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Comparison Between Two Different Approaches To Resum Large Treshold Logarithms in Drell-Yan Rapidity Distribution

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Sommario

Il calcolo all'ordine fisso non è spesso sufficiente per raggiungere la precisione richiesta per via di grandi contributi che possono emergere in particolari configurazioni cinematiche dello stato finale, rompendo la convergenza dello sviluppo perturbativo. Per questa ragione sono state sviluppate tecniche in grado di includere questi grandi contributi a tutti gli ordini perturbativi, almeno in certe regioni cinematiche. In particolare una di queste tecniche è il soggetto di studio di questa tesi, la risommazione dei grandi logaritmi di soglia.

Mentre la risommazione dei grandi logaritmi di soglia è ben definita per sezioni d'urto differenziali in una variabile, le sezioni d'urto differenziali in due variabili, come per il caso della distribuzione in rapidità, sono recentemente sotto i riflettori, in fatti al momento esistono due approcci per includere i grandi logaritmi di soglia, l'approccio Mellin-Mellin e l'approccio Mellin-Fourier.

In questo lavoro è stato fatto uno studio matematico dei due approcci e sono state messe in luce le principali differenze. In particolare è stato notato che i due approcci differiscono per termini che sono quadraticamente soppressi quando la massa dell'oggetto rivelato è vicina all'energia delle particelle incidenti. Questo risultato è particolarmente interessante perché lascia intendere che la sezione d'urto in due variabili non abbia o include già termini linearmente soppressi quando si raggiunge la regione di soglia.

Contents

Introduction

Nowadays the physics of the Standard Model has been deeply probed thanks to successful experiments performed at LEP and SLC and Tevatron, and in recent years at the Large Hadron Collider (LHC), where it was possible to do some of the most remarkable detections in the field of particle physics. For example, during the LHC RUN I it has been possible to detect one of the most fundamental bricks of the Standard Model, the Higgs boson, necessary to explain why the $W^{\pm}, Z$ bosons, mediators of the weak interaction, are massive, and consequently from where the mass of all the other particles come from. This discovery leads to a better understanding of the already accepted physics that governs particle interactions described by the Standard Model, and gives important clues on particles that are predicted by models which try to go beyond the Standard Model by the introduction of new particles in order to solve intrinsic problems of the Standard Model; the literature presents many examples of theories that aim at extending the Standard model, at the price of predicting the existence of new particles. An important process has taken place recently: the production of the Higgs boson with a top-antitop quark pair has been observed. This process is very rare and difficult to detect because it is hidden by the multitude of decay products. This observation has been possible thanks to the high energy that the accelerator can reach, 13-14 TeV, and the high precision of the detector used, that are constantly upgraded.

In this context the theory that more than other is under the attention by theoretical particle physicist is *quantum chromodynamics* (QCD), which is the accepted quantum field theory that describes strong interactions. QCD has a peculiar intrinsic property, the so-called asymptotic freedom, which arises from the non-abelian nature of the invariance group which defines the theory. In practice, this means that the coupling constant effectively decreases with increasing energy of the process under consideration. This allows us to use perturbation theory to make predictions for relevant observables.

The possibility to reach a high precision in the experiments requires very accurate theoretical predictions. This is why much of the theoretical efforts have been employed to calculate as many perturbative orders as possible for the processes of interest. For example the cross section for the Drell-Yan process, chosen as the test process in this work for its experimental importance, has been computed up to next-to-next-to leading order (NNLO).

Fixed order calculations are often not sufficient to achieve the required precision, because of large contributions that arise in special configurations of the final-state kinematics, and spoil the convergence of the perturbative expansion. For this reason, techniques to include contributions at all orders in perturbation theory, at least in some particular kinematic region, have been developed, and are by now quite established. In particular one of these techniques is the subject of study of this thesis, the *resummation* of large threshold logarithms.

Let us consider the production process of a heavy, weakly interacting final state, such as for example a Higgs boson or a lepton-anti lepton pair. The mechanism of cancellation of soft singularities produces terms in the perturbative coefficients that become large when the mass of the detected object is close to the energy of the colliding particles; in fact, such terms grow as powers of the logarithm of $1 - M^2/E^2$, where M is the mass of the final state and E the available energy, with the power of logs growing with the perturbative order. The peresence of such large logarithms spoil the perturbative behavior of the series. We are then forced to resum all of these large logarithmic contributions to obtain reliable prediction from the perturbative expansion.

It turns out that, because of the energy and momentum conservation constraint, an all order resummation in the space of physical variables appears not to be possible; it is instead relatively easy to develop a resummation program in the conjugated Mellin space, i.e. to resum the Mellin transform of the partonic cross section with respect to the ratio $x = M^2/E^2$, and then go back to the physical space. This approach, which is adopted in the present work, was pioneered by S. Catani and L. Trentadue [1] and by G. Sterman [2].

It should be mentioned that a different resummation technique, based on an effective field theory of QCD usually refer to as SCET (Soft and Collinear Effective Theory), has also been developed and widely applied. We will not consider further this approach in this thesis.

Threshold resummation has been long studied and applied to semi-inclusive processes, such as the invariant mass distribution of Drell-Yan pairs or the transverse momentum distribution of massive systems, but only recently it has been applied to double differential observables, such as invariant mass and rapidity distributions. The resummation for the double semi-inclusive observables in the literature is present with two different approaches: the Mellin-Mellin approach [3], and the Mellin-Fourier approach [4]. The subject of this thesis is a detailed mathematical comparison between the two approaches. In particular, in ref. [3] a numerical analysis of the two methods is presented, and interesting differences between the two approaches are found. The authors explain this difference with the hypothesis that the two approaches resum different kind of logarithms due to the different variables used. At the end of this work it will be proposed a different, and more plausible, origin of this difference.

This work is organized as follows. In Chapter 1 we present a brief history of the theory of the strong interaction and a brief introduction to quantum chromodynamics, in order to enlighten the particular aspects that make this theory the perfect candidate to describe the strong interaction. In Chapter 2 we study studied in detail the Drell-Yan process and we present the explicit calculation of the first perturbative correction in the case of the invariant mass distribution and the double differential distribution in terms of the mass invariant and the rapidity. This process has been chosen for its practical importance; in fact, it is the ground process to study physics at the high energy scale that can be reached. Furthermore, the production of Drell-Yan pairs is an extremely clean signal. In Chapter 3 we present a detailed review of the resummation techniques, following the main steps of ref. [1], showing the problematics that can arise using this resummation program and a brief discussion on how to solve it In particular, we will illustrate the so-called Minimal Prescription

[5], a way to circumvent some difficulties originated by the low-energy behavior of the theory. In Chapter 4 a detailed study on the two above-mentioned approaches for the resummation of large logarithms in double semi-inclusive observables is presented, showing that they are the same approach up tu net-to-next-to-leading-power terms.

In this work all expressions are written in natural units.

Chapter 1

Theory of the Strong Interaction

This chapter will introduce the theory that, for its features, has been chosen to describe strong interactions. The first attempt to find this theory had the target of explaining the binding nuclear forces between the Proton and the Neutron. Experiments shown that this force is not only very strong, but also acts at very short distances. Yukawa found a phenomenological way to describe this force choosing as mediator of the force a scalar massive boson called the Pion. The attempt to find this new particle and test important aspect of the particles physics like the beta decaying and other electroweak phenomena led to build high precise particle detector able to reveal out short living particles like the wanted Pion ($\sim 2.6 \times 10^{-8}$s).
Before the 50s the only known particles were Electron, Muon, Proton, and Neutron, but after those years the "particle zoo"[1] was open and a lot of new particles enter in the game. The first new particles detected was, clearly, the Pion discovered in 1947 in the studies of cosmic rays but in that year also the Kaon was discovered, it also discovered in the cosmic rays. Since then many new particles were discovered, but the multitude of these particles and the fact that some of these are mass excitations of the lighter particles like the Proton and the Neutron lead to the hypothesis that these particles can be made of sub-particles and the hadrons are a possible combination of these.
After the 60s Murray Gell-Mann and others proposed two classification scheme for some of these particles, one called the "Eightfold Way", to classify particles like the Pion, the Kaon (i.e. mesons), the Proton and the Neutron (i.e. 1/2-spin baryons) and the second called the "Decuplet Way" to classify 3/2-spin baryons. The classifications are showed in Figure 1.1. These classifications were explain better in 1964 introducing the quark model. This model postulated the existence of three elementary particles with 1/2-spin and fractional electric charge named Up (u), Down (d) and Strange (s)[2] and use them to classify the hadrons in group-theoretical terms as multiplets of the group SU(3). This model provided all the hadrons observed at that time. Later on, the development of the electroweak theory suggested that the quarks have six flavors instead of three, with the relative antiparticles, extending the quark model. The complete list of the quarks is shown in Table 1.1 with the relative masses and electric charge. In particular, the discovery of the Δ^{++} in 1951 and

[1] Name given to the multitude of particles detected in those years.

[2] The names of these particles derive from the situation observed at that time and the idea behind the classification, in fact the two quarks, Up and Down, take the names from the two components of the isospin and the quark Strange take its name from the "strange" long lifetime of some particles

its peculiar properties; the 3/2-spin and the $2e$ electric charge (e is the elementary electric charge $e = 0.3028\sqrt{\hbar c}$), makes the wave function completely symmetric,

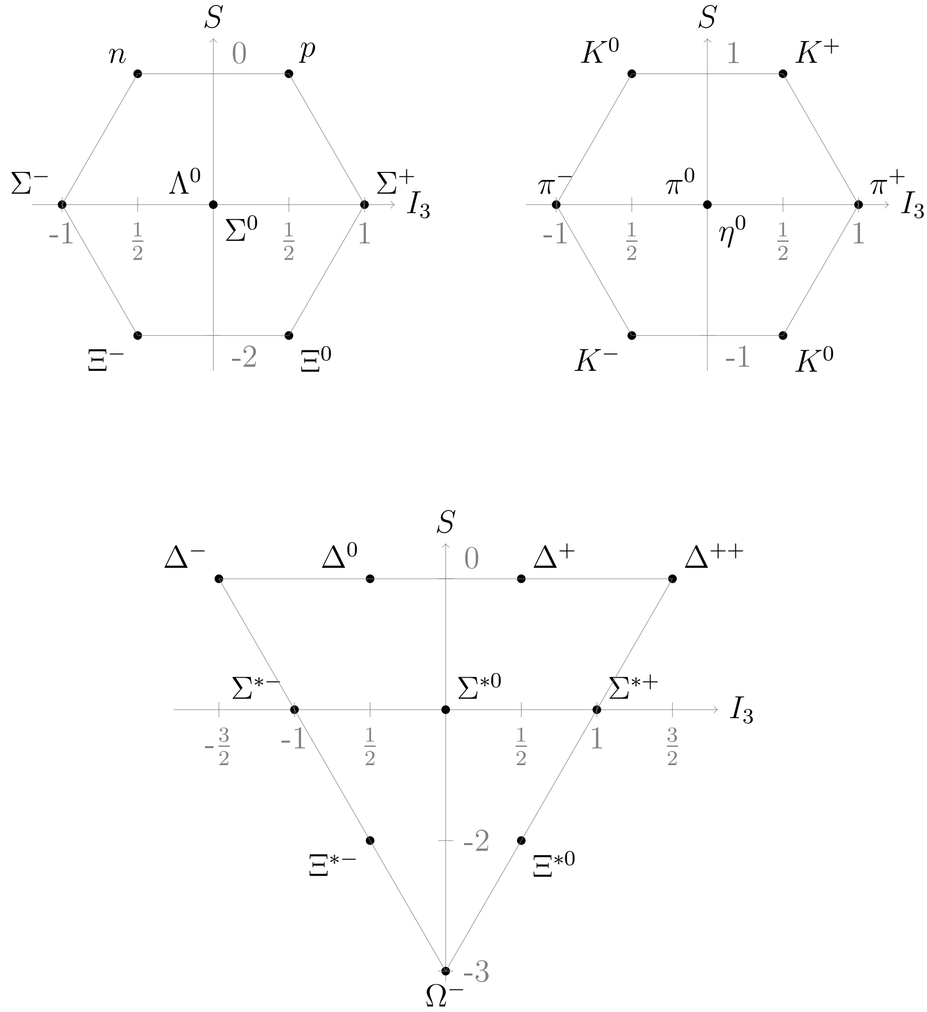


Figure 1.1: The first two schemes are, respectively, the baryon and meson octet, while the third is the baryon decuplet. On the abscissa there is the third component of the isospin defined by the difference $I_3 = [(n_u - n_{\bar{u}}) - (n_d - n_{\bar{d}})]/2$ where the n's are the number of the quark up, down and their anti-particle. On the ordinate there is the "strangeness" that take into account the number of strange quark inside the hadron and it is defined as $S = -(n_s - n_{\bar{s}})$.

$$|\Delta^{++}\rangle = |\ u_\uparrow\ u_\uparrow\ u_\uparrow\ \rangle. \tag{1.1}$$

This configuration obviously violates the basic principle of the fermionic quantum theory, the Pauli exclusion principle, that states that two identical quantum particle, with half-integer spin, cannot stay in the same state, i.e. cannot have the same quantum numbers at the same time. To justify the existence of the baryon Δ^{++} Gell-Mann and others were forced to introduce a new quantum number, today called the color charge, to give anti-symmetry to that state, in fact, if the quarks have this new quantum number the state can be written in an anti-symmetric form,

$$|\Delta^{++}\rangle = \frac{1}{\sqrt{6}} \sum_{ijk=1}^{3} \epsilon_{ijk} | \; u_{i\uparrow} \; u_{j\uparrow} \; u_{k\uparrow} \; \rangle, \tag{1.2}$$

where ϵ_{ijk} is the Levi-Civita tensor ($\epsilon_{123} = 1$), that it is obvously anti-symmetric in the exchange of any color index. It is interesting to note that if one assumes that the color charge has the SU(3) symmetry than the baryons transform as an SU(3) singlet, where the quarks are irreducible representation of the group; this means that the quarks are assembled to have "vanishing" color charge, in the language of color this means that the hadrons are white. This assumption is very important for two reason; first, there is the possibility to describe the manifestation of this charge via the special unitary group SU(3) in a similar fashion of the electroweak theory where the electric and weak charge are the manifestation of the U(1)$\otimes$SU(2) group, and the second reason is that the observable states[3] are not colored, thus we are not able to directly measure the color charge. All the information about the color seems to be confined inside the hadron and to see the effect of this charge it is necessary to probe it with enough energy to solve the constituents.

The success of the quark model opened the road to find an elementary theory able to predict this kind of elementary particles, i.e. fermions with fractional electric charge and a color charge; in addition it must show two peculiar properties of the strong interaction, the color confinement, that close all the information about the color charge inside the hadron, in fact, also if we try to separate its constituents from it, they will acquire enough energy to form new colorless states with other quarks produced by vacuum excitation; furthermore the asymptotic freedom discovered in the 1973 by Gross and Wilczek following the suggestion of the Bjorken studies on the scaling behavior of the structure function of the deep inelastic scattering process between an Electron and a nucleon; later we will discuss this property in more detail. The second property is very important to study high energy phenomena because it means that at high energies (typically at energies much grater than the hadronic scale $\sim$ 200 MeV) the constituent of the hadrons, more generally called partons, do not interact each other and they are essentially free (not free to leave that tiny space but anyway free); the "absence" of interaction inside the hadron permit us to study the strong interaction effect using perturbation theory; but, as will be shown later, the perturbative correction breaks the scaling behavior and the contribution at all order of this scale-breaking terms can lead to a great violation of the asymptotic freedom; this is an impossible situation because the existence of this property is experimentally proved. The only theory that fits all of these request is Quantum Chromo-Dynamics, QCD, a non-abelian gauge theory with the SU(3) symmetry where the strong force is mediated by eight spin-1 mass-less vector bosons called Gluons.

[3] There is no mathematical proof that the hadronic states are the only directly observable states, but we have not been able yet to reveal free quarks. This problem is the confinement of the quark. Below will be shown this aspect in more detail.

Quark	u	d	c	s	t	b
Mass (MeV/c^2)	~ 2.3	~ 4.8	~ 1275	~ 95	~ 173210	~ 4180
e	2/3	-1/3	2/3	-1/3	2/3	-1/3

Table 1.1

1.1 Quantum Chromodynamics

To build up this theory it is necessary to write a lagrangian that show all the symmetries asked for the theory. In this case we have, in addition to the obvious Lorentz covariance, required for any physical system, we take into account the local special unitary group of symmetry SU(3)[4] and the local gauge invariance, which is actually related to the unitary group. The property of the theory to be gauge invariant is one of the most important because it is responsible for the renormalizability of the theory, beyond that it simplifies a lot the problem taken into account.
To take into account the SU(3) symmetry let me introduce a multiplet of three Dirac spinor:

$$\Psi(x) = \begin{pmatrix} \psi_r(x) \\ \psi_g(x) \\ \psi_b(x) \end{pmatrix}, \tag{1.3}$$

where the subscript represent the three color charges (red, green, blue). The infinitesimal action of this group on the multiplet can be write as follow:

$$\Psi(x) \mapsto \Psi'(x) = e^{ig_s\omega^A t^A}\Psi(x) = U(x)\Psi(x) \tag{1.4}$$

where $U(x)$ satisfy the unitary relation

$$U^{-1}(x) = e^{-ig_s\omega^A t^A} = U^\dagger(x), \tag{1.5}$$

and t^A, $(A = 1, 2, \ldots, 8)$ represent the generator of the group.

A set of generators of the group is made of eight hermitian and traceless square matrices of rank three (there are eight generator instead of nine because the group is special, the requirement detU = 1 remove one degree of freedom), named Gell-Mann matrices:

$$t^A = \frac{1}{2}\lambda^A, \tag{1.6}$$

[4]This symmetry is not a geometrical/spatial symmetry as the Lorentz symmetry but it is an internal symmetry, thus it does not affect the position of the evaluation of the field, also if this symmetry is local.

with

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{1.7}$$

such that,

$$\mathrm{Tr}\left[t^A t^B\right] = \frac{1}{2}\delta^{AB}. \tag{1.8}$$

The Gell-Mann matrices were normalized to obtain the 1/2 factor in the above trace so we will be able to write the lagrangian of the theory with a similar form of the QED case. From the relation (1.8) follow another important relation of this matrices, that is;

$$\sum_A t^A_{ab} t^B_{bc} = \frac{4}{3}\delta_{ac}, \tag{1.9}$$

or in general ,

$$\sum_A t^A_{ab} t^B_{cd} = \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \frac{1}{3}\delta_{ab}\delta_{cd}\right). \tag{1.10}$$

This relation are important because usually when we calculate the M matrix element in the perturbation theory, i.e. the Feynman diagrams, we mediate over the initial color and sum over the final color of the quarks involved in the process, following the idea that we don't have methods to measure the color of the quarks. This matrices permits to calculate the structure function of the algebra of the group from the usual relation;

$$[t^A, t^B] = i f^{ABC} t^C, \tag{1.11}$$

where f^{ABC} is real and completely antisymmetric.

$$\begin{gathered} f^{123} = 1, \qquad f^{257} = f^{147} = f^{246} = f^{345} = \frac{1}{2}, \\ f^{156} = f^{367} = -\frac{1}{2}, \qquad f^{485} = f^{678} = \frac{\sqrt{3}}{2}, \end{gathered} \tag{1.12}$$

while all the other components are null.

A way to build a gauge invariant lagrangian is to generalize the minimal coupling from the electromagnetic case to this one. I said generalize because in the electromagnetic case the generator of the group are simply numbers, in fact, they commute, i.e. the gauge theory is an abelian theory, and the vector potential has the usual gauge transformation law, $A_\mu(x) \mapsto A_\mu(x) - \partial_\mu \omega(x)$; in the case of QCD the generators do not commute as we can see from Eq. (1.11). The generalized minimal coupling is reached by setting up a matrix-valued derivative operator D_μ that, applied to the multiplets Ψ, transform like Ψ itself, thus it is a representation of the SU(3) group. Following the idea in the QED case the covariant derivative operator is,

$$D_\mu = \partial_\mu - i g_s A^B_\mu(x) t^B, \tag{1.13}$$

where we have introduced the non-abelian real vector potential. Its transformation under the SU(3) group is defined by imposing that $D_\mu \Psi(x) \mapsto U(x)(D_\mu \Psi(x))$;

$$A^B_\mu(x) t^B \mapsto U(x) \left[A^B_\mu(x) t^B \right] U^\dagger(x) - \frac{i}{g_s} [\partial_\mu U(x)] U^\dagger(x), \tag{1.14}$$

that has a very more complicate form than the abelian case.
Now we are able to write the free lagrangian for the dynamics of the multiplets of spinors that is completely invariant under the Lorentz transformation and SU(3) transformation:

$$\mathcal{L}_M = \overline{\Psi}(x)(i\not{D} - M)\Psi(x). \tag{1.15}$$

To give to the vector potential the role of a dynamical field we need to add to the above lagrangian a term that describe the dynamics of the potential fields, i.e. terms proportional to $\left(\partial^\mu A^B_\mu(x)\right)^2$, and invariant under the action of the group SU(3). The only valid term to this purpose is the square of the non-abelian field strength antisymmetric valued matrix tensor defined by;

$$F^B_{\mu\nu}(x) = \partial_\mu A^B_\nu(x) - \partial_\nu A^B_\mu(x) - i g_s [A^B_\mu(x), A^B_\nu(x)], \tag{1.16}$$

that, multiplied by the generator t^A, transform under the action of the SU(3) group like the adjoint representation;

$$F_{\mu\nu}(x) = F^A_{\mu\nu}(x) t^A \mapsto U(x) F^A_{\mu\nu}(x) U^\dagger(x) = U(x) \left[F^A_{\mu\nu}(x) t^A \right] U^\dagger(x), \tag{1.17}$$

in fact, it is usual to say that the gluons live in the adjoint representation of the SU(3) group while the quarks live in the fundamental represetation. The trace of the square of $F_{\mu\nu}(x)$ is obviously invariant for the cyclic property of the trace and the unitary of $U(x)$.

We can now write down the lagrangian used to describe the strong interaction, that take into account all the symmetries required;

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}(x)F^{\mu\nu}(x)\right] + \overline{\Psi}(x)(i\not{D} - M)\Psi(x). \tag{1.18}$$

From the above lagrangian we can appreciate a new properties not present in an abelian gauge theory like QED, the vector bosons field, that from now we will refer to it with the name of gluons, are self-interacting as we can see from the definition (1.16) where we have a new term compared with the electrodynamic case, $ig_s[A^B_\mu(x), A^B_\nu(x)]$; that represent the presence of the self interaction of the gluons and it breaks the field strength gauge invariance, only its form in the (1.18) is gauge invariant thanks to the cyclic property of the trace.

Like any other local gauge invariance required to be satisfied by the lagrangian, it is forbidden to add the mass term for the gluons, like $m^2 A_\mu A^\mu$, because it is not gauge invariant, thus the gluons must be massless, like they seems to be. Beside this the mass term for the quark field it's still gauge invariant and then admissible.

On the lagrangian (1.18) we can let act discrete operators, the C (charge conjugation), P (spatial inversion, parity transformation) and T (time reversal) symmetry. These operators represent three variations that should not affect the theory, i.e. the theory should be trivially invariant under those transformation, in fact, for example, why should a field be different if we observe it reversing the time direction? It should not be different, but the nature not always work as we aspect, in fact there can be situation where these symmetries are broken, the most famous one is the parity violation in the electroweak theory. In the case of QCD, i.e. the lagrangian in the form (1.18), is well known that the three discrete symmetries are respected; however it is possible to add a new term in the theory that preserve all the other required symmetries but breaks both the CP symmetry, the term is,

$$g_s\theta F^A_{\mu\nu}(x)\tilde{F}^{\mu\nu A}(x), \tag{1.19}$$

where $\tilde{F}(x)$ is the dual of the field strength tensor,

$$\tilde{F}^{\mu\nu A}(x) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^A_{\rho\sigma}(x). \tag{1.20}$$

this is an uncomfortable situation because also if we set the θ parameter in such a way to be negligible it can still be generated by the CP-violating effects of the weak interaction. In the case of high energy physics this terms can be simply omitted because it it is a total divergence, in fact,

$$F^A_{\mu\nu}(x)\tilde{F}^{\mu\nu A}(x) = \partial_\mu\left[2\epsilon^{\mu\nu\rho\sigma}A^B_\nu(x)\left(\partial_\rho A^B_\sigma(x) - \frac{2}{3}g_s f^{ABC}A^B_\rho(x)A^C_\sigma(x)\right)\right]; \tag{1.21}$$

that does not give any contribution in the context of perturbation theory. However the possibility to put that term in a 4-divergence form does not help in the non-perturbative region in fact, the topology of the QCD vacuum can be non-trivial and boundary terms can not be neglected. This is a still open problem called "strong CP problem".

The lagrangian (1.18) is not the most general, because, by symmetry, also other terms like,

$$\overline{\Psi}(x)\left(D_\mu D^\mu\right)^2\Psi(x), \quad \left(\overline{\Psi}(x)F_{\mu\nu}D^\mu D^\nu\Psi(x)\right)^3, \quad \left(F_{\mu\nu}F^{\mu\nu}\right)^4, \tag{1.22}$$

are permitted, in fact, they are all invariant under Lorentz and gauge transformation, but they are not admissible in the theory because they make it non renormalizable, in fact, by power counting method only the terms with a positive coupling's mass dimension could be permitted. This request can be understood by an easy dimension analysis. The divergencies can arise in the perturbation terms of a certain object, for example the propagator, by combination of 1PI (one particle irreducible) loop diagrams. For a general object with a coupling λ of mass dimension a and divergent contribution given by only one 1PI loop diagram at a generic order n we can write the correction to this object in the following way;

$$\Delta = \Delta_0 + \Delta_1\lambda^n \int^\infty \mathrm{d}l\ l^{D-1}. \tag{1.23}$$

The loop contribution gives a divergent contribution only if $D > 0$, than we can see that if we call the mass dimension of Δ and Δ_1 respectively δ and δ_1 we get the following equation for D,

$$D = -an + (\delta - \delta_1). \tag{1.24}$$

The dimension δ and δ_1 depend only on the process take in consideration, i.e. in the external leg configuration. From the equation (1.24) it is trivial that if $a > 0$ we have a finite number of possible divergent diagram in the perturbation series, otherwise, if $a < 0$ there are divergencies at each perturbative order of increasing power; this situation is far to be practical and the amount of divergencies are not tractable, this makes the theory non renormalizable. This is not a rigorous mathematical demonstration that coupling constant with negative dimension are not renormalizable, this is a simple example to show the idea behind the power counting techniques. To find more about this see the book [6], [7].

We are now sure that the theory (1.18) is the most general one that is invariant under Lorentz transformation, SU(3) gauge symmetry and maybe[5] renormalizable. The last step left to do is its quantization, For our purpose, this means to write down the Feynman rules for this theory. These rules are a powerful tool to study the theory in the perturbative way and these, thanks to the path integral formalism, can be deducted directly from the lagrangian (1.18). In addition they will provide the propagator of the fields involved in the theory but since the theory is gauge invariant we can't define the gluon propagator, thus we must add to the lagrangian a "gauge-fixing term" and an associated ghost field to eliminate extra degrees of freedom that will arise by gauge-fix lagrangian; for a more detailed explanation on the ghost field see the reference [8, 9] . The most used gauge-fixing term is the R_ξ gauge;

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2, \tag{1.25}$$

[5] The power counting method is just a guide to know if a theory could be renormalizable, because also if $D \leq 0$ there can be logarithm divergencies at all order.

that is a generalization of the Lorentz gauge; while the associated ghost is,

$$\mathcal{L}_{\text{ghost}} = \partial_\mu \overline{B}^A D^\mu_{AC} B^C, \tag{1.26}$$

where B^A is a complex scalar field. Now the Feynman rules are quite straightforward from the lagrangian (1.18) plus the gauge fixing terms. The inverse of propagator of the various fields, in the momentum space, can be found substituting $\partial_\mu \mapsto -ip_\mu$, in fact, for the three fields we have the following inverse propagator and the respective propagator:

$$\text{Quark :} \qquad -i\delta_{ab}(\not{p} - M) \quad \Rightarrow \quad i\frac{\not{p}+M}{p^2-M^2}\delta_{ab} \tag{1.27}$$

$$\text{Gluon :} \quad i\delta_{AB}\left[p^2 g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)p_\mu p_\nu\right] \quad \Rightarrow \quad \frac{i}{p^2}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]\delta^{AB} \tag{1.28}$$

$$\text{Ghost :} \qquad -i\delta_{AB}p^2 \quad \Rightarrow \quad \frac{i}{p^2}\delta^{AB}. \tag{1.29}$$

Finally the expression for the vertex are:

$$(q\ q\ g) \qquad -ig_s t^A \gamma^\mu \tag{1.30}$$

$$(g\ g\ g) \qquad -g_s f^{ABC}\left[(p-k)^\mu g^{\nu\rho} + (k-l)^\nu g^{\rho\mu} + (l-p)^\rho g^{\mu\nu}\right] \quad (\text{where } \ p+k+l=0) \tag{1.31}$$

$$\begin{aligned}(g\ g\ g\ g) \qquad & -ig_s^2[f^{FAC}f^{FBD}(g^{\mu\nu}g^{\sigma\rho} - g^{\mu\rho}g^{\sigma\nu}) \\ & f^{FAD}f^{FBC}(g^{\mu\nu}g^{\sigma\rho} - g^{\mu\sigma}g^{\nu\rho}) \\ & f^{FAB}f^{FCD}(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})]\end{aligned} \tag{1.32}$$

$$(b\ b\ g) \qquad g_s f^{ABC} p^\mu, \tag{1.33}$$

where the last one expression is for the ghost-ghost-gluon vertex with p^μ the ghost outgoing momenta.

1.2 Asymptotic Freedom

The lagrangian (1.18) seems to be a perfect candidate to describe the strong interaction, but beyond the symmetry that the theory can have, experience teaches us that the strong interaction have two main features, that are in some way linked to each other, color confinement and asymptotic freedom. The first one is very difficult to deal with because it's a phenomenon that we can observe at "low energy" (at maximum at the typical hadronic scale) where the theory can't be treated with the perturbation theory, this means that to be sure that the lagrangian (1.18) has such behavior we must find an exact solution that nowadays hasn't been found. Fortunately the technology is by our side and we are able to "simulate" the dynamics of the theory, especially at low energy, with very interesting results, in fact, a lot of simulation has been done to predict the masses of the lightest mesons and baryons and the accordance with the experimental evidence is stunning, as showed in figure 1.2. This is an indirect proof that we could be on the right track, but a very hard one. Fortunately the second property, the asymptotic freedom, gives us a great help, in fact, it states that at high energies, or equivalently at very short distances, the partons are essentially free. This means that at high energies we are allowed to use perturbation theory to make predictions. However also the perturbation theory has

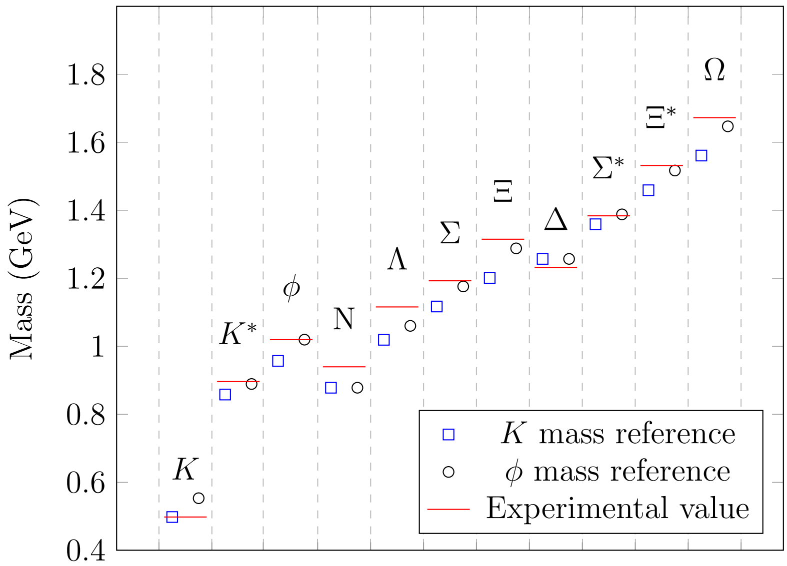


Figure 1.2: In the picture are shown the masses of carious mesons and baryons predicted by numerical solution of QCD and are compared with the experimental value. The parameter in the numerical analysis are only the coupling strength and the Strange-quark mass. The results seems to show a "good" accordance with the experimental value with a magnitude of deviance of about 11%. This results has been exposed in the reference [10]

its own problems, in fact, usually the perturbative series of some objects, like the Green function, can asymptotically diverge, the reason that will be clear soon. The asymptotic divergence is a problem to deal with when we do calculation in high energy QCD. Another problem of the perturbative series is the presence of the loop diagrams that can be divergent; but thanks to the renormalizability of the theory we can redefine (renormalize) the coupling constant and the fields present in the theory to get a series in power of the new coupling constant with finite terms. Once the coupling constant has been renormalized it will depend on the ratio of the energy scale of the process (that we will named M) and another energy scale chosen by ourselves (that it is usually named μ). This dependence on the scale is called the "running coupling constant". The choice of μ must be made to get a ratio M/μ as near as possible the unity. This request follow from the logarithmic dependence of the renormalized coupling constant on the ration of the two energies. To appreciate the last sentence we can go on following two ways, the first need to calculate some diagrams to appreciate the fact that the UV divergencies, in this case, i.e. for a adimensional coupling, appear as logarithms of that ratio (this method may be too long), otherwise we can use the renormalization group approach that gives us the energy evolution equation of the coupling constant. We now show this way because it's more interesting and efficient. When we regularize the divergencies that appears in the loop integral we are forced to introduce a parameter, like μ, to control it. For example the most trivial way is to add a UV cut-off in the integral or, like in the case of the dimensional regularization, the parameter must be added to preserve the original dimension of the physical object, because the unphysical dimensions added in the procedure. The new variable is not physical, so the observable must not depend on it. Here enters the renormalization procedure, performing a scaling of the bare quantities[6] and adding counterterms to remove the divergencies and the μ dependence. We can rewrite the physics, or better the perturbative expansion, without divergent terms using the renormalized coupling constant throwing also the dependence from the unphysical variable in it. The advantage to move the dependence of μ in the coupling constant is that in the experiment what is actually measured is the normalized coupling, thus we can fix μ at the same energy at which the coupling is experimentally known and than do prediction letting the coupling "run". To appreciate it it's enough to see that an observable, that mustn't depends on the renormalization scale, satisfy the equation:

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} G(M^2/\mu^2, \alpha_s) = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right) G(M^2/\mu^2, \alpha_s) = 0. \tag{1.34}$$

The equation can be solved introducing a new function, the "running coupling" $\alpha_s(M^2)$, that satisfy the following relations:

$$\mu^2 \frac{\partial \alpha_s(M^2)}{\partial \mu^2} = -\beta(\alpha_s(M^2)), \qquad \frac{\partial \alpha_s(M^2)}{\partial \alpha_s} = \frac{\beta(\alpha_s(M^2))}{\beta(\alpha_s)}, \tag{1.35}$$

where $\beta(\alpha_s(M^2))$ is the famous "beta function" defined as:

$$\beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}. \tag{1.36}$$

The perturbation theory allow us to expand $G(M^2/\mu^2, \alpha_s)$ in power of α_s, but the expressions (1.35) teach us that objects proportional to $\alpha_s(M^2)$ are solution of the (1.34); this means that we can rewrite $G(M^2/\mu^2, \alpha_s)$ only in terms of the running coupling constant, $G(1, \alpha_s(M^2))$. This states that the interacting behavior of the theory depends on the energetic scale we are, thus if we want use perturbation theory we must be sure that the coupling is small, but now, "small" is connected with the energy involved. To understand how the coupling constant change with the energy (how it "runs") it's enough to solve:

$$M^2 \frac{\partial \alpha_s(M^2)}{\partial M^2} = -\beta(\alpha_s(M^2)). \tag{1.37}$$

[6]The bare quantities are non renormalized object, i.e. divergent quantities.

The solution of this equation is not trivial because we don't have a finite form for the beta function, in fact, we are only able to calculate the perturbative coefficient of the expansion in power of $\alpha_s(M^2)$;

$$\beta(\alpha_s(M^2)) = \alpha_s^2(M^2)\sum_{n=0}^{\infty}\beta_n\left[\alpha_s(M^2)\right]^n, \tag{1.38}$$

through the Feynman diagrams. For example the first three coefficient are:

$$\beta_0 = \frac{33-2n_f}{12\pi}, \tag{1.39}$$

$$\beta_1 = \frac{153-19n_f}{24\pi^2}, \tag{1.40}$$

$$\beta_2 = \frac{77139-15099n_f+325n_f^2}{3456\pi^3}, \tag{1.41}$$

where n_f is the number of "activated"[7] flavors. The value of this coefficient must be directly calculated from the diagram. For a detailed explanation see references [11, 12].

The positivity of the coefficients and the minus sign in the (1.37) are the mark of the non abelian character of the theory that leads to the asymptotic freedom property. For example we can easily solve the first order of the (1.37);

$$M^2\frac{\partial\alpha_s(M^2)}{\partial M^2} = -\alpha_s^2(M^2)\beta_0 \tag{1.42}$$

that is,

$$\alpha_s(M^2) = \frac{\alpha_s(\mu^2)}{1+\beta_0\alpha_s(\mu^2)\log(M^2/\mu^2)}, \tag{1.43}$$

where μ is arbitrarily chosen but it must be in the perturbative region of energy. Usually it's chosen as the Z-boson mass and it is used the notation $\alpha_s(M_Z)=\alpha_s$. From this expression we can see that the coupling in the limit of very high energy, i.e. when $M^2 \gg \mu^2$, goes to zero, this means that the colored particle are essentially free, they don't feel the presence of the other colors. On the other side the coupling diverge at energy near to;

$$\Lambda^2 = M_Z^2\exp\left[-\frac{1}{\beta_0\alpha_s}\right]. \tag{1.44}$$

This behavior is a rude proof of the color confinement problem, in fact, if we try to separate two colored particle the energy around each other grows, until it is sufficient to extract a particle from the vacuum and form a white bound state. In addition

[7] A flavor is "active" if there is enough energy to produce its mass.

the perturbative approach tends to fail when we move towards that energy scale, also before the coupling diverge, in fact, to break down the perturbation theory it's enough that the coupling is near to the unity. Fortunately the coupling diverge very fast and the energy at which the the coupling approaches unity and where it diverges are very near, in fact, we can see that at the first order;

$$\frac{\Lambda_\infty}{\Lambda_{\mathbb{I}}} = e^{-\frac{1}{2\beta_0}} \quad \Rightarrow \quad \text{for } n_f = 5 \quad \Lambda_\infty = 0.94\ \Lambda_{\mathbb{I}}, \tag{1.45}$$

where Λ_∞ is the energy at which the coupling diverge, and $\Lambda_{\mathbb{I}}$ the energy at which the coupling approaches to unity.

From the equation (1.39), (1.40) and (1.41) we can see that the way the coupling run depends on the number of flavor activated. The activation of other flavor makes $\alpha_s(M^2)$ not differentiable in the points where the flavors are activated, in addition, the energy at which the coupling diverges will depend on the number of flavors involved (we can think about it as the number of flavors in an hadrons determine its dimension), thus each value of n_f has its own Λ-energy. For example when we use as reference energy the Z-boson mass we must use $n_f = 5$, because only the top quark is not activated. This means that when we perform calculation using the running constant we must chose the correct one of it based on the energy of the process. Imposing the continuity of the coupling through the activation of other quarks masses we can get the following relation that link the energy limit λ at different number of flavor, this relation are true for the one loop approximation, i.e. the lowest order of $\beta(\alpha_s)$:

$$n_f = 5 \leftrightarrow 6; \qquad \Lambda_6 = \Lambda_5 \left(\frac{\Lambda_5}{m_t}\right)^{\frac{2}{21}} \qquad \Lambda_5 = \Lambda_6 \left(\frac{m_t}{\Lambda_6}\right)^{\frac{2}{23}}, \tag{1.46}$$

$$n_f = 4 \leftrightarrow 5; \qquad \Lambda_5 = \Lambda_4 \left(\frac{\Lambda_4}{m_b}\right)^{\frac{2}{23}} \qquad \Lambda_4 = \Lambda_5 \left(\frac{m_b}{\Lambda_5}\right)^{\frac{2}{25}}, \tag{1.47}$$

$$n_f = 3 \leftrightarrow 4; \qquad \Lambda_4 = \Lambda_3 \left(\frac{\Lambda_3}{m_c}\right)^{\frac{2}{25}} \qquad \Lambda_3 = \Lambda_4 \left(\frac{m_c}{\Lambda_4}\right)^{\frac{2}{27}}. \tag{1.48}$$

At the moment is available also the exact solution of the (1.37) up to the fifth order. For example the two loop solution of the (1.37) can be found by solving an implicit equation;

$$\frac{1}{\alpha_s(M^2)} - \frac{1}{\alpha_s} + \frac{\beta_1}{\beta_0} \log\left(\frac{\alpha_s(M^2)}{1 + \frac{\beta_1}{\beta_0}\alpha_s(M^2)}\right) - \frac{\beta_1}{\beta_0} \log\left(\frac{\alpha_s}{1 + \frac{\beta_1}{\beta_0}\alpha_s}\right) = \beta_0 \log\frac{M^2}{M_Z^2}, \tag{1.49}$$

that can be numerically solved at any desired accuracy or, at this level, it is still possible to give an approximated solution to the equation (1.49), that is;

$$\alpha_s(M^2) = \frac{\alpha_s}{1+\beta_0\alpha_s\log\frac{M^2}{M_Z^2}}\left[1-\frac{\frac{\beta_1}{\beta_0}\alpha_s\log\left(1+\beta_0\alpha_s\log\frac{M^2}{M_Z^2}\right)}{1+\beta_0\alpha_s\log\frac{M^2}{M_Z^2}}\right]. \tag{1.50}$$

We can use this expression (just because it's more accurate than the (1.43)) to give an estimation of the energy Λ, at which the coupling diverge. Take $M_Z = 91,19$ GeV and $\alpha_s(M_Z^2) = 0.1184$. The energy limit, at the second order is given by

$$\Lambda = M_Z \exp\left\{-\frac{1}{2\beta_0\alpha_s} - \frac{\beta_1}{2\beta_0^2}\log\frac{\alpha_s}{1+\frac{\beta_1}{\beta_0}\alpha_s}\right\}, \tag{1.51}$$

that gives $\Lambda = 180$ MeV, a very low energy for the subnuclear physics in fact it is the energy scale of the lighter mesons. This means that we can effectively try to study what happen inside an hadron with the help of the perturbation theory.

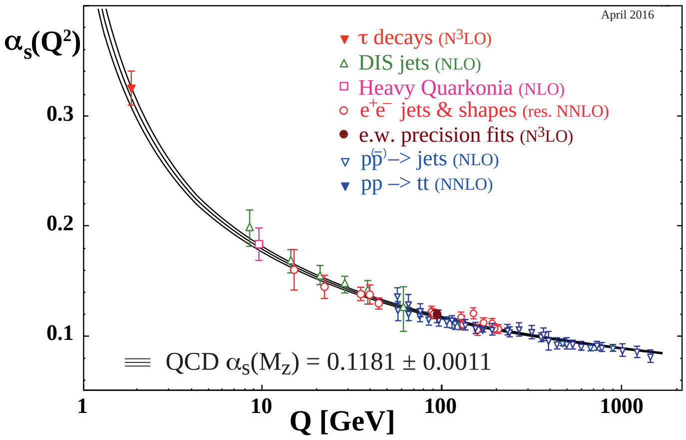


Figure 1.3: We can see the running of the coupling α_S from a summary of measurements of it as a function of the respective energy scale Q. The measurements show good accordance with the theoretical value, moreover the two little variations are due to the change of the number of flavor involved and they are situated at the value of the quark charm mass (∼1.29 GeV) and the quark bottom mass (∼ 4.20 GeV). The plot is from the PDG [13]

1.3 Beyond the Perturbative Expansion

The new definition of the coupling constant permits us to rewrite physical quantity in term of it. For example, in the high energy regime we can use perturbation theory and express physical quantity as a power series of $\alpha_s(M^2)$;

$$G(\alpha_s(M^2);\mu) = \alpha_s(M^2)G_1 + \alpha_s^2(M^2)G_2 + \ldots \tag{1.52}$$

Thus $\alpha_s(M^2)$ tells us how $G\left(\alpha_s(M^2);\mu\right)$ varies with the energy, we can make this dependence explicitly using, for simplicity, the (1.43), that since $|\beta_0\alpha_s(\mu^2)\log M^2/\mu^2| < 1$, it can be expanded as a geometrical series. The requirement at this level is valid up to some hundred TeV,

$$\beta_0\alpha_s(M_Z^2)\log\frac{M^2}{M_Z^2} = 1 \qquad \Rightarrow \qquad M = M_Z \exp\left\{\frac{1}{2\beta_0\alpha_s(M_Z^2)}\right\}$$
$$M \simeq 200\text{TeV}. \tag{1.53}$$

Anyway since what matter is not only the value of $\alpha_s(\mu^2)$ but also the ratio of M^2/μ^2 in the logarithm, if we change the scale μ^2 we can still fall in the region of convergence of the geometric series. We can write any physical quantity as function of the running coupling and thanks to the perturbation theory it can be written as a power series of the coupling. Studying perturbatively a preocess we can understand a lot about the theory in a wide range of energies but anyway limited. A way to get more information from the theory going away from the perturbative region is necessary but not easy. The main problem are the divergences that arise from the theory, like the well known Landau pole. For example expand $\alpha_S(M^2)$ at the first order,

$$\alpha_s(M^2) = \alpha_s(\mu^2)\left(1 - \beta_0\alpha_s(\mu^2)\log\frac{M^2}{\mu^s} + \ldots\right), \tag{1.54}$$

then $G(\alpha_s(M^2);\mu)$ can be written as;

$$G(\alpha_s(M^2);\mu) = G_1\alpha_s(\mu^2)\sum_{n=0}^{\infty}\left(-\alpha_s(\mu^2)\beta_0\log\frac{M^2}{\mu^2}\right)^n + \ldots \tag{1.55}$$

We can understand the nature of $G(\alpha_s(M^2);\mu)$ more from the running coupling than the function itself; it is then important to get a well understanding of the running coupling. If we want to push $G\left(\alpha_s(M^2);\mu\right)$ towards the Landau pole for example we can see that $G\left(\alpha_s(M^2);\mu\right)$ diverge because $\alpha_s(M^2)$ diverge, but at the same time the terms of the perturbative series are finite; the perturbative series is an asymptotic one and can be approximated to a convergent series that does not diverge at the energy near the Landau pole . In other words the usual perturbation expansion holds only inside the hadron, this means that when we expand a physical quantity in terms of the renormalized coupling constant, we can not describe its behavior further and further away from the perturbative region; going on with the perturbative series it is possible to fall in the non perturbative region, where the series badly diverge. Thus if we want to have a finite expression for a physical quantity we must stop the series at the right point to be finite and such that it can still be predictive for the theory. This is not an obvious work to do, because we must find an expression that behaves like (1.52) in the perturbative region, but it is finite away from it. An analysis of this situation has been done by 't Hooft [14]; here I report the crucial steps.

Our aim is to find a finite expression for the (1.52) also away from the perturbative region. Since physical quantity can be expressed in terms of Green functions, the two point Green function will be taken as example; in addition because $\beta(\alpha_s(M^2))$ is an infinite series we truncate the series a the second perturbative order then by now $\alpha_s(M^2)$ will be understood as the solution of the;

$$M^2 \frac{\partial \alpha_s(M^2)}{\partial M^2} = -\beta_0 \alpha_s^2(M^2) - \beta_1 \alpha_s^3(M^2), \tag{1.56}$$

that is well defined.

In terms of this coupling the Callan-Symanzik equation for the two point green function reads;

$$\left(M^2 \frac{\partial}{\partial M^2} + \beta\left(\alpha_s\left(M^2\right)\right) \frac{\partial}{\partial \alpha_s(M^2)} + \gamma\left(\alpha_s\left(M^2\right)\right)\right) G\left(k^2, M^2, \alpha_s\left(M^2\right)\right) = 0, \tag{1.57}$$

where $\gamma(\alpha_s(M^2))$ is the mass anomalous dimension and k^2 is the mass of the propagating particle, unlike the (1.34) in this expression has been introduced also the mass anomalous dimension $\gamma(\alpha_s(M^2))$ to be more general. The first two member of the equation generate a vector field; if we choose to move the green function on that flow, the above equation can be rewritten as follows,

$$\begin{aligned} \frac{\mathrm{d} \log\ G}{\mathrm{d}\alpha_s(M^2)} \left(k^2, M^2, \alpha_s\left(M^2\right)\right) &= -\frac{\gamma\left(\alpha_s\left(M^2\right)\right)}{\beta\left(\alpha_s\left(M^2\right)\right)} \\ &= \frac{z_0}{\alpha_s^2(M^2)} + \frac{z_1}{\alpha_s(M^2)} + z_2 + \dots \end{aligned} \tag{1.58}$$

The anomalous dimension is still an infinite series, but the finiteness of the truncated beta function permits us to write it down in a Laurent-series form. The differential equation (1.58) doesn't depend on k^2, thus we can factorize it. The choice of the curve over which we integrate the differential equation will introduce the scale dependence μ^2, also this contribution is put in the factorized part; than it is possible to rewrite the green function as:

$$G\left(k^2, M^2, \alpha_s\left(M^2\right)\right) = Z(\alpha_s(M^2))\overline{G}(k^2/\mu^2), \tag{1.59}$$

where the green function's physical dimension force it to depend on the ration k^2/μ^2.

The solution to the equation (1.58) is given by;

$$G\left(k^2, M^2, \alpha_s\left(M^2\right)\right) = \overline{G}(k^2/\mu^2) \exp\left\{ -\frac{z_0}{\alpha_s(M^2)} - z_1 \log \alpha_s(M^2) - z_2 \alpha_s(M^2) - \dots \right\}; \tag{1.60}$$

the coefficients z_0 and z_1 are the only interesting terms because in the high energy limit, when $\alpha_s(M^2) \to 0$, they give a contribution different from zero, while the other terms can be absorbed in the coefficients of the perturbative series. This is a tentative to resum the infinite series of $G\left(k^2, M^2, \alpha_s\left(M^2\right)\right)$ using only a truncated beta function. Let's now see what tell us this solution; in a compact form reads,

$$G^R(k^2/M^2, \alpha_s) = \alpha_s^{z_1} e^{-\frac{z_0}{\alpha_s}}\ \overline{G}(k^2/\mu^2), \tag{1.61}$$

where has been used $\alpha_s = \alpha_s(M^2)$ to simplify the notation and R stay for "resummed" green function. It is important to study the analytical behavior of these objects to understand if and how analytical continuation can be used in order to extend the function domain of convergence. Analytic functions are strongly dominated by their poles and only knowing that we can have a lot of information about the character of the function, a clear knowledge of the singularity of the (1.61) is needed. More precisely, how the coupling affect the observable is what to look for. The first information that we can get form (1.61) is the a branch-cut of the negative real axes due to the term $\alpha_s^{z_1}$, this is not a bad thing because the coupling is positive and the branch-cut can be bypassed on the Riemann surface. Other information are hidden inside $\overline{G}(k^2/\mu^2)$. To understand the effect of these singularities it's convenient to write the ratio k^2/μ^2 using the solution of the (1.56) neglecting, for simplicity, the term proportional to β_1, than we have;

$$x = \log\frac{k^2}{\mu^2} = \frac{1}{\alpha_s} + \beta_0 \log\frac{k^2}{M^2}, \tag{1.62}$$

as argument of $\overline{G}(x)$. To study the analytic structure of $\overline{G}(x)$ assumes x complex, than, from experience, it is known that the green function has a pole when the particle is on-shell, than, if k^2 is assumed to be complex, the singularity in the x complex plane is situated in,

$$x = \frac{1}{\alpha_s} + \beta_0 \log\frac{|k|^2}{M^2} + 2\pi\beta_0 i n, \tag{1.63}$$

where $n \in \mathbb{Z}$. This mean that the singularity lies all on the real positive axis. These singularities, from the point of view of a complex coupling α_s, give a problematic situation, in fact, supposing that α_s is complex the on-shell divergence occur when;

$$\alpha_s = \frac{\mathrm{Re}[\alpha_s] - 2\pi i\beta_0 n}{\mathrm{Re}^2[\alpha_s] + (2\pi\beta_0 n)^2}, \tag{1.64}$$

that means that all the on-shell singularities that arise from the multi-valued complex logarithm condensate in the region where $\alpha_s \to 0$. This accumulation of divergencies in the asymptotic freedom region is the worst situation to have; analytic continuation will never work in this situation. A stronger method to get a finite result for resummed divergent series must be found. A possible solution is to use well defined resummation techniques for asymptotic series, like the Borel summation. This method states that the resummed green function must have a form like,

$$G^R(k^2, \alpha_s) = \int_0^\infty \mathrm{d}z \; F(z) e^{-\frac{z^2}{\alpha_s}}, \tag{1.65}$$

where $F(z)$ is,

$$F(x) = a_0\delta(z) + \sum_{n=0}^{\infty} \frac{z^n}{n!} a_{n+1}. \tag{1.66}$$

The advantage to use this method is that $F(z)$ converges much faster than $G(k^2, \alpha_s)$

and it may have a very well defined radius of convergence. Than if $F(z)$ can be analytically continued to all real positive z and the integral converge, not necessarily for every α_s, than we can have a finite form for the resummed green function. Also this method fails to give a finite solution for the resummed green function, by the way, it fail "better" than the previous one, let's see how it work and why it fails.

Take for example the correlation function of the vacuum,

$$Z_0[\phi] = \int D\phi \; e^{-S[\phi]}, \tag{1.67}$$

that is the green function of the vacuum, where $S[\phi]$ is the action of the theory in its euclidean formulation, and $D\phi$ is the usual infinite dimensional integration of the path integral approach. To simplify the following step it is useful to scale the field to get:

$$Z_0[\phi'] = \int D\phi' \; e^{-\frac{S[\phi']}{\alpha_s}}. \tag{1.68}$$

To write it in terms of the Borel sum, the function $F(z)$ has the following form;

$$F(z) = \int D\phi' \; \delta(z - S[\phi']) = \sum_i \left| \frac{\delta S[\phi]}{\delta\phi} \right|^{-1}_{\phi_i(z)}, \tag{1.69}$$

where $\phi_i(z)$ is the i-th solution of $S[\phi] = z$. All the singular structure of $F(z)$ is encoded in the equation of motion, i.e. when $\delta S[\phi]/\delta\phi = 0$, in addition if $\overline{z} = S\left[\overline{\phi}\right]$ such that $\overline{\phi}$ satisfy the equation of motion, the (1.69) can be expanded around $\overline{z}$ to get;

$$\frac{\delta S[\phi]}{\delta\phi} \quad \to \quad \sqrt{2\frac{\delta^2 S[\phi]}{\delta\phi^2}\Big|_{\overline{\phi}} (z - \overline{z})}, \tag{1.70}$$

thus $F(\overline{z})$ is not only singular but it also has a branch cut towards the negative real axes if $\overline{\phi}$ is a minimum of the action, or towards the positive real axes is $\overline{\phi}$ is a maximum of the action. This kind of singularities are called instantons which physics is well known and there are much ways to handle it, so they are not a problem. In particular the case of QCD has singularity among the real positive axes, but anyway this kind of singularities are well understood and procedure to find a finite result are possible. A worst situation arise for other kind of singularities due to the procedure of renormalization, in fact, the divergent contribution of loop integral has the following form,

$$\alpha_s C^n \int \frac{\mathrm{d}^4 k}{k^{2a}} (\log k^2)^n, \tag{1.71}$$

where the exponent a depends on what is observed, i.e. the process studied, and it is at least three of greater; to obtain a not UV divergent integral; and C should be proportional to the first non-trivial $\beta(\alpha_s)$ coefficient. A general solution of the integral is proportional to

$$(a-2)^{-n-1} C^n n!, \tag{1.72}$$

that means that the green function can be written as,

$$G(\alpha_s) \simeq \sum \frac{(g_s C)^n}{(a-2)^{-n-1}} n! \tag{1.73}$$

and, consequently the $F(z)$, in the limit of high n,

$$F(z) \simeq \sum \left(\frac{zC}{a-2} \right)^n , \tag{1.74}$$

it is obviously divergent in the case that all coefficient are all equal 1, i.e. there are singularity in

$$\overline{z} = \frac{a-2}{C}. \tag{1.75}$$

In the case of an asymptotic freedom theory, like QCD, where C is negative, this singularity are on the real negative axes and does not create any problem until the region of interest is the real positive one. Anyway in QCD there could be IR singularities that are take into account for value of a smaller than three, projecting the singularities on the real positive axes opening a peculiar problem for the QCD because unlike to the UV singularities these are not well understood and a safe treatment is developed. A better understanding of the IR divergencies is needed.

1.4 Parton Model

The theory exposed above is not enough to describe real events because it's not still possible to control free quarks in collision experiments. Since there is only the possibility to let collide hadrons, QCD seems to be useless. Anyway in high energy experiment it's possible to reach energy high enough to penetrate the hadron and the hard collision can be figured to happen through a parton and another external particle; in the case of hadron-hadron collision, between a parton from each hadron, breaking the stable initial structure and producing a cluster of new hadrons in the final state. To get information from the multitude of hadrons generated in the final state is very hard indeed it's more wise to be inclusive over all hadronic final state and take information from other observables, for example, in the DIS the information needed are carried by the scattered electron, in the Drell-Yan process by the final state lepton pair and in the annihilation of leptons is used the angle of jet productions, but not the component of the jets itself. The procedure to describe the collision between with hadrons like a collision between partons can be applied only if the internal hadron's characteristic time is much smaller than the time of the hard collision; this is required to get detailed information on the internal structure of the hadron, it's like to take a picture of it. The tentative to study a process in such a way is the so called "naive parton model". It states that an hadron is composed by partons (quarks and gluons) that carries a fraction of the hadron's momentum, everyone directed in the hadron's direction, with zero transverse momentum (this assumption it's obviously false, indeed the real situation is more near to a transverse component with a gaussian distribution peaked around zero) and the information on distribution inside of it are in the "parton distribution function" (PDF); if x is the fraction of the hadronic momentum carried by a parton, $f(x)$ tell us the probability that the parton has a momentum xP, where P is the hadronic momentum. The PDF is specific for every hadron, i.e. they does not depend on the process taken in consideration; it is only necessary to measure it since it is still not possible to obtain it from from the elementary theory, and use the data to make prediction with the theory. In addition the conservation of the total hadronic momentum holds,

$$\int_0^1 \mathrm{d}x \sum_i x f_i(x) = 1, \tag{1.76}$$

where i runs over the partons flavor. In this sense the differential cross section for this process can be written as,

$$\mathrm{d}\sigma(p) = \sum_i \int_0^1 \mathrm{d}z \; f_i(z) \mathrm{d}\hat{\sigma}(zp). \tag{1.77}$$

An important consequence of this model is that it is valid only at very high energies because if the parton carries hadron's momentum fraction means that,

$$\hat{p} = xp \quad \Rightarrow \quad \hat{m}^2 = x^2 m^2. \tag{1.78}$$

where $\hat{}$ indicate partonic variables. The second relation means that the mass on the parton, an elementary particle, is not constant; it depends on the fraction of energy carried by the parton, this has no sense, but in the limit where the mass can be neglected the equality holds and the model can be applied, furthermore this assumption is in accordance with the request that the hadronic characteristic internal time is negligible compared to the time of interaction indeed if the mass is negligible the characteristic partonic time is very dilated and we can consider the partons as frozen during the interaction, then it is possible to go back to the x variable after the detection.

Luckily this model can be approximately derived from QCD. To show it, it is more convenient to take into account the DIS, where an hadron is probed by an electron. Consider the following process,

$$e^-(k) + H(p) \to e^-(k') + X(p'), \tag{1.79}$$

where e^- is an electron, H the initial hadron and X the final cluster of hadrons; through the exchange of a virtual photon with momentum $Q^\mu = p^\mu - p'^\mu$, and we defines,

$$x = -\frac{Q^2}{2pQ}, \quad y = \frac{pQ}{pk}, \quad s = (p+k)^2 = -\frac{Q^2}{xy}, \tag{1.80}$$

where all the mass are neglected except for the hadronic final state. The y variable varies in the region $0 \leq y \leq 1$ while x is limited by the positiveness of the final hadronic invariant mass,

$$M_X^2 = (p+q)^2 = Q^2 + 2pQ = -Q^2 \frac{1-x}{x} \geq 0, \tag{1.81}$$

than $0 \leq x \leq 1$. The electron it is used to probe the internal structure of the hadron, i.e. to define the structure function. The differential cross section for this process can be written separating the leptonic part from the hadronic ones, i.e.

$$\sigma \propto L_{\mu\nu} H^{\mu\nu}, \tag{1.82}$$

where $L_{\mu\nu}$ is the leptonic tensor and $H^{\mu\nu}$ the hadronic one. The leptonic tensor it is completely determined by QED and its form is well known,

$$L_{\mu\nu} = 2\left(2k_\mu k_\nu + k_\mu Q_\nu + k_\nu Q_\mu + g_{\mu\nu}\frac{Q^2}{2}\right); \tag{1.83}$$

the hadronic tensor is instead completely unknown, but some informations can be extrapolated from it. Inside the hadronic tensor there are all the information on how the electromagnetic field interact with it; these can be understood writing the tensor in a more convenient form taking advantage of the electromagnetic current conservation ($Q_\mu H^{\mu\nu} = Q_\nu H^{\mu\nu} = 0$) and the symmetry in the exchange of the tensorial indices. Then the hadronic tensor that satisfy these two requirements can be written as;

$$H^{\mu\nu}(p,Q) = \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) H_1(x,Q^2) + \left(p^\mu + \frac{1}{2x}Q^\nu\right)\left(p^\nu + \frac{1}{2x}Q^\mu\right) H_2(x,Q^2), \tag{1.84}$$

then the differential cross section reads,

$$\begin{aligned} \mathrm{d}\sigma = \frac{1}{2s} L_{\mu\nu}(k,k') H^{\mu\nu}(p,Q) \mathrm{d}\Phi_X &= \frac{4\pi\alpha^2}{Q^4}\left[y^2 H_1(x,Q^2) + \frac{1-y}{x} H_2(x,Q^2)\right] \mathrm{d}Q^2 \mathrm{d}x \\ &= \frac{4\pi\alpha^2}{Q^4} F(x,y,Q^2) \mathrm{d}Q^2 \mathrm{d}x. \end{aligned} \tag{1.85}$$

The phase space can be easily written in terms of Q^2 and x using the relation (1.80) and setting the reference frame as the hadronic center of mass frame. Consider now the partonic sub-process:

$$e^-(k) + q(\hat{p}) \to e^-(k') + q(\hat{p}'), \tag{1.86}$$

where $q(\cdot)$ is a quark that carry a momentum $\hat{p} = zp$. The differential cross section for this process is similar to (1.85) with the difference that in this case the functions $H_i(x,Q^2)$ are completely known because the quark does not have an internal structure, they are point like object, and are completely determined from QCD; thus the partonic tensor, “dual” of the hadronic tensor, is;

$$\begin{aligned}\hat{H}^{\mu\nu}(\hat{p},Q) &= \frac{e_q^2}{8\pi}\int \frac{\mathrm{d}^3\hat{p}'}{(2\pi)^3 2\hat{p}'_0}(2\pi)^4\delta^{(4)}(\hat{p}+Q-\hat{p}')\sum_{pol}\bar{u}(\hat{p}')\gamma^{\mu}u(\hat{p})\bar{u}(\hat{p})\gamma^{\mu}u(\hat{p}')\\ &= e_q^2\delta(1-\hat{x})\left[-\frac{1}{2}\left(g^{\mu\nu}-\frac{Q^{\mu}Q^{\nu}}{Q^2}\right)+\frac{2\hat{x}}{Q^2}\left(\hat{p}^{\mu}+\frac{1}{2\hat{x}}Q^{\mu}\right)\left(\hat{p}^{\nu}+\frac{1}{2\hat{x}}Q^{\nu}\right)\right],\end{aligned} \tag{1.87}$$

where $\hat{x} = Q^2/2\hat{p}Q$. The differential partonic cross section thus reads;

$$\mathrm{d}\hat{\sigma} = \frac{4\pi\alpha^2}{Q^4}L_{\mu\nu}(k,k')\hat{H}^{\mu\nu}(\hat{p},Q)\frac{y^2}{2Q^2}\mathrm{d}Q^2\mathrm{d}\hat{x}. \tag{1.88}$$

The two cross section are linked through the parton model formula (1.77), than it is possible to link $H^{\mu\nu}$ to $\hat{H}^{\mu\nu}$

$$H^{\mu\nu}(p,Q) = \sum_q \int_x^1 \frac{\mathrm{d}z}{z}f_q(z)\hat{H}^{\mu\nu}(zp,Q), \tag{1.89}$$

where the sum is done over all the quark flavor activated in the process. The lower extreme of integration is constrained by the relation,

$$(zp+Q)^2 \geq 0 \quad \Rightarrow \quad z \geq -\frac{Q^2}{2pQ} = x. \tag{1.90}$$

Putting the (1.84) and (1.87) in the (1.89), is it possible to write a relation for the $H_i(x,Q^2)$, indeed,

$$H_1(x,Q^2) = \frac{1}{2}\sum_q e_i^2 f_q(x), \tag{1.91}$$

$$H_2(x,Q^2) = \sum_q e_i^2 x f_q(x). \tag{1.92}$$

The structure function of the hadron is completely determined by the PDF. Furthermore the relations (1.91) and (1.92) prove the Callan-Gross relation for a 1/2-spin particle. The Callan-Gross relation states that the ratio between the longitudinal and the transversal cross section is:

$$\frac{\sigma_L}{\sigma_T} = 1 - \frac{F_2(x,Q^2)}{2xF_1(x,Q^2)}, \tag{1.93}$$

where $F_i(x,Q^2)$ is the structure function of the studied object. A null ratio means that the longitudinal and transversal part are not linked and as consequence, that the object is a 1/2-spin particle. Indeed the relations (1.91) and (1.92) produce a null ratio;

$$H_2(x,Q^2) - 2xH_1(x,Q^2) = 0. \tag{1.94}$$

This is a proof that the constituent of an hadron are 1/2-spin particles; this is not exactly true because there is a sea of virtual particles, antiparticles and gluons that not alter the overall characteristic of the hadron, in fact, the Callan-Gross relation fails at low x.

With this model it is still possible to do prediction using QCD, also if in the process appear only hadrons. An important consequence of the parton model is the scaling behavior, also known as the Bjorken scaling. It is a direct proof that the partons, inside the hadron, are essentially free, in fact, the structure functions (1.91) and (1.92) don't depend on Q^2, i.e. the internal structure is the same at each energy it is observed, it only depends on adimensional quantities. This is the most important property of this model. Anyway this is true only for the leading order of the QCD perturbative expansion, the other terms will break the scaling because they will introduce soft and collinear singularities; the tentative to remove it introduce dependence on the energy scale. Let's see how this dependence arise.

Defining the Born cross section for the DIS, where the initial quark has momentum $\hat{p}$;

$$\hat{H}^{\mu\nu}_{(0)}(\hat{p}, Q) = \frac{1}{2}\, \mathcal{M}^{\mu}(\hat{p})\, \not{\hat{p}}\, \overline{\mathcal{M}}^{\nu}(\hat{p}), \tag{1.95}$$

where C is a constant specific for the process. Than consider an initial state gluon emission where the emitted gluon has momentum l. Since we are interested to the infrared and collinear singularities it is convenient to write the gluon momentum using the Sudakov parametrization;

$$l^{\mu} = (1-x)\hat{p}^{\mu} + l^{\mu}_{T} + \xi\eta^{\mu}, \tag{1.96}$$

where η is an arbitrary longitudinal ($\eta\hat{p} = 1$) vector with null mass. Choose as reference frame the c.o.m. along the z axes, such that,

$$\hat{p} = \hat{p}(1, 0, 0, 1), \quad \eta = \frac{1}{2\hat{p}}(1, 0, 0, -1), \quad l_T = (0, \vec{l}_T, 0), \tag{1.97}$$

as consequence it is possible to define ξ,

$$\xi = \frac{|l^2_T|}{2(1-x)}, \tag{1.98}$$

that goes to zero as $|l^2_T|$ does. The terms proportional to ξ can be neglected as long as only the divergent part is required.

The matrix element for the initial state emission with the above parametrization of the emitted gluon momenta is;

$$\begin{aligned} & g_s\mathcal{M}^{\mu}(\hat{p}-l)\frac{\not{\hat{p}} - \not{l}}{(\hat{p}-l)^2}\gamma^{\mu}u(\hat{p})\epsilon_{\rho}(l) \\ &= g_s\mathcal{M}^{\mu}(\hat{p}-l)\frac{2xl^{\mu}_T + (1-x)\not{l}_T\gamma_{\mu}}{|l_T|^2}\gamma^{\mu}u(\hat{p})\epsilon_{\mu}(l), \end{aligned} \tag{1.99}$$

where the equation of motion ($l^{\mu}\epsilon_{\mu}(l) = 0$, $\not{\hat{p}}u(\hat{p}) = 0$) has been used. The phase space of the outgoing gluon has an important property, it is divergent itself in the soft region, in fact it can be written,

$$\frac{\mathrm{d}^2 l}{(2\pi)^3 2|l|} = \frac{\mathrm{d}^2 l_T}{2(2\pi)^3} \frac{\mathrm{d}x}{(1-x)}. \tag{1.100}$$

It is interesting to note that in the matrix element there isn't trace of the soft singularity, it arises only in the phase space, this suggest that to understand this kind of divergence a detailed structure of the phase space at any order is needed, but, cause the momentum conservation, the structure of it is not trivial.

The square matrix element of the gluon emission mediated over the polarization than is;

$$\hat{H}_R^{\mu\nu}(\hat{p}, Q, |l_T|^2) = g_s^2 \frac{1}{|l_T|^2}(1+x^2)\mathcal{M}^\mu(\hat{p}-l)\, \hat{\not{p}}\, \overline{\mathcal{M}}^\nu(\hat{p}-l), \tag{1.101}$$

in the limit of soft gluon emission the argument of the matrix element can be written as $x\hat{p}$, in fact, $\hat{p} - l = x\hat{p} - l_\perp - \xi\eta \approx x\hat{p}$, than the cross section reads,

$$\hat{H}_R^{\mu\nu}(\hat{p}, Q) = \frac{2\alpha_s}{3\pi} \int \frac{\mathrm{d}l_T^2}{l_T^2} \frac{\mathrm{d}x}{x} \frac{1+x^2}{1-x} \hat{H}_{(0)}^{\mu\nu}(x\hat{p}, Q), \tag{1.102}$$

that present soft $x \to 1$ and collinear $|l_\perp|^2 \to 0$ divergencies. At this order must take into account also the virtual emission that summed with the (1.102) gives the total first order correction for the real emission,

$$\hat{H}_{(1)}^{\mu\nu}(\hat{p}, Q) = \frac{2\alpha_s}{3\pi} \int \frac{\mathrm{d}l_T^2}{l_T^2} \frac{\mathrm{d}x}{x} \frac{1+x^2}{1-x} [\hat{H}_{(0)}^{\mu\nu}(x\hat{p}, Q) - \hat{H}_{(0)}^{\mu\nu}(\hat{p}, Q)]. \tag{1.103}$$

It is clear that virtual contribution cancels the divergence at $x = 1$ while the collinear one it's still in. This not happen for example in process of e^+e^- annihilation where the hadrons are present only in the final state; in this case the gluon emission in the final state factorize a Born cross section that depends on $\hat{p}$ instead of $z\hat{p}$, this remove the entire divergent term and a complete cancellation of the infrared and collinear divergencies happen.

The presence of divergencies seems to break the parton model but it do not scare us because in quantum field theory is usual to find physical object that present divergent terms. This situation can be arranged by redefining quantities that must be measured, like in the case of the running coupling constant, to get a finite result; we can do the same procedure also in the parton model redefining the PDFs, that are not predicted by any theory, because we can only use it through data interpolation, then we consider that the real PDF that is measured is the redefined one.

When the gluon is emitted it is still inside the hadron; this means that, because the confinement, the energy of the gluon can not be too small, thus it is reasonable to put a lower cut off in the transverse momentum integration of order Λ, in such a way the integration produce a logarithmic dependence on the energy scale. In this context is useful to introduce the plus distribution (for more detail on the $\cdot_+$ see Appendix B), and define the so called Altarelli-Parisi splitting function:

$$P_{qq}(x) = \frac{4}{3}\left[\frac{1+x^2}{1-x}\right]_+, \tag{1.104}$$

that is a universal function of a pure quark initial state; moreover, in order to lighten the notation, it is convenient to express the results through the convolution product;

$$
\begin{aligned}
f(x) &= (f_1 \otimes f_2 \otimes \cdots \otimes f_n)(x) = \\
&= \int \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \; f_1(x_1) f_2(x_2) \ldots f_n(x_n) \delta(x - x_1 x_2 \ldots x_n),
\end{aligned} \tag{1.105}
$$

for which the following properties are true:

$$
(\mathbb{I} \otimes f)(x) = f(x), \tag{1.106}
$$

$$
(f_1 \otimes f_2)(x) = (f_2 \otimes f_1)(x) \tag{1.107}
$$

$$
((f_1 \otimes f_2) \otimes f_3)(x) = (f_1 \otimes (f_2 \otimes f_3))(x) = (f_1 \otimes f_2 \otimes f_3)(x) \tag{1.108}
$$

then the complete form for next to leading order approximation to the partonic tensor, using the notation (1.104) and (1.105) is:

$$
\hat{H}^{\mu\nu}(\hat{p}, Q) = \left(\mathbb{I} + \frac{\alpha_s}{2\pi} P_{qq} \log \frac{Q^2}{\mu^2} \right) \otimes \left(\mathbb{I} + \frac{\alpha_s}{2\pi} P_{qq} \log \frac{\mu^2}{\Lambda^2} \right) \otimes \hat{H}^{\mu\nu}_{(0)}(\hat{p}, Q), \tag{1.109}
$$

where the logarithm has been separated introducing an arbitrary scale μ and the terms of order $\mathcal{O}(\alpha_s^2)$ has been neglected. With this formalism can be written the hadronic tensor (1.89),

$$
\begin{aligned}
H^{\mu\nu}(p, Q) &= \sum_q f_q \otimes \hat{H}^{\mu\nu}(p, Q) \\
&= \sum_q f_q \otimes \left(\mathbb{I} + \frac{\alpha_s}{2\pi} P_{qq} \log \frac{\mu^2}{\Lambda^2} \right) \otimes \hat{H}^{\mu\nu}_{(0)}(p, Q) \otimes \left(\mathbb{I} + \frac{\alpha_s}{2\pi} P_{qq} \log \frac{Q^2}{\mu^2} \right) \\
&= \sum_q f_q(\mu^2) \otimes \hat{H}^{\mu\nu}(p, Q^2/\mu^2),
\end{aligned} \tag{1.110}
$$

where,

$$
f_q\left(\mu^2\right) = f_q \otimes \left(\mathbb{I} + \frac{\alpha_s}{2\pi} P_{qq} \log \frac{\mu^2}{\Lambda^2} \right), \tag{1.111}
$$

$$
\hat{H}^{\mu\nu}\left(p, Q^2/\mu^2\right) = \hat{H}^{\mu\nu}_{(0)}(p, Q) \otimes \left(\mathbb{I} + \frac{\alpha_s}{2\pi} P_{qq} \log \frac{Q^2}{\mu^2} \right), \tag{1.112}
$$

where the PDF and the partonic tensor have been redefined acquiring the dependence on the arbitrary scale μ. This is the same consequence that we have in the case of the ultraviolet divergencies, removed redefining the coupling constant and make it acquire a dependence on arbitrary energy scale. In the form (1.110) the parton model is known as the QCD-improved parton model, the possibility to use perturbation theory and the parton model together is open. The logarithm has been separated in that way to absorb the collinear divergence in the PDFs. To remove the divergence has been used a low energy cut-off Λ, it can be interpreted with the Heisenberg indetermination principle to say that in the PDF are hidden contributions from something happened long before the interaction, when the hadron were a free moving body. The choice to put the dependence on the low energy cut-off in the PDFs have sense since the PDFs are universal functions and that describe an intrinsic property of the hadron. Furthermore the large contribution from $\log \mu^2/\Lambda^2$ it is no longer a problem because it is impossible to give a precise mathematical form to this functions then, as already said, they con only be used from the interpolation of experimental data. The partonic tensor, instead, take into account only what happen at "short distances", thus chosen μ^2 sufficiently near to Q^2, to do not make large contribution appear, the process can be studied via perturbation theory. The scale μ^2 introduced here is not the same the one introduced in the renormalization procedure, this one is called "factorization scale" and it's usually indicated with μ_F^2.

Can be shown, through not trivial step [15] that (1.110) is valid at all order in perturbation theory; this result is called the "factorization theorem" and the direct calculation of perturbative orders can be done as an alternative proof to the theorem.

Also if the PDFs are not theoretically calculable their scale dependence can still be found, in fact, $H^{\mu\nu}(p, Q)$ is independent of μ because its product with the leptonic (scale independent) tensor is proportional to the hadronic (observable) cross section; then it is possible to write:

$$\mu^2 \frac{\partial}{\partial \mu^2} H^{\mu\nu}(p, Q) = 0 \;\Rightarrow\; \left[\mu^2 \frac{\partial}{\partial \mu^2} f_q(\mu^2)\right] \otimes \hat{H}^{\mu\nu}(p, \mu^2) + f_q(\mu^2) \otimes \left[\mu^2 \frac{\partial}{\partial \mu^2} \hat{H}(p, \mu^2)\right] = 0 \tag{1.113}$$

then using the equation (1.112) at the order α_s the second relation in (1.113) can be written as:

$$\mu^2 \frac{\partial}{\partial \mu^2} f_q(\mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} f_q(\mu^2) \otimes P_{qq}, \tag{1.114}$$

since the hard process is arbitrary. The equation(1.114) in its extended form is:

$$\mu^2 \frac{\partial}{\partial \mu^2} f_q(z, \mu^2) = \frac{\alpha_s}{2\pi} \int_z^1 \frac{\mathrm{d}x}{x} P_{qq}\left(\frac{z}{x}\right) f_q(x, \mu^2), \tag{1.115}$$

known as the GLAP (Gribov-Lipatov-Altarelli-Parisi) equation. In this form the GLAP equation is valid only at the first order in perturbation theory and for process where are involved initial quark state and the emission of a gluon. A more detailed derivation of the GLAP equation shows that the splitting functions can be written as a power series in $\alpha_s(\mu^2)$, where a the first order it is valid the (1.115). The general form of the GLAP equation, taking in consideration also all possible splitting situation is:

$$\mu^2 \frac{\partial}{\partial \mu^2} f_i(z, \mu^2) = \int_z^1 \frac{\mathrm{d}x}{x} \sum_j P_{ij}\left(\frac{z}{x}, \alpha_s(\mu^2)\right) f_j(x, \mu^2), \tag{1.116}$$

where the indices i and j run over the partonic flavor and,

$$P_{ij}\left(x,\alpha_s(\mu^2)\right) = \frac{\alpha_s(\mu^2)}{2\pi} P_{ij}(x) + \left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2 P_{ij}^{(2)}(x) + \dots \tag{1.117}$$

The component of the splitting matrix $P_{ij}\left(x,\alpha_s(\mu^2)\right)$, at the first order, are:

$$P_{qq}(x) = P_{\bar{q}\bar{q}}(x) = \frac{4}{3}\left[\frac{1+x^2}{1-x}\right]_+ \tag{1.118}$$

$$P_{qg}(x) = P_{\bar{q}g}(x) = \frac{1}{2}\left[x^2 + (1-x)^2\right] \tag{1.119}$$

$$P_{gq}(x) = P_{g\bar{q}}(x) = \frac{4}{3}\left[\frac{1+(1-x)^2}{x}\right] \tag{1.120}$$

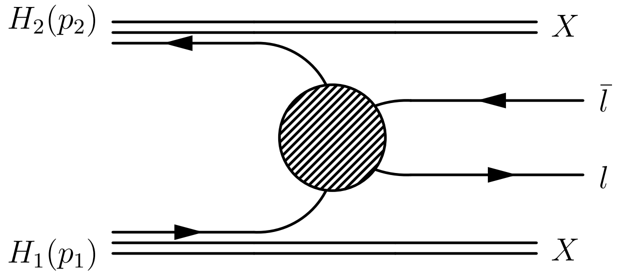
$$P_{gg}(x) = 6\left\{x\left[\frac{1}{1-x}\right]_+ + \frac{1-x}{x} + x(1-x) + \left(\frac{11}{12} - \frac{n_f}{6}\right)\delta(1-x)\right\}. \tag{1.121}$$

Not all of this splitting function arise in the calculation of the NLO of some process, like for the DIS where the lowest perturbative correction can only have a quark state from the hadron that emit a gluon or a gluon state that decay in a quark-antiquark pair, i.e. the equations (1.118), (1.119) and (1.120), while at the NNLO the initial partonic state can also be a gluonic state that decay in other gluons, i.e. appear the (1.121).

To end this section it is important to remark that at high energies the parton model has good comparison with the experiments and show the correct scaling behavior experimentally observed; the most important property of the model. Adding perturbative correction the scaling behavior break down but the way it is broken at every perturbative order can lead to a big violation of it, it has been proved that the way QCD breaks the scaling it is acceptable such that we can still use the model in the perturbative framework. Furthermore, the divergent quantities are used to redefine the PDFs that, like in the case of the coupling constant in the renormalization procedure, will be experimentally defined and we can use it as a finite function. The difficulty of giving a precise mathematical form to these functions lies in the fact that they are connected to the confinement of the quarks, a problem that is still open in quantum chromodynamics.

Chapter 2

Drell Yan Process



$$H_1(p_1) + H_2(p_2) \to l + \bar{l} + X \tag{2.1}$$

The parton model is the main way to predict an event in high energy particle physics involving hadrons and due to the recent successful prediction such as the discovery of the Higgs boson or more older one like the discovery of the Z and W bosons, the Drell-Yan mechanism is a powerful tool to probe new physics and to reach better understanding of the already known phenomena. This is possible thanks to the production of two not colored fermions that carry information about hard sub-process. For this reason, this mechanism has been chosen in this work as an example to show how observables, like the cross-section, are affected by infrared singularities and how these can be handled to give a finite result for the hadronic cross section. This mechanism comprises the production of high invariant lepton-pair mass called Drell-Yan pair, $M^2 = \left(l + \bar{l}\right)^2 \gg 1$ GeV^2, in a hadron-hadron collision.

To use the parton model we must working in the high energy regime $M^2 \gg \Lambda^2_{QCD}$ (where M^2 is the transfer momentum and the Λ_{QCD} is the typical strong interaction's energy scale of about 100 MeV), i.e. it is possible to neglect the masses of all the particle involved in the process.

The hadronic cross section for this kind of process in the parton model formulation can be written as follow

$$\sigma = \sum_{a,b} \int_0^1 dx_1 \int_0^1 dx_2 \; f_a(x_1) f_b(x_2) \; \hat{\sigma}_{q\bar{q}\to l\bar{l}}, \tag{2.2}$$

where the sum runs over the hadron constituents that can take place in the hard subprocess, $f_a(x_1)$ and $f_b(x_2)$ are the parton distribution function, $\hat{\sigma}_{q\bar{q}\to l\bar{l}}$ is the hard partonic cross section. In the following we evaluate the first perturbative order of the hard sub-process in the case where the particle exchanged is a virtual-photon with transferred momentum M, and we show how to handle the soft and collinear divergences to get a finite result for the hadronic cross section.

2.1 The hard sub-process

In the partonic process are involved two partonic initial state that carries hadron momentum's fraction, called in the usual literature notation $\hat{p}_1 = x_1 p_1$ and $\hat{p}_2 = x_2 p_2$, that annihilate into an electroweak boson with momentum Q that successively decay in the leptonic final state with momentum l_1 and l_2. To produce a $l\bar{l}$ pair at the lowest order in $\alpha = e^2/4\pi$ only the Z^0 boson and a virtual photon are possible exchanged particle, for this reason the expression given in this section are valid for only these two cases. The study of this process can be simplified separating the quark part by the leptonic part in $|M(\hat{p}_1, \hat{p}_2; l_1, l_2)|^2$ and integrating over the leptonic tensor, in such way we can write the partonic cross section as follow

$$\mathrm{d}\hat{\sigma} = \frac{1}{4\hat{p}_1\hat{p}_2}\left[\frac{1}{(Q^2 - M_B^2)^2} H^{\mu\nu} L_{\mu\nu}\right] \mathrm{d}\Phi_L \mathrm{d}\Phi_S, \tag{2.3}$$

where the term M_B represent the mass of the boson exchanged, i.e. for the photon the mass is 0 and for the Z^0 boson is about 91.19 GeV. In this case is not important which gauge is chosen in the electroweak propagator because the leptonic tensor is still gauge invariant and the tensorial part of the propagator reduces to the Minkowski metric tensor. The term $\mathrm{d}\Phi_S$ takes into account of the degrees of freedom due to the emission of soft-gluons that are associate to the hadronic part. For example in the case of a single emission,

$$\mathrm{d}\Phi_S = \int \frac{\mathrm{d}M^2}{2\pi}\left[\frac{\mathrm{d}\vec{k}}{(2\pi)^3 2|\vec{k}|}\frac{\mathrm{d}\vec{Q}}{(2\pi)^3 2\omega_{\vec{Q}}}(2\pi)^4\delta^{(4)}\left(\hat{p}_1 + \hat{p}_2 - k - Q\right)\right]. \tag{2.4}$$

Where $\omega^2_{\vec{Q}} = |\vec{Q}|^2 + M^2$. Multiple gluons emission are more difficult to work with because the momenta are linked to each other by the momentum conservation then it impossible to find a factorize form. This situation can be avoided working in the Mellin space and in the limit of soft-emission.

The leptonic tensor is,

$$L_{\mu\nu} = e^2 \,\mathrm{Tr}\left[\not{l}_2\gamma_\nu(g_1 - g_2\gamma^5)\not{l}_1\gamma_\mu(g_1 - g_2\gamma^5)\right] \tag{2.5}$$

$$\mathrm{d}\Phi_L = \frac{\mathrm{d}\vec{l}_1}{(2\pi)^3 2|\vec{l}_1|}\frac{\mathrm{d}\vec{l}_2}{(2\pi)^3 2|\vec{l}_2|}(2\pi)^4\delta^{(4)}\left(Q - l_1 - l_2\right), \tag{2.6}$$

where e is the positron charge and the constant g_1 and g_2 depends on the flavour of the fermions and the boson exchanged. For the Z^0 boson they are

$$g_1 = \frac{T_3 - 4e_l \sin^2\theta_W}{2\sin\theta_W\cos\theta_W}, \quad g_2 = \frac{T_3}{2\sin\theta_W\cos\theta_W},$$

with $T_3 = \sigma_3/2$ third generator of the $SU(2)$ group, θ_W the weak mixing angle and e_l the electric charge's fraction of the lepton. For the photon the constants are $g_1 = -1$ and $g_2 = 0$.

Using the equation (2.5) and (2.6) we can see that is possible to integrate pver the leptonic tensor contribution. To this purpose is useful to write the leptonic tensor using its property such as tensorial symmetry and the Ward identity[1] ($Q^\mu L_{\mu\nu} = 0 = Q^\nu L_{\mu\nu}$), thus,

$$L_{\mu\nu} = A g_{\mu\nu} + B Q_\mu Q_\nu. \tag{2.7}$$

This is the most general form for a two degrees of freedom tensor satisfying the above properties in fact, for a [4,4] tensor, 6 degrees are solved by the symmetry property and 8 degrees are solved by the gauge invariance, leaving only 2 free degrees.

To determine the coefficient A and B we need two find two equation to fix the two free degrees; an handy choice is to solve the following system of two equations,

$$\begin{cases} Q^\mu L_{\mu\nu} & = A Q_\nu + B Q^2 Q_\nu = 0, \\ L^\mu{}_\mu & = 4A + M^2 B = -4e^2(g_1^2 + g_2^2) Q^2. \end{cases}$$

The solution of this system gives the expression the the leptonic tensor

$$L_{\mu\nu} = \frac{4}{3} e^2 (g_1^2 + g_2^2) \left(-g_{\mu\nu} Q^2 + Q_\mu Q_\nu\right). \tag{2.8}$$

To perform the integration over the leptonic phase space it is enough to evaluate the volume of the phase space since the leptonic tensor, as write in the Eq.(2.8), does not depend explicitly on the leptonic momenta but just on the transferred momentum. The volume of the leptonic phase space is $1/(8\pi)$, thus the leptonic tensor is

$$L_{\mu\nu} = \frac{e^2(g_1^2 + g_2^2)}{6\pi} \left(-g_{\mu\nu} + \frac{Q_\mu Q_\nu}{Q^2}\right) Q^2. \tag{2.9}$$

It is interesting to note that the tensor part of the Eq.(2.9) have the same form of the sum over the polarization of a massive spin-1 boson, like in the case of an off-shell photon where,

$$\sum_{pol} \varepsilon^a_\mu(\vec{Q}) \varepsilon^{*a}_\nu(\vec{Q}) = -g_{\mu\nu} + \frac{Q_\mu Q_\nu}{Q^2}. \tag{2.10}$$

[1]The Ward identity is still valid for the weak boson until the leptons are massless.

This similarity permit us to write the cross section as follow

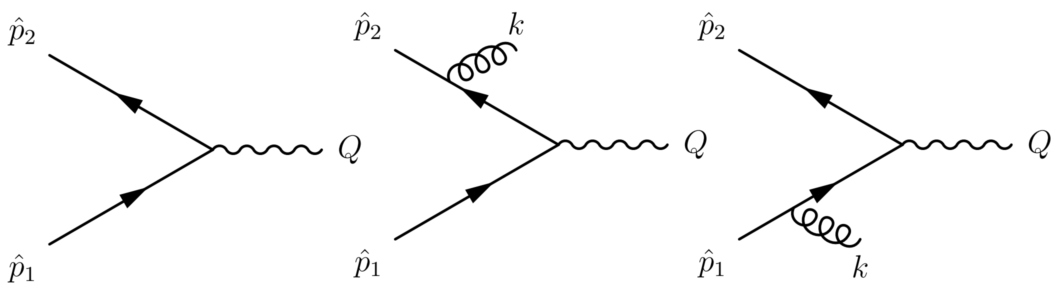
$$
\begin{aligned}
\mathrm{d}\hat{\sigma} &= \frac{e^2(g_1^2+g_2^2)}{6\pi}\frac{Q^2}{4\hat{p}_1\hat{p}_2}\left[\frac{1}{\left(Q^2-M_B^2\right)^2}H^{\mu\nu}\left(-g_{\mu\nu}+\frac{Q_\mu Q_\nu}{Q^2}\right)\right]\mathrm{d}\Phi_S \\
&= \frac{e^2(g_1^2+g_2^2)Q^2}{6\pi\left(Q^2-M_B^2\right)^2}\left[\frac{\mathrm{d}\Phi_S}{4\hat{p}_1\hat{p}_2}\sum_{pol}H^{\mu\nu}\varepsilon_\mu^a(\vec{Q})\varepsilon_\nu^{*a}(\vec{Q})\right] \\
&= \frac{e^2(g_1^2+g_2^2)Q^2}{6\pi\left(Q^2-M_B^2\right)^2}\,\mathrm{d}\hat{\sigma}_H. \qquad (2.11)
\end{aligned}
$$

In this way we just need to study the diagram in Figure 2.1 to evaluate the hadronic cross section.

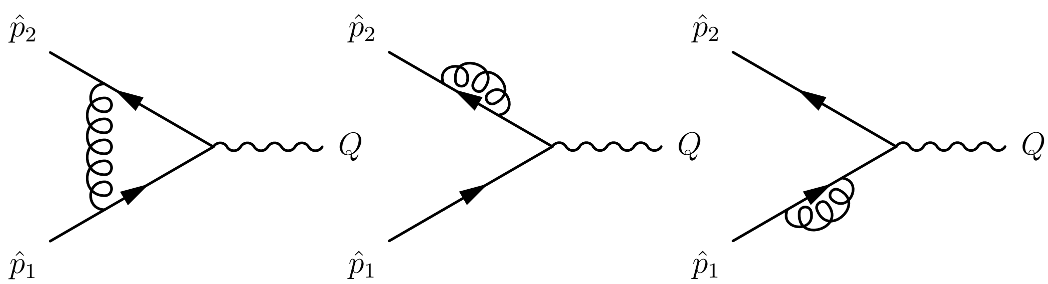
In the rest of the chapter we evaluate the first perturbative chromodinamic correction for the differential and double differential partonic cross section. In the case of the double differential partonic cross section we will be focused on the rapidity distribution, defined as:

$$
y = \frac{1}{2}\log\frac{Q^0+Q^3}{Q^0-Q^3}. \qquad (2.12)
$$

The sub-process examined is, for simplicity, the annihilation of a quark pair in a virtual photon.



(a) Born (b) Real



(c) Virtual (d) External leg correction

Figure 2.1

2.2 Leading order calculation

Let's show a direct calculation of the NLO; the first diagram in the Figure 2.1. The amplitude of the process is

$$\mathcal{M}(\hat{p}_1, \hat{p}_2; Q) = -iee_q\, \bar{v}(\hat{p}_2)\not{\varepsilon}^*(Q)u(\hat{p}_1), \tag{2.13}$$

so we can write the square matrix element

$$\mathcal{M}(\hat{p}_1, \hat{p}_2; Q)\overline{\mathcal{M}}(\hat{p}_1, \hat{p}_2; Q) = (ee_q)^2\Big(\bar{v}(\hat{p}_2)\gamma^\mu u(\hat{p}_1)\Big)\Big(\bar{u}(\hat{p}_1)\gamma^\nu v(\hat{p}_2)\Big)\varepsilon_\mu(Q)\varepsilon^*_\nu(Q). \tag{2.14}$$

Now we average over the initial polarization of the quark and their colour and sum over the final polarization of the virtual photon, thus we obtain

$$\sum_{pol} |\mathcal{M}(\hat{p}_1, \hat{p}_2; Q)|^2 = \frac{1}{12}(ee_q)^2 \mathrm{Tr}\left[\not{\hat{p}}_2\gamma^\mu\not{\hat{p}}_1\gamma^\nu\right](-g_{\mu\nu}) = \frac{1}{3}(ee_q)^2\hat{s} \tag{2.15}$$

for the sake of clarity it is good to specify the operation done in the above expression. The average over the initial polarization gives us a $1/4$ factor and the trace of the spinor matrix, because

$$\sum_{\sigma=1,2} \bar{u}(\hat{p}_1;\sigma)u(\hat{p}_1;\sigma) = \not{\hat{p}}_1, \tag{2.16}$$

$$\sum_{\sigma=1,2} \bar{v}(\hat{p}_2;\sigma)v(\hat{p}_2;\sigma) = \not{\hat{p}}_2, \tag{2.17}$$

from this relation follow the trace above. The average over the color of the quark gives a factor $1/3$ because the electro-weak vertex does not change the color, in fact the electro-weak vertex, in the simple case of a virtual photon, is

$$-\,iee_q\gamma^\mu\delta^{ab}, \tag{2.18}$$

where a and b are color index with three possible value. The average over the color is then the total number of colored states over the sum of possible colored states of each quark, i.e. $3/9 \to 1/3$. In this way the total averaging procedure gives the factor $1/12$. The sum over the photon polarization is reduced to the metric tensor cause the gauge invariance indeed can be trivially shown that

$$Q_\mu Q_\nu \mathrm{Tr}\left[\not{\hat{p}}_2\gamma^\mu\not{\hat{p}}_1\gamma^\nu\right] = 0. \tag{2.19}$$

In the last term of the equation (2.15) is the square energy of the partonic process $\hat{s} = (\hat{p}_1 + \hat{p}_2)^2 = 2\hat{p}_1\hat{p}_2$.

At this level all the outgoing momenta has already been integrated, thus the expression (2.11) is not the differential cross section but it is the total cross section:

$$\hat{\sigma}_H = \frac{1}{2\hat{s}}|\mathcal{M}(\hat{p}_1, \hat{p}_2; Q)|^2 = \frac{(ee_q)^2}{6}, \tag{2.20}$$

and the total hadronic cross section reads

$$\hat{\sigma} = \frac{4\pi\alpha^2}{3\hat{s}}\frac{e_q^2}{3} = \frac{e_q^2}{3}\,\sigma_0(\hat{s}), \tag{2.21}$$

where α is the famous fine-structure constant defined as $\alpha = e^2/4\pi$ in natural unit and we have defined $\sigma_0(\hat{s}) = 4\pi\alpha^2/3\hat{s}$.

It is useful to write the differential cross section for the lepton pair mass,

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}M^2} = \sigma_0(M^2)\frac{e_q^2}{3}\,\delta(\hat{s} - M^2), \tag{2.22}$$

while the double differential cross section for the rapidity and the transverse momentum at the partonic level reads respectively,

$$\frac{\mathrm{d}^2\hat{\sigma}}{\mathrm{d}M^2\mathrm{d}Y} = \sigma_0(M^2)\frac{e_q^2}{3}\,\delta(\hat{s} - M^2)\,\delta(Y - y) \tag{2.23}$$

$$\frac{\mathrm{d}^2\hat{\sigma}}{\mathrm{d}M^2\mathrm{d}Q_T} = \sigma_0(M^2)\frac{e_q^2}{3}\,\delta(\hat{s} - M^2)\,\delta(Q_T). \tag{2.24}$$

With these expression we can write the LO of the parton-model differential cross section for this process. It is important to recall that the partonic momenta are a fraction of the hadronic one; $\hat{p}_1 = x_1 p_1$ and $\hat{p}_2 = x_2 p_2$ such that $\hat{s} = x_1 x_2 s$, in this way:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}M^2} = \frac{\sigma_0(M^2)}{3}\sum_q \int_0^1 dx_1 \int_0^1 dx_2\; f_q(x_1) f_{\bar{q}}(x_2)\; e_q^2\; \delta(x_1 x_2 s - M^2). \tag{2.25}$$

From this expression it easy to see that to multiply the equation (2.25) by M^4 introducing the variable $\tau = M^2/s$, the cross-section exhibit a scaling behavior in the τ variable:

$$\begin{aligned} M^4\frac{\mathrm{d}\sigma}{\mathrm{d}M^2} &= \frac{4\pi\alpha^2}{9}\tau\sum_q \int_0^1 dx_1 \int_0^1 dx_2\; f_q(x_1) f_{\bar{q}}(x_2)\; e_q^2\; \delta(x_1 x_2 - \tau) \\ &= \frac{4\pi\alpha^2}{9}\tau F(\tau). \end{aligned} \tag{2.26}$$

This result is analogue to the approximate Bjorken scaling observed in the deep inelastic scattering that, together with the concepts of partons, has been a first hint of the asymptotic freedom property of the strong interacting particle that conducted to the formulation of quantum chromodynamics. In fact, the scaling means that when we are at high energies, or better, at energies greater than the typical hadronic scale ($\Lambda_S \sim 100$ MeV), the cross section written in terms of adismensional kinematical variables like τ show an energy dependence like its canonical dimension, in fact, the

(2.26) can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{4\pi\alpha^2}{9M^2}\tau F(\tau), \tag{2.27}$$

this means that the cross section goes to zero as the energy increase, then if we probe the hadron deeper and deeper we will only see a sea of non-interacting elementary particle, the partons. The possible interaction energy between them would give a cross section that will not go exactly to zero but has some mass energy structure, then the observation of the scaling phenomenon is a proof of the asymptotic freedom of the strong interaction. This is an amazing property because permits us to probe the hadronic structure as deep as we want. However, this scaling can be violated by the perturbative treatment of the theory as shown in the previous chapter. Will be shown in this chapter that the scale is broken also for the Drell-Yan process already at the first perturbative order.

To end this section we give also the form of the hadronic double differential cross section in the invariant mass and rapidity variable. To write the rapidity distribution is useful to express the Eq. (2.12) in the center of momentum of the two hadrons, i.e.

$$2Q^\mu = \sqrt{s}(x_1 + x_2, 0, 0, x_1 - x_2) \tag{2.28}$$

$$y = \frac{1}{2}\log\frac{x_1}{x_2}. \tag{2.29}$$

The rapidity distribution is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}M^2\mathrm{d}y} = \frac{\sigma_0(M^2)}{3s}\sum_q e_q^2\, f_q\left(\sqrt{\tau}\, e^y\right) f_{\bar{q}}\left(\sqrt{\tau}\, e^{-y}\right). \tag{2.30}$$

Using this distribution it is in principle possible to measure the parton distribution function for the quark and the relative anti-quark, anyway the best method is still to measure it thought the deep inelastic scattering.

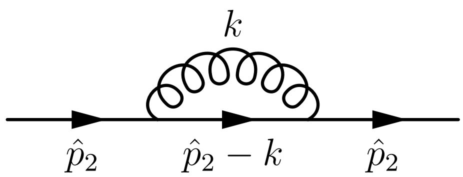


Figure 2.2: Contribution to the quark self energy by strong interaction.

2.3 External leg correction

The last two diagrams in Figure 2.1d do not give contribution to the final result because they are identically zero until we are at energy high enough to have vanishing quarks masses. We show as example the direct calculation of the first diagram in Figure 2.1d with the loop momentum set like in Figure 2.2.

The matrix element of this correction is;

$$\begin{aligned}\mathcal{M}(\hat{p}_1, \hat{p}_2; Q) &= \bar{v}(\hat{p}_2)\left(-ig_s t^A \gamma^\mu\right)\left(i\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\not{k} - \hat{\not{p}}_2}{(k^2+i\eta)\big((\hat{p}_2 - k)^2 + i\eta\big)}\right)\times \\ &\times (-ig_s t^A \gamma_\mu)\frac{i\hat{\not{p}}_2}{\hat{p}_2^2 + i\eta}(-iee_q\gamma^\rho)u(\hat{p}_1)\varepsilon^*_\rho(Q) \\ &= ee_q\bar{v}(\hat{p}_2)\big(-i\,\Sigma(\hat{p}_2)\big)\frac{\hat{\not{p}}_2}{\hat{p}_2^2 + i\eta}\not{\varepsilon}^*(Q)\,u(\hat{p}_1).\end{aligned} \tag{2.31}$$

Now we show that the term $-i\,\Sigma(\hat{p}_2)$ is proportional to $\hat{\not{p}}_2$, thus for the equation of motion $\bar{v}(\hat{p})\hat{\not{p}} = 0$ it makes vanish the matrix element. The corrective term is

$$\Sigma(\hat{p}_2) = g_s^2 C_F \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\gamma^\mu(\not{k} - \hat{\not{p}}_2)\gamma_\mu}{(k^2+i\eta)\big((\hat{p}_2 - k)^2 + i\eta\big)}. \tag{2.32}$$

We can use the property of the gamma matrices in the numerator of the integrand to get $\gamma^\mu \not{p}\gamma_\mu = -2\not{p}$, and use the Feynman trick to simplify the denominator, we also omit the $i\eta$ term to lighten the notation, then we get,

$$\frac{1}{k^2(\hat{p}_2 - k)^2} = \int_0^1 \mathrm{d}x \frac{1}{(k^2 - 2x\hat{p}_2 k)^2} = \int_0^1 \mathrm{d}x \frac{1}{(k - x\hat{p}_2)^4} \tag{2.33}$$

$$l = k - x\hat{p}_2 \quad \Rightarrow \quad \int_0^1 \mathrm{d}x\,\frac{1}{l^4}, \tag{2.34}$$

with these operation we have;

$$\begin{aligned}&2g_s^2 C_F \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{(1-x)\hat{\not{p}}_2 - \not{l}}{l^4} \\ &= \hat{\not{p}}_2\; g_s^2 C_F \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{l^4},\end{aligned} \tag{2.35}$$

where, in the numerator, the integral of the term proportional to $\not{l}$ is zero because it is an odd function integrated over an even domain.

From this result the proportionality to the external momentum is clear, and the matrix element is proved to be zero. Someone can ask how the divergent integral in the (2.35) affect the result, indeed it can be shown that, if regularized with dimensional regularization (for the main aspect of the dimensional regularization see Appendix A), the integral is null; indeed with the dimensional regularization, i.e.

passing from a 4-dimensional space-time to a $d = 4-2\epsilon$ dimensional one, and introducing also a mass regulator μ in the denominator in order to get its limit $\mu \to 0$ after the integration we have,

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2-\mu^2)^2} \sim \left(\frac{1}{\mu}\right)^{\epsilon}. \tag{2.36}$$

The limit $\mu \to 0$ makes the integral vanish only for $\epsilon < 0$ while it would be divergent everywhere else, but continue analytically the solution we can say that the integral is zero everywhere. This result is true also for the other leg. The fact that the integral of a generic power of the momentum, integrated over the whole domain, is zero. This is an important and also convenient property of the dimensional regularized integrals.

Now we show an example that highlights the vanishing of an integral like the one in (2.35). we use the dimensional regularization on the integral,

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{M^2}{l^2(l^2-M^2)} = \int \frac{\mathrm{d}^d l}{(2\pi)^d} \left[\frac{1}{l^2-M^2} - \frac{1}{l^2}\right]. \tag{2.37}$$

The integral on the left side of the equation can be solved using the Feynman parameters in order to write it in a more convenient form;

$$\begin{aligned} M^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{1}{(k^2-xM^2)^2} &= -i \left(\frac{M}{4\pi}\right)^{\frac{d}{2}} \Gamma(\epsilon) \int_0^1 \mathrm{d}x \; x^{-\epsilon} \\ &= -i \left(\frac{M}{4\pi}\right)^{\frac{d}{2}} \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-\epsilon)} \\ &= i \left(\frac{M}{4\pi}\right)^{\frac{d}{2}} \Gamma(\epsilon-1). \end{aligned} \tag{2.38}$$

This is the same solution as the first integral on the right side of the (2.37), in fact it is

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{l^2-M^2} = i \left(\frac{M}{4\pi}\right)^{\frac{d}{2}} \Gamma(\epsilon-1), \tag{2.39}$$

the general formula to solve this integral is reported in Appendix A.

These results show that the dimensional regularization take into account that integrals like the one in (2.35) are null. This is an intrinsic feature of this regularization scheme. The generalization to any other power of the integrated momentum it is not straightforward that we will not report it here, a detailed explanation can be found in [16].

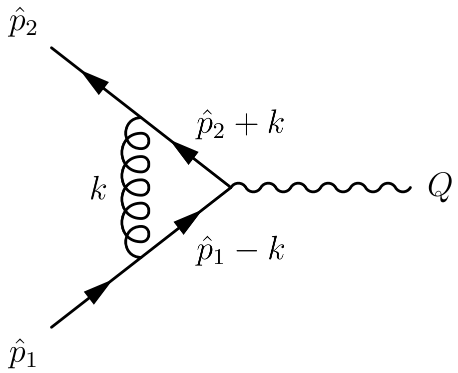


Figure 2.3: Correction due to the exchange of a virtual gluon.

2.4 Virtual contribution

This diagram is already of order α_s indeed its contribution enter by interference with the Born level diagram weighted by a factor 2 for symmetry. The loop momentum is set as in Figure 2.3. The contribution from the virtual process is;

$$\begin{aligned}\sum_{pol} |\mathcal{M}(\hat{p}_1, \hat{p}_2; Q)|^2 = \sum_{pol} \; & 2\Big[\bar{v}(\hat{p}_2)(-ig_s t^A \gamma^\rho) \int \frac{d^4k}{(2\pi)^4} \frac{-i(\hat{\not{p}}_2 + \not{k})}{(\hat{p}_2 + k)^2 + i\eta} (-iee_q\gamma^\mu) \\ & \frac{i(\hat{\not{p}}_1 - \not{k})}{(\hat{p}_1 - k)^2 + i\eta} \frac{-i}{k^2 + i\eta} (-ig_s t^A \gamma_\rho) u(\hat{p}_1)\Big] \Big[\bar{u}(\hat{p}_1)(iee_q\gamma^\nu)v(\hat{p}_2)\Big](-g_{\mu\nu}) \\ = \; & -\frac{1}{6}(ee_q)^2 \; \mathrm{Tr}[\hat{\not{p}}_2 \Gamma^\mu(\hat{p}_1, \hat{p}_2)\hat{\not{p}}_1 \gamma_\mu], \end{aligned} \tag{2.40}$$

where $\Gamma^\mu(\hat{p}_1, \hat{p}_2)$ is

$$\Gamma^\mu(\hat{p}_1, \hat{p}_2) = ig_s^2 C_F \int \frac{d^4k}{(2\pi)^4} \gamma^\rho \frac{(\hat{\not{p}}_2 + \not{k})}{(\hat{p}_2 + k)^2 + i\eta} \frac{\gamma^\mu}{k^2 + i\eta} \frac{(\hat{\not{p}}_1 - \not{k})}{(\hat{p}_1 - k)^2 + i\eta} \gamma_\rho. \tag{2.41}$$

The factor 1/6 in front of this expression, we are taking into account the sum over the colors of the two quarks. Differently from the Born cross section, where the only electroweak vertex conserves the color charge, we have also two qqg (quark-quark-gluon) vertex that could not conserve the color, in fact, the qqg vertex have also a Gell-Mann matrix inside (the infinitesimal generators of the SU(3) group) instead of a colour delta; but if two qqg vertex are linked by the same gluon propagator the gluonic charge is conserved then we have

$$t^A_{ab} \; \delta^{AB} \; t^B_{bc} = \mathrm{Tr}[t^A_{ab} t^A_{bc}] = C_F \delta_{ac} = \frac{4}{3}\delta_{ac}, \tag{2.42}$$

then the sum over the color states become the same as the Born cross section where the delta in the last expression preserve the quark color.

In this process the gauge invariance for the electromagnetic interaction is trivially respected because it is respected for the Born level diagram; the sum over the photon polarization is reduced to $-g_{\mu\nu}$;

$$Q_\nu \overline{\mathcal{M}}^\nu_{real} = 0 \tag{2.43}$$

$$\mathcal{M}^\mu_{virtual}\left(-g_{\mu\nu} + \frac{Q_\mu Q_\nu}{Q^2}\right)\overline{\mathcal{M}}^\nu_{real} = \mathcal{M}^\mu_{virtual}(-g_{\mu\nu})\overline{\mathcal{M}}^\nu_{real}. \tag{2.44}$$

The propagator for the virtual gluon has been chosen in the Feynman gauge to simplify the calculation; in any case, other choices of the gauge do not change the result. In fact for a generic gauge the gluon propagator reads,

$$D^{\mu\nu}(k) = \left[-g^{\mu\nu} + (1+\xi)\frac{k^\mu k^\nu}{k^2 + i\eta}\right]\frac{i}{k^2 + i\eta}. \tag{2.45}$$

The second term in the propagator gives a contribution to the corrected vertex proportional to

$$\Gamma^\mu \propto \gamma^\mu \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{k^4}. \tag{2.46}$$

This integral gives a null contribution in the dimensional regularization scheme, that will be later used to regularize the divergencies of the integral, as explained in the previous section. This can be seen as a proof that the dimensional regularization does not break the gauge invariance, and thus it is a very powerful regularization scheme.

Now our purpose is to evaluate the corrected vertex $\Gamma^\mu(\hat{p}_1, \hat{p}_2)$. The first thing to note is that the final result will be proportional to the Born cross section because the massless quark and the electromagnetic gauge invariance constrain the corrected vertex to be proportional to γ^μ, in fact we can see from the numerator of the integrand in (2.40) that

$$\begin{aligned}\gamma^\rho(\hat{\not{p}}_2 + \not{k})\gamma^\mu(\hat{\not{p}}_1 - \not{k})\gamma_\rho &= -2(\hat{\not{p}}_1 - \not{k})\gamma^\mu(\hat{\not{p}}_2 + \not{k}) - 2\epsilon\not{k}\gamma^\mu\not{k} \\ &= 4(1-\epsilon)k^\mu\not{k} + 2(\hat{s} + 2(\hat{p}_1 - \hat{p}_2)k - (1-\epsilon)k^2)\gamma^\mu,\end{aligned} \tag{2.47}$$

where just the first term is not proportional to γ^μ, but we will show using the Feynman parameters that, for symmetry, also that term is proportional to the gamma matrices.

In the equation (2.47) we have omitted the terms proportional to $\hat{\not{p}}_1$, $\hat{\not{p}}_2$, p_1^μ and p_2^μ because they give null contribution to the final result cause the zero mass of the particles, in fact massless particles have $\hat{\not{p}}\hat{\not{p}} = 0$. The calculation has been done in dimensional regularization, extending the dimensions from 4 to $d = 4 - 2\epsilon$ with $\epsilon \in \mathbb{C}$. For a complete list of gamma matrices property in dimensional regularization see Appendix A.

A regularization scheme is necessary because the integral is obviously divergent. Thus once regularized we can give a well defined finite part to it that in order to get physical information and to write the singular terms component in a more convenient way such that they do not affect the physical meaning of the solution.

To perform the integration (2.41) we write the integrand's denominator in a more efficient way using the Feynman parameters. We omit the term $i\eta$ during the calculation to simplify the notation;

$$\begin{aligned}\frac{1}{k^2(k^2-2\hat{p}_1k)(k^2+2\hat{p}_2k)} &= 2\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}z\frac{\delta(1-x-y-z)}{\big(zk^2+y(k^2+2\hat{p}_2k)+x(k^2-2\hat{p}_1k)\big)^3}\\ &= 2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{1}{\big((k+y\hat{p}_2-x\hat{p}_1)^2+xy\hat{s}\big)^3}\end{aligned}$$

$$l=k+y\hat{p}_2-x\hat{p}_1,\quad \Delta=-xy\hat{s}\quad\Rightarrow\quad 2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{1}{(l^2-\Delta)^3}. \tag{2.48}$$

Now we can write the Eq. (2.41) in terms of the Feynman parameters and the shifted momentum l^μ in the dimensional regularization scheme;

$$\Gamma^\mu(\hat{p}_1,\hat{p}_2)=4i\frac{g_s^2C_F}{\mu^{d-4}}\gamma^\mu\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{F(l)}{(l^2-\Delta)^3}, \tag{2.49}$$

where

$$F(l)=(1-\epsilon)\left(\frac{2}{d}-1\right)l^2+\hat{s}(1-y-x+xy(1-\epsilon)). \tag{2.50}$$

To get the function $F(l)$ in this form we used the relation $l^\mu l^\nu\sim g^{\mu\nu}l^2/d$. This relation is true until the integration is performed over the whole Minkowski space, in fact, the denominator of the integrand is an even function while the numerator is an odd one for the terms outside the trace of $l^\mu l^\nu$; thus we can say, in sense of integration, the two expressions are equivalent.

From this change of variable one can note that the corrected vertex is proportional to γ^μ as required by the zero mass of the quark. To perform the integral over the momentum we need to pass to the euclidean space performing the Wick rotation, $l_E^o=-il^0$, then

$$-i\int\frac{\mathrm{d}^dl_E}{(2\pi)^d}\,\frac{\frac{(1-\epsilon)^2}{2-\epsilon}l_E^2-\hat{s}(1-y-x+xy(1-\epsilon))}{(l_E^2+\Delta)^3}, \tag{2.51}$$

then we can write

$$\int\frac{\mathrm{d}^dl_E}{(2\pi)^d}\frac{A\;l_E^2}{(l_E^2+\Delta)^3}+\int\frac{\mathrm{d}^dl_E}{(2\pi)^d}\frac{B}{(l_E^2+\Delta)^3}, \tag{2.52}$$

with

$$A=-\frac{(1-\epsilon)^2}{2-\epsilon},\qquad B=\hat{s}(1-y-x+xy(1-\epsilon)). \tag{2.53}$$

The two integral can be solved using the expressions in the Appendix A;

$$\int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{A\, l_E^2}{(l_E^2+\Delta)^3} = \frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{4\pi}{\Delta}\right)^{\epsilon}\Gamma(\epsilon)\ (\epsilon-2)A \tag{2.54}$$

$$\int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{B}{(l_E^2+\Delta)^3} = \frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{4\pi}{\Delta}\right)^{\epsilon}\Gamma(\epsilon)\ \frac{\epsilon}{\Delta}B. \tag{2.55}$$

The final result of the integration over the momentum is

$$\frac{1}{2}\frac{i}{(4\pi)^2}\left(-\frac{4\pi}{xy\hat{s}}\right)^{\epsilon}\Gamma(\epsilon)\left[\frac{\epsilon}{xy}(1-y-x+xy(1-\epsilon))-(1-\epsilon)^2\right]. \tag{2.56}$$

The corrected vertex then reads;

$$\begin{aligned}\Gamma^{\mu}(\hat{p}_1,\hat{p}_2) =& -\frac{\alpha_s}{2\pi}C_F\gamma^{\mu}\left(-\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\Gamma(\epsilon)\times\\ &\times\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\ [\epsilon\ (x^{-1-\epsilon}y^{-1-\epsilon}-x^{-1-\epsilon}y^{-\epsilon}-x^{-\epsilon}y^{-1-\epsilon}+x^{-\epsilon}y^{-\epsilon}(1-\epsilon))-\\ &-(1-\epsilon)^2x^{-\epsilon}y^{-\epsilon}].\end{aligned} \tag{2.57}$$

To perform the integration over the Feynman parameters is convenient to use the Euler beta function defined as follow,

$$\beta(\alpha_1,\alpha_2) = \int_0^1 dx\ x^{\alpha_1-1}(1-x)^{\alpha_2-1} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}. \tag{2.58}$$

The integral over the Feynman parameters can be generalized as,

$$\int_0^1 \mathrm{d}x\ x^{\alpha}\int_0^{1-x}\mathrm{d}y\ y^{\beta} = \frac{1}{\beta+1}\int_0^1 \mathrm{d}x\ x^{\alpha}(1-x)^{\beta+1} = \frac{1}{\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+3)}. \tag{2.59}$$

The dependence in the $\Gamma^{\mu}(\hat{p}_1,\hat{p}_2)$ on the quark momentum enters only through the squared energy of the partonic process, thus from now it is convenient to write it as an only function of it, i.e. $\Gamma^{\mu}(\hat{s})$.
Using the expression (2.59) we can finally evaluate the Eq.(2.57);

$$\begin{aligned}\Gamma^{\mu}(\hat{s}) =& -\frac{\alpha_s}{2\pi}C_F\gamma^{\mu}\left(-\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\Gamma(\epsilon)\Big[-\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}+\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}-\\ &-\frac{\epsilon}{1-\epsilon}\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}-(1-2\epsilon)\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\Big]\\ =& -\frac{\alpha_s}{4\pi}C_F\gamma^{\mu}\left(-\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8\right).\end{aligned} \tag{2.60}$$

Where we used the Laurent expansion of the gamma function

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \frac{\epsilon}{12}(6\gamma^2 + \pi^2) + \mathcal{O}(\epsilon^2) \tag{2.61}$$
$$\Gamma(-\epsilon) = -\frac{1}{\epsilon} - \gamma - \frac{\epsilon}{12}(6\gamma^2 + \pi^2) + \mathcal{O}(\epsilon^2), \tag{2.62}$$

where γ is the Eulero-Mascheroni constant $\gamma = 0.5772\ldots$, and the factorial gamma function property

$$\Gamma(1-\epsilon) = -\epsilon\,\Gamma(-\epsilon). \tag{2.63}$$

Then in the equation (2.60) we can simplify the following term using the above expansions,

$$\begin{aligned}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} =& \frac{1}{\Gamma(1-\epsilon)}\big(1+\mathcal{O}(\epsilon^3)\big)\big(1+\frac{\pi^2}{6}\epsilon^2+\mathcal{O}(\epsilon^3)\big)\\ =& \frac{1}{\Gamma(1-\epsilon)}\big(1+\frac{\pi^2}{6}\epsilon^2+\mathcal{O}(\epsilon^3)\big),\end{aligned} \tag{2.64}$$

and expand the term $(-1)^\epsilon$;

$$(-1)^\epsilon = 1 + i\pi\epsilon - \frac{\pi^2}{2}\epsilon^2, \tag{2.65}$$

but since we have only the real part of this contribution we can forget about the term $i\pi\epsilon$, and get

$$\begin{aligned}\Gamma^\mu(\hat{s}) =& -\frac{\alpha_s}{4\pi}C_F\gamma^\mu\left(\frac{4\pi\mu^2}{\hat{s}}\right)^\epsilon\frac{1}{\Gamma(1-\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8-\pi^2\right).\\ =& -\frac{\alpha_s}{4\pi}\,V(\hat{s};\epsilon)\gamma^\mu,\end{aligned} \tag{2.66}$$

where,

$$V(\hat{s};\epsilon) = \left(\frac{4\pi\mu^2}{\hat{s}}\right)^\epsilon\frac{C_F}{\Gamma(1-\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8-\pi^2\right). \tag{2.67}$$

Inserting Eq. (2.66) in Eq. (2.40) we have the final form of the square matrix element of this diagram;

$$\begin{aligned}\sum_{pol}|\mathcal{M}|^2 =& \frac{\alpha_s}{2\pi}\,V(\hat{s};\epsilon)\left(\frac{(ee_q)^2}{12}\,\mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma_\mu]\right)\\ =& -\frac{\alpha_s}{2\pi}\,V(\hat{s};\epsilon)\mathcal{M}_0^2,\end{aligned} \tag{2.68}$$

where

$$\mathcal{M}_0^2 = \frac{1}{3}(ee_q)^2(1-\epsilon)\hat{s}, \tag{2.69}$$

that is the Born level square matrix element dimensionally regularized, in fact it has the same form of the Eq. (2.15). At the Born level we do not take into account the term $(1-\epsilon)$ because there are no poles; thus in the limit $\epsilon \to 0$ that term approach unity.

The total partonic cross section of the virtual diagram is proportional to the Born cross section and it reads;

$$\hat{\sigma}^V = -\sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}V(\hat{s};\epsilon). \tag{2.70}$$

This contribution does not add any new information to the differential cross section for neither the partonic level nor the hadronic one. The total partonic cross section can be wrote by adding this contribution to the Born level cross section;

$$\hat{\sigma} = \sigma_0(\hat{s})\frac{e_q^2}{3}\left(1 - \frac{\alpha_s}{2\pi}V(\hat{s};\epsilon)(1-\epsilon) + \mathcal{O}(\alpha_s^2)\right) = \sigma_0(\hat{s})\frac{e_q^2}{3}\sum_{i=0}^{\infty}\left(\frac{\alpha_s}{2\pi}\right)^i \hat{\sigma}_i, \tag{2.71}$$

where we identify,

$$\hat{\sigma}_1 = -V(\hat{s};\epsilon)(1-\epsilon) \tag{2.72}$$

and the phase space is made by Dirac deltas. The above expression it is not measurable because the term $V(\hat{s};\epsilon)$ is divergent in the infrared region but the Kinoshita-Lee-Nauenberg (KLN) theorem stated that any observables must be infrared finite, in fact, this divergence will be removed by taking into account also the possibility that an initial state quark emits a gluon before the collision. We close this section with the expression for the differentials partonic cross section;

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}M^2} = \sigma_0(M^2)\frac{e_q^2}{3}\left(1 + \frac{\alpha_s}{2\pi}\hat{\sigma}_1 + \mathcal{O}(\alpha_s^2)\right)\ \delta(\hat{s}-M^2), \tag{2.73}$$

$$\frac{\mathrm{d}^2\hat{\sigma}}{\mathrm{d}M^2\mathrm{d}y} = \sigma_0(M^2)\frac{e_q^2}{3}\left(1 + \frac{\alpha_s}{2\pi}\hat{\sigma}_1 + \mathcal{O}(\alpha_s^2)\right)\ \delta(\hat{s}-M^2)\ \delta(y-Y). \tag{2.74}$$

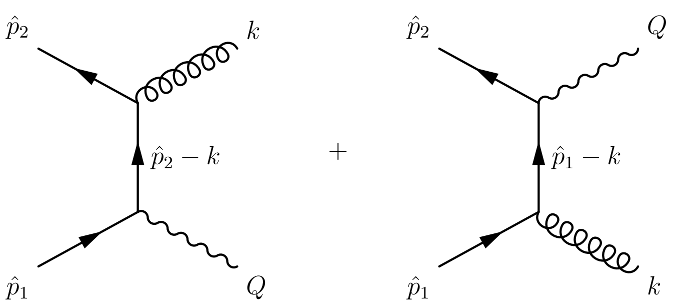


Figure 2.4: Real gluon emission from initial state quarks

2.5 Real emission

We need to take into account this process for two reason; first, its squared matrix element is of order $\mathcal{O}(\alpha_s)$; second, we cannot know if the quark has radiates gluons before the annihilation and since we are inclusive all the possible hadronic final state we must take it in consideration.

We start by writing the matrix element for these process;

$$\begin{aligned}\mathcal{M}(\hat{p}_1, \hat{p}_2, k; Q) &= \bar{v}(\hat{p}_2)\Big[(-ig_s t^A \gamma^\rho)\frac{-i(\hat{\not{p}}_2 - \not{k})}{(\hat{p}_2 - k)^2 + i\eta}(-iee_q\gamma^\mu) + \\ &\quad + (-iee_q\gamma^\mu)\frac{i(\hat{\not{p}}_1 - \not{k})}{(\hat{p}_1 - k)^2 + i\eta}(-ig_s t^A \gamma^\rho)\Big]u(\hat{p}_1)\ \varepsilon^*_\rho(k)\varepsilon^*_\mu(Q) \\ &= -iee_q g_s t^A \left[\mathcal{M}_1^{\mu\rho} - \mathcal{M}_2^{\mu\rho}\right]\ \varepsilon^*_\rho(k)\varepsilon^*_\mu(Q). \end{aligned} \tag{2.75}$$

At this order quantum chromodynamics have a behavior similar to quantum electrodynamics because there are not contributions from the self-interaction of the gluons; this means that the Ward identity for the external gluon is trivially valid and the sums over the gluon polarization reduce to the electrodynamic case;

$$\sum_{pol} \varepsilon_\rho(k)\varepsilon^*_\sigma(k) = -g_{\rho\sigma}. \tag{2.76}$$

It is important to specify that the Ward identity is valid only for the sum of the two amplitudes, $\mathcal{M}_1^{\mu\rho}$ and $\mathcal{M}_2^{\mu\rho}$, but and not for the two amplitudes individually. At this order it is easy to show the validity of the Ward identity both for the external photon and for the external gluon. The external photon carries a momentum $Q = \hat{p}_1 + \hat{p}_2 - k$, thus the identity can be proved as follows;

$$\begin{aligned}Q_\mu(\mathcal{M}_1^{\mu\rho} - \mathcal{M}_2^{\mu\rho}) &= \bar{v}(\hat{p}_2)\left[\gamma^\rho\frac{(\hat{\not{p}}_2 - \not{k})}{(\hat{p}_2 - k)^2}\not{Q} - \not{Q}\frac{(\hat{\not{p}}_1 - \not{k})}{(\hat{p}_1 - k)^2}\gamma^\rho\right]u(\hat{p}_1) \\ &= \bar{v}(\hat{p}_2)\left[\gamma^\rho\frac{(\hat{\not{p}}_2 - \not{k})}{(\hat{p}_2 - k)^2}(\hat{\not{p}}_2 - \not{k}) - (\hat{\not{p}}_1 - \not{k})\frac{(\hat{\not{p}}_1 - \not{k})}{(\hat{p}_1 - k)^2}\gamma^\rho\right]u(\hat{p}_1), \\ &= \bar{v}(\hat{p}_2)\left[\gamma^\rho\frac{(\hat{p}_2 - k)^2}{(\hat{p}_2 - k)^2} - \frac{(\hat{p}_1 - k)^2}{(\hat{p}_1 - k)^2}\gamma^\rho\right]u(\hat{p}_1) \\ &= 0 \end{aligned} \tag{2.77}$$

where we have used the equations of motion $\hat{\not{p}}_2\bar{v}(\hat{p}_2) = 0$ and $\hat{\not{p}}_1 u(\hat{p}_1) = 0$ and the gamma matrix properties $\not{A}\not{A} = A^2$. Thus the gauge invariance is respected by the emitted virtual photon. We now show that it is also true for the radiated gluon:

$$
\begin{aligned}
k_\rho(\mathcal{M}_1^{\mu\rho} - \mathcal{M}_2^{\mu\rho}) &= \bar{v}(\hat{p}_2)\left[\not{k}\frac{(\hat{\not{p}}_2 - \not{k})}{(\hat{p}_2 - k)^2}\gamma^\mu - \gamma^\mu\frac{(\hat{\not{p}}_1 - \not{k})}{(\hat{p}_1 - k)^2}\not{k}\right]u(\hat{p}_1) \\
&= \bar{v}(\hat{p}_2)\left[\frac{\not{k}\hat{\not{p}}_2}{(\hat{p}_2 - k)^2}\gamma^\mu - \gamma^\mu\frac{\hat{\not{p}}_1\not{k}}{(\hat{p}_1 - k)^2}\right]u(\hat{p}_1) \\
&= \bar{v}(\hat{p}_2)\left[-\frac{2k\hat{p}_2}{2k\hat{p}_2}\gamma^\mu + \gamma^\mu\frac{2k\hat{p}_1}{2k\hat{p}_1}\right]u(\hat{p}_1) \\
&= 0,
\end{aligned}
\tag{2.78}
$$

We still used the equation of motion and the zero mass of the on-shell gluon, i.e. $\not{k}\not{k} = 0$. The identity for the external gluon is then proved.

We can now proceed with the evaluation of the squared matrix element performing the sum over the final bosons polarization and mediating over the initial fermion polarization;

$$
\begin{aligned}
\sum_{pol}|\mathcal{M}(\hat{p}_1, \hat{p}_2, k; Q)|^2 =& \frac{(ee_q)^2 g_s^2}{12}C_F\left[\mathcal{M}_1^{\mu\rho}\mathcal{M}_{1\mu\rho}^* + \mathcal{M}_2^{\mu\rho}\mathcal{M}_{2\mu\rho}^* - 2\mathrm{Re}(\mathcal{M}_1^{\mu\rho}\mathcal{M}_{2\mu\rho}^*)\right] \\
=& \frac{(ee_q)^2 g_s^2}{12}C_F\left[M_1 + M_2 - M_3\right].
\end{aligned}
\tag{2.79}
$$

We can now calculate the M_1 and M_3 terms. We omit the calculation of the M_2 term because it is the same for the M_1 terms, and from the definition of $\mathcal{M}_1^{\mu\rho}$ and $\mathcal{M}_2^{\mu\rho}$ we can see that we can get the the $\mathcal{M}_2^{\mu\rho}$ form by the exchange $\hat{p}_1 \leftrightarrow \hat{p}_2$.

The term M_1 reads,

$$
\begin{aligned}
M_1 =& \sum_{pol}\left[\bar{v}(\hat{p}_2)\gamma^\rho\frac{\hat{\not{p}}_2 - \not{k}}{(\hat{p}_2 - k)^2 + i\eta}\gamma^\mu u(\hat{p}_1)\right]\left[\bar{u}(\hat{p}_1)\gamma_\mu\frac{\hat{\not{p}}_2 - \not{k}}{(\hat{p}_2 - k)^2 + i\eta}\gamma_\rho v(\hat{p}_2)\right] \\
=& \mathrm{Tr}\left[\hat{\not{p}}_2\gamma_\rho\frac{\hat{\not{p}}_2 - \not{k}}{(\hat{p}_2 - k)^2 + i\eta}\gamma_\mu\hat{\not{p}}_1\gamma^\mu\frac{\hat{\not{p}}_2 - \not{k}}{(\hat{p}_2 - k)^2 + i\eta}\gamma^\rho\right].
\end{aligned}
\tag{2.80}
$$

In the calculation we neglect the term $i\eta$ and we work in the dimensional regularization scheme. First of all we have to calculate the trace;

$$
\begin{aligned}
&\mathrm{Tr}\left[\hat{\not{p}}_2\gamma_\rho(\hat{\not{p}}_2 - \not{k})\gamma_\mu\hat{\not{p}}_1\gamma^\mu(\hat{\not{p}}_2 - \not{k})\gamma^\rho\right] \\
=&4(1-\epsilon)^2\mathrm{Tr}\left[\hat{\not{p}}_2(\hat{\not{p}}_2 - \not{k})\hat{\not{p}}_1(\hat{\not{p}}_2 - \not{k})\right] \\
=&4(1-\epsilon)^2\mathrm{Tr}\left[\hat{\not{p}}_2\not{k}\hat{\not{p}}_1\not{k}\right] = 32(1-\epsilon)^2(\hat{p}_1 k)(\hat{p}_2 k).
\end{aligned}
\tag{2.81}
$$

The same result is be obtained for the M_2 term. The expression (2.81) can be express in term of cinematic invariant such as the Mandelstam variable;

$$\hat{s} = (\hat{p}_1 + \hat{p}_2)^2 = 2\hat{p}_1\hat{p}_2, \tag{2.82}$$

$$\hat{t} = (\hat{p}_2 - k)^2 = -2\hat{p}_2 k, \tag{2.83}$$

$$\hat{u} = (\hat{p}_1 - k)^2 = -2\hat{p}_1 k, \tag{2.84}$$

$$\hat{s} + \hat{t} + \hat{u} = M^2. \tag{2.85}$$

In therm of the Mandelstam variable the M_1 term reads:

$$M_1 = 8(1-\epsilon)^2 \frac{\hat{u}}{\hat{t}}. \tag{2.86}$$

While the M_2 term;

$$M_2 = 8(1-\epsilon)^2 \frac{\hat{t}}{\hat{u}}. \tag{2.87}$$

Now it is the time to evaluate the M_3 term. The procedure to evaluate the trace is very long thus we jump some trivial algebraic operations. The M_3 term became,

$$\begin{aligned}
M_3 &= 2\sum_{pol}\left[\bar{v}(\hat{p}_2)\gamma^{\rho}\frac{\hat{\not{p}}_2 - \not{k}}{(\hat{p}_2-k)^2}\gamma^{\mu}u(\hat{p}_1)\right]\left[\bar{u}(\hat{p}_1)\gamma_{\rho}\frac{\hat{\not{p}}_1 - \not{k}}{(\hat{p}_1-k)^2}\gamma_{\mu}v(\hat{p}_2)\right] \\
&= 2\ \mathrm{Tr}\left[\hat{\not{p}}_2\gamma_{\mu}\frac{\hat{\not{p}}_1 - \not{k}}{(\hat{p}_1-k)^2}\gamma_{\rho}\hat{\not{p}}_1\gamma^{\mu}\frac{\hat{\not{p}}_2 - \not{k}}{(\hat{p}_2-k)^2}\gamma^{\rho}\right] \\
&= \frac{2}{\hat{t}\hat{u}}\ \mathrm{Tr}\left[\hat{\not{p}}_2\gamma_{\mu}(\hat{\not{p}}_1 - \not{k})\gamma_{\rho}\hat{\not{p}}_1\gamma^{\mu}(\hat{\not{p}}_2 - \not{k})\gamma^{\rho}\right] \\
&= -\frac{4}{\hat{t}\hat{u}}\left(\mathrm{Tr}\left[\hat{\not{p}}_2\hat{\not{p}}_1\gamma_{\rho}(\hat{\not{p}}_1 - \not{k})(\hat{\not{p}}_2 - \not{k})\gamma^{\rho}\right] - \epsilon\mathrm{Tr}[\hat{\not{p}}_2\gamma_{\rho}(\hat{\not{p}}_1 - \not{k})\gamma_{\rho}\hat{\not{p}}_1(\hat{\not{p}}_2 - \not{k})\gamma^{\rho}]\right) \\
&= -\frac{4}{\hat{t}\hat{u}}\left(2\hat{s}\mathrm{Tr}[(\hat{\not{p}}_1 - \not{k})(\hat{\not{p}}_2 - \not{k})] - 2\epsilon(\hat{s}+\hat{t})\mathrm{Tr}[\hat{\not{p}}_1(\hat{\not{p}}_2 - \not{k})] + 2\epsilon^2\mathrm{Tr}[\hat{\not{p}}_2\not{k}\hat{\not{p}}_1\not{k}]\right) \\
&= -\frac{16}{\hat{t}\hat{u}}\Big(\hat{s}(1-\epsilon)M^2 - \hat{t}\hat{u}(1-\epsilon)\epsilon\Big),
\end{aligned} \tag{2.88}$$

where we have used the contraction of the gamma matrices in the dimensional regularization as reported in the Appendix A and in the final step the property of the Mandelstam variables (2.85).

By adding all the three contributes we get the final expression for the squared matrix element;

$$\sum_{pol}|\mathcal{M}(\hat{p}_1,\hat{p}_2,k;Q)|^2 = \frac{2}{3}(ee_q)^2 g_s^2 C_F(1-\epsilon)\frac{\hat{s}}{\hat{t}\hat{u}}\left((1+z^2)\hat{s} - \epsilon(1-z)^2\hat{s} - 2\frac{\hat{t}\hat{u}}{\hat{s}}\right), \tag{2.89}$$

where $z = M^2/\hat{s}$, that is the fraction of the total energy taken by the lepton pair. The squared matrix element is manifestly Lorentz invariant, thus to continue the calculation we can put the system in any inertial frame we desire without affect the solution; for simplicity it can be chosen as the partonic center of momentum frame. The presence of $\hat{t}\hat{u}$ in the denominator produces collinear singularity. We can note that a this level there is no trace of infrared singularity, necessary to remove the one produced from the virtual contribution, indeed it is a property of the phase space as stated by the KLN theorem; we show its validity for this perturbative order here below. The cancellation of the collinear singularity in a similar way the infrared one cancels is not possible, in fact, to remove it, the only way is to redefine the PDFs in order to incorporate it, like for the renormalization of the coupling constant. The cancellation of infrared divergencies are very important because it can generate terms that can spoil the perturbative aspect of the series then we are forced to find a resummed form for these large terms. This is the purpose of any resummation techniques.

Now we define the cinematic of this process as the partonic center of momentum:

$$\hat{p}_1^\mu = \frac{\sqrt{\hat{s}}}{2}(1, \vec{0}, 1), \qquad \hat{p}_2^\mu = \frac{\sqrt{\hat{s}}}{2}(1, \vec{0}, -1), \qquad k^\mu = (k^0, \vec{k}_T, k^3). \tag{2.90}$$

In this frame the Mandelstam variables reads:

$$\hat{t} = -\sqrt{\hat{s}}(k^0 - k^3), \tag{2.91}$$

$$\hat{u} = -\sqrt{\hat{s}}(k^0 + k^3), \tag{2.92}$$

using this expressions we get

$$|k_T|^2 = \frac{\hat{t}\hat{u}}{\hat{s}}. \tag{2.93}$$

To calculate the cross section we need to define properly the phase space using dimensional regularization. From the relation (2.93) we can note that the square matrix element for the real emission depends only on the variables z and $|k_T|^2$ thus we can express the phase space in terms of these variables. Using the Eq. (2.4) in $d = 4 - 2\epsilon$ dimensions the phase space is

$$\mathrm{d}\Phi_S = \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{|k_T|^2}\right)^\epsilon \left(\frac{\sqrt{\hat{s}}}{\sqrt{\hat{s}(1-z)^2 - 4|k_T|^2}}\right) \frac{\mathrm{d}z}{16\pi^2} \mathrm{d}|k_T|^2. \tag{2.94}$$

As anticipated, the infrared singularity ($z \to 1$) is introduced by the phase space, in fact, the infrared limit costrains the transverse momentum to go to zero, then the singularity for $z = 1$ appear.

The differential partonic cross section for the real emission reads;

$$\begin{aligned}\frac{\mathrm{d}\hat{\sigma}^R}{\mathrm{d}z\ \mathrm{d}|k_T|^2} = \sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}C_F\frac{1}{z\hat{s}|k_T|^2}\left((1+z^2)\hat{s} - \epsilon(1-z)^2\hat{s} - 2|k_T|^2\right)\times \\ \times \frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{|k_T|^2}\right)^\epsilon \left(\frac{\sqrt{\hat{s}}}{\sqrt{\hat{s}(1-z)^2 - 4|k_T|^2}}\right).\end{aligned} \tag{2.95}$$

To write the rapidity distribution we need to link the transverse momentum and the rapidity; to this purpose we can express the energy and the z-direction momentum of the virtual photon in terms of the rapidity using the definition (2.12):

$$\begin{cases} Q^0 = m_T \cosh y \\ Q^3 = m_T \sinh y, \end{cases} \tag{2.96}$$

where $m_T^2 = M^2 + |k_T|^2$. The energy of the virtual photon as function of the variable z is;

$$\left(Q^0\right)^2 = \frac{\hat{s}}{4}(1+z)^2 \tag{2.97}$$

then the dependence of $|k_T|^2$ from the rapidity follow, and it is,

$$|k_T|^2 = \frac{\hat{s}}{4}\left[\frac{1+z^2-2z\cosh 2y}{\cosh^2 y}\right]. \tag{2.98}$$

It is possible to write the double differential cross section for the real emission in terms of the rapidity and the transferred momentum's energy fraction inserting (2.98) in (2.95), and we get,

$$\begin{aligned}\frac{\mathrm{d}\hat{\sigma}^R}{\mathrm{d}z\mathrm{d}y} = \sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}C_F\frac{1+z}{z\cosh^2 y}\left((1+z)^2(\tanh^2 y+1)-2\epsilon(1-z)^2\right)\times \\ \times\frac{1}{\Gamma(1-\epsilon)}\left(\frac{16\pi\mu^2}{\hat{s}}\right)^{\epsilon}\left(\frac{\cosh^2 y}{1+z^2-2z\cosh 2y}\right)^{1+\epsilon}.\end{aligned} \tag{2.99}$$

This distribution does not have a best mathematical form one can desire, but we need it for two reasons; the rapidity it's easy to measure in experiments and it's invariant under Lorentz boost making it a useful mathematical quantity.

From the expressions (2.98), (2.99) we can get information on the rapidity variable. The first expression give us information about the domain of the it, in fact;

$$|k_T|^2 \geq 0 \quad \Rightarrow \quad y \in \left[-\frac{1}{2}\log\frac{1}{z}\ ;\ \frac{1}{2}\log\frac{1}{z}\right], \tag{2.100}$$

while from the last term of the (2.99) we can see that the divergence of this process sits on the boundaries of the region, in fact, that term is singular in $y = \pm\log(\sqrt{z})$.

To end this section we give the expression for the energy distribution. We do not give the expression for the total cross section because we need the virtual contribution to cancel the infrared singularity.

The energy distribution is obtained integrating over the transverse momentum the (2.95):

$$\frac{\mathrm{d}\hat{\sigma}^R}{\mathrm{d}z} = -\sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{2}{z\epsilon}\frac{1+z^2}{(1-z)^{1+2\epsilon}}, \tag{2.101}$$

where we have neglected the terms of order $\mathcal{O}(\epsilon)$. We can do an additional step by introducing the so called *plus distribuiton*. This distribution help to handle divergence that arise from the limit $z\to 1$ of $(1-z)^{-1-2\epsilon}$ let them appear like poles in the complex plane of ϵ. The distribution can be introduces as follow;

$$\int_0^1 \mathrm{d}z\frac{f(z)}{(1-z)^{1+2\epsilon}} = \int_0^1 \mathrm{d}z\frac{f(z)-f(1)}{(1-z)^{1+2\epsilon}} + f(1)\int_0^1 \mathrm{d}z\frac{1}{(1-z)^{1+2\epsilon}}, \tag{2.102}$$

where $f(z)$ is a test function. The idea is to separate the divergent part from the finite one; in this way the first integral is finite and it contains every information about $f(z)$. The whole divergence is given by the second integral. The denominator of the first integral can be expand in power of ϵ to get a more handle ϵ-dependence, while the last integral can be simply solved using the beta function producing a pole $1/\epsilon$ as we expected. Thus the integrand of the equation (2.102) can be written as;

$$\begin{aligned}\int_0^1 \mathrm{d}z \frac{f(z)}{(1-z)^{1+2\epsilon}} &= -\frac{1}{2\epsilon}\int_0^1 \mathrm{d}z f(z)\delta(1-z) + \int_0^1 \mathrm{d}z f(z) \sum_{n=0}^{\infty} \frac{(-2\epsilon)^n}{n!}\left[\frac{\log(1-z)^n}{1-z}\right]_+ \\ &= \int_0^1 \mathrm{d}z f(z) \left\{-\frac{1}{2\epsilon}\delta(1-z) + \left[\frac{1}{1-z}\right]_+ - 2\epsilon\left[\frac{\log(1-z)}{1-z}\right]_+ + \mathcal{O}(\epsilon^2)\right\}.\end{aligned} \tag{2.103}$$

Inserting (2.103) in (2.101) and expanding the gamma function,

$$\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \frac{\pi^2}{6}\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{2.104}$$

we get ;

$$\begin{aligned} z\frac{\mathrm{d}\hat{\sigma}^R}{\mathrm{d}z} = \sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\times \\ \times\left\{\left(\frac{2}{\epsilon^2}-\frac{\pi^2}{3}\right)\delta(1-z) - \frac{2}{\epsilon}\frac{1+z^2}{[1-z]_+} + 4(1+z^2)\left[\frac{\log(1-z)}{1-z}\right]_+\right\}.\end{aligned} \tag{2.105}$$

2.6 Final remarks

To end this section we sum the real and virtual contribution to get the Drell-Yan cross section and its rapidity distribution. We start with the simplest case, the energy distribution. This is the simplest case because the divergencies are already in form of poles, thus the two expression, virtual and real contributions, are consistent each other. The NLO correction of the Drell-Yan partonic sub-process is:

$$\begin{aligned}\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}z} = \frac{\mathrm{d}\hat{\sigma}^R}{\mathrm{d}z} + \frac{\mathrm{d}\hat{\sigma}^V}{\mathrm{d}z} = \ & \sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\frac{1}{z}\times \\ \times\left\{\left(\frac{2}{3}\pi^2-8\right)\delta(1-z) - \frac{2}{\epsilon}\left(\frac{1+z^2}{[1-z]_+} + \frac{3}{2}\delta(1-z)\right) + 4(1+z^2)\left[\frac{\log(1-z)}{1-z}\right]_+\right\}.\end{aligned} \tag{2.106}$$

As it is aspected the quadratic pole has been removed, this means that the infrared divergence have been canceled, but there are still collinear divergencies. We can identify the quadratic pole as the infrared divergence because the soft limit imply the collinear one, then we have a pole of degrees two. The collinear divergence can be absorbed in the PDF as the factorization theorem states. This operation is similar to the procedure of the renormalization of the coupling constant, i.e. a renormalization scheme is needed. In this context is used the $\overline{\text{MS}}$ scheme because it works well with the dimensional regularization. The dimensional regularization produces extra universal constant that we link to the divergent part in order to do not get their contribution in the finite part.

Defining;

$$P_{qq}(z) = C_F \left(\frac{1+z^2}{[1-z]_+} + \frac{3}{2}\delta(1-z) \right) \tag{2.107}$$

$$\begin{aligned} D_q(z) = C_F \Bigg[& \left(\frac{2}{3}\pi^2 - 8 \right) \delta(1-z) + 4(1+z^2) \left[\frac{\log(1-z)}{1-z} \right]_+ \\ & - 2\frac{1+z^2}{1-z} \log z \Bigg], \end{aligned} \tag{2.108}$$

where $P_{qq}(z)$ is the Altarelli-Parisi splitting function. The hadronic energy distribution follows;

$$\begin{aligned} z\frac{\mathrm{d}\sigma}{\mathrm{d}z} = & \sum_{a,b} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \; f_a(x_1) f_b(x_2) \sigma_0(x_1 x_2 s) \frac{e_q^2}{3} \times \\ & \times \left\{ \delta(1-z) + \frac{\alpha_s}{2\pi} D_q(z) - \frac{\alpha_s}{\pi} P_{qq}(z) \left[\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{M^2} \right) \right] \right\}. \end{aligned} \tag{2.109}$$

Where μ^2 is taken in the $\overline{\mathrm{MS}}$ scheme, thus it is convenient to change the definition of the energy reference scale, $\mu^2 \to 4\pi\mu^2/e^{\gamma_E}$.

The first and the last terms in the curly brackets can be taken together to redefine the PDF giving to it a dependence on the scale of the process; in fact, neglecting the $\mathcal{O}(\alpha_s^2)$ terms, it is possible to rewrite the PDF as follows,

$$f_q(x_1, M^2/\mu^2) = f_q(x_1) - \frac{\alpha_s}{2\pi} \left[\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{M^2} \right) \right] P_{qq}(z) f_q(x_1) + \mathcal{O}(\alpha_s^2), \tag{2.110}$$

With this two definition the Drell-Yan energy distribution has a form similar to the one of the parton model with some little but important differences. The energy distribution in terms of the variable τ and "renormalized" PDF, reads;

$$\begin{aligned} \tau\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = & \sum_{q} \int_0^1 \frac{\mathrm{d}x_1}{x_1} \int_0^1 \frac{\mathrm{d}x_2}{x_2} \; f_q(x_1, M^2/\mu^2) f_{\bar{q}}(x_2, M^2/\mu^2) \sigma_0(x_1 x_2 s) \frac{e_q^2}{3} \times \\ & \times \left[\delta\left(1 - \frac{\tau}{x_1 x_2} \right) + \frac{\alpha_s}{2\pi} D_q\left(\frac{\tau}{x_1 x_2} \right) \right]. \end{aligned} \tag{2.111}$$

In this form the cross section is safe from divergences and can be used to predict events. Anyway the "renormalized" PDF now depends on the scale chosen to study the process, this means that the scaling property is violated. However the way QCD effects break the scaling is not dangerous in fact, QCD effects add mass dependence through logarithms of the ratio of M^2 and μ^2 but since there is no constrain on the choice of the scale μ^2 it can be chosen high enough to preserve the asymptotically freedom property of QCD. Inside the renormalized PDF there is the contribution of the long distance interaction, but more important, the renormalized PDF has the same form of the one obtained in the DIS; this prove the universality of the PDF, i.e. the independence of the PDF from the process under study, as stated from the factorization theorem. Unfortunately there is still no way to calculate it by first principle, the only way is to get it in a phenomenological way. However QCD can predict very well the short distance interaction through the perturbative approach;

$$\hat{\sigma} = \hat{\sigma}^0 + \frac{\alpha_s}{2\pi}\hat{\sigma}^1 + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^2 + \dots. \tag{2.112}$$

We can recognize the first two terms in this expression as the correction to the Born cross section in the expression (2.111);

$$\tau\frac{\mathrm{d}\hat{\sigma}^0}{\mathrm{d}\tau} = \sigma_0(x_1x_2s)\frac{e_q^2}{3}; \tag{2.113}$$

$$\tau\frac{\mathrm{d}\hat{\sigma}^1}{\mathrm{d}\tau} = \sigma_0(x_1x_2s)\frac{e_q^2}{3}D_q\left(\frac{\tau}{x_1x_2}\right). \tag{2.114}$$

The coefficients for the perturbation series of the partonic cross section are not easy to obtain cause the multitude of process due to the self-interaction of the gluons in fact, for the for the Drell-Yan rapidity distribution these are known up to the NNLO. The expression for the NNLO has been calculated in the article [17] but they are not reporter here because of their length.

To sum the real and virtual contribution of the rapidity distribution it is important that the two expression are consistent each other, i.e. the divergencies must be expressed in the same formalism to make the cancellation possible; thus it is useful to let them manifest as poles in the real contribution. The virtual contribution has already the divergencies in that form.

To write the (2.99) in a more practical form it is convenient to write the double differential cross section in terms of the variable;

$$v = \frac{1+\cos\theta}{2}, \tag{2.115}$$

that reads;

$$\begin{aligned} z\frac{\mathrm{d}^2\hat{\sigma}^R}{\mathrm{d}z\mathrm{d}v} =& \sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}C_F\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\times \\ &\times z^{\epsilon}\left[\frac{(1+z^2)-(1-z)^2\epsilon}{(1-z)^{1+2\epsilon}}v^{-1-\epsilon}(1-v)^{-1-\epsilon} - 2(1-z)\right], \end{aligned} \tag{2.116}$$

where θ is the angle between the direction of the partons and the virtual photon. The rapidity can be written as function of this variable because the third component of the virtual photon is $Q^3 = |Q| \cos\theta$, where $|Q| = \sqrt{\hat{s}}(1-z)/2$. The rapidity reads;

$$y = \frac{1}{2} \log t, \qquad \text{where} \qquad t = \frac{z + (1-z)v}{1-(1-z)v}. \tag{2.117}$$

To simplify the notation it is convenient to use the variable t instead of y.

We can now perform the integration of the equation (2.116) over the v variable adding the constrain of the rapidity to get the final expression in terms of it. The equation (2.116) became;

$$z\frac{\mathrm{d}^2\hat{\sigma}^R}{\mathrm{d}z\mathrm{d}t} = \sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}C_F\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon} \times$$
$$\times \int_0^1 \mathrm{d}v\, \delta\left(t - \frac{z+(1-z)v}{1-(1-z)v}\right) z^{\epsilon}\left[\frac{(1+z^2)-(1-z)^2\epsilon}{(1-z)^{1+2\epsilon}}v^{-1-\epsilon}(1-v)^{-1-\epsilon} - 2(1-z)\right], \tag{2.118}$$

The term $v^{-1-\epsilon}(1-v)^{-1-\epsilon}$ can be expressed using the *plus distribution* as follows;

$$\begin{aligned}\frac{1}{v^{1+\epsilon}(1-v)^{1+\epsilon}} =& \frac{1}{v^{1+\epsilon}(1-v)^{\epsilon}} + \frac{1}{v^{\epsilon}(1-v)^{1+\epsilon}} \\ =& -\frac{1}{\epsilon}\delta(v) + \frac{1}{[v]_+} - \epsilon\left(\left[\frac{\log v}{v}\right]_+ + \frac{\log(1-v)}{v}\right) + (v \leftrightarrow (1-v)).\end{aligned} \tag{2.119}$$

When the rapidity constrain is multiplied by the $\delta(1-z)$ we get a delta independent of v, $\delta(t-z)$; this happen specially for the terms:

$$\left[\frac{\log v}{v}\right]_+ + \left[\frac{\log(1-v)}{1-v}\right]_+ + \frac{1}{\epsilon}\left(\frac{1}{[v]_+} + \frac{1}{[1-v]_+}\right), \tag{2.120}$$

that gives zero contribution, and for the terms

$$\frac{\log(1-v)}{v} + \frac{\log v}{1-v}, \tag{2.121}$$

that integrated gives $-\pi^2/3$.

Finally we get the real part of the rapidity distribution with the divergences expressed as poles of ϵ,

$$\begin{aligned}z\frac{\mathrm{d}^2\hat{\sigma}^R}{\mathrm{d}z\mathrm{d}t} =& \sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\frac{\alpha_s}{2\pi}C_F\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\left[-2\frac{1+z}{(1+t)^2} + \right. \\ &+\left(\delta(t-z)+\delta\left(t-\frac{1}{z}\right)\right)\left(\delta(1-z)\left(\frac{1}{\epsilon^2}+\frac{\pi^2}{3}\right) - \frac{1}{\epsilon}\frac{1+z^2}{[1-z]_+} - \frac{1+z^2}{1-z}\log z + \right. \\ &\left.\left. +2(1+z^2)\left[\frac{\log(1-z)}{1-z}\right]_+ + 1 - z\right) + \frac{1+z}{(1+t)^2}\frac{1+z^2}{[1-z]_+}\left(\left[\frac{1+t}{t-z}\right]_+ + \left[\frac{1+t}{zt-1}\right]_+\right)\right].\end{aligned} \tag{2.122}$$

It is possible now to write down the total double differential partonic cross section adding to the (2.122) the virtual and the Born terms (2.74),

$$z\frac{\mathrm{d}^2\hat{\sigma}}{\mathrm{d}z\mathrm{d}t} = \sigma_0(\hat{s})(1-\epsilon)\frac{e_q^2}{3}\Bigg\{\delta(1-z)\delta(1-t) + \frac{\alpha_s}{2\pi}D_q(z,t) \\ - \frac{\alpha_s}{2\pi}P_{qq}(z)\left[\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{M^2}\right)\right]\left(\delta(t-z) + \delta\left(t-\frac{1}{t}\right)\right),\Bigg\} \tag{2.123}$$

where and μ^2 it is expressed in the $\overline{\mathrm{MS}}$ scheme while $D_q(z,t)$ is:

$$D_q(z,t) = C_F\left(\delta(t-z) + \delta\left(t-\frac{1}{z}\right)\right)\left(\delta(1-z)\left(\frac{\pi^2}{3}-4\right) + 2(1+z^2)\left[\frac{\log(1-z)}{1-z}\right]_+ - \\ - \frac{1+z^2}{1-z}\log z + 1 - z\right) + \frac{1+z}{(1+t)^2}\frac{1+z^2}{[1-z]_+}\left(\left[\frac{1+t}{t-z}\right]_+ + \left[\frac{1+t}{zt-1}\right]_+\right) - \\ - 2\frac{1+z}{(1+t)^2}. \tag{2.124}$$

The divergent part has the same form of the more inclusive case exposed above. i.e. the mass invariant distribution; thus, for the factorization theorem, also in case it can be absorbed in the PDF to get a finite result. In this way the rapidity distribution for the finite hadronic cross section can be integrated over the variables z and v instead of x_1 and x_2 performing a simple change of variables and expressing the double differential hadronic cross section as function of the hadronic variables $\tau = M^2/s$ and T, the logarithm of the hadronic rapidity, that are:

$$T = \frac{x_1}{x_2}t\ , \quad \tau = zx_1x_2 \quad \Rightarrow \quad \begin{cases} x_1 = \sqrt{\frac{\tau T}{zt}} \\ x_2 = \sqrt{\frac{\tau t}{zT}} \end{cases} \tag{2.125}$$

where T is the hadronic rapidity. Than the double differential hadronic cross section reads;

$$2\tau T\frac{\mathrm{d}^2\sigma}{\mathrm{d}\tau\mathrm{d}T} = \sum_q\int_\tau^1\frac{\mathrm{d}z}{z}\int_z^{1/z}\frac{\mathrm{d}t}{t}\ f_q(x_1,M^2/\mu^2)f_{\bar{q}}(x_2,M^2/\mu^2)\ \sigma_0\left(\frac{\tau}{z}s\right)\frac{e_q^2}{3}\times \\ \left(\delta(1-z)\delta(1-t) + \frac{\alpha_s}{2\pi}D_q(z,t)\right). \tag{2.126}$$

Also, in this case, it has been possible to put the divergences in the PDF to get a finite result, and the finite perturbative correction modifies the leading order parton model cross section.

We did not consider every possible diagram at this order in fact, the partons from the hadrons can not only be a pair of quark-antiquark states but they can also be gluonic states; then partonic process like $qg \to \gamma^* g$ are possible to happen and must be taken into account. The reason why we did not expose its calculation is because that processes at this order do not present any infrared singularity, they only produce collinear singularity that are innocuous since we are free to redefine the PDFs. The infrared singularity is not so innocuous because the cancellation between the real and virtual contribution left their trace in a finite contribution expressed terms of logarithms of the radiated gluons energy. This terms, when we approach to the threshold region ($z \sim 1$), i.e. when the energy of the radiated gluons goes to zero, can became sufficiently large to spoil the perturbative aspect of the series making the perturbative approach unreliable. For this reason we have focused out attention on the calculation of the quark-antiquark annihilation to understand how these terms arise. In the expression (2.124) we can identify this term by $[\log(1-z)/(1-z)]_+$.

It is interesting to note that in processes that involved initial gluonic state there are not only the absence of soft singularities but also this kind of processes at this order do not present virtual correction, unlike the annihilation of a quark-antiquark pair. The virtual contribution is needed to cancel the soft singularities if these are present; if they are not present also the virtual diagram will not be. Usually the number of bremsstrahlung process and the virtual process at a certain order compensate each other to perform a complete cancellation of the soft singularity, leaving their trace as large logarithmic terms.

We do not show the entire calculation to get the initial gluon states contribution but we just give the form of the finite part and the equivalent of the Altarelli-Parisi splitting function for the quark-antiquark process that will be added to the redefinition of the PDF in a similar way we do with P_{qq}. The finite part is:

$$\begin{aligned} D_g^{(1)}(z,t) = \delta(t-z)\left[\left(z^2+(1-z)^2\right)\left(\log\frac{(1-z)^2}{z}\right)+2z(1-z)\right]+ \\ +\frac{1+z}{(1+t)^2}\left(z^2+(1-z)^2\right)\left[\frac{1+t}{t-z}\right]_+ +\frac{(1-z)^2}{1+z}\left(t-z-2z(1+t)\right) \end{aligned} \tag{2.127}$$

$$\begin{aligned} D_g^{(2)}(z,t) = \delta\left(t-\frac{1}{z}\right)\left[\left(z^2+(1-z)^2\right)\left(\log\frac{(1-z)^2}{z}\right)+2z(1-z)\right]+ \\ +\frac{1+z}{(1+t)^2}\left(z^2+(1-z)^2\right)\left[\frac{1+t}{zt-1}\right]_+ +\frac{(1-z)^2}{1+z}\left(zt-1-2z(1+t)\right). \end{aligned} \tag{2.128}$$

In this case, we have two finite contributions because we are considering that the initial gluonic state can came both from the first hadron or from the second. In the parton model formula the two contribution must be multiplied to the correct PDF:

$$\sum_a \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 (f_g(x_1) f_a(x_2) D_g^{(1)}(z,t) + f_a(x_1) f_g(x_2) D_g^{(2)}(z,t)). \tag{2.129}$$

The gluon splitting function is:

$$P_{qg} = T_F \left[z^2 + (1-z)^2\right] \tag{2.130}$$

where $T_F = 1/2$ is got mediating over the possible initial gluon.

Chapter 3

Resummation

In the previous section has been shown how the cancellation procedure of the soft and collinear divergencies leaves terms that grow like double logarithms. Especially in the threshold region, these terms can produce a significant contribution, thus breaking the perturbative treatment of the series. This happens because the cancellation of the soft divergence, get by adding the real and virtual contribution together is unbalanced at the threshold energy regime, i.e. when the collision is near the elasticity ($z \to 1$) and the soft real emission is strongly suppressed, such that the virtual term gives space to large contribution. An all order resummation of these large logarithms is necessary to obtain accurate predictions for the experiments thus, in this section, we will review the main step to building up the resummation formula enlightening the main problem and difficulties. A lot of literature is present on this subject, see for example two milestone articles [1, 2]. The all order resummation formula has been used in many different processes and situations confirming its validity.

The terms that produce such large correction are of the form $\alpha_S^n[\log^m(1-z)/(1-z)]_+$ with $m \leq 2n-1$. It is not possible to find an explicit resummed formula of these logarithm in the physical z-space because it is not possible to factorize the phase space, while it can be naturally factorized in the Mellin moment space (or N-space). We will show that in this space the resummation is straightforward. In the Mellin space the large logarithms in z become large logarithms in N for which the threshold limit is recovered by pushing N to infinity. This limit permits us to neglect sub-leading terms simplifying the exponentiation formula and giving an easier expression with which go back in the z space. An important problem of the resummed expression is that its inverse does not exist because of the presence of the Landau singularity. To circumvent this issue, a well-defined prescription must be used when we go be to the physical space. The main prescription used is the minimal prescription (MP) proposed in the article [5] by Catani and Trentadue, that consist on setting the real value of N at which we go back to the z space between the rightmost singularity before the Landau pole and the Landau pole itself; in this way the resummed formula in the z space do not diverge and, if we truncate the series, the error done neglecting higher order terms is suppressed by a factor that is stronger than any power suppression.

3.1 Drell-Yan Resummation Formula

To build up a resummation formula for the Drell-Yan process let is start considering the partonic annihilation process with a single gluon emission;

$$q(\hat{p}_1) + \bar{q}(\hat{p}_2) \to \gamma^*(Q) + g(k), \tag{3.1}$$

in the limit of soft emission the matrix element can be obtained from equation (2.75) setting $k \to 0$, then we have:

$$\begin{aligned}\mathcal{M}^\mu(\hat{p}_1, \hat{p}_2; k, Q) &= -ig_s e e_q t^A \bar{v}(\hat{p}_2)\gamma^\mu u(\hat{p}_1)\left[\frac{\hat{p}_2^\rho}{\hat{p}_2 k} - \frac{\hat{p}_1^\rho}{\hat{p}_1 k}\right]\epsilon^*_\rho(k) \\ &= g_s t^A \mathcal{M}_0^\mu(\hat{p}_1, \hat{p}_2; Q) J^\rho(\hat{p}_1, \hat{p}_2; k)\varepsilon^*_\rho(k),\end{aligned} \tag{3.2}$$

where $\mathcal{M}_0^\mu(\hat{p}_1, \hat{p}_2; Q)$ is the Born level matrix element and $J^\rho(\hat{p}_1, \hat{p}_2; k)$ is the eikonal current. It is called eikonal because we are describing the emission of n-gluons like n copies of the single emission. We can see from the Eq. (3.2) the contribution of the soft emission perfectly factorizes the Born cross section, this feature is very important because it is true at every perturbative order, in fact, for a generic multiple soft emission with n final soft gluons we have;

$$\begin{aligned}\mathcal{M}_n^\mu(\hat{p}_1, \hat{p}_2; K, k_n, Q) &= g_s t^A \bar{v}(\hat{p}_2)\Bigg[\widetilde{\mathcal{S}}^\mu_{n-1}(\hat{p}_1, \hat{p}_2; K, Q)\frac{\hat{\not{p}}_1 - \not{k}_n}{(\hat{p}_1 - k_n)^2}\gamma^{\rho_n} - \\ &- \gamma^{\rho_n}\frac{\hat{\not{p}}_2 - \not{k}_n}{(\hat{p}_2 - k_n)^2}\widetilde{\mathcal{S}}^\mu_{n-1}(\hat{p}_1, \hat{p}_2; K, Q)\Bigg]u(\hat{p}_1)\varepsilon^*_{\rho_n}(k_n) \\ &= g_s t^A \mathcal{M}^\mu_{n-1}(\hat{p}_1, \hat{p}_2; K, Q) J^{\rho_n}(\hat{p}_1, \hat{p}_2; k_n)\varepsilon^*_{\rho_n}(k_n),\end{aligned} \tag{3.3}$$

where $\bar{v}(\hat{p}_2)\widetilde{\mathcal{S}}^\mu_{n-1}(\hat{p}_1, \hat{p}_2; K, Q)u(\hat{p}_1) = \mathcal{M}^\mu_{n-1}(\hat{p}_1, \hat{p}_2; K, Q)$ that take into account the matrix element for the emission of $n-1$ soft gluons. Then the square matrix element for the emission of n soft gluons, summed over the final gluons polarization and mediating over the initial quark color, is:

$$\begin{aligned}|\mathcal{M}_n(\hat{p}_1, \hat{p}_2; K, k_n, Q)|^2 &= \frac{4}{3}g_s^2|\mathcal{M}_{n-1}(\hat{p}_1, \hat{p}_2; K, Q)|^2 J^2(\hat{p}_1, \hat{p}_2; k_n) \\ &= \frac{4}{3}g_s^2|\mathcal{M}_0(\hat{p}_1, \hat{p}_2; Q)|^2\prod_{i=1}^n J^2(\hat{p}_1, \hat{p}_2; k_i)\end{aligned} \tag{3.4}$$

where the sum over the n-th gluon polarization has been taken in the same way as in the previous section, see Eq. (2.76), and $K = \sum_{i=1}^{n-1} k_i$ is the total momentum of the more internal soft gluons emitted. From Eq. (3.4) we can see that the probability of the emission of n soft gluons is the probability of zero gluon emission times the squared eikonal current for each gluon and since they do not change the color state of the emitting particle and are soft, the probability emission of n soft gluons from

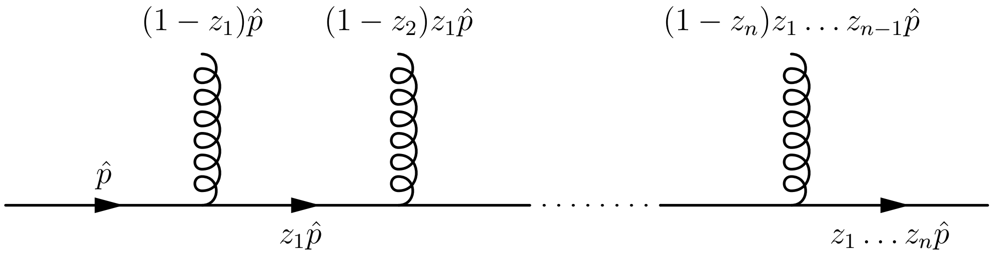


Figure 3.1

a single line can be written as

$$\sigma_0 \mathrm{d}W_n(K) = |\mathcal{M}_n(\hat{p}_1, \hat{p}_2; K, Q)|^2 \mathrm{d}\Phi_n(K) \simeq \frac{\sigma_0}{n!} \prod_{i=1}^{n} \mathrm{d}W_1(k_i), \tag{3.5}$$

where $\mathrm{d}W_1(k)$ is the one soft gluon emission differential probability, and the factor $n!$ is needed not to overcount the possible final gluons state configuration, i.e. identical boson states. Eq. (3.5) is very useful because we need to know only the single soft emission diagram, but it still presents some difficulties, in fact, the phase space constrain of n final soft gluons it is not obvious and easy to work with.

In the semi-inclusive case where we are interested only on the energy distribution then the phase space constraint can be written as [1],

$$\delta\left(Q^2 - M^2\right) = \delta\left((\hat{p}_1 + \hat{p}_2 - K)^2 - M^2\right) \overset{k_i \to 0}{\simeq} \delta(z - z_1 \dots z_n), \tag{3.6}$$

where $z = M^2/\hat{s}$ and z_i is the fraction of the parton energy after the emission of the i-th soft gluon, as shown in Figure 3.1.

A method to simplify the phase space (3.6) is to work directly in the Mellin moment space, in fact the transform of the phase space constraint is,

$$\int_0^1 \mathrm{d}z \; z^{N-1} \delta(z - z_1 \dots z_n) = z_1^{N-1} \dots z_n^{N-1}. \tag{3.7}$$

The factorization of the phase space together with the factorization of the n soft gluons emission differential probability permits us to write the partonic cross section for the emission of n soft gluons in the Mellin moment space, i.e.

$$\begin{aligned} W(N, \alpha_s) &= \int z^{N-1} \mathrm{d}W_n(K, \alpha_s) = \left[1 + \frac{1}{n!} \sum_{n=1}^{\infty} \left(\int \mathrm{d}W_1(k, \alpha_s) z^{N-1}\right)^n\right] \\ &= \exp\left\{\int \mathrm{d}W_1(k, \alpha_s) z^{N-1}\right\}. \end{aligned} \tag{3.8}$$

In this space has been possible to get a simple expression to resum the multiple soft gluons emission studying only the case of single emission. Other step must be done, in fact, $\mathrm{d}W_1(k)$ has an infrared singularity that can be removed by adding the contribution of the virtual diagram and thanks to the exponentiation and factorization of the single emission it is enough to study what happen at the NLO[1].

At the first perturbative order the virtual contribution in the soft limit is,

$$
\begin{aligned}
\mathcal{M}_V^\mu(\hat{p}_1,\hat{p}_2;Q) &= -ig_s t^A \mathcal{M}_0^\mu(\hat{p}_1,\hat{p}_2;Q)\int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{d}k^0\frac{\hat{p}_1\hat{p}_2}{k^2(\hat{p}_1k)(\hat{p}_2k)} \\
&= g_s t^A \mathcal{M}_0^\mu(\hat{p}_1,\hat{p}_2;Q)\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left[\frac{\hat{p}_1\hat{p}_2}{(\hat{p}_1k)(\hat{p}_2k)} - \frac{2k^0}{k^3|k_T|^2}\right], \qquad (3.9)
\end{aligned}
$$

where the second integrand of the last expression is a pure imaginary term that cancel when we take the square modulus of the amplitude, while the first one is exactly the opposite of the integrated squared eikonal current, that is,

$$
J^2(\hat{p}_1,\hat{p}_2;k) = -\frac{\hat{p}_1\hat{p}_2}{(\hat{p}_1k)(\hat{p}_2k)}. \qquad (3.10)
$$

Collecting the soft and virtual contributions of the cross section we get;

$$
\hat{\sigma}^{(1)}(z) = g_s^2\frac{4}{3}\sigma_0\int \mathrm{d}W_1(k)(\delta(1-z)-\delta(x-z)), \qquad (3.11)
$$

whose Mellin transform gives us,

$$
\tilde{\sigma}^{(1)}(N) = -g_s^2\frac{4}{3}\sigma_0\int \mathrm{d}W_1(k)(x^{N-1}-1), \qquad (3.12)
$$

and the one gluon soft emission differential probability is,

$$
\int \mathrm{d}W_1(k) = -\int \frac{\mathrm{d}^3k}{(2\pi)^3k^0}\frac{\hat{p}_1\hat{p}_2}{(\hat{p}_1k)(\hat{p}_2k)} = -\frac{1}{\pi}\int_0^1 \mathrm{d}z\frac{1}{1-z}\int_{\mu_F^2}^{(1-z)M^2}\frac{\mathrm{d}q^2}{q^2}, \qquad (3.13)
$$

where has been introduced an infrared cut-off (μ_F^2) for the virtuality of the parton after the emission of the gluon and we have performed the variable change with the following relations,

$$
\begin{aligned}
k_3 &= \frac{\hat{s}}{4}(1-z)\sqrt{1-2(1-z)\frac{q^2}{\hat{s}}} \simeq \frac{\hat{s}}{4}(1-z)\left(1-(1-z)\frac{q^2}{\hat{s}}\right) \qquad (3.14)\\
q^2 &= (\hat{p}_i-k)^2 \simeq \frac{k_T^2}{1-z}. \qquad (3.15)
\end{aligned}
$$

We used the soft limit such that $k_T^2/(1-z)^2 \ll 1$.

[1]The possibility to see only what happen at the first perturbative order it is not a theorem but a suggestion to understand how to resum the large contribution, in fact, at higher perturbative order are present a lot of diagrams that are not taken into account by this procedure but they are indispensable to fix the coefficients of the higher logarithmic order.

Finally a complete all order resummation in the N-space is done and we have,

$$\begin{aligned}\tilde{\sigma}(N) &= \sigma_0 \exp\left\{\alpha_s \frac{8}{3\pi}\int_0^1 \mathrm{d}z\ \frac{z^{N-1}-1}{1-z}\int_{\mu_F^2}^{(1-z)M^2}\frac{\mathrm{d}q^2}{q^2}\right\} \\ &= \sigma_0 \exp\left\{\alpha_s \frac{8}{3\pi}\int_0^1 \mathrm{d}z\ z^{N-1}\left(\left[\frac{\log(1-z)}{1-z}\right]_+ + \frac{1}{[1-z]_+}\log\frac{M^2}{\mu_F^2}\right)\right\}. \end{aligned} \tag{3.16}$$

The equation (3.16) show that the probability of soft emission produce exactly the large logarithm behavior we are looking for. It is the same we identified at the end of the previous chapter as the one that breaks the perturbative aspect of the series in the soft limit. Solving the integrals in the (3.16) we can see that large logarithms in the z space became large logarithms in the N space, in fact,

$$\int_0^1 \mathrm{d}z\ z^{N-1}\left[\frac{\log(1-z)}{1-z}\right]_+ = \frac{1}{2}\left(\psi_N^{(0)2} + 2\gamma_E\psi_N^{(0)} - \psi_N^{(1)} + \zeta(2) + \gamma_E^2\right), \tag{3.17}$$

$$\int_0^1 \mathrm{d}z\ z^{N-1}\left[\frac{1}{1-z}\right]_+ = -\left(\psi_N^{(0)} + \gamma_E\right), \tag{3.18}$$

where $\psi_N^{(n)}$ is the polygamma function, that in the large N limit became,

$$\tilde{\sigma}(N) = \sigma_0 \exp\left\{\alpha_s \frac{4}{3\pi}\left(\log^2 N + 2\left(\gamma_E - \log\frac{M^2}{\mu_F^2}\right)\log N + \zeta(2) + \gamma_E^2\right)\right\}. \tag{3.19}$$

In Ref. [18] is showed that in the large N limit the integrals like (3.17) and (3.18) can be computed in the NLL approximation by the change $z^{N-1}-1 \to -\theta(1-N_0/N-z)$ where $N_0 = e^{-\gamma_E}$, in fact we can see from the (3.17);

$$-\int_0^{1-N_0/N} \mathrm{d}z\ \frac{\log(1-z)}{1-z} = \log^2\left(\frac{N_0}{N}\right) = \log^2 N + \gamma_E^2 + 2\gamma_E \log N, \tag{3.20}$$

that the correct logarithmic forms has been produced. This method can be generalized to any logarithmic accuracy by adding a new term in the change exposed above, that is as follows,

$$z^{N-1} - 1 \to -\Gamma\left(1 - \frac{\partial}{\partial \log N}\right)\theta(1-1/N-z) + \mathcal{O}\left(\frac{1}{N}\right) \tag{3.21}$$

with,

$$\begin{aligned}\Gamma\left(1 - \frac{\partial}{\partial\log N}\right) &= 1 + \gamma_E\frac{\partial}{\partial\log N} + \frac{1}{2}\left(\gamma_E^2 + \zeta(2)\right)\left(\frac{\partial}{\partial\log N}\right)^2 \\ &\quad + \frac{1}{6}\left(\gamma_E^3 + 3\gamma_E\zeta(2) + 2\zeta(3)\right)\left(\frac{\partial}{\partial\log N}\right)^3 + \ldots, \end{aligned} \tag{3.22}$$

where each term of the series represent a logarithmic accuracy, i.e. the first term represents the LL, the second represents the NLL, the third represents the NNLL and so on. As example we can see that the expansion (3.22) produce all the terms got from the direct calculation of the (3.17), i.e.

$$-\Gamma\left(1-\frac{\partial}{\partial\log N}\right)\int_0^{1-1/N}\mathrm{d}z\,\frac{\log(1-z)}{1-z}=\frac{1}{2}\left(\log^2 N+2\gamma_E\log N+\zeta(2)+\gamma_E\right). \tag{3.23}$$

The expression (3.16) is complete for the massless QED; than the only way large logarithms can arise is through soft emission from external fermionic legs; in the case of QCD the situation is more intricate because the self interaction of the gluons that, at any order, can produce sub-leading terms, i.e. like $\alpha_s^n\log^m(1-z)/(1-z)$ with $m\leq n$. The expression (3.16) can be generalized including the running coupling effect [19] to resum also the less dominant power of $\log N$ getting the well-known resummed partonic cross section;

$$\begin{aligned}\tilde{\sigma}^{res}(N,\alpha_s)=\sigma_0\exp\Bigg\{&2\int_0^1\mathrm{d}z\,\frac{z^{N-1}-1}{1-z}\int_{\mu_F^2}^{(1-z)M^2}\frac{\mathrm{d}q^2}{q^2}A\left(\alpha_s((1-z)q^2)\right)\\&+2\int_0^1\mathrm{d}z\,\frac{z^{N-1}-1}{1-z}B\left(\alpha_s((1-z)M^2)\right)\Bigg\},\end{aligned}\tag{3.24}$$

where the function $A(\alpha_s)$ is the anomalous cusp dimension and it is an intrinsic function of the soft gluons emission, i.e. it is universal, and $B(\alpha_s)$ is a process dependent function. Both functions are expressed as power series of α_s and reads;

$$\begin{aligned}A(\alpha_s)&=\sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n A^{(n)},\\A^{(1)}&=\frac{4}{3},\quad A^{(2)}=2\left(\frac{67}{18}-\frac{\pi^2}{6}\right)-\frac{10}{9}n_f,\\A^{(3)}&=28\Bigg[9\left(\frac{245}{24}-\frac{67}{54}\pi^2+\frac{11}{6}\zeta(3)+\frac{11}{180}\pi^4\right)-\frac{4}{3}n_f\left(\frac{55}{24}-2\zeta(3)\right)\\&\quad+3n_f\left(\frac{10}{54}\pi^2-\frac{209}{108}-\frac{7}{3}\zeta(3)\right)-n_f^2\left(\frac{1}{27}\right)\Bigg],\\B(\alpha_s)&=\sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n B^{(n)}.\end{aligned}\tag{3.25}$$

The expression (3.24) can be written in a more convenient form using the beta function and the (3.22); then we can write,

$$
\begin{aligned}
\log \frac{\tilde{\sigma}^{res}(N,\alpha_s)}{\sigma_0} &= 2\Gamma\left(1-\frac{\partial}{\partial \log N}\right)\left[\int_{1/N}^{1}\frac{\mathrm{d}z}{z}\int_{\mu_F^2}^{zM^2}\frac{\mathrm{d}q^2}{q^2}A\left(\alpha_s(zq^2)\right)\right. \\
&\quad \left. + \int_{1/N}^{1}\frac{\mathrm{d}z}{z}B\left(\alpha_s(zM^2)\right)\right] \\
&= 2\Gamma\left(1-\frac{\partial}{\partial \log N}\right)\left[\int_{\alpha_s(M^2/N)}^{\alpha_s(M^2)}\frac{\mathrm{d}\alpha'}{\alpha'\beta(\alpha')}\int_{\alpha'(\mu_F^2)}^{\alpha'}\frac{\mathrm{d}\alpha}{\alpha\beta(\alpha)}A(\alpha)+\right. \\
&\quad \left. + \int_{\alpha_s(M^2/N)}^{\alpha_s(M^2)}\frac{\mathrm{d}\alpha}{\alpha\beta(\alpha)}B(\alpha)\right] = \Phi(\lambda,\alpha_s), \qquad (3.26)
\end{aligned}
$$

where $\lambda = \alpha_s\beta_0 \log N$ and $\Phi(\lambda,\alpha_s)$ is the resummed function that can be expressed as a series of α_s in the following way,

$$
\Phi(\lambda,\alpha_s) = \log N g_1(\lambda) + \sum_{n=0}^{\infty}\alpha_s^n g_{n+2}(\lambda). \qquad (3.27)
$$

The function $g_i(\lambda)$ are known up to the fourth order. In a more conventional form the (3.26) it is written as;

$$
\tilde{\sigma}^{res}(N,\alpha_s) = \sigma_0 e^{\Phi(\lambda,\alpha_s)}. \qquad (3.28)
$$

It is important to point out that in this way we are trying to predict the form, at least in the Mellin space, of the terms that make the series a divergent one in the z space; but since we are not able to know exactly the form of each perturbative order and we are limited in the knowledge of just few of it there is somewhat freedom in the choice on which terms keep and which not. Different choice are called prescription and the most famous one is the minimal prescription, exposed in the article [1], that we are going to expose in the next section. However the resummed cross section is an approximation of the series studied via Feynman diagrams method then if we are able to know the fixed order solution of a certain process it is wise to combine knowledge of an exact expression with our ability to predict large contribution at all orders with the resummation formula subtracting from the resummed formula the exactly known perturbative order. The combination of the two contribution gives the following partonic cross section;

$$
\tilde{\sigma}^{N^kLL+N^pLO}(N,\alpha_s) = \tilde{\sigma}^{res}(N,\alpha_s) + \sum_{j=0}^{p}\left(\frac{\alpha_s}{\pi}\right)^j\tilde{\sigma}_j(N) - \sum_{j=0}^{p}\frac{\alpha_s^j}{j!}\left[\frac{\mathrm{d}^j\tilde{\sigma}(N,\alpha_s)}{\mathrm{d}\alpha_s}\right]_{\alpha_s=0}, \qquad (3.29)
$$

where the N^kLL approximation it is provided by the expansion of the function $A((1-z)q^2)$ until the α_s^k and the function $B((1-z)M^2)$ the order α_s^{k-1}.

3.2 Minimal Prescription

From the equations (3.24) we can note the first problem of the N space resummation formula, i.e. the integration of both integrals hit the Landau pole at $z = 1 - \Lambda^2/M^2$ making $\tilde{\sigma}^{res}(N, \alpha_s)$ divergent for every N, however if we take the resummed exponent in the form (3.27) we have a finite expression that is divergent up to the very large point $N_L = \exp\{1/(2\alpha_s\beta_0)\}$. To give an idea of the magnitude of N_L we have for the coupling evaluated at the Z-boson energy scale, $\alpha_s(M_Z) \simeq 0.118$ and with 5 flavor activated, $N_L \simeq 1038$.

The branch-cut present in (3.27) is due to the simplification $z^{N-1} - 1 \to -\theta(1 - z - 1/N)$ with which we have exluded the Landau pole from the integration for $N < N_L$. We must underline that using the theta function we are neglecting sub-leading terms in the large-N limit and then the finiteness of the (3.27) means that the divergences in (3.24) are due to sub-leading terms that are removed by other of the same nature who are themselves sub-leading in the large-N limit.

The branch-cut along the positive real axis of the N-plane that go from N_L to ∞ is due to the dependence on N through $\alpha_s(M^2/N)$ of the resummed exponent, as we can see from the expression (3.26), which expansion in power of $\alpha_s(M^2)$ converges only for $N < N_L$. This means that $\tilde{\sigma}^{res}(N, \alpha_s)$ has no convergence abscissa and then it can not be the Mellin transform of any function, nevertheless if we expand the $\tilde{\sigma}^{res}(N, \alpha_s)$ in power of $\alpha_s(M^2)$ we can perform the inverse transform, terms by terms, because they are simply polynomials of $\log N$. Hence the partonic cross section in the z-space can be written as,

$$\hat{\sigma}^{res}(z, \alpha_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \hat{\sigma}_n(z), \tag{3.30}$$

where,

$$\hat{\sigma}_n(z) = \int_{\bar{N}-i\infty}^{\bar{N}+i\infty} \frac{\mathrm{d}N}{2\pi i}\, z^{-N} \tilde{\sigma}_n(N), \tag{3.31}$$

and $\tilde{\sigma}_n(N)$ the coefficients of the expansion in $\alpha_s(M^2)$ of $\tilde{\sigma}^{res}(N, \alpha_s)$. At this point another problem arise, the series (3.30) do not converge because the terms (3.31) integrated with the PDF provides the hadronic cross section with a factorial growing behavior [5], it can not be possible because it must be finite. To give an example we can take the double logarithm approximation, i.e. $\Phi(\lambda) = \alpha_s \log^2 N$, then the partonic cross section in the z-space is,

$$\hat{\sigma}^{res}(z, \alpha_s) = \int \frac{\mathrm{d}N}{2\pi i} z^{-N} e^{\alpha_s \log^2 N} = -\frac{\mathrm{d}}{\mathrm{d}x} \left(\theta(1 - x - \epsilon) e^{\alpha_s \log^2(1-z)}\right) + \mathrm{NLL}, \tag{3.32}$$

a complete derivation of this formula can be found in the article [5], then the total cross section is,

$$\sigma^{res}(\tau, \alpha_s) = \int_{\tau}^{1} \mathrm{d}z\; e^{\alpha_s \log^2(1-z)} \frac{\mathrm{d}}{\mathrm{d}z} \mathcal{L}\left(\frac{\tau}{z}\right), \tag{3.33}$$

where,

$$\mathcal{L}(z) = \int_z^1 \frac{\mathrm{d}x}{x} f\left(\frac{z}{x}\right) f(x), \tag{3.34}$$

is the luminosity. The expression (3.33) is divergent for every τ because the exponential diverge faster then any power when $z \to 1$ and if we expand the total cross section in power of α_s we can see that the series has a factorial divergence, in fact, since $\mathcal{L}(z)$ is a smooth function in $z \simeq 1$ the nature of the divergence is totally encoded in the exponential, thus for sake of example we can see that the integral of the exponential only has a factorial divergence, i.e.

$$\int_0^1 \mathrm{d}z \; e^{\alpha_s \log^2(1-z)} = \sum_{n=0}^{\infty} \frac{\alpha_s^n}{n!} \int_0^1 \mathrm{d}z \; \log^{2n}(1-z) = \sum_{n=0}^{\infty} \frac{\alpha_s^n (2n)!}{n!}, \tag{3.35}$$

that is an asymptotic series that for large n grows like $(4\alpha_s)^n n!$. The only way to deal with an asymptotic series is to truncate it when the next term is of the same size of the current one. In the article [5] is shown that the same result that one get truncating the series can be achieved insterting an unphysical cut-off in the upper bound of integration.

The factorial growth has nothing to do with IR renormalon, that cause the same growth in the perturbative expansion, indeed this situation is not possible since the physical cross section must be finite. Conceptually this problem reflect that a complete resummation of the large logarithms in the z-space is impossible cause the conservation of momentum, that factorize in the Mellin space, but when we try to go back to the physical space, i.e. the z-space, we are forcing the resummed expression in the N-space to became a resummed expression in the z-space, this operation violates the conservation of the momentum infinite times giving a divergent results. To be more specific the factorial growth in the physical result arise because in the expression (3.26) we have neglected sub-leading terms, that kept, would have produced factorial growing terms necessary to cancel the the one produced by the dominant logarithms. It is then important to underline that a good resummation program does not only have to give a complete resummed form but it also have to deal with these other problem that arise when we try to go back to the physical space, and make sure that the precision of the prediction must not be so much influenced by the choice of the logarithmic accuracy.

A way to handle sub-leading terms such that the hadronic cross section is finite is exposed in the article [5], where Catani et al. show that

$$\begin{aligned} \sigma(\tau, \alpha_s) &= \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \mathrm{d}N \; \tau^{-N} f_N^2 \; \tilde{\sigma}^{res}(N, \alpha_s) \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \alpha_s^n \int_{C-i\infty}^{C+i\infty} \mathrm{d}N \; \tau^{-N} f_N^2 \; P_n(\log N), \end{aligned} \tag{3.36}$$

where $P_n(\log N)$ are the polynomials of $\log N$, is convergent if C is taken at left of the Landau pole N_L and at right of all the other singularities. They also show that in that strip of convergence no spurious factorial growing term are produced and the error of truncating the expansion at the minimum, cause it is an asymptotic expansion, is of order

$$e^{-H\frac{M(1-\tau)}{\Lambda}} \tag{3.37}$$

where H is a slowly varying positive function. It is interesting to note that when $M(1-\tau) \to \Lambda$ the argument of the exponential approaches to 1, that means that the mass of the radiation is approaching the Landau energy Λ spoiling the perturbative treatment of the problem as we would expect.

Also with this prescription it is very hard or quite impossible to fine an explicit resummed formula in the physical space, then the only way to get $\sigma(\tau, \alpha_s)$ is by numerical computation.

This prescription is widely used to implement the threshold effects in fixed order prediction for different process like, for example, the Drell-Yan process, Jet cross section, heavy flavor cross section.

Chapter 4

Resummation of the Rapidity Distribution

In the previous section we have shown how to include the contribution of the logarithmically enhanced terms that are present at any order for the semi-inclusive cross section and the related problems that arise in the conversion of the Mellin transform, which can only be performed on the basis of some prescription.

In this Chapter, we consider the resummation for the double semi-inclusive Drell-Yan cross section in the energy and rapidity variable. We will study two different way to approach the problem. The two main way are exposed in the articles [4, 20]. The main difference between the two approaches lies in the conjugated space where the large logarithms are resummed. In the first article a Mellin-Fourier transform of the partonic cross section is performed, with respect to the variables z and y, respectively the energy fraction of the virtual photon or Z boson and the rapidity of the pair in the partonic center-of-mass frame, as defined in (2.12), while in the second article a double Mellin transform in x_1 and x_2, the momentum fractions carried by the incoming partons, is performed.

4.1 Double Mellin Transform Resummation

In Ref. [3] the authors present a new way to get a general resummed formula; the method was developed for the first time in Ref. [21, 22], while in Ref. [3] it is extended to less inclusive processes. In this section we review the main steps of their way to build a resummed formula.

In the approach of Ref. [3] the partonic cross section is written in terms of the variables z_1 and z_2, defined as:

$$z_1 = \sqrt{\frac{x_1^0}{x_1}} e^Y \tag{4.1}$$

$$z_2 = \sqrt{\frac{x_2^0}{x_2}} e^{-Y}, \tag{4.2}$$

where $\tau = x_1^0 x_2^0$ and $Y = \log \sqrt{x_1^0/x_2^0}$. With these variables the threshold region corresponds to $z_i \to 1$, with $i = 1, 2$. This method is based on the use of the renormalization group invariance of certain quantities, mass factorization and the resummation concepts exposed in the previous chapter as guiding principles. An interesting point is that they work directly in $d = 4 + 2\epsilon$ dimensions, i.e. in the dimensional regularization scheme; this choice permits to work with ordinary functions, avoiding the ambiguities that can arise by the direct use of distributions, especially if with more than one variable.

The differential cross section for the Drell-Yan process has the well known form:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\tau\mathrm{d}Y}(x_1^0, x_2^0, M^2) = \hat{\sigma}^0(x_1^0, x_2^0) \sum_{a,b} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \; f_a(x_1, \mu_F^2) f_b(x_2, \mu_F^2) \times$$
$$\times \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \; \delta(x_1^0 - x_1 z_1)\delta(x_2^0 - x_2 z_2)\Delta_d(z_1, z_2, M^2, \mu_F^2, \mu_R^2), \tag{4.3}$$

where μ_F is the factorization scale. It is called *factorization scale* because it takes into account the energy separation between the short distance part and the long distance one that are factorized in these processes; while μ_R is the scale dependence that arises from the renormalization procedure. In (4.3), $\Delta_d(z_1, z_2, M^2, \mu_F^2, \mu_R^2)$ represents the partonic differential cross section; the subscript "d" is for doubly differential (as in the case of the energy-rapidity distribution).

To get an infrared safe hadronic cross section also the partonic cross section must be. In Chapter 2 it has been shown that an infrared safe result can be obtained by summing the real and virtual contributions. The partonic cross section can be written separating the contribution from the *soft* and *virtual* (SV) part as follows:

$$\Delta_d(z_1, z_2, M^2, \mu_F^2, \mu_R^2) = \Delta_d^{\mathrm{hard}}(z_1, z_2, M^2, \mu_F^2, \mu_R^2) + \Delta_d^{SV}(z_1, z_2, M^2, \mu_F^2, \mu_R^2), \tag{4.4}$$

where the hard part contains everything that it is not generated by the cancellation between the real and virtual contributions, while the SV part is the subject of the resummation procedure that we are going to discuss. In the articles [23, 24] it is shown how to get the hard part by direct calculation of the fixed orders. It is well known that the resummed formula of the large logarithmic terms has an exponential form in the conjugated space; then we ask for the the soft and virtual part of the partonic cross section to have the following form:

$$\Delta_d^{SV}(z_1, z_2, M^2, \mu_F^2, \mu_R^2) = \mathcal{C} \exp\left\{\Psi_d(z_1, z_2, M^2, \mu_F^2, \mu_R^2, \epsilon)\right\}\Big|_{\epsilon=0}, \tag{4.5}$$

where ϵ is the dimensional regularization parameter, while $\mathcal{C}$ means *convolution*, i.e.

$$\mathcal{C}e^{f(z_1,z_2)} = \delta(1 - z_1)\delta(1 - z_2) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\bigotimes_{i=1}^{n} f(z_1, z_2)\right], \tag{4.6}$$

with the outer product as the double Mellin convolution product, see Eq. (1.105), i.e. the functions convoluted with respect to the variables z_1 and z_2 separately. After the limit $\epsilon \to 0$ is taken, the function $\Psi_d(z_1, z_2, M^2, \mu_F^2, \mu_R^2, \epsilon)$ is a combination of distributions of the type $\delta(1 - z_i)$ or $[\log^n(1 - z_i)/(1 - z_i)]_+$. This choice is due to the fact that the delta distributions are always present for phase space constrains while the plus distributions of those functions must be taken into account because they are the responsible for the logarithms enhancement.

The function $\Psi_d(z_1, z_2, M^2, \mu_F^2, \mu_R^2, \epsilon)$ has four different contributions that can be written as follow

$$\begin{aligned}\Psi_d(z_1, z_2, M^2, \mu_F^2, \mu_R^2, \epsilon) = &\left[\log\left(Z(\hat{\alpha}_S, \mu^2, \mu_R^2, \epsilon)\right)^2 + \log\left(\hat{F}(\alpha_S, M^2, \mu^2, \epsilon)\right)^2\right] \times \\ &\times \delta(1-z_1)\delta(1-z_2) + 2\Phi_d(\hat{\alpha}_s, M^2, \mu^2, z_1, z_2, \epsilon) - \\ &- \mathcal{C} \log \Gamma_{qq}(\hat{\alpha}_s, \mu^2, \mu_F^2, z_1, \epsilon)\delta(1-z_2) - \\ &- \mathcal{C} \log \Gamma_{qq}(\hat{\alpha}_s, \mu^2, \mu_F^2, z_2, \epsilon)\delta(1-z_1),\end{aligned} \tag{4.7}$$

where $\hat{\alpha}_s$ is the bare coupling constant and

- $\left|\hat{F}(\alpha_S, M^2, \mu^2, \epsilon)\right|^2$ is the bare squared form factor of the the Drell-Yan process,
- $Z(\alpha_S, \mu^2, \mu_R^2, \epsilon)$ is the overall operator renormalisation constant, which, for lepton pair production, is unitary,
- $\Phi_d(\alpha_s, M^2, \mu^2, z_1, z_2, \epsilon)$ is the soft distribution function, free of collinear singularities; all the logarithmic enhancement is contained in this term,
- $\Gamma_{qq}(\alpha_s, \mu^2, \mu_F^2, z_i, \epsilon)$ is the mass factorization kernel in the $\overline{\text{MS}}$ scheme. This function factorizes the collinear singularity; the qq subscript refers to the Drell-Yan process with a quark-antiquark pair in the initial state.

All of these functions can be expanded in powers of the bare coupling constant, that in the dimensional regularization scheme, i.e. in $d = 4 + 2\epsilon$ dimensions, can be written as

$$S_\epsilon \hat{\alpha}_s = Z(\mu_R^2)\alpha_s(\mu_R^2)\left(\frac{\mu^2}{\mu_R^2}\right)^\epsilon, \tag{4.8}$$

where

$$S_\epsilon = \exp\{\epsilon(\gamma_E - \log 4\pi)\}, \tag{4.9}$$

is the so-called spherical factor.

Since the bare coupling constant is independent of the renormalization scale, we can write the renormalization group equation in d dimensions as

$$\mu_R^2 \frac{\mathrm{d}}{\mathrm{d}\mu_R^2} \log \alpha_s(\mu_R^2) = \epsilon - \frac{\beta(\alpha_s(\mu_R^2))}{\alpha_s(\mu_R^2)}. \tag{4.10}$$

This relation will be crucial in the following to capture the correct singular behavior.

What we have to do now is to understand the structure of the above functions in order to enlighten the form of the soft contribution in the $\Phi_d(\hat{\alpha}_s, M^2, \mu^2, z_1, z_2, \epsilon)$

function. Let us start with the form factor $\hat{F}(\hat{\alpha}_s, M^2, \mu^2, \epsilon)$. Since it is non-renormalized, it satisfies the following Sudakov-type integro differential equation:

$$q^2 \frac{\mathrm{d}}{\mathrm{d}q^2} \log \hat{F}(\hat{\alpha}_S, q^2, \mu^2, \epsilon) = \frac{1}{2}\left[K\left(\hat{\alpha}_s, \frac{\mu_R^2}{\mu^2}, \epsilon\right) + G\left(\hat{\alpha}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right)\right]. \tag{4.11}$$

This statement follows from gauge and renormalization group invariances, as shown in Ref. [25, 26, 27]. The function $K\left(\hat{\alpha}_s, \mu_R^2/\mu^2, \epsilon\right)$ contains all the poles in ϵ while $G\left(\hat{\alpha}_s, q^2/\mu_R^2, \mu_R^2/\mu^2, \epsilon\right)$ all the terms that that are regular as $\epsilon \to 0$. Since the bare form factor does not depend on the renormalization scale, the following differential equations hold:

$$\mu_R^2 \frac{\mathrm{d}}{\mathrm{d}\mu_R^2} K\left(\hat{\alpha}_s, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = -A\left(\alpha_s(\mu_R^2)\right) \tag{4.12}$$

$$\mu_R^2 \frac{\mathrm{d}}{\mathrm{d}\mu_R^2} G\left(\hat{\alpha}_s, \frac{M^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = A\left(\alpha_s(\mu_R^2)\right), \tag{4.13}$$

where the function $A(\alpha_s(\mu_R^2))$ is the cusp anomalous dimensions and can be expanded in power of $\alpha_s(\mu_R^2)$:

$$A(\alpha_s(\mu_R^2)) = \sum_{n=0}^{\infty} \hat{\alpha}_s^n(\mu_R^2) A_n, \tag{4.14}$$

then the equation (4.12) and (4.13) can be solved in powers of the bare coupling constant, i.e. writing

$$K\left(\hat{\alpha}_s, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = \sum_{n=1}^{\infty} \hat{\alpha}_s^n \left(\frac{\mu_R^2}{\mu^2}\right)^{n\epsilon} S_\epsilon^n K_n(\epsilon) = \sum_{n=1}^{\infty} \hat{\alpha}_s^n(\mu_R^2) Z^n(\alpha_S, \mu^2, \mu_R^2, \epsilon) K_n(\epsilon). \tag{4.15}$$

For example the first term in the (4.15) is easily found; in fact, the differential equation (4.12) at the lowest order is

$$K_1 \mu_R^2 \frac{\mathrm{d}}{\mathrm{d}\mu_R^2} \log \alpha_s(\mu_R^2) = -\alpha_s(\mu_R^2) A_1, \tag{4.16}$$

and using the renormalization group equation (4.10) at the lowest order we find

$$K_1 = -\frac{1}{\epsilon} A_1, \tag{4.17}$$

and as stated it contains a singular term in ϵ.

The function $G\left(\hat{\alpha}_s, M^2/\mu_R^2, \mu_R^2/\mu^2, \epsilon\right)$ is given by

$$G\left(\hat{\alpha}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = G(\alpha_s(q^2), 1, \epsilon) + \int_{q^2/\mu_R^2}^{1} \frac{\mathrm{d}\lambda^2}{\lambda^2} A(\alpha_s(\lambda^2 \mu_R^2)), \tag{4.18}$$

where the finite function $G(\alpha_s(q^2), 1, \epsilon)$ can be expanded in powers of $\alpha_s(q^2)$ with coefficients $G_n(\epsilon)$ that are finite function of ϵ. Collected these two results, it is now possible to integrate the differential equation (4.11) and express the form factor as a series in $\hat{\alpha}_s$,

$$\log \hat{F}(\hat{\alpha}_s, q^2, \mu^2, \epsilon) = \sum_{n=1}^{\infty} \hat{\alpha}_s \left(\frac{q^2}{\mu^2}\right)^{n\epsilon} S_\epsilon^n \hat{\mathcal{F}}^{(n)}(\epsilon), \tag{4.19}$$

where for example at the first order the coefficient $\hat{\mathcal{F}}(\epsilon)$ for the Drell-Yan process is

$$\hat{\mathcal{F}}^{(1)}(\epsilon) = \frac{1}{2}\left[\frac{A_1}{\epsilon}\left(\frac{\mu_R}{q}\right)^{2\epsilon}\left(\int \frac{\mathrm{d}\lambda}{\lambda^{1-\epsilon}} - \frac{1}{\epsilon}\right) + \frac{1}{\epsilon}G_1(\epsilon)\right] \tag{4.20}$$

The coefficients $G_n(\epsilon)$ can be found in Ref. [28]. They are expressed in terms of two other functions, the first one is indicated by B_n and are flavor independent terms, i.e. they are the same of every process; the second is indicated by f_n and it is an analogous of the cusp anomalous dimension, in fact, they are necessary to predict the single pole of the logarithm of the form factors up to two-loop level. Finally, even if not every value of the coefficient A_n, B_n, f_n are known, we are able to predict the pole structure for the form factor at every order in $\hat{\alpha}_s$.

To factorize the collinear contribution is requested to find the structure of the function $\Gamma_{qq}(\hat{\alpha}_s, \mu^2, \mu_F^2, z_i, \epsilon)$. This function satisfies a DGLAP-like equation:

$$\mu_F^2 \frac{\mathrm{d}}{\mathrm{d}\mu_F^2}\Gamma_{qq}(\hat{\alpha}_s, \mu^2, \mu_F^2, z_i, \epsilon) = \frac{1}{2}P_{qq}(z_i, \mu_F^2) \otimes \Gamma_{qq}(\hat{\alpha}_s, \mu^2, \mu_F^2, z_i, \epsilon), \tag{4.21}$$

where $P_{qq}(z_i, \mu_F^2)$ is the Altarelli-Parisi splitting function that can be expanded as a power series in $\alpha_s(\mu_F^2)$:

$$P_{qq}(z_i, \mu_F^2) = 2\sum_{n=1}^{\infty} \alpha_s^n(\mu_F^2)\left[B_{n+1}\delta(1-z_i) + A_{n+1}D_0(z_i)\right] + P_{qq,reg}^{(n)}(z_i), \tag{4.22}$$

where it is clear that $D_0(z_i)$ gives the main collinear structure to $\log \Gamma_{qq}(\hat{\alpha}_s, \mu^2, \mu_F^2, z_i, \epsilon)$, and the constants A_n and B_n are the same introduced above for the form factor. Also in this case we give as an example the first order in $\alpha_s(\mu_F^2)$ for the mass factorization kernel:

$$\Gamma_{qq}(z_i, \mu_F^2, \epsilon) = \delta(1-z_i) + \sum_{n=1}^{\infty} \alpha_s(\mu_F^2)\Gamma_{qq}^{(n)}(z_1, \epsilon), \tag{4.23}$$

where,

$$\Gamma_{qq}^{(1)}(z_i, \epsilon) = \frac{1}{\epsilon}\left(2\left(B_1\delta(1-z_i) + A_1 D_0(z_i)\right) + P_{qq,reg}^{(1)}(z_i)\right). \tag{4.24}$$

It is straightforward that to get a finite and soft form for the SV part of the partonic cross section the function $\Phi_d(\hat{\alpha}_s, M^2, \mu^2, z_1, z_2, \epsilon)$ must contain the terms that will remove the non-finite and non-soft contribution arising from the other function. It must contain only the soft structure of the process then every term in $\Phi_d(\hat{\alpha}_s, M^2, \mu^2, z_1, z_2, \epsilon)$ proportional to D_0 will remove the one coming from the mass factorization kernel.

In Ref. [22] to study of the form of $\Phi_d(\hat{\alpha}_s, M^2, \mu^2, z_1, z_2, \epsilon)$ is required that it must satisfy a Sudakov-type integro-differential equation as the previous one used to study the structure of the form factor (4.11), i.e. it satisfies

$$q^2 \frac{\mathrm{d}}{\mathrm{d}q^2} \Phi_d(\hat{\alpha}_s, q^2, \mu^2, z_1, z_2, \epsilon) = \frac{1}{2}\left[\overline{K}\left(\hat{\alpha}_s, \frac{\mu_R^2}{\mu^2}, z_1, z_2, \epsilon\right) + \overline{G}\left(\hat{\alpha}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z_1, z_2, \epsilon\right)\right]. \tag{4.25}$$

Also in this case the function $\overline{K}\left(\hat{\alpha}_s, \mu_R^2/\mu^2, z_1, z_2, \epsilon\right)$ contains all the poles in ϵ while $\overline{G}\left(\hat{\alpha}_s, q^2/\mu_R^2, \mu_R^2/\mu^2, z_1, z_2, \epsilon\right)$ has only positive powers of ϵ. The procedure to find the structure of this function is the same as done for the form factor with the only difference that in this case the K and G functions depends also on z_1 and z_2, then it is asked for them to satisfy the following two renormalization group equations,

$$\mu_R^2 \frac{\mathrm{d}}{\mathrm{d}\mu_R^2} \overline{K}\left(\hat{\alpha}_s, \frac{\mu_R^2}{\mu^2}, z_1, z_2, \epsilon\right) = -\overline{A}\left(\alpha_s(\mu_R^2)\right)\delta(1-z_1)\delta(1-z_2), \tag{4.26}$$

$$\mu_R^2 \frac{\mathrm{d}}{\mathrm{d}\mu_R^2} \overline{G}\left(\hat{\alpha}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z_1, z_2, \epsilon\right) = \overline{A}\left(\alpha_s(\mu_R^2)\right)\delta(1-z_1)\delta(1-z_2). \tag{4.27}$$

If $\Phi_d(\hat{\alpha}_s, q^2, \mu^2, z_1, z_2, \epsilon)$ contains the right terms to remove the poles from the other functions, $\overline{A}\left(\alpha_s(\mu_R^2)\right)$ must be the opposite of the cups anomalous dimension of the form factor $A\left(\alpha_s(\mu_R^2)\right)$, i.e. $\overline{A}\left(\alpha_s(\mu_R^2)\right) = -A\left(\alpha_s(\mu_R^2)\right)$ and then the solution of the (4.27) is,

$$\begin{aligned}\overline{G}\left(\hat{\alpha}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z_1, z_2, \epsilon\right) &= \overline{G}\left(\alpha_s(q^2), 1, z_1, z_2, \epsilon\right) - \\ &- \delta(1-z_1)\delta(1-z_2)\int_{q^2/\mu_R^2}^{1} \frac{\mathrm{d}\lambda^2}{\lambda^2} A\left(\alpha_s(\lambda^2\mu_R^2)\right),\end{aligned} \tag{4.28}$$

where the second term is exactly the opposite of the one found in the form factor.

As done for the form factor we can write the soft function in powers of the bare coupling constant. Using the renormalization group invariance and changing $q^2 \to q^2(1-z_1)(1-z_2)$ also $\Phi_d(\hat{\alpha}_s, q^2(1-z_1)(1-z_2), \mu^2, \epsilon)$ will be a solution of the Sudakov-type integro-differential equation (4.25). Then we can write the expansion of the solution as follows,

$$\begin{aligned}\Phi_d(\hat{\alpha}_s, q^2(1-z_1)(1-z_2), \mu^2, \epsilon) &= \sum_{n=1}^{\infty} \hat{\alpha}_s^n \left(\frac{q^2(1-z_1)(1-z_2)}{\mu^2}\right)^{\epsilon n} S_\epsilon^n \times \\ &\times \left(\frac{\epsilon n}{4(1-z_1)(1-z_2)}\right)\left[\overline{K}^{(n)}(\epsilon) + \overline{G}^{(n)}(\epsilon)\right],\end{aligned} \tag{4.29}$$

where the functions $\overline{G}^{(n)}(\epsilon)$ only contain finite contributions for $\epsilon \to 0$ and are unknown. Ravindran et al. showed in Ref. [21] that the (4.29) can be recast in the more useful form:

$$
\begin{aligned}
\Phi_d(\hat{\alpha}_s, q^2, \mu^2, z_1, z_2, \epsilon) = &\frac{1}{2}\delta(1-z_2)\left[\frac{1}{1-z_1}\left(\int_{\mu_R^2}^{q^2(1-z_1)} \frac{\mathrm{d}\lambda^2}{\lambda^2} A\left(\alpha_s(\lambda^2)\right)\right.\right. \\
&\left.\left. + \overline{G}(\alpha_s(q^2(1-z_1), \epsilon)\right)\right]_+ \\
&+ \left[\frac{1}{4(1-z_1)(1-z_2)}\left(A(\alpha_s(q^2(1-z_1)(2-z_2)))\right.\right. \\
&\left.\left. + \frac{\mathrm{d}\overline{G}\left(\alpha_s\left(q^2(1-z_1)(1-z_2)\right), \epsilon\right)}{\mathrm{d}\log(q^2(1-z_1)(1-z_2))}\right)\right]_+ \\
&+ \frac{1}{2}\delta(1-z_1)\delta(1-z_2)\sum_{n=1}^{\infty}\hat{\alpha}_s^n\left(\frac{q^2}{\mu^2}\right)^{\epsilon n} S_\epsilon^n\left[\overline{K}^{(n)}(\epsilon) + \overline{G}^{(n)}(\epsilon)\right] \\
&+ \frac{1}{2}\delta(1-z_1)\left[\frac{1}{1-z_2}\right]_+ \sum_{n=1}^{\infty}\hat{\alpha}_s^n\left(\frac{\mu_R^2}{\mu^2}\right)^{\epsilon n} S_\epsilon^n \overline{K}^{(n)}(\epsilon) \\
&+ (z_1 \leftrightarrow z_2),
\end{aligned}
\tag{4.30}
$$

with,

$$
\begin{aligned}
\overline{G}(\alpha_s(q^2 f(z_1, z_2)), \epsilon) &= \sum_{n=1}^{\infty}\hat{\alpha}_s^n\left(\frac{q^2 f(z_1, z_2)}{\mu^2}\right)^{\epsilon n} S_\epsilon^n \overline{G}^{(n)}(\epsilon) \\
&= \sum_{n=1}^{\infty}\hat{\alpha}_s^n(q^2 f(z_1, z_2))\overline{\mathcal{G}}^{(n)}(\epsilon)
\end{aligned}
\tag{4.31}
$$

Quite surprisingly, in Eq. (4.30) we can find the same expression found by Catani and Trentadue in their original article [1] but still not Mellin transformed. This means that with this method we are able to capture the structure of the soft behavior of the partonic cross section and the generality of the method permits to find similitude between well known processes like the Drell-Yan process and a more tough one like the Higgs production in hadronic collisions.

From the Eq. (4.30) we can see that in the fifth line has been produces the correct pole structure to eliminate the one coming from the form factor and the deltas part of the ass factorization kernel, while the collinear contributions are correctly removed by the terms of the sixth line. We finally have the expression for the exponentiated function of the soft and virtual contribution of the partonic cross section in the limit $\epsilon \to 0$:

$$
\begin{aligned}
\Psi_d(z_1, z_2, q^2, \mu^2) = \delta(1-z_2)\Bigg[\frac{1}{1-z_1}\Bigg(\int_{\mu_F^2}^{q^2(1-z_1)} \frac{\mathrm{d}\lambda^2}{\lambda^2} A\left(\alpha_s(\lambda^2)\right) \\
+ \overline{G}(\alpha_s(q^2(1-z_1))\Bigg)\Bigg]_+ \\
+ \Bigg[\frac{1}{2(1-z_1)(1-z_2)}\Bigg(A(\alpha_s(q^2(1-z_1)(2-z_2))) \\
+ \frac{\mathrm{d}\overline{G}\left(\alpha_s\left(q^2(1-z_1)(1-z_2)\right)\right)}{\mathrm{d}\log(q^2(1-z_1)(1-z_2))}\Bigg)\Bigg]_+ \\
+ \frac{1}{2}\delta(1-z_1)\delta(1-z_2)\log(g_0(\alpha_s(\mu_F^2))) \\
+ (z_1 \leftrightarrow z_2),
\end{aligned}
\tag{4.32}
$$

where we have set $\mu_R = \mu_F$. Several ways on how to evaluate the coefficients of the expansion of the function $\overline{G}(\alpha_s(q^2 f(z_1, z_2))$ can be found in the articles [21, 29, 30].

We can now perform the double Mellin transformation of the soft and virtual partonic cross section:

$$
\begin{aligned}
\tilde{\Delta}_d^{SV}(\omega) &= \int_0^1 \mathrm{d}z_1 z^{N_1-1} \int_0^1 \mathrm{d}z_2 z^{N_2-1} \Delta_d^{SV}(z_1, z_2, M^2, \mu_F^2, \mu_R^2) \\
&= g_0(\alpha_s)\exp\left[g(\alpha_s, \omega)\right],
\end{aligned}
\tag{4.33}
$$

where $\omega = \alpha_s\beta_0\log(\overline{N}_1\overline{N}_2)$ and $\overline{N}_i = e^{\gamma_E}N_i$ according with the article [31]; this result will be proved in the section 4.3. The function $g(\alpha_s, \omega)$ can be expanded in power of α_s in the usual way:

$$
g(\alpha_s, \omega) = g(\omega)\log(\overline{N}_1\overline{N}_2) + \sum_{n=0}^{\infty}\alpha_s^n g_{n+1}(\omega). \tag{4.34}
$$

Using the results of the above discussion it is possible to evaluate the resummed coefficients for the Drell-Yan rapidity distribution; in Ref. [20, 3] this method has been used to evaluate the function $g(\alpha_s, \omega)$ up to NNLO+NNLL respectively for the production of a Higgs scalar boson and for the Drell-Yan process. Below we report the first three terms in the expansion of $g(\alpha_s, \omega)$ in order to match it with the one that we will get in the next section for the Mellin-Fourier resummation,

$$
g_1(\omega) = \frac{A_1}{\beta_0\omega}[(1-\omega)\log(1-\omega) + \omega], \tag{4.35}
$$

$$
\begin{aligned}
g_2(\omega) = &\frac{A_2}{\beta_0^2}[(-\omega) - \log(1-\omega)] + \frac{A_1}{\beta_0}[\log(1-\omega)L_{QR} + \omega L_{FR}] + \frac{\mathcal{G}_1}{\beta_0}\log(1-\omega) \\
&+ \frac{A_1\beta_1}{\beta_0^2}\left[\frac{1}{2}\log^2(1-\omega) + \log(1-\omega) + \omega\right],
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
g_3(\omega) =& \frac{1}{2\beta_0^3}(A_3 - A_1\beta_2 + A_1\beta_1^2 - A_2\beta_1)\frac{-\omega^2}{1-\omega} \\
&+ \frac{A_1\beta_1^2}{\beta_0^3}\frac{\log(1-\omega)}{1-\omega}\left[1 + \frac{1}{2}\log(1-\omega)\right] + \frac{A_1\beta_2 - A_1\beta_1^2}{\beta_0^3}\log(1-\omega) \\
&- \left(\frac{A_2\beta_1}{\beta_0^3} - \frac{\mathcal{G}_1\beta_1}{\beta_0^2}\right)\left[\frac{\omega}{1-\omega} + \frac{\log(1-\omega)}{1-\omega}\right] \\
&+ \left(\frac{A_1\beta_2}{\beta_0^3} + \frac{\mathcal{G}_2}{\beta_0^2}\right)\frac{\omega}{1-\omega} + \left[\left(\frac{A_2 - A_1\beta_1}{\beta_0^2} - \frac{\mathcal{G}_1}{\beta_0}\right)\frac{-\omega}{1-\omega} + \frac{A_1\beta_1}{\beta_0^2}\frac{\log(1-\omega)}{1-\omega}\right]L_{QR} \\
&+ \frac{A_2}{\beta_0^2}\omega L_{FR} + \frac{A_1}{2\beta_0}\left[-\omega L_{FR}^2 + \frac{\omega}{1-\omega}L_{QR}^2\right]. \qquad (4.37)
\end{aligned}
$$

where we have defined $L_{QR} = \log Q^2/\mu_R^2$ and $L_{FR} = \log \mu_R^2/\mu_F^2$.

4.2 Mellin-Fourier Transform Resummation

The Mellin-Fourier approach discussed in this section is based on the classical approach of the resummation technique expressed in the article [1], where the resummation is performed, for the semi-inclusive case, in the Mellin space, this is due to the fact that the resummation of the rapidity distribution mainly consist in the observation that the Mellin-Fourier transform of the partonic cross section, in the limit $z \to 1$ can factorize the contribution of the rapidity trough a delta, i.e. $\Delta(z, y) \simeq \delta(y)\Delta(z)$, and then we have the possibility to perform the resummation of the rapidity integrated partonic cross section in the well known semi-inclusive case. This intuitive approach was proposed for the first time in the article [32] were Laenen and Sterman showed, via the phase space constraint, the possibility to factorize the z and y contribution trough a $\delta(y)$ in the case of a single soft gluon emission. To appreciate how this factorization works let us starts from the hadronic rapidity distribution;

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\tau\mathrm{d}Y}(\tau, Y, M^2) = \sum_{a,b}\int_{x_1^0}^1 \frac{\mathrm{d}x_1}{x_1}\int_{x_2^0}^1 \frac{\mathrm{d}x_2}{x_2} f_a(x_1, \mu_F^2) f_b(x_2, \mu_F^2) \times \\
\times \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}z\mathrm{d}y}\left(z, y, \alpha_s(\mu_R^2), \frac{M^2}{\mu_R^2}, \frac{M^2}{\mu_F^2}\right), \qquad (4.38)
\end{aligned}
$$

and since there is no ambiguity we omit to write the dependence of the differential partonic cross section from the running coupling, the renormalization and the factorization scales; in such a way it is useful to rename the differential partonic cross section as,

$$
C(z, y) = \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}z\mathrm{d}Y}(z, y) \qquad (4.39)
$$

and operate Mellin-Fourier transform in the hadronic variables τ and Y such that,

$$
\begin{aligned}
\int_0^1 \mathrm{d}\tau \ \tau^{N-1} \int_{-\infty}^{\infty} \mathrm{d}Y e^{ibY} \frac{\mathrm{d}\sigma}{\mathrm{d}\tau \mathrm{d}Y}(\tau, Y, M^2) \\
&= \sum_{a,b} \int_{x_1^0}^1 \mathrm{d}x_1 \ x_1^{N+\frac{ib}{2}} f_a(x_1, \mu_F^2) \int_{x_2^0}^1 \mathrm{d}x_2 \ x_2^{N-\frac{ib}{2}} f_a(x_1, \mu_F^2) \\
&\times \int_0^1 \mathrm{d}z \ z^{N-1} \int_{\log\sqrt{z}}^{-\log\sqrt{z}} \mathrm{d}y \ e^{iby} C(z, y),
\end{aligned}
\tag{4.40}
$$

where the hadronic and parotnic rapidity are linked by the following relation:

$$
Y - y = \frac{1}{2} \log \frac{x_1}{x_2} \tag{4.41}
$$

where the term $\log\sqrt{x_1/x_2}$ is the hadronic rapidity in the center of momentum frame. The the boundaries limits in the y and z integration are due to kinematical constraints. The reason for the boundary constraints in the rapidity integration is exposed in the Chapter 2.

In this case we are only interested on the partonic resummation, we then just analyze,

$$
C(N, b) = \int_0^1 \mathrm{d}z \ z^{N-1} \int_{\log\sqrt{z}}^{-\log\sqrt{z}} \mathrm{d}y \ e^{iby} C(z, y). \tag{4.42}
$$

In the article [4] is shown that if we take only the Fourier transform of the parton rapidity distribution and we expand the exponential we find

$$
\begin{aligned}
\tilde{C}(z, b) &= \int_{\log\sqrt{z}}^{-\log\sqrt{z}} \mathrm{d}y \ e^{iby} C(z, y) = \int_{\log\sqrt{z}}^{-\log\sqrt{z}} \mathrm{d}y \ C(z, y) \left[1 + \mathcal{O}(y)\right] \\
&= C(z) \left[1 + \mathcal{O}(1 - z)\right],
\end{aligned}
\tag{4.43}
$$

where $C(z)$ is the rapidity integrated partonic cross section, obtained because in the limit $z \to 1$ the integration region collapse to 0 and the terms of the expansion of the exponential goes as $\log|z| = 1 - z + \mathcal{O}\left((1 - z)^2\right)$, then $\tilde{C}(z, b)$ is independent from the parameter b up to correction of order $\mathcal{O}(1 - z)$. The factorization of the delta is straightforward by taking the inverse Fourier of $C(z)$, in fact,

$$
C(z, y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}b}{2\pi} e^{-iby} \tilde{C}(z, b) = C(z)\delta(y)[1 + \mathcal{O}(1 - z)] \tag{4.44}
$$

then up tu subdominant terms the partonic rapidity distribution in the Mellin-Fourier space can be written as,

$$
C(N, b) = \int_0^1 \mathrm{d}z \ z^{N-1} C(z)[1 + \mathcal{O}(1 - z)], \tag{4.45}
$$

and then the resummed partonic cross section is performed in the usual way,

$$C^{res}(\alpha_s, N) = g_0(\alpha_s)\exp[g(\alpha_s, \lambda)], \tag{4.46}$$

with,

$$g(\alpha_s, \lambda) = \frac{1}{2\alpha_s\beta_0}g_1(\lambda) + g_2(\lambda) + 2\alpha_s\beta_0 g_3(\lambda) + \dots \tag{4.47}$$

where, as anticipated, $C(\alpha_s, N)$ is the Mellin transform of the rapidity integrated partonic cross section.

We conclude this section giving the first three terms of the expansion of the resummed function,

$$g_1(\lambda) = \frac{2A_1}{\beta_0}[(1+\lambda)\log(1+\lambda) - \lambda], \tag{4.48}$$

$$\begin{aligned} g_2(\lambda) = &\frac{A_2}{\beta_0^2}[\lambda - \log(1+\lambda)] + \frac{A_1}{\beta_0}[\log(1+\lambda)L_{QR} - \lambda L_{FR}] - 2\frac{A_1}{\beta_0}\gamma_E\log(1+\lambda) \\ &+ \frac{A_1\beta_1}{\beta_0^2}\left[\frac{1}{2}\log^2(1+\lambda) + \log(1+\lambda) - \lambda\right], \end{aligned} \tag{4.49}$$

$$\begin{aligned} g_3(\lambda) = &\frac{1}{4\beta_0^3}(A_3 - A_1\beta_2 + A_1\beta_1^2 - A_2\beta_1)\frac{\lambda^2}{1+\lambda} \\ &+ \frac{A_1\beta_1^2}{2\beta_0^3}\frac{\log(1+\lambda)}{1+\lambda}\left[1 + \frac{1}{2}\log(1+\lambda)\right] + \frac{A_1\beta_2 - A_1\beta_1^2}{2\beta_0^3}\log(1+\lambda) \\ &+ \left(\frac{A_1\beta_1}{\beta_0^2}\gamma_E + \frac{A_2\beta_1}{2\beta_0^3}\right)\left[\frac{\lambda}{1+\lambda} - \frac{\log(1+\lambda)}{1+\lambda}\right] \\ &- \left(\frac{A_1\beta_2}{2\beta_0^3} + \frac{A_1}{\beta_0}(\gamma_E^2 + \zeta_2) + \frac{A_2}{\beta_0^2}\gamma_E - \frac{\bar{D}_2}{4\beta_0^2}\right)\frac{\lambda}{1+\lambda} \\ &+ \left[\left(\frac{A_1}{\beta_0}\gamma_E + \frac{A_2 - A_1\beta_1}{2\beta_0^2}\right)\frac{\lambda}{1+\lambda} + \frac{A_1\beta_1}{2\beta_0^2}\frac{\log(1+\lambda)}{1+\lambda}\right]L_{QR} \\ &- \frac{A_2}{2\beta_0^2}\lambda L_{FR} + \frac{A_1}{4\beta_0}\left[\lambda L_{FR}^2 - \frac{\lambda}{1+\lambda}L_{QR}^2\right]. \end{aligned} \tag{4.50}$$

4.3 Comparison of the Two Approaches

In this section we analyze the difference between the two approaches showing that they differ for *next-to-next-to-leading power*, i.e. terms suppressed by $(1-z)^2$. This equality teach us a very important lesson because in the Mellin-Fourier approach we have a resummed exponent that is independent from the Fourier variable making the partonic cross section to depend on the rapidity in the threshold regime only through a Dirac delta as shown in the previous section. Then in the threshold region the dependence on the rapidity is entirely given by the PDF, indeed, we can see from the fixed order calculation in Chapter 2, that the non-trivial term in the equation (2.124) that depends on the rapidity is,

$$
\begin{aligned}
&C_F \frac{1+z^2}{[1-z]_+}\left(\left[\frac{1+t}{t-z}\right]_+ + \left[\frac{1+t}{tz-1}\right]_+\right) \\
&= \left(P_{qq}(z) - \frac{3}{2}C_F\delta(1-z)\right)\left(\left[\frac{1+t}{t-z}\right]_+ + \left[\frac{1+t}{tz-1}\right]_+\right) \\
&= P_{qq}(z)\left(\left[\frac{1+t}{t-z}\right]_+ + \left[\frac{1+t}{tz-1}\right]_+\right) + 3C_F\delta(1-z)\frac{1+t}{[1-t]_+} + 3C_F\delta(1-z)\delta(1-t),
\end{aligned}
\tag{4.51}
$$

where the third term is harmless, the first factorize the Altarelli-Parisi splitting function and can be absorbed in the PDF definition, carrying with him correction due to the collinear terms, and the second term vanish when we perform the Mellin-Fourier transformation, in fact, we can see that

$$
\int_0^1 \mathrm{d}z\; z^{N-1}\delta(1-z)\int_{\log\sqrt{z}}^{-\log\sqrt{z}} \mathrm{d}y\; e^{iby}\frac{1+e^{-y}}{[1-e^{-y}]_+} = 0, \tag{4.52}
$$

because the region of integration of the rapidity is reduced to a point by the action of the delta, and since the integrand of the rapidity integration is a regular function the whole operation is well defined and the contribution from this terms is null.

We can now start with the analysis of the two approaches. In the previous sections we gave the expressions for the resummed coefficients (4.35)-(4.37) and (4.48)-(4.50), got from the two methods, in order to make the comparison easier at first sight, in fact, we can easily observe that they differ only for constant terms that depend on the definition of the expansion coefficients of the $D(\alpha_s)$ function and the definition of the resummed variable, in fact, we can see that performing the substitution $\omega \leftrightarrow -\lambda$ the functions $g_i(\cdot)$, with $i \neq 0$, have the same structure.

We first analyze the equality of the logarithmic structure of the two approaches starting by demonstrating the not trivial result (4.33). We derive the resummed exponent (4.33) omitting the constant function, i.e. the $g_0(\alpha_s)$ function, and the $\overline{G}(\alpha_s)$ since both of them do not change the following result and we can use a lighter notation. The result can be easily extended to the $\overline{G}(\alpha_s)$ function. The Double-Mellin transform of Eq. (4.32) is,

$$
\begin{aligned}
g(\alpha_s, N_1, N_2) = &\int_0^1 \mathrm{d}z_1\, \frac{z_1^{N_1-1}-1}{1-z_1}\int_{\mu_F^2}^{(1-z_1)q^2}\frac{\mathrm{d}\lambda}{\lambda}A(\alpha_s(\lambda)) \\
&+ \int_0^1 \mathrm{d}z_2\, \frac{z_2^{N_2-1}-1}{1-z_2}\int_{\mu_F^2}^{(1-z_2)q^2}\frac{\mathrm{d}\lambda}{\lambda}A(\alpha_s(\lambda)) \\
&+ \int_0^1 \mathrm{d}z_1\int_0^1 \mathrm{d}z_2\, \frac{z_1^{N_1-1}-1}{1-z_1}\frac{z_2^{N_2-1}-1}{1-z_2}A(\alpha_s((1-z_1)(1-z_2)q^2)),
\end{aligned}
\tag{4.53}
$$

and using the usual relation (3.21) to solve the integral in the large N_i limit, with $i = 1, 2$, the expression (4.53) became,

$$
\begin{aligned}
&g(\alpha_s, N_1, N_2) = \\
&= -\Gamma\left(1 - \frac{\partial}{\partial \log N_1}\right)\Gamma\left(1 - \frac{\partial}{\partial \log N_2}\right)\left[\int_{1/N_1}^{1} \frac{\mathrm{d}x_1}{x_1} \int_{\mu_F^2}^{x_1 q^2} \frac{\mathrm{d}\lambda}{\lambda} A(\alpha_s(\lambda))\right. \\
&\left. + \int_{1/N_2}^{1} \frac{\mathrm{d}x_2}{x_2} \int_{\mu_F^2}^{x_2 q^2} \frac{\mathrm{d}\lambda}{\lambda} A(\alpha_s(\lambda)) - \int_{1/N_1}^{1} \frac{\mathrm{d}x_1}{x_1} \int_{1/N_2}^{1} \frac{\mathrm{d}x_2}{x_2} A(\alpha_s(x_1 x_2 q^2))\right]. \qquad (4.54)
\end{aligned}
$$

The last term in the above expression can be written in more convenient way as follows,

$$
\frac{1}{2}\int_{1/N_1}^{1} \frac{\mathrm{d}x}{x}\left[\int_{\mu_F^2}^{xq^2} - \int_{\mu_F^2}^{xq^2/N_2}\right]\frac{\mathrm{d}\lambda}{\lambda} A(\alpha_s(\lambda)) + (N_1 \leftrightarrow N_2), \qquad (4.55)
$$

where $\lambda = x_1 x_2 q^2$. Then we put the (4.55) in the (4.54) getting,

$$
\begin{aligned}
&-\frac{1}{2}\Gamma\left(1 - \frac{\partial}{\partial \log N_1}\right)\Gamma\left(1 - \frac{\partial}{\partial \log N_2}\right)\left[\int_{1/N_1}^{1} \frac{\mathrm{d}x}{x} \int_{\mu_F^2}^{xq^2} \frac{\mathrm{d}\lambda}{\lambda} A(\alpha_s(\lambda))\right. \\
&\left. + \int_{(N_1 N_2)^{-1}}^{1/N_1} \frac{\mathrm{d}x}{x} \int_{\mu_F^2}^{xq^2} \frac{\mathrm{d}\lambda}{\lambda} A(\alpha_s(\lambda)) + (N_1 \leftrightarrow N_2)\right], \qquad (4.56)
\end{aligned}
$$

thus since we are in the large N_1, N_2 limit we have that $(N_i)^{-1} > (N_1 N_2)^{-1}$, with $i = 1, 2$, and then we can join the integration domain of the first integral to get the form we are looking for; we put it directly in a more useful form

$$
-\Gamma\left(1 - \frac{\partial}{\partial \log N_1}\right)\Gamma\left(1 - \frac{\partial}{\partial \log N_2}\right)\int_{\alpha_s\left(\frac{q^2}{N_1 N_2}\right)}^{\alpha_s(q^2)} \frac{\mathrm{d}\alpha'}{\alpha'\beta(\alpha')} \int_{\alpha_s(\mu_F^2)}^{\alpha'} \frac{\mathrm{d}\alpha}{\alpha\beta(\alpha)} A(\alpha). \qquad (4.57)
$$

The running coupling depend on the two Mellin variables through the logarithm of their product then the integration is a function of $\log N_1 + \log N_2$ and then we can simplify the action of the Gamma function like,

$$
-\Gamma^2\left(1 - \frac{\partial}{\partial \log N_1 N_2}\right)\int_{\alpha_s\left(\frac{q^2}{N_1 N_2}\right)}^{\alpha_s(q^2)} \frac{\mathrm{d}\alpha'}{\alpha'\beta(\alpha')} \int_{\alpha_s(\mu_F^2)}^{\alpha'} \frac{\mathrm{d}\alpha}{\alpha\beta(\alpha)} A(\alpha). \qquad (4.58)
$$

If we replace $\gamma_E \to 2\gamma_E$ and $\zeta(n) \to 2\zeta(n)$ we recover the same resummed coefficients get in the (3.26) because, Taylor expanding the logarithm of the Gamma function we can write,

$$
\Gamma^2\left(1 - \frac{\partial}{\partial \log N_1 N_2}\right) = \exp\left\{2\gamma_E \frac{\partial}{\partial \log N_1 N_2} + \sum_{n=2}^{\infty} \frac{2\zeta(n)}{n}\left(\frac{\partial}{\partial \log N_1 N_2}\right)^n\right\}. \qquad (4.59)
$$

Finally we can write the resummed exponent as a function of $\log N_1 N_2$ as used in the article [20].

We can now turn our attention on the Mellin-Fourier approach. First of all let us derive the relation between the two set of the conjugated variables, to this purpose we need the relations between the physical variables, that are,

$$z = z_1 z_2, \tag{4.60}$$
$$y = \frac{1}{2}\log\frac{z_1}{z_2}, \tag{4.61}$$

then the double Mellin variables and the Mellin-Fourier variables are linked in the following way,

$$\begin{aligned}\int_0^1 \mathrm{d}z_1\ z_1^{N_1-1}\int_0^1 \mathrm{d}z_2\ z_2^{N_2-1} &= \int_0^1 \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}y\ \left(\sqrt{z}e^{y}\right)^{N_1-1}\left(\sqrt{z}e^{-y}\right)^{N_2-1}\\ &= \int_0^1 \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}y\ z^{\frac{N_1+N_2}{2}-1}e^{(N_1-N_2)y}\\ &= \int_0^1 \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}y\ z^{N-1}e^{iby};\end{aligned} \tag{4.62}$$

we see that the variables of the Mellin-Mellin space are complex numbers in the Mellin-Fourier variables,

$$\begin{cases} N_1 = N + \frac{ib}{2}, \\ N_2 = N - \frac{ib}{2}, \end{cases} \tag{4.63}$$

since $N_1, N_2, N \in \mathbb{C}$ we can say that the Fourier variable act just like a translation in the imaginary direction of the complex Mellin variable N. The resummed variable of the Mellin-Mellin approach can be written in terms of the variables (N, b) and we can see that if b is finite, the large-N limit let the two approaches be the same, in fact,

$$\log(N_1N_2) = \log\left(N^2 + \frac{b^2}{4}\right) = \log N^2 + \log\left(1 + \frac{b^2}{4N^2}\right) \simeq \log N^2 + \mathcal{O}\left(\frac{b^2}{4N^2}\right), \tag{4.64}$$

then in the large N limit the result seems to be independent on b. We can do a better estimation of the neglected terms, in fact, the logarithms resummed in the in Mellin-Mellin approach are like,

$$\alpha_s^n \frac{\log^{m_1}(1-x_1)}{1-x_1}\frac{\log^{m_2}(1-x_2)}{1-x_2}, \tag{4.65}$$

where $m_1 + m_2 \leq 2n$. The Mellin-Mellin transform of these logarithms expressed in the (N, b) variables is

$$\alpha_s^n \int_0^1 \mathrm{d}x_1\, x_1^{N+i\frac{b}{2}-1} \frac{\log^{m_1}(1-x_1)}{1-x_1} \int_0^1 \mathrm{d}x_2\, x_2^{N-i\frac{b}{2}-1} \frac{\log^{m_2}(1-x_2)}{1-x_2} \tag{4.66}$$

then we are free to expand the variable exponentiated by b around $x_i = 1$, thus we get,

$$\begin{aligned}
&\alpha_s^n \left(\sum_{k_1=0}^{\infty} (-1)^{k_1} \binom{i\frac{b}{2}}{i} \int_0^1 \mathrm{d}x_1\, x_1^{N-1} \frac{\log^{m_1}(1-x_1)}{(1-x_1)^{1-k_1}} \right) \times \\
&\times \left(\sum_{k_2=0}^{\infty} (-1)^{k_2} \binom{-i\frac{b}{2}}{k_2} \int_0^1 \mathrm{d}x_2\, x_2^{N-1} \frac{\log^{m_2}(1-x_2)}{(1-x_2)^{1-k_2}} \right) \\
&= \alpha_s^n \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, x_1^{N-1} x_2^{N-1} \left[\frac{\log^{m_1}(1-x_1)}{1-x_1} \frac{\log^{m_2}(1-x_2)}{1-x_2} \right. \\
&\qquad\qquad \left. + \mathcal{O}\left(\frac{b^2}{4} \log^{m_1}(1-x_1) \log^{m_2}(1-x_2) \right) \right] \\
&= \alpha_s^n \log^{m_1+m_2+2} N + \mathcal{O}\left(\frac{b^2}{4N^2} \log^{m_1+m_2+2} N \right)
\end{aligned} \tag{4.67}$$

where the resummation of the first term in the last expression give the same resummed function get in the Mellin-Fourier approach, while the neglected terms to get the Mellin-Fourier approach result from the Mellin-Mellin one are *next-to-next-to-leading power* terms.

The above relation is true only if we take the large-N limit before that any action on the Fourier variable is done, in fact, if we keep both N and b it is not obvious that the Fourier variable does not give any contribution, especially when we perform the Fourier inverse transformation. At this point it is no longer obvious the equality of the two approaches because the possibility to neglect the Fourier variable in the large-N limit is not obvious itself. We have to understand better the structure of the large logarithms in the (N, b) space.

Now we show a way to recover the Mellin-Fourier approach from the Mellin-Mellin results. We express the resummed partonic cross section of the Mellin-Mellin approach in terms of the Mellin-Fourier variables, getting,

$$\hat{\sigma}(z, y) = g_0(\alpha_s) e^{g\left(\alpha_s, N^2 + \frac{b^2}{4}\right)} = g_0(\alpha_s) e^{\frac{b^2}{4}\frac{\partial}{\partial N^2}} e^{g\left(\alpha_s, N^2\right)}. \tag{4.68}$$

Through the double inverse transformation we can see that the result of the Mellin-Fourier approach can be reached, in fact,

$$\tilde{\sigma}(N, b) = -\frac{i}{4\pi^2} g_0(\alpha_s) \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \mathrm{d}b\, e^{-iby} \left(\frac{b}{2}\right)^{2n} \int_{c-i\infty}^{c+i\infty} \mathrm{d}N\, z^{-N} \left(\frac{\partial}{\partial N^2}\right)^n e^{g\left(\alpha_s, N^2\right)}, \tag{4.69}$$

in the integral over the Mellin variable we can perform n-times an integration by parts passing the n derivative from the resummed function to the z^{-N} term, this operation will generate non trivial boundary terms that will vanish only in the soft limit, in fact, the integration by parts will produce the following terms,

$$z^{-N}\log^{k-1} z\left(\frac{\partial}{\partial N^2}\right)^{n-k} e^{g\left(\alpha_s,N^2\right)}\Big|_{c-i\infty}^{c+i\infty} \xrightarrow{z\to 1} \begin{cases} k=1 & \left(\frac{\partial}{\partial N^2}\right)^{n-1} e^{g\left(\alpha_s,N^2\right)}\Big| \\ k\neq 1 & 0 \end{cases}, \tag{4.70}$$

where $1\leq k\leq n$ and $n>0$. For $k\neq 1$ the boundary terms go to zero because of the $\log z$, while for $k=1$ the boundary terms are analytic at the infinity point, thus,

$$\lim_{L\to\infty}\left(\frac{\partial}{\partial N^2}\right)^{n-1} e^{g\left(\alpha_s,N^2\right)}\Big|_{-iL}^{iL} = 0. \tag{4.71}$$

This show that the boundary terms can be neglected only at the threshold limit otherwise the evaluation of the resummed function at the boundary of the integration will contribute to the final results since it do not go to zero at imaginary infinity.

The rest of the Eq. (4.69) after the integration by parts is,

$$\begin{aligned}
\hat{\sigma}(z,y) &= -\frac{i}{4\pi^2} g_0(\alpha_s)\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{R}} \mathrm{d}b\ e^{-iby}\left(\frac{b}{2}\right)^{2n}\int_{c-i\infty}^{c+i\infty}\mathrm{d}N\ z^{-N}\log^n z e^{g\left(\alpha_s,N^2\right)} \\
&= -\frac{i}{4\pi^2} g_0(\alpha_s)\int_{\mathbb{R}} \mathrm{d}b\ e^{-iby+\frac{b^2}{2}\log\sqrt{z}}\int_{c-i\infty}^{c+i\infty}\mathrm{d}N\ z^{-N}\log^n z\ e^{g\left(\alpha_s,N^2\right)} \\
&= -\frac{i}{4\pi^2} g_0(\alpha_s)\sqrt{\frac{4\pi}{\log 1/z}}\ e^{-\frac{y^2}{\log 1/z}}\int_{c-i\infty}^{c+i\infty}\mathrm{d}N\ z^{-N}\log^n z\ e^{g\left(\alpha_s,N^2\right)},
\end{aligned} \tag{4.72}$$

in the last expression we can see that in the soft limit the delta of the rapidity appear, in fact,

$$\sqrt{\frac{4\pi}{\log 1/z}}\ e^{-\frac{y^2}{\log 1/z}} \xrightarrow{z\to 1} 2\pi\delta(y), \tag{4.73}$$

recovering the result of the previous section, i.e.

$$\hat{\sigma}(z,y) = g_0(\alpha_s)\delta(y)\int_{c-i\infty}^{c+i\infty}\mathrm{d}N\ z^{-N} e^{g(\alpha_s,N^2)}. \tag{4.74}$$

With this procedure we have shown that the two approaches are the same in the soft limit but they can be very different if we move away from it. We can assert that the two approaches differ for collinear terms, in fact, at the beginning of this section we have proved that is possible to factorize the delta of the rapidity at the first perturbative order by putting the collinear contribution in the PDF definition. This calculation also confirm that the contribution of the Fourier variable come mainly from the PDFs giving us the possibility to simplify the problem by taking the soft partonic cross section of the form,

$$\hat{\sigma}^S(z,y) = \delta(y)\hat{\sigma}^S(z), \tag{4.75}$$

where $\hat{\sigma}^S(z)$ is the soft partonic cross section for the mass invariant distribution.

At the beginning of this section, we mentioned a difference between the two approaches that rely on the choice of the resummed variable, $e^{\gamma_E}N$ or simply N. We have to underline that the choice of the resummed variable does not depend on the approach used but only from the logarithmic accuracy required, in fact at the LL and NLL the Mellin integral is solved using $z^{N-1} - 1 \to -\theta(1 - z - e^{-\gamma_E}/N)$ to recover the correct result that we get by the whole integration, i.e. without using the theta function, as shown at the end of the section 3.1, while at the NNLL the method used to get resummed coefficients is independent from the choice of the resummed variable. This situation can be understand by the following example. Take the α_s order in the perturbation theory, i.e.

$$\alpha_s \int_0^1 \mathrm{d}z \; (z^{N-1} - 1)\frac{\log(1-z)}{1-z} \tag{4.76}$$

then we have, respectively, at the LL, NLL, NNLL, using N,

$$\begin{gathered}\alpha_s \log^2 N \\ \alpha_s(\log^2 N + 2\gamma_E \log N) \\ \alpha_s\left(\log^2 N + 2\gamma_E \log N + \gamma_E^2 + \zeta(2)\right),\end{gathered} \tag{4.77}$$

while, using $e^{\gamma_E}N$, the LL already has a NNLL structure,

$$\log^2(e^{\gamma_E}N) = \log^2 N + 2\gamma_E \log N + \gamma_E^2, \tag{4.78}$$

in fact, at a generic perturbative order the LL of $e^{\gamma_E}N$ gives the LL, $\alpha_s^n \log^m N$ with $n < m \le 2n$, NLL with $n = m$ and NNLL with $m < n$ in the variable N. This means that the two choice produce different results at the LL and NLL while are the same at the NNLL up to constant terms that can be adjusted by redefining the coefficients of $g_0(\alpha_s)$. For example we can see how the two choice affect the first perturbative order of the resummed partonic cross section. The coefficient $g_0^{(1)}$ of the expansion $g_0(\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^n g_0^{(n)}$ is, respectively in the N and $e^{\gamma_E}N$ case,

$$A_1\left(4\zeta(2) - 4 + 2\gamma_E^2\right) \tag{4.79}$$

$$A_1\left(4\zeta(2) - 4\right) \tag{4.80}$$

and the first perturbative order of the $\tilde{\sigma}(\omega)$ is,

$$\tilde{\sigma}(\omega) = g_0(\alpha_s)e^{\alpha_s \mathcal{S} + \dots} = 1 + \left(\mathcal{S} + g_0^{(1)}\right)\alpha_s + \dots. \tag{4.81}$$

The difference of the first perturbative order $|\tilde{\sigma}(\log N^2) - \tilde{\sigma}(e^{2\gamma_E}\log N^2)|$ in the two cases are

$$\text{LL} \qquad A_1\gamma_E(\gamma_E + 2\log N) \tag{4.82}$$
$$\text{NLL} \qquad A_1\gamma_E^2 \tag{4.83}$$
$$\text{NNLL} \qquad 0. \tag{4.84}$$

There is no difference in the NNLL accuracy while at lowest order the two case are not comparable especially at the LL order. This situation also affect the prediction of the theory, indeed in a recent article [3] has been shown a numerical comparison of the two cases, more precisely in that article the authors show that there is a numerical differences between the Mellin-Mellin approach and Mellin-Fourier approach ascribing to this differences the difference nature of the terms we are resumming in the two approaches, in fact, since in the second approach we are resumming over the rapidity integrated partonic cross section, cause the presence of the $\delta(y)$ in the (4.75), it is reasonable the idea that we have different logarithms and then the prediction of the two approaches can present some differences but in that analysis there is hidden also the numerical difference between the two choice of N, in fact, the comparison has been made taking into consideration the results from the article [4] where they resum large $\log N$, while the authors of [3] resum large $\log e^{\gamma_E} N$. The numerical results, see Figure (4.1), exactly show the behavior exposed above, i.e. at the LO+LL the two approaches produce very different correction to the fixed order prediction while tends to be the same for NLO+NLL and they are also more similar for the NNLO+NNLL.

y	$(\frac{\mu_R}{M_Z}, \frac{\mu_F}{M_Z})$	LO	$\text{LL}_{\text{M-F}}$	$\text{LL}_{\text{M-M}}$	NLO	$\text{NLL}_{\text{M-F}}$	$\text{NLL}_{\text{M-M}}$	NNLO	$\text{NNLL}_{\text{M-F}}$	$\text{NNLL}_{\text{M-M}}$
0.0	$(2,2)$	72.626	+0.988	+3.219	73.450	+1.639	+1.796	70.894	+ 0.630	+0.646
0.0	$(2,1)$	63.197	+0.768	+2.595	70.625	+0.761	+1.017	70.360	+0.292	+0.317
0.0	$(1,2)$	72.626	+1.095	+3.577	73.535	+1.912	+1.760	70.509	+0.510	+0.395
0.0	$(1,1)$	63.197	+0.851	+2.887	71.395	+0.858	+0.901	70.537	+0.248	+0.167
0.0	$(1,1/2)$	53.241	+0.621	+2.216	67.581	+ 0.156	+0.140	69.834	- 0.001	- 0.094
0.0	$(1/2,1)$	63.197	+0.953	+3.278	72.355	+0.945	+0.681	70.266	+0.091	- 0.015
0.0	$(1/2,1/2)$	53.241	+0.695	+2.504	69.259	+0.102	- 0.154	70.283	- 0.039	- 0.146

Figure 4.1: In the Figure is reported the comparison of resummed results between Mellin-Mellin approach and Mellin-Fourier approach using the minimal prescription scheme at y = 0 for various choices of the scales.

In the table 4.1 are reported the absolute value of the difference of the correction to the fixed order calculation given from the resummation procedure in the two approaches. We have called, $\delta_{N^kLL} = |\varepsilon_{MF} - \varepsilon_{MM}|$, where obviously ε_{MF} and ε_{MM} correspond to the resummation contribution to the fixed order. We can see from this Table that the mean difference between the correction of the fixed order given

$(\mu_R/M, \mu_F/M)$	δ_{LL}	δ_{NLL}	δ_{NNLL}
(2,2)	2.231	0.157	0.016
(2,1)	1.827	0.256	0.025
(1,2)	2.482	0.152	0.115
(1,1)	2.036	0.043	0.081
(1,1/2)	1.595	0.016	0.093
(1/2,1)	2.325	0.264	0.106
(1/2,1/2)	1.809	0.256	0.107
Mean Difference	2.043	0.163	0.078

Table 4.1

by the resummation procedure of the two approaches decrease when the logarithmic accuracy increase. This can be take as a proof that the main difference of the two approaches used in the article [4] and [3] resides in the choice of the resummed variable more than the approach itself. In fact, in this section the equality of the two approaches in the soft limit has been proved and we have stressed that a considerable difference can arise from different choice of the resummed variable, like in this case where two approaches has been compared using different choice of the resummed variable.

Conclusion

In this work we have studied two different approaches to the resummation of the large logarithms that arise by the cancellation of the soft and collinear singularities for the rapidity distribution of the Drell-Yan process. This particular process is experimentally very important because its probe, the leptonic pair, can be detected with great accuracy; it is then very important to have precise theoretical prediction to make possible new discoveries.

The large logarithms that are presents in the physical space, i.e. function of the soft radiated energy fraction, can be resummed to all orders in the conjugated Mellin space, because only upon Mellin transformation the cross section (squared amplitude and phase space measure) take a factorized form that leads to exponentiation of the leading contributions. In the case studied in this thesis the large logarithms are functions of the soft radiated energy fraction and the rapidity; hence it is necessary to take an integral transform in both variables to obtain a resummed formulation. In a recent work [3] a resummed formula for this observable was proposed, which relies on a Mellin-Mellin transformation. The result has been numerically compared with the result of a previous work [4] where the resummation is performed in a Mellin-Fourier space.

In ref. [3] an interesting difference between the two approaches is shown, at different logarithmic accuracy; the results reported in Table 4.1 show that the two methods differ sizeably at the leading logarithmic accuracy, but the difference tends to reduce with increasing logarithmic accuracy; at the next-to-next-to-leading logarithm accuracy the two methods give essentially the same result. The authors of the ref. [3] associate this difference to the different kind of logarithms that are resummed or to a simply accidental coincidence.

In this thesis we have investigated this discrepancy in greater detail, and we have obtained a first mathematical answer to this question. In fact we have shown that the logarithms that are resummed in the two approaches differ for *next-to-next-to-leading power*, i.e. terms that are suppressed by terms that go to zero as the square of the radiated energy in the soft limit. It is unlikely that such terms originate the discrepancy found in [3] since such terms are systematically neglected in the calculation of the soft partonic cross section.

The bulk of the numerical difference in Table 4.1 can be instead traced back to the argument of the logarithms which are resummed. Indeed, while the resummation of ref. [4] considers logarithms of the Mellin variable N, the Authors of ref. [3] resum a rescaled version of their Mellin variables. At any given logarithmic order, the resummation of rescaled and un-rescaled logarithms clearly differ by subleading terms. Thus, if we compare the LL resummation of ref. [4] versus the LL one of ref. [3], we find that the two differ by sizeable NLL contributions. Analogously, the two NLL resummations differ by NNLL contributions, and so on. Thus, we expect the difference between the two approaches to grow smaller as the logarithmic accuracy of the calculation is increased. This is indeed what we observe. Finally, let us note that in related contexts, rescaling the argument of the logarithms by an arbitrary factor is used as a way of estimating theoretical uncertainties due to missing higher logarithmic contributions.

Appendix A

Dimensional Regularization

This is the best and easy methods to deal with divergent integrals that are common in the perturbative field theories. The method consist in change the number of dimension from the usual Minkowski space $\mathbb{R}^{(1+3)}$ to a space with $d = 4-2\epsilon$ dimension with $d \in \mathbb{C}$. With this substitution it is possible to give precise meanings to divergent integrals because the divergent aspect of the integral appear as a pole with the same degree of the divergent integral for $d \to 4$ while all the other informations, i.e. the finite parts, are understood.

Changing the dimensions of the process two particular object will be more affected then others, these are the Gamma matrices and the phase space measure. In this appendix we will give their form in d dimensions. Let's start with the Gamma matrices. These became $d \times d$ matrices in a d-dimensional space-time with the Minkowski signature $g^{\mu\nu} = (+, -, \dots, -)$ where the only positive component is the temporal, i.e. the component g^{00}. The trace property needful to calculate the scattering matrix elements remains the same;

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu} \tag{A.1}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\rho\nu} - g^{\mu\rho} g^{\sigma\nu}). \tag{A.2}$$

A great difference comes with the contractions of the Gamma matrices,

$$\gamma^\mu \gamma^\rho \gamma_\mu = -2(1-\epsilon)\gamma^\rho \tag{A.3}$$

$$\gamma^\mu \gamma^\rho \gamma^\sigma \gamma_\mu = 4g^{\rho\sigma} - 2\epsilon \gamma^\rho \gamma^\sigma \tag{A.4}$$

$$\gamma^\mu \gamma^\rho \gamma^\sigma \gamma^\nu \gamma_\mu = -2\gamma^\nu \gamma^\sigma \gamma^\rho + 2\epsilon \gamma^\rho \gamma^\sigma \gamma^\nu. \tag{A.5}$$

We now analyze how the change of dimensions affect the phase space. The differential measure of a euclidean d-dimensional vector, $\mathrm{d}^d p_E$, can be written in spherical coordinates, namely

$$\begin{cases} p_1 & = p\cos\theta_1 \\ p_2 & = p\sin\theta_1 \cos\theta_2 \\ & \vdots \\ p_{d-1} & = p\cos\theta_1 \cos\theta_2 \dots \cos\theta_{d-2} \cos\phi \\ p_d & = p\cos\theta_1 \cos\theta_2 \dots \cos\theta_{d-2} \sin\phi \end{cases} \tag{A.6}$$

with $0 \leq \theta_n \leq \pi$ for $n = 1, 2, \ldots, d-2$ and $0 \leq \theta_n \leq 2\pi$ while $p = |p_E|$. Then the measure become,

$$\mathrm{d}^d p_E = \mathrm{d}p \; p^{d-1} \sin^{d-2}\theta_1 \mathrm{d}\theta_1 \ldots \sin\theta_{d-2}\mathrm{d}\theta_{d-2}\mathrm{d}\phi = \mathrm{d}p \; p^{d-1}\mathrm{d}\Omega_{d-1}, \tag{A.7}$$

where $\mathrm{d}\Omega_{d-1}$ is the measure of a $d-1$ sphere that can be evaluated with the help of the Gamma function,

$$\begin{aligned}\int \mathrm{d}\Omega_d &= 2\pi \prod_{i=1}^{d-1} \int_0^{\pi} \mathrm{d}\theta_i \sin^{d-i}\theta_i = 2\pi \prod_{i=1}^{d-1} \int_0^1 \mathrm{d}x \; x^{-1/2}(1-x)^{\omega-1-i/2} \\ &= \frac{2\pi^{d/2}\Gamma(1)\Gamma(3/2)\ldots\Gamma(d/2-1/2)}{\Gamma(3/2)\ldots\Gamma(d/2-1/2)\Gamma(d/2)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}.\end{aligned} \tag{A.8}$$

For a generic final n-body phase space we can implement the dimensional regularization as follows,

$$\begin{aligned}\mathrm{d}\Phi_n(P; p_1, p_2, \ldots, p_n) &= (2\pi)^d \delta^{(d)}\left(P - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{\mathrm{d}^d p_i}{(2\pi)^{d-1}} \; \delta^+\left(p_i^2 + M_i^2\right) \\ &= (2\pi)^d \delta^{(d)}\left(P - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{\mathrm{d}p_i \; p_i^{d-1}}{(2\pi)^{d-2} 2p_i^0} \mathrm{d}\Omega_{d-2}^{(i)}\end{aligned} \tag{A.9}$$

with M_i the mass of the i-th particle; it can be written in a recursive form factorizing the 2-body phase space,

$$\mathrm{d}\Phi_n(P; p_1, p_2, \ldots, p_n) = \int \frac{\mathrm{d}Q^2}{2\pi} \mathrm{d}\Phi_2(Q; p_1, p_2)\mathrm{d}\Phi_{n-1}(P; Q, p_3, \ldots, p_n), \tag{A.10}$$

that can be thought like to introduce an intermediate "virtual" momentum. Then it is important to give the explicit expression for 2-body phase space and 1-body phase space since with this only two measure any other can be constructed. The 2-body phase space can be written easily in the C.o.M. frame, i.e. for $\vec{Q} = 0$, then we have,

$$\begin{aligned}\mathrm{d}\Phi_2(Q; p_1, p_2) &= \frac{1}{(2\pi)^{d-2}} \delta\left(M_2^2 - M_1^2 - Q^2 + 2Qp_1^0\right) \frac{\mathrm{d}p_1 \; p_1^{d-2}}{2p_1^0} \mathrm{d}\Omega_{d-2}^{(1)} \\ &= \frac{1}{8}\left(\frac{1}{2\pi}\right)^{d-2} Q^{d-4}\left((1 - x_1 - x_2)^2 - 4x_1x_2\right)^{(d-3)/2} \mathrm{d}\Omega_{d-2}^{(1)},\end{aligned} \tag{A.11}$$

where $x_1 = M_1^2/Q^2$ and $x_2 = M_2^2/Q^2$, we can now evaluate the angular measure introducing the usual variable y and integrating over the remaining angular dimensions because the cross section is usually independent of those variables,

$$y = \frac{1 - \cos\theta}{2} \tag{A.12}$$

$$\mathrm{d}\Omega_{d-2} = \mathrm{d}\theta \sin^{d-3}\theta \mathrm{d}\Omega_{d-2} = 2^{d-3}[y(1-y)]^{(d-4)/2}\mathrm{d}y\mathrm{d}\Omega_{d-2}, \tag{A.13}$$

finally the measure of a two-body phase space is,

$$\mathrm{d}\Phi_2(Q;p_1,p_2) = \frac{1}{8\pi}\left(\frac{4\pi}{Q^2}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\frac{\sqrt{(1-x_1-x_2)^2-4x_1x_2}}{((1-x_1-x_2)^2-4x_1x_2)^{\epsilon}}[y(1-y)]^{-\epsilon}\mathrm{d}y. \tag{A.14}$$

The one-body phase space is,

$$\mathrm{d}\Phi_1(Q;p) = 2\pi\delta(Q^2-M^2). \tag{A.15}$$

To close the appendix we give a generic useful integrals solved with the dimensional regularization,

$$\begin{aligned}\mu^{2\epsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{k^a}{(k^2+\Delta)^n} &= \frac{1}{16\pi^2}\frac{(4\pi\mu^2)^{\epsilon}}{\Gamma(1-\epsilon)}\int_0^{\infty}\mathrm{d}k\frac{k^{d-1+a}}{(k^2+\Delta)^n}\\ &= \frac{1}{16\pi^2}\frac{(4\pi\mu^2)^{\epsilon}}{\Gamma(1-\epsilon)}\Delta^{\frac{d-a}{2}-n}\frac{\Gamma\left(\frac{d+a}{2}\right)\Gamma\left(n-\frac{d+a}{2}\right)}{\Gamma(n)}.\end{aligned} \tag{A.16}$$

with $a \geq 0$, and the integral in $\mathrm{d}k$ is a simply Beta function.

Appendix B

Plus Distribution

This distribution is a useful tool to work with the soft divergencies that arise from the bremsstrahlung diagram. It is defined as:

$$\int_{x_1}^{x_2} dx\ g(x)[f(x)]_+ = \int_{x_1}^{x_2} dx\ \ (g(x) - g(x_i))\, f(x), \tag{B.1}$$

where $g(x)$ is a test function (continuous in the set $[x_1, x_2]$) and x_i is the extreme of the domain where $f(x)$ is singular at most like a polynomial. In the following, where will be omitted the test function and the integral to simplify the notation, it is important to remember that the expression has sense only when it's integrated with some regular function.

The distribution is request because it is possible to encounter a function $f(x)$ that is singular in a point of integration, than to separate the divergent contribution from the regular one it is useful to go on with the following procedure;

$$\begin{aligned}\int_{x_1}^{x_2} dx\ f(x)g(x) &= \int_{x_1}^{x_2} dx\ \left(f(x)g(x) - f(x)g(x_i) + f(x)g(x_i)\right) \\ &= \int_{x_1}^{x_2} dx\ g(x)[f(x)]_+ + g(x)\delta(x - x_i)\int_{x_1}^{x_2} dx'\ f(x'),\end{aligned} \tag{B.2}$$

now the first terms is regular and contains the information of the convolution of the two function in the domain, excluded the divergent point, while the second one is contains only the divergent contribution. Usually the last term is thrown away in some measurable function. For example if $f(x) = x^{-1-\epsilon}$ with $\epsilon \ll 1$ in the region $[0, 1]$, we have;

$$\int_0^1 dx\ \frac{g(x)}{x^{1+\epsilon}} = \int_0^1 dx\ g(x)\sum_{n=0}^{\infty}(-1)^n\frac{\epsilon^n}{n!}\left[\frac{\log^n x}{x}\right]_+ - \delta(x)g(0)\frac{1}{\epsilon}. \tag{B.3}$$

Following this example, the distribution can be used to Laurent expand the function $f(x)$ around the singularity:

$$x^{-1+\epsilon} = \delta(x)\frac{1}{\epsilon} + \left[\frac{1}{x}\right]_+ + \epsilon\left[\frac{\log x}{x}\right]_+ + \frac{\epsilon^2}{2}\left[\frac{\log^2 x}{x}\right]_+ + \ldots. \tag{B.4}$$

This distribution can be defined also in more dimension for a function singular in

$(x_1^0, \ldots, x_N^0)$ as follows;

$$\int_\Omega \mathrm{d}x_1 \ldots \mathrm{d}x_N \; g(x_1, \ldots, x_N)[f(x_1, \ldots, x_N)]_+ = \\ = \int_\Omega \mathrm{d}x_1 \ldots \mathrm{d}x_N \; f(x_1, \ldots, x_N) \left(g(x_1, \ldots, x_N) - g(x_1^0, \ldots, x_N^0) \right) . \tag{B.5}$$

This way to separate the singular term from the regular one can also be applied to functions that are divergent over a sub-variety throwing away the contribution from that region, instead of a single point.

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To be or not to be. I
Have to be grateful to
A place where
Not to be, is the
Key to be.

You are the vessel
Over the sea haze, the
Untouchable Friend.

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