \end{cases}$$

□

Corollary A.6. *We have*

$$U(q) = \left(\sum_{n \geq 1} (-1)^{n-1} q^{\binom{n+1}{2}} \right) \prod_{i \geq 1} \frac{1}{(1-q^i)^2}.$$

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