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# The DFSZ axion: analysis and generalization

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*A mio nonno Vincenzo,*
*che non ha mai smesso di restarmi vicino.*

# Sommario

Lo scopo di questo progetto è quello di prendere in esame alcune estensioni del modello standard capaci di incorporare al loro interno l'assione QCD.

Nel primo capitolo vengono analizzate da zero le motivazioni per l'introduzione del concetto di assione. Il problema della violazione CP nelle interazioni forti è introdotto a partire dalla non banale struttura topologica del gruppo di simmetria di colore in QCD.

Nel secondo capitolo viene studiato un particolare modello di assione, ossia quello DFSZ. Vengono quindi sviluppati il suo contenuto di campi e il suo spettro di massa. In particolare, viene considerata una versione leggermente differente di modello DFSZ, dove il termine quartico misto del potenziale di Higgs è sostituito da un contributo cubico.

Una piccola frazione di questo lavoro è inoltre dedicata a riassumere il problema dei *domain walls*. Grande importanza è data alla costruzione di modelli teorici che possano mettere al sicuro la teoria da una catastrofe cosmologica per mezzo di una oculata scelta dell'attribuzione delle cariche di Peccei-Quinn.

Per concludere, gli accoppiamenti degli assioni con i fermioni sono presi in esame. Grande attenzione viene rivolta alla generalizzazione del modello richiedendo la simultanea soppressione degli accoppiamenti assionici a protoni, neutroni ed elettroni. Una applicazione importante di questa impostazione è la capacità di indebolire diversi vincoli astrofisici, permettendo di raggiungere la cosiddetta finestra di massa pesante per l'assione. Viene argomentato come una condizione necessaria perchè questo avvenga sia l'introduzione di una assegnazione di cariche di Peccei-Quinn non universale per i quark e i leptoni del modello standard. A seguire, viene identificata una classe minimale di questi assioni DFSZ, dove le proprietà di nucleofobia ed elettrofobia possono essere implementate.

# Abstract

The aim of this project is to examine some extensions of the SM model able to embed the QCD axion particle.

In the first chapter the motivations for the introduction of the axion concept are analyzed from scratch. The strong CP problem is introduced starting from the non-trivial topological structure of the color symmetry of QCD.

In the second chapter we study a particular axion model: the DFSZ model. So, its field content and its mass spectrum are developed. In particular, we consider a slighlty different version of DFSZ theory, where the mixed quartic term of the Higgs potential is replaced by a cubic one.

A small part of this project is also devoted to summarize the *domain wall problem*. Great importance is given to how model building can save the theory from that cosmological catastrophe through an accurate choice of the Peccei-Quinn charge pattern.

Finally, the axion couplings to fermions are taken into account. We expecially investigate a generalization of the model requiring conditions to simultaneously suppress the axion coupling to protons, neutrons and electrons. An important application of this setup is the relaxation of various astrophysical bounds, which allows to reach the so-called heavy axion mass window. It is shown that a necessary condition for that to happen is the introduction of a non-universal Peccei-Quinn charge assignment for the standard model quarks and leptons. Next, it is identified a minimal class of these non-universal DFSZ axions, where nucleophobia and electrophobia are feasible.

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# Contents

# Introduction

Nowadays, there are compelling evidences that our insight of particle physics and cosmology is unsatisfactory and that some extensions or, even worse, gross modifications are called for. Both theoretical requirements and experimental data suggest there is still a huge number of problems whose solution escapes our understanding. The dark matter and dark energy existence, the baryon asymmetry conundrum or the break of flavour symmetry are just some open questions belonging to a long list. The path to pursue in order to improve our knowledge of nature is clearly non-unique: it is drawn by theoretical imagination, dammed by experimental results.

In this scenario, an extremely appealing idea is the concept of axion. The introduction of this new particle in the standard model spectrum seems to be highly motivated from a theoretical point of view, because it will enable us to sort out two appearently unrelated issues at once: the strong CP problem, troubling particle physics, and the dark matter puzzle, which plagues astrophysics and cosmology at their foundation. Moreover, the axion way has all the aesthetical requirements of simplicity and naturalness that, as Dirac said, make a theory more likely to be true.

The history of axion is quite lengthy and convoluted. Maybe, we can track back its origin in the 1970s, when, with the development of physics of strong interactions, the $U(1)_A$-*problem* shows up for the first time. The QCD effective theory involving up and down quarks predicted the $U(1)_A$ symmetry group to be spontaneously broken by a chiral condensate operator, just as the $SU(2)_A$ one: for the latter, this break gives rise to some pseudo-goldstone bosons (i.e the pions), but no particle of the QCD spectrum seems to be associated to the $U(1)_A$ group. This missing goldstone boson was the bedrock of the problem, which was completely swept out in 1976 by 't Hooft, who realized the $U(1)_A$ symmetry was actually not a symmetry of QCD at all. It was understood that the non-abelian nature of the $SU(3)_c$ color symmetry lead to highly non-trivial properties of QCD, which could violate classical symmetries at quantum level.

Nevertheless, the solving of one problem was the beginning of a new one. The awareness of the complex topological structure of QCD vacuum resulted in the appearence of a new term in the Lagrangian: the so-called $\theta$-*term*. In its turn, this new contribution introduced, through a parameter $\theta$, sources of CP violation in strong interactions, which were known to respect parity with great accuracy. By measuring the neutron electric dipole momentum, the staggering constraint of $|\theta| < 1.3 \times 10^{-10}$ emerged. But if the introduction of this new parameter is well justified, as it is, how could we accept this unnatural fine-tuning of a variable, which is, in principle, free to take its values in the whole set $[0, 2\pi[$? Until now, a conclusive answer to the strong CP problem does not exist yet.

By the way, among a plethora of possible ways-out that were proposed, in 1977 Peccei and Quinn suggested a pretty easy and viable possibility. They noticed how the $\theta$ parameter could have been made unphysical and erased from the theory by simply endowing the SM Lagrangian with an extra global and anoumalous phase symmetry. However, in 1978 Weinberg and Wilczek figured out that this interesting path had a side-effect. Indeed, if the Peccei-Quinn symmetry had turned out to be spontaneously broken by the VEV of a scalar operator, a new particle would have come out: this awareness was the real birth of axion.

From that moment on, theoretical models trying to embed axions started to thrive. Despite that, it was immediately realized how difficult it could be to cope with phenomenology. A first class of models (PQWW), which identified the PQ symmetry breaking scale with the electroweak one, was quickly ruled out by experimental results and, in particular, by the tight bound $\mathcal{B}(K^+ \to \pi^+ + \text{nothing}) < 3.8 \times 10^{-8}$ (which an updated measurement made even stronger $\mathcal{B}(K^+ \to \pi^+ + \text{nothing}) < 7.3 \times 10^{-11}$). Thereby, these *visible axion models* were abandoned, together with their associated axion mass window, around $10KeV$. The axion should have been an even lighter and more weakly interacting particle in order to agree with experimental outcomes. Consequently, between 1979 and 1981, a new set of theories was designed, predicting the existence of an *invisible axion*, referring to two paradigmatic classes: the KSVZ and DFSZ models. The former is probably the easiest invisible axion model which can be thought of. Nonetheless, despite the simplicity, it is affected by some cosmological drawbacks, requiring some more elaborated versions. The latter is instead more involved just from the onset. In spite of that, both of them are characterized by a new scalar field (a Higgs singlet), whose VEV defines a further energy scale, directly related to all axion features.

After the introduction of the axion concept, it was also quickly understood how this particle, interacting very feebly with common matter and light, could be a natural dark matter candidate. The need for dark matter is extremely motivated by astrophysical observations: the unexpected behaviour of rotation curve of galaxies, the mass discrepancy emerging from gravitational lensing or the inconsistency between baryon and matter abundance measured in cosmic microwave background, just to mention some of them. Moreover, we know how cosmological data support the idea of a flat universe, but that, at the same time, the amount of energy density that we observe is not enough to pursue this scenario. Nevertheless, the SM does not offer any possible dark matter particle: that was the reason why the proposal of axion to solve the strong CP problem sounded like an interesting opportunity. It was not an *ad hoc* introduction, made to explain something we can not see or understand, but a possibility stemming directly from a phenomenological quantum field theory problem.

All of these theoretical motivations justify the great effort coming both from particle physics and astrophysical research in order to better constrain and unveil axion properties. Of course, because of its own nature, directly revealing axions turns out to be incredibly challenging, exactly like the detection of neutrinos after their theoretical prediction by Pauli. An important step forwards was done by Sikivie in 1983 with a paper of him, in which he proposed two of the most fruitful techniques to search for invisible axions: the axion helioscope, to detect the flux of axions supposed to be emitted from the Sun, and the axion haloscope, to probe the presence of axions from the hypothetical DM galactic halos. However, nowadays, a great number of experiments has been designed, including some to generate axions in laboratories, too, such as the *light shining through walls* (LSW) approach or the *vacuum magnetic birefringence* (VMB) procedure. In 2013 a further way of disclosing the axion mystery was advanced by Graham and Rajendran: it consisted in employing potential fluctuations of the neutron electric dipole momentum, that might be induced by axion background oscillations.

In the latest years a lot of data have been collected, constraining more and more the space of parameters where axions can hide. Considerations about a hot cosmological axion population set an upper bound on axion mass of $m_a < 0.8eV$, while black hole superradiance phenomena require a lower bound of $m_a > 10^{-10}eV$. The evolution of stars of horizontal branch of globular clusters imposes a limit on axion coupling to photons of $g_{a\gamma} < 6.6 \times 10^{-11}GeV^{-1}$, which is the strongest limit applicable to a wide range of axion masses: more severe constraints were obtained by considering the TeV $\gamma$ rays trasparency of our universe, but only for axion masses of $m_a < 10^{-7}$ or much lighter. An even higher sensitivity to $g_{a\gamma}$ is expected to be achieved by the next generation of axion helioscopes, such as IAXO and IAXO+. On the other hand, white dwarfs and supernovae rates of cooling were employed to set a bound respectively on the axion-electron coupling of $|g_{ae}| < 2.7 \times 10^{-13}$ and on the axion-nucleons interaction as $g_{ap}^2 + g_{an}^2 < 3.6 \times 10^{-19}$.

Furthermore, an axion particle must be embodied in a suitable extension of the standard model, so that other constraints on axion theories will be related to the field content of these BSM constructions. Together with the direct detection of new particles with some new generations of accelerators (that might be able to explore the ultra-TeV region), the most powerful way of confining the intrinsic freedom of these SM generalizations is through the *electroweak precision test*. By comparing their prediction of some of the most precisely known electroweak quantities with the corresponding experimental values, a good amount of information can be extracted. Among the most famous and useful parameters, we have the ratio between gauge boson masses $\rho$, which turns out to be $\rho = 1.00037 \pm 0.00023$.

Even if little room seems to be left to axions to stay hidden, their existence is still very well motivated. What we are going to do in this text is exactly to explore some of the residual freedom that is left to theory to predict axion properties respecting all of experimental data. After all, because of their growing precision, they can not be evaded or bypassed any more. That will give us the opportunity to realize how profitable is the axion idea, which is able to relate, with a simple concept, profoundly different research areas.

# Chapter 1

# The strong CP problem

The strong CP problem is a very well-known issue in the quantum chromodynamic scenery, naturally arising from the presence of a parity odd term inside the Lagrangian (the so-called *theta term*). According to the basic principle of a quantum field approach, any kind of term respecting the general symmetries of the theory, together with unitarity and renormalizability, enjoys the same right to enter the Lagrangian as any other. Therefore, some particular explanation for our incapability of detecting it (or better the consequences of it) must be given. It will be shown how the Peccei-Quinn mechanism offers a very elegant way to emerge from this problem, but that it has, as a side-effect, the prediction of a new kind of particle: the axion.

## 1.1 The $U(1)_A$ problem

Quantum chromodynamic is the theory describing the strong interaction, one of the three elementary forces of nature that can be understood employing a quantum field theory approach. Between those three, the strong force is also the most intense at low energy (because of the peculiar behavior of the $\beta$-function), making any kind of perturbative calculation useless in this regime. Moreover, it is known that, among matter fields, only quarks perceive this force, because of a color charge that can assume three different nuances (red, blue and green), as proved by scattering experiments. Following the general scheme of QFT, these color-charged particles will interact with each other through the exchange of quanta of the strong field, i.e gluons.

The Lagrangian of this theory can be easily obtained starting from a free fermionic model involving six triplets of Dirac fields $\psi$ (just as the six known quark flavours), enjoying a $SU(3)_c$ global symmetry of color, which mixes the three components of the triplets themselves. These fields belong to the fundamental representation of the group, as dictated by phenomenology. Then, following the gauge principle, this symmetry is made local through the introduction of a matrix-valued vector field $A_\mu$ of the adjoint representation. In so doing, the most general interaction between matter and force fields can be accounted for by the theory. These building blocks can be put together in order to obtain all possible Lorentz invariant terms with quartic mass dimensions. The possible list of them will be narrowed down by requiring unitarity, renormalizability and local $SU(3)_c$ invariance. The QCD Lagrangian compatible with these requirements has the unique form

$$\mathcal{L}_{QCD} = \sum_{f=1}^{6} [\bar{\psi}_f (i\not{\mathcal{D}} - M)\psi_f] - \frac{1}{2} tr[\mathcal{G}^{\mu\nu}\mathcal{G}_{\mu\nu}] \tag{1.1}$$

where $f$ is a flavour index and color indeces have been clearly understood. $\mathcal{D}_\mu$ is the matrix-valued covariant derivative operator

$$\mathcal{D}_\mu \psi \equiv (\partial_\mu - igA^a_\mu \tau^a_F)\psi \tag{1.2}$$

with $a$ the color index running from one to three and $\tau^a_F$ the infinitesimal hermitian generators in the fundamental representation (that is the Gell-Mann matrices $\lambda_a$), fulfilling the correct commutation relation of the group. Finally, $\mathcal{G}_{\mu\nu}$ is the field strength tensor, with the well-known formula

$$\mathcal{G}_{\mu\nu} \equiv \frac{i}{g}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{1.3}$$

Being $g$ the strong coupling, the last term tells us that the field which mediates the strong force is charged itself: this fact gives rise to self-interactions. For the kinetic term of the field strength tensor the normalization $tr(\tau^a_F \tau^b_F) = \frac{1}{2}\delta^{ab}$ has been used, where the trace acts upon color indeces.

With the introduction of a non-abelian gauge field, the $SU(3)_c$ symmetry is made local. Hence, under a transformation $U_\omega(x) = exp\{ig\omega^a(x)\tau^a_F\}$ of the group acting on the Dirac field triplets, the covariant derivative will transform homogeneously as

$$\mathcal{D}_\mu \psi(x) \longmapsto U_\omega(x)\mathcal{D}_\mu \psi(x), \tag{1.4}$$

provided that we require the following non-linear transformation law for the gauge field

$$A_\mu(x) \longmapsto U_\omega(x) A_\mu(x) U^\dagger_\omega(x) - \frac{i}{g}[\partial_\mu U_\omega(x)] U^\dagger_\omega(x), \tag{1.5}$$

and so

$$\mathcal{G}_{\mu\nu}(x) \longmapsto U_\omega(x)\mathcal{G}_{\mu\nu}(x) U^\dagger_\omega(x) \tag{1.6}$$

Consequently, the field strength is not invariant under a gauge transformation and, as it is for any non-abelian gauge theory, it will not be a physical object. Therefore, it is quite evident that the concept of color magnetic and color electric field, naively imported from QED, can not be pursed in this context (at least, not in the way we are accustomed to it).

In the expression (1.1), we have also inserted a mass term for quark fields including a six-by-six mass matrix $M$. This latter will be diagonal, if we are considering the physical states of the theory and not simply the gauge ones. In the standard model, this term will be generated by a Higgs mechanism, starting from the Yukawa-term interactions. Anyway, one should observe that nothing compels this mass matrix to be real and in general it will not be so.

It is worthy of note that, if we assume the compatibility with symmetries as the sole guiding principle to guess the form of the Lagrangian, there is still some freedom in choosing $\mathcal{L}_{QCD}$. Nevertheless, we will be able to develop this aspect better in what follows.

Once reminded the underlying non-trivial structure of the color gauge group, it will be interesting to clarify how the theory enjoys some extra global symmetries. To fathom the essence of this issue, it will be enough to analyse a low energy regime, where all quark degrees of freedom are frozen, with the exception of the up-like and down-like ones: after all, they are the two lightest quarks comprising the standard model. Focusing only on the quark sector of the Lagrangian, one will get:

$$\mathcal{L} = \bar{u}(\not{\mathcal{D}} - m_u)u + \bar{d}(\not{\mathcal{D}} - m_d)d. \tag{1.7}$$

Moreover, in most of physical applications, the very light masses $m_u$ and $m_d$ can be disregarded, too: thus, we will momentarily set them to zero. In this low energy approximation, we see that the Lagrangian possesses an extra global $U(2)$ symmetry mixing up and down quarks. The absence of mass terms, which couple the right and left chiral parts, lets this $U(2)$ group act independently on the left and right components:

$$\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \longmapsto \psi'_L = U_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad \psi_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \longmapsto \psi'_R = U_R \begin{pmatrix} u_R \\ d_R \end{pmatrix}. \tag{1.8}$$


Figure 1.1: On the left side, a couple of quark and anti-quark endowed with positive helicity is shown: the net momentum and angular momentum is zero. The same is true for the couple on the right side, whose particles have negative helicity. We remind that the match between chirality and helicity holds solely in the null mass limit.

As usual, we will have two conserved abelian currents ($j^\mu_L = \bar{\psi}_L \gamma^\mu \psi_L$ and $j^\mu_R = \bar{\psi}_R \gamma^\mu \psi_R$) and six non-abelian ones ($j^{\mu a}_L = \bar{\psi}_L \gamma^\mu \sigma^a \psi_L/2$ and $j^{\mu a}_R = \bar{\psi}_R \gamma^\mu \sigma^a \psi_R/2$), in addition to the sixteen color currents. They will be associated to some classical charges $Q_{L/R}$ and $Q^a_{L/R}$, which respect the algebra

$$[Q^a_{L/R}, Q^b_{L/R}] = i\epsilon^{abc} Q^c_{L/R} \qquad [Q^a_L, Q^b_R] = 0 \tag{1.9}$$

if $\epsilon^{abc}$ is the $SU(2)$ structure constant.

We can actually parametrize transformations of the $U(2)$ group in a pretty formal but useful way. Indeed, we can try to decompose this global symmetry not as $U(2)_L \times U(2)_R$, but with the improper notation $U(2)_V \times U(2)_A$, where $U(2)_V$ is the vector part of the group, being a group itself. This one is obtained by choosing $U_L = U_R$ and so acting in the same way on the right and left fermions. The corresponding currents are obtained as $j^\mu_V = j^\mu_L + j^\mu_R = \bar{\psi}\gamma^\mu\psi$ and $j^{\mu a}_V = j^{\mu a}_L + j^{\mu a}_R = \bar{\psi}\gamma^\mu \sigma^a \psi/2$. On the other hand, $U(2)_A$ is the set of axial transformations, simply built as $U(2)_A = U(2) - U(2)_V$ (or equally well setting $U_L = U_R^\dagger$). In this case, we can readily write $j^\mu_A = j^\mu_R - j^\mu_L = \bar{\psi}\gamma^\mu\gamma^5\psi$ and $j^{\mu a}_A = j^{\mu a}_R - j^{\mu a}_L = \bar{\psi}\gamma^\mu\gamma^5\sigma^a\psi/2$. Of course, $U(2)_A$ will not be a real group, because it will not be close under composition of two of its elements. This is perfectly described by means of the associated algebra of conserved charges:

$$[Q^a_V, Q^b_V] = i\epsilon^{abc} Q^c_V \qquad [Q^a_A, Q^b_A] = i\epsilon^{abc} Q^c_V \qquad [Q^a_V, Q^b_A] = i\epsilon^{abc} Q^c_A \tag{1.10}$$

Finally, if we separate $U(1)$ phase transformations from special unitary ones, we will be able to rewrite the original global transformation in the form $SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$.

In this very physical limit, in which up and down quarks are massless, the Lagrangian seems to enjoy a great number of symmetries. But the $U(1)_V$ and $SU(2)_V$ groups are actually approximate symmetries of nature: the first one is associated with the concept of *baryon number* and the second one with the *isospin* idea. We know, for example, how bound states of quarks can be classified in irreducible representations of $SU(2)_V$. The non-perfect realization of them is mainly due to the small difference between the two masses of up and down quarks (not to their appearing in the Lagrangian). A completely different discussion is called for while dealing with the remaining two symmetries, for which we do not have an explicit realization in nature. If this were the case, a $SU(2)_A$ symmetry could show up by associating to each isospin multiplet an axial counterpart of opposite parity (because of the presence of $\gamma^5$), for example.

Therefore, the first resonable idea is that the vacuum state of some scalar operator breaks spontaneously the $SU(2)_A \times U(1)_A$ symmetry. As well explained in [1], this fact is attributable to QCD vacuum properties. Just because the two quark masses are very light and the strong force tremendously intense, the energy cost to create couples comprising a particle and an anti-particle with zero momentum will be very little. Moreover, approaching the limit of null masses (where particles' energy and momentum are related by $E = |p|$), we know that Weyl equations hold, i.e $i\partial_t \psi_L = i\vec{\sigma} \cdot \vec{\nabla}\psi_L$ and $i\partial_t\psi_R = -i\vec{\sigma} \cdot \vec{\nabla}\psi_R$, from which $h\psi_L = -\psi_L$ and $h\psi_R = +\psi_R$ (with $h = \vec{\sigma} \cdot \hat{p}$ the helicity operator). So, the more energetically convinient choise is to produce a couple of particles endowed with zero angular momentum, too, which can be approximately obtained in the null mass limit by creating them with equal chirality (see figure 1.1). Consequently, we can imagine the QCD vacuum like a state swarming with particles created and immediately annihilated: this physical situation is referred to as *chiral condensate*. The name is due to the fact that these couples of particles will have a net chiral charge. Indeed, if we identify this quantum state with the easiest scalar operator achievable in our theory, e.g $\psi_{L\alpha i}\bar{\psi}^{R\alpha j}$ (where the greek indeces are the spinor ones and the latin ones are family indeces), we will readily see that its vacuum expectation value changes under $U(2)_L \times U(2)_R$ as:

$$\langle 0|(\psi_{L\alpha i}\bar{\psi}^{R\alpha j})'|0\rangle = U_{Li}{}^{k}U^{*j}_{R\ m}\langle 0|\psi_{L\alpha k}\bar{\psi}^{R\alpha m}|0\rangle = -v_\psi^3 U_{Li}{}^{k}U^{*j}_{R\ m}\delta_k{}^m \tag{1.11}$$

That immediately tells us how this operator is not invariant under an axial transformation, but it is unchanged by a vector one. It turns out to be a color singlet, not to break the $SU(3)_c$. Furthermore, we point out how the quantity $v_\psi$ must be a constant with dimensions of a mass. Thus, the chiral part of the symmetry is spontaneously broken by chiral condensates and, according to *Goldstone theorem*, we expect to find four massless goldstone bosons associated with the broken generators of the chiral group. By looking at the mass spectrum of QCD, the experimentally lightest particles that can be spotted are pions, coming in a triplet of $SU(2)_V$. The idea of spontaneous symmetry breaking applied in this context was a revolutionary one, because paves the way for the understanding of the extremely light pion masses. Those can be explained identifing the pions with pseudo-goldstone bosons of the broken $SU(2)_A$ symmetry (where the use of "pseudo" will be clarified later).

Thereby, our working hypothesis is that the QCD low energy Lagrangian is $SU(2)_V \times SU(2)_A$ symmetric: it is just the VEV that breaks the axial part of the group. So, we can build our Lagragian employing the massless quark doublets $\psi_i$ ($i = L, R$) and the quark condensate operator $\psi_\alpha\bar{\psi}^\alpha$ and inserting all possible terms respecting the above symmetry. Taking into account that $\psi_i$ transforms according to (1.8) and $(\psi_\alpha\bar{\psi}^\alpha)' = U_L(\psi_\alpha\bar{\psi}^\alpha)U_R^\dagger$, we can write [2]

$$\begin{aligned}\mathcal{L}_{chiral} =& \bar{\psi}i\partial\!\!\!/\psi + \frac{1}{4f_\pi^2\mu^2}tr[\partial_\mu(\psi_\alpha\bar{\psi}^\alpha)^\dagger\partial^\mu(\psi_\beta\bar{\psi}^\beta)] + V_c(\psi_\alpha\bar{\psi}^\alpha) + \\ &+ y(\bar{\psi}_L(\psi_\alpha\bar{\psi}^\alpha)\psi_R + \bar{\psi}_R(\psi_\alpha\bar{\psi}^\alpha)^\dagger\psi_L)\end{aligned} \tag{1.12}$$

where the trace in the kinetic term is performed over flavour indeces and $V_c$ is the chiral potential providing the symmetry breaking scenario. It worth noticing that, for reasons of convenience, we have introduced two new energy scales $f_\pi$ and $\mu$, which are related to $v_\psi$ as $v_\psi^3 = f_\pi^2\mu$: $f_\pi$ is a constant associated to pion physics, that we will specify later on, and $\mu$ is a different way of encoding quark condensate phenomena.

As first described in the Gell-Mann Levy model [3], the four degrees of freedom of the scalar chiral operator $\psi_{L\alpha}\bar{\psi}^{R\alpha}$ can be parametrized in a non-linear way as

$$(\psi_\alpha\bar{\psi}^\alpha)(x) = f_\pi^2\sigma(x)\Sigma(x) \tag{1.13}$$

Just because $\bar{\psi}\psi$ is a composite scalar operator, it has the non-customary dimensions of a cubic mass, while the scalar field $\sigma$ is a parity even component with simple mass dimensions. $\Sigma$ is a unitary matrix containing the pion fields. It is clear that, to recover the previous transformation law of the condensate operator, $\sigma$ must be a singlet and $\Sigma$ has to belong to the adjoint representation, i.e $\Sigma(x) \longmapsto U_L\Sigma(x)U_R^\dagger$. This latter can be explicitly written down as

$$\Sigma(x) = \exp\left\{i\sigma^a\frac{\pi^a(x)}{f_\pi}\right\}, \tag{1.14}$$

In this matrix, pion fields clearly play the roles of phases multiplied by the three Pauli matrices $\sigma^a$, which have been used to express the generators of $SU(2)$. $f_\pi$ is exactly a dimensional constant called *pion decay constant*, because it can be computed through the $\pi^-$ weak decay (i.e $\pi^- \to \mu^- \bar{\nu}_\mu$, to remind the main decay channel not suppressed by the helicity contraint), getting a result of $92MeV$. Nevertheless, in literature its value can change by a factor of $\sqrt{2}$, according to the convention adopted.

The symmetry breaking situation described above will be mirrored here simply requiring $\langle 0|\sigma|0\rangle = -\mu$ and $\langle 0|\Sigma|0\rangle = \mathbb{1}_{2\times 2}$ (or $\langle 0|\pi^a|0\rangle = 0$), achievable through a chiral invariant potential of the form [2]

$$V_{chiral}(\psi_\alpha \bar{\psi}^\alpha) = -btr[(\psi_\alpha \bar{\psi}^\alpha)^\dagger (\psi_\beta \bar{\psi}^\beta)] + \lambda (tr[(\psi_\alpha \bar{\psi}^\alpha)^\dagger (\psi_\beta \bar{\psi}^\beta)])^2 \tag{1.15}$$

with $b$ and $\lambda$ some real positive constants, such that $b/\lambda = 2v_\psi^6$.

Using (1.14) and the small field fluctuation hypothesis, we can recast (1.13)

$$(\psi_\alpha \bar{\psi}^\alpha)(x) \approx f_\pi^2 \sigma(x) \mathbb{1}_{2\times 2} + f_\pi \mu \pi^a \sigma^a \tag{1.16}$$

Under a $SU(2)_A$ rotation, we can also state

$$\begin{aligned}(\psi_\alpha \bar{\psi}^\alpha)'(x) = e^{i\beta\cdot\sigma/2}(\psi_\alpha \bar{\psi}^\alpha)(x) e^{i\beta\cdot\sigma/2} \approx (\psi_\alpha \bar{\psi}^\alpha)(x) + i\{\delta\beta\cdot\sigma/2, (\psi_\alpha \bar{\psi}^\alpha)(x)\} = \\ = (\psi_\alpha \bar{\psi}^\alpha)(x) + i f_\pi^2 (\delta\beta\cdot\sigma)\sigma(x) + i\frac{f_\pi}{2}\mu\pi^b \delta\beta^a \{\sigma^a, \sigma^b\}\end{aligned} \tag{1.17}$$

and if we match it with the variation $\delta(\psi_\alpha \bar{\psi}^\alpha)(x) \approx f_\pi^2 \delta\sigma(x) \mathbb{1}_{2\times 2} + f_\pi \mu \delta\pi^a \sigma^a$, we eventually end up with

$$\delta\sigma(x) = i\frac{\mu}{f_\pi}\delta\beta\cdot\pi \qquad \delta\pi^a = i\frac{f_\pi}{\mu}\delta\beta^a \sigma(x) \tag{1.18}$$

These formulae clarify how pion fields are associated to the broken part of $SU(2)_L \times SU(2)_R$: they possess non-zero fluctuations around the VEV, given by $\delta\pi^a = i\frac{f_\pi}{\mu}\delta\beta^a \langle\sigma(x)\rangle = -i\frac{f_\pi}{\mu}\delta\beta^a \mu \neq 0$.

If we rewrite (1.12) in terms of this new parametrization, we obtain

$$\mathcal{L}_{chiral} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{f_\pi^2}{2\mu^2}\partial_\mu \sigma \partial^\mu \sigma + \frac{f_\pi^2}{4\mu^2}\sigma^2 tr[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + V_c(\sigma) + y\sigma(\bar{\psi}_L \Sigma \psi_R + \bar{\psi}_R \Sigma^\dagger \psi_L) \tag{1.19}$$

It is interesting to emphasize how the Gell-Mann Levy model predicts the existence of an effective contribution to quark masses, even without an explicit term in the Lagrangian. As a matter of fact, when $\sigma$ and $\Sigma$ settle into their VEVs, the initially massless light quarks get a mass by means of the last term of (1.19). That turns out to be equal to $m_u^{eff} = m_d^{eff} = y\mu^2/f_\pi$, where the extra factor $(\mu/f_\pi)$ stems from the rescaling of $\sigma(x)$, used to correctly normalize its kinetic contribution. This mass will be generated while quarks propagate through the QCD vacuum. The up and down quarks interact with the vacuum through the same coupling $y$ and, thereby, their masses will be equal and $SU(2)_V$ will be preserved.

Without any claim to describing in full details this model here, we just mention that a clear problem of this theory is the identification of the $\sigma'$ fluctuations around the VEV $\langle\sigma\rangle$ with some of the known particles. Therefore, after the spontaneous breaking of the $SU(2)_A$ symmetry, the mass of the sigma field is sent to infinity and the sigma oscillations are integrated out through their equation of motion. If, furthermore, quark fields at low energy are hidden by confinement, we are left with a model where only pions appear. In this new scenario, fluctuations of the chiral operator around the vacuum will be simply given by:

$$\langle 0|(\psi_\alpha \bar{\psi}^\alpha)(x)|0\rangle = -v_\psi^3 \Sigma(x) \tag{1.20}$$

The Lagrangian involving solely the pseudo-scalar part of the quark condensate field after symmetry breaking can be immediately read off from (1.12) to be

$$\mathcal{L} = \frac{1}{4} f_\pi^2 tr[\partial^\mu \Sigma^\dagger \partial_\mu \Sigma], \tag{1.21}$$

Even if we have derived it from our previous reasoning, it is worthy of note that, because of the property $\Sigma^\dagger\Sigma = 1$, the only viable non-trivial terms include derivatives of $\Sigma$ field.

Upon expanding $\Sigma$ in powers of the pion field, we obtain kinetic and quartic interaction terms for pions:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \pi^a \partial_\mu \pi^a - \frac{1}{6} f_\pi^{-2} (\pi^a \pi^a \partial^\mu \pi^b \partial_\mu \pi^b - \pi^a \pi^b \partial^\mu \pi^a \partial_\mu \pi^b) + ... \tag{1.22}$$

However, up to now, pions are perfectly massless goldstone bosons. Next, we can explore the effects of introducing back the original mass term for quarks (that coming from a would-be Higgs mechanism). We have again to consider the most general achievable contribution. A useful rule of thumb can be to couple the Dirac fields with an external scalar source $s$, which transforms as $s \longmapsto LsR^\dagger$, and to employ it to build up an extra invariant term. Then, by setting $s = M$ (where $M$ is a two-by-two mass matrix containing the up and down masses), the symmetry is correctly broken. This new contribution turns out to be:

$$\mathcal{L}_{\text{mass}} = (\bar{\psi}_L^{\alpha i} M_i{}^j \psi_{R\alpha j} + h.c.) = -(M_i{}^j \psi_{R\alpha j} \bar{\psi}_L^{\alpha i} + h.c.) = -tr[M\psi_\alpha \bar{\psi}^\alpha] \tag{1.23}$$

where we made use of the grassmann nature of spinor fields in the next-to-last passage. The plus sign in front of $\bar{\psi}_L^{\alpha i} M_i{}^j \psi_{R\alpha j} + h.c.$ is necessary, because that leads us to a minus in $-tr[M\psi_\alpha\bar{\psi}^\alpha]$, consistently with the corresponding plus of the kinetic term. It worth pointing out that the matrix $M$ is generally complex. With a $SU(2)_L \times SU(2)_R$ transformation, we can make this matrix diagonal, but it will always survive a complex phase in the most general case. The role of it and its consequences are the core of all this work and they will be extensively discussed later. What it emerges from experiments is, anyway, that this complex phase should be very small or, at least, consistent with a zero value. So, we will assume for now that $M = M^\dagger$. Then, if we substitute the $VEV$ of the theory inside the previous formula and we expand it in powers of pion field fluctuations as before, what we get is

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= v_\psi^3 tr[M(\Sigma + \Sigma^\dagger)] = -\frac{v_\psi^3}{f_\pi^2} tr[M\sigma^a\sigma^b]\pi^a\pi^b + ... = \\ &= -\frac{1}{2}\frac{v_\psi^3}{f_\pi^2} tr[M\{\sigma^a, \sigma^b\}]\pi^a\pi^b + ... = -\frac{v_\psi^3}{f_\pi^2} tr[M]\pi^a\pi^a + ... \end{aligned} \tag{1.24}$$

where we neglected the constant term in the expansion. In the last expression, clearly, a mass term for pions shows up, given by the so-called *Gell-Mann-Oakes-Renner relation*:

$$m_\pi^2 = 2\frac{(m_u + m_d)v_\psi^3}{f_\pi^2}. \tag{1.25}$$

It is noteworthy that the contribution to pion masses (up to now equal for all of pions) comes solely from the introduction of a small mass term for quarks, which *explicitly* breaks the axial $SU(2)_A$ symmetry: this is the final reason why we speak of pions as *pseudo-goldstone bosons*. As already stressed, the isospin symmetry $SU(2)_V$ will be violated just by the mass difference between the two quarks (which renders the matrix $M$ not diagonal any more) and by some electromagnetic corrections (the up and down particles possess distinct electric charges). Nevertheless, there will still be evidences of this imperfect symmetry, if these discrepancies are not too large.

But we started our discussion speaking of four goldstone bosons associated with the breaking of the axial part of the $U(2)_L \times U(2)_R$ global symmetry of the QCD Lagrangian. The reason why we neglected the possible boson of the $U(1)_A$ symmetry (and so an extra phase term in (1.20)) is that there are no experimental evidences for it. Indeed, the lightest pseudoscalar ($J^P = 0^-$) mesons that have been measured are (see table 1.1): three pions, well explained by the presented effective model and other five mesons of larger masses ($K^{\pm}$, $K^0$, $\bar{K}^0$ and $\eta$). These latter can be theoretically understood adding the strange quark to our description and enlarging our global group of symmetry to $SU(3)_L \times SU(3)_R$. The number of mesons equals that of $SU(3)$ generators, so that the matrix (1.14) is suitably replaced by

$$\Sigma = exp\left\{\frac{i}{f_\pi}\left(\sum_{a=1}^{8}\pi^a\lambda^a + \mathbb{1}_{3\times3}\frac{\eta_0}{\sqrt{2}}\right)\right\} \quad \text{with} \quad \sum_{a=1}^{8}\pi^a\lambda^a = \begin{pmatrix} \pi^0 + \frac{\eta_8}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{\eta_8}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -2\frac{\eta_8}{\sqrt{3}} \end{pmatrix} \tag{1.26}$$

where $\lambda^a$ are the eight Gell-Mann matrices and where the possible $U(1)_A$ singlet $\eta_0$ has been highlighted. As justified in [4], the constant $f_\pi$ can still be assumed to be the pion decay constant by convention. Of course, the assumption of a massless strange quark is less justified here, if one considers that

$$\frac{m_d - m_u}{m_u + m_d} \approx 0.29 \qquad \frac{m_s}{m_u + m_d} \approx 25 \tag{1.27}$$

and just that is at the origin of the higher masses of the additional mesons of the octet. The imperfect completion of the $SU(3)_V$ symmetry is responsible for a further complication: states with the same quantum numbers can mix with each other. Just because isospin is a pretty good symmetry group of nature, the mixing will affect the neutral $S = 0$ pseudo-scalar mesons $\eta_8$ and $\eta_0$: the $\pi^0$ is essentially protected, because one can prove that possible mixings are proportional to the tiny factor $m_d - m_u$. The physical states $\eta$ and $\eta'$ will be obtained rotating the previous ones as

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} cos\theta_m & -sin\theta_m \\ sin\theta_m & cos\theta_m \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix} \tag{1.28}$$

with a mixing angle $\theta_m \approx 17°$ [5]. As a matter of fact, one should notice from table 1.1 how the $\eta$ mass slightly departs from the kaon ones. Nevertheless, the experimental mass value of the remaining $\eta'$ is even larger: this suggests that it can not be identified with our alleged $U(1)_A$ meson, unless something is missing in our theory. The conundrum of this expected but unobserved golstone boson was dubbed by Weinberg $U(1)_A$ *problem*. Weinberg himself suggested that the $U(1)_A$ transformation should not have been a symmetry of the low energy Lagrangian at all [6]. Then, this fact was shown by t'Hooft [7], who explained how the QCD vacuum structure should have been more complicated than assumed. This would have made the $U_A(1)$ group not a symmetry anymore, because of quantum corrections.

## 1.2 The QCD vacuum structure

We have already noticed that QCD, as a non-abelian gauge theory, is very different from the QED counterpart: we pointed out how the field strength tensor is not gauge invariant here or how gauge fields carry a charge themself. But there are much more subtle aspects that need to be highlighted in QCD.

Everybody knows that a good way to build a Lagrangian is to include all terms which respect the symmetries of the theory and its basic assumptions. So, writing down the QCD Lagrangian,

| meson | state | mass (MeV) | strangeness |
|---|---|---|---|
| $\pi^0$ | $\frac{\bar{u}u-\bar{d}d}{\sqrt{2}}$ | $134.9770 \pm 0.0005$ | 0 |
| $\pi^+$ | $u\bar{d}$ | $139.57061 \pm 0.00024$ | 0 |
| $\pi^-$ | $d\bar{u}$ | $139.57061 \pm 0.00024$ | 0 |
| $K^+$ | $u\bar{s}$ | $493.677 \pm 0.016$ | -1 |
| $K^-$ | $s\bar{u}$ | $493.677 \pm 0.016$ | +1 |
| $K^0$ | $d\bar{s}$ | $497.611 \pm 0.013$ | -1 |
| $\bar{K}^0$ | $s\bar{d}$ | $497.611 \pm 0.013$ | +1 |
| $\eta$ | $\sim \frac{\bar{u}u+\bar{d}d}{\sqrt{2}}$ | $547.862 \pm 0.017$ | 0 |
| $\eta'$ | $\sim \bar{s}s$ | $957.78 \pm 0.06$ | 0 |

Table 1.1: List of the first lighest pseudo-scalar ($J^P = 0^-$) mesons with related masses and strangeness number [8]. It is remarkable how the composition of the mesons $\eta_8 = \frac{\bar{u}u+\bar{d}d-2\bar{s}s}{\sqrt{6}}$ and $\eta_0 = \frac{\bar{u}u+\bar{d}d+\bar{s}s}{\sqrt{3}}$ is modified into that of the states reported in this table after the mixing induced by $\theta_m$.

we have been superficial on the possible existence of an extra contribution. In both abelian and non-abelian theories, there are two Lorentz invariant terms that can be obtained starting from the general field strength $\mathcal{F}^{\mu\nu}$, i.e $tr[\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}]$ and $\epsilon^{\mu\nu\rho\sigma}tr[\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}]$. The first term is parity even, while the second one parity odd, because of the presence of the pseudo-tensor $\epsilon^{\mu\nu\rho\sigma}$. But one can directly verify, after some algebraic manipulations, that the latter is a total derivative:

$$\epsilon^{\mu\nu\rho\sigma}tr[\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}] = \partial_\mu\big(4\epsilon^{\mu\nu\rho\sigma}tr[A_\nu\partial_\rho A_\sigma - \frac{2ig}{3}A_\nu A_\rho A_\sigma]\big) = 2\partial_\mu J^\mu_{CS} \tag{1.29}$$

where the four-vector $J^\mu_{CS}$ is named the topological *Chern-Simons current.* It is known that the Lagrangian is always defined up to a total derivative, provided that fields will go to zero at infinity fast enough. Indeed, after a Wick rotation, just because the Euclidean action is the integral of the Lagrangian over coordinate space, we could claim

$$\sim \int d^4x_E \epsilon_{\mu\nu\rho\sigma}tr[\bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\rho\sigma}] = \int d^4x_E 2\partial_\mu \bar{J}^{CS}_\mu = \int_{S^3} d\sigma_{E\mu}2\bar{J}^{CS}_\mu \tag{1.30}$$

where in the last expression we are integrating over spatial infinity of the euclidean space, topologically equivalent to a 3-sphere. The $\sim$ symbol stands for a correction factor, due to the euclidean transition, which will be specified later on. We remind how, in euclidean formalism, all indeces are reported downward by convention. Moreover, we know that:

$$\epsilon^{\mu\nu\rho\sigma}tr[\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}] = -i\epsilon_{\mu\nu\rho\sigma}tr[\bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\rho\sigma}] \qquad \text{and} \qquad \partial_0 = i\partial_4 \qquad \partial_i = \partial_i \tag{1.31}$$

and hence we can derive, by direct inspection of (1.29), the transformation for the pseudo-four-vector $\bar{J}^{CS}_\mu = (-J^{CS}_0, -i\vec{J}^{CS})$ with $k \in \{1,2,3\}$.

Thus, what makes the difference is the asymptotic behavior of $\bar{J}^{CS}_\mu$. From QED we are accustomed to set to zero this term and the theory works pretty well with this assumption. But nothing assures us that a straightforward generalization exists for the non-abelian case. To see that, it will be needed to go through the topic in more details.

### 1.2.1 The boundary conditions

A first mathematical notion, which will prove to be useful, is that of *homotopy classes*, the set of all functions connected by a continuous transformation. For example, we can write a map $S^1 \to U(1) \sim S^1$ as $g_\beta(\theta) = exp[i(\nu\theta + \beta)]$. All possible maps are parametrized by means of the two variables $\nu \in \mathbb{Z}$ and $\beta \in \mathbb{R}$. But all functions with different values of $\beta$ belong to the same homotopy class, because they can be transformed in each other with this homotopy:

$$H(\theta, s) = exp[i(\nu\theta + (1-s)\beta_1 + s\beta_2)] \tag{1.32}$$

which describes all maps with the same $\nu$ for varying $s \in \mathbb{R}$. The same thing can not be done for distinct values of $\nu$, which therefore identifies separate homotopy classes. $\nu$ is an integer number, named *winding number*, which can be directly derived from the map itself:

$$\nu = \frac{-i}{2\pi}\int_0^{2\pi} d\theta g(\theta)^{-1}\frac{dg(\theta)}{d\theta} \tag{1.33}$$

Anyway, a possible non-trivial generalization can be obtained for more complicated maps $S^n \to SU(m)$. We will be particularly interested in the case $n = 3$ and $m = 3$, for obvious reasons. In this situation different homotopy classes are labelled by a $\nu$, whose euclidean formula is given by:

$$\nu = \frac{1}{24\pi^2}\int d^3\theta\epsilon^{ijk} tr[(V(\theta)\partial_i V^\dagger(\theta))(V(\theta)\partial_j V^\dagger(\theta))(V(\theta)\partial_k V^\dagger(\theta))] \tag{1.34}$$

which is invariant under a change of coordinates and under smooth deformations of $V(\theta)$ itself. Moreover, it is remarkable that this topological object, sometimes dubbed *Pontryagin index*, is gauge invariant. It can also be noticed that, multiplying two maps each other, one gets a new map whose winding number is the sum of the two individual winding numbers of the original maps.

This topological structure will have profound consequences on QCD. Indeed, if we go back to our euclidean action, we can write

$$S_E = -\frac{1}{2}\int d^4x_E tr[\bar{\mathcal{F}}_{\mu\nu}\tilde{\bar{\mathcal{F}}}_{\mu\nu}] = -\frac{1}{2}\int d\Omega\int_0^\infty dr r^3 tr[\bar{\mathcal{F}}_{\mu\nu}(r,\Omega)\tilde{\bar{\mathcal{F}}}_{\mu\nu}(r,\Omega)] \tag{1.35}$$

where $d\Omega$ is the angular measure in a 3-sphere. In order for this integral to be finite, we just need that $\bar{\mathcal{F}}_{\mu\nu} = \mathcal{O}(1/r^3)$, going to zero faster than $1/r^2$ for large $r$. Keeping in mind the definition of the field strength tensor, this entails that $\bar{A}_\mu = \mathcal{O}(1/r^2)$, which therefore goes to zero at spatial infinity more rapidly than $1/r$. But it was first noticed by t'Hooft [7] that the correct boundary condition for the four-potential should have fixed $\bar{A}_\mu$ in the form of a pure gauge field, since any gauge transformation acting on it can leave the required behaviour of the field strength untouched. This subtlelty can be equally well understood by considering that, in (1.30), the four-divergence of the Chern-Simons current is gauge-invariant, hence it can be included as a contribution to the QCD Lagrangian. However, the current itself, which has a role in the boundary problem, is not. This is a key point, which ultimately leads to the solution of the $U(1)_A$ puzzle.

Having said this, reminding that $\bar{A}_\mu = (\bar{A}_4 = -iA_0, \vec{A})$, we could state

$$\bar{A}_i(x) \underset{r\to\infty}{\longrightarrow} \frac{i}{g}V(\Omega)\partial_i V(\Omega)^{-1} + \mathcal{O}(1/r^2) \qquad \bar{A}_4(x) \underset{r\to\infty}{\longrightarrow} \frac{i}{g}V(\Omega)\partial_4 V(\Omega)^{-1} + \mathcal{O}(1/r^2) \tag{1.36}$$

where $V(\Omega)$ is a continuous and differenziable map, which depends only on the angular variables ($V : S^3 \to SU(3)$). Moreover, if we act on the potential through a gauge transformation $U(x)$, using the inverse of the gauge transformation (1.5), we will modify the asymptotic formula of $\bar{A}_\mu$ as

$$\bar{A}_\mu(x) \underset{r\to\infty}{\longrightarrow} \frac{i}{g}U(\Omega)V(\Omega)\partial_\mu V(\Omega)^{-1}U(\Omega)^{-1} + \frac{i}{g}U(\Omega)\partial_\mu U(\Omega)^{-1} + \mathcal{O}(1/r^2) = \\ = \frac{i}{g}(U(\Omega)V(\Omega))\partial_\mu(U(\Omega)V(\Omega))^{-1} + \mathcal{O}(1/r^2) \tag{1.37}$$

Here is the crux of our discussion: if we were able to set $U(\Omega) = V(\Omega)^{-1}$ at spatial infinity using the gauge freedom, that would restore the trivial boundary condition $\bar{A}_\mu \sim \mathcal{O}(1/r^2)$. But the problem is that this choice can not be generally pursued. Indeed, $U$ and $V$ are functions of different nature. $U$ is a customary gauge transformation, which has to be continuous and correctly defined at each point. Hence, not to be singular, it will not depend on angular variables at the origin, but it will assume a constant value at $r = 0$. Starting from this point, it is always possible to continuously deform $U$ while approaching $r \to \infty$. But that means the function $U(\Omega)$ belongs to the same homotopy class of the identity ($\nu = 0$), which maps the spatial infinity to the unit element of $SU(3)$ (up to a constant). On the other hand, $V$ is a function solely of the angular coordinates and it can belong to any homotopy class. So, in general, we are not allowed to choose $U(\Omega) = V(\Omega)^{-1}$, because of four-potential configurations corresponding to non-zero winding numbers. It is just this fact that prevents us from simply removing the term $tr[\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu}]$ from the Lagrangian for a non-abelian gauge theory. In the abelian case, we have no problem of this sort: a mathematical result states there is only one trivial homotopy class for the map $S^3 \to U(1)$.

More explicitly, rewriting the expression for the Chern-Simons current, which can be read off from (1.29), in a slightly more useful form, we will get

$$\bar{J}^{CS}_\mu = 2\epsilon_{\mu\nu\rho\sigma}tr\left[\bar{A}_\nu\partial_\rho\bar{A}_\sigma - \frac{2ig}{3}\bar{A}_\nu\bar{A}_\rho\bar{A}_\sigma\right] = 2\epsilon_{\mu\nu\rho\sigma}tr\left[\frac{\bar{A}_\nu}{2}(\partial_\rho\bar{A}_\sigma - \partial_\sigma\bar{A}_\rho) - \frac{ig}{3}\bar{A}_\nu[\bar{A}_\rho,\bar{A}_\sigma]\right] = \\ = \epsilon_{\mu\nu\rho\sigma}tr\left[\bar{A}_\nu\bar{\mathcal{F}}_{\rho\sigma} + \frac{2ig}{3}\bar{A}_\nu\bar{A}_\rho\bar{A}_\sigma\right] \underset{r\to\infty}{\longrightarrow} \frac{2}{3g^2}\epsilon_{\mu\nu\rho\sigma}tr[(V\partial_\nu V^{-1})(V\partial_\rho V^{-1})(V\partial_\sigma V^{-1})] \tag{1.38}$$

where we have eventually considered the asymptotic behaviour both of the field strength tensor and of the four-potential at spatial infinity. Clearly, $\bar{\mathcal{F}}_{\rho\sigma}$ goes to zero for $r \to +\infty$, so that the first term in the last passage does not contribute. For $\bar{A}_\mu$ we used (1.36). Now taking into account the definition (1.34) of winding number, we can recast it as a surface integral over a surface at infinity in four-dimensional euclidean space [9]:

$$\nu = -\frac{1}{24\pi^2}\int d\sigma_{E\mu}\epsilon_{\mu\nu\rho\sigma}tr[(V(\theta)\partial_\nu V^\dagger(\theta))(V(\theta)\partial_\rho V^\dagger(\theta))(V(\theta)\partial_\sigma V^\dagger(\theta))] \tag{1.39}$$

in which the overall minus sign is due to the change of coordinates. As a matter of fact, the jacobian factor from cartesian to spherical coordinates is given by $\mathcal{J} = -r^3 sin\theta_1 sin^2\theta_2$, so that $\epsilon^{x_1x_2x_3x_4} = 1$ and $\epsilon^{r\theta_1\theta_2\theta_3} = -1$.

Finally, keeping in mind that $tr[\bar{\mathcal{F}}_{\mu\nu}\tilde{\bar{\mathcal{F}}}_{\mu\nu}] = \epsilon_{\mu\nu\rho\sigma}tr[\bar{\mathcal{F}}_{\mu\nu}\tilde{\bar{\mathcal{F}}}_{\rho\sigma}]/2$, one gets

$$\int d^4x_E tr[\bar{\mathcal{F}}_{\mu\nu}\tilde{\bar{\mathcal{F}}}_{\mu\nu}] = \int_{S^3} d\sigma_{E\mu}\frac{2}{3g^2}\epsilon_{\mu\nu\rho\sigma}tr[(V\partial_\nu V^{-1})(V\partial_\rho V^{-1})(V\partial_\sigma V^{-1})] = -\frac{16\pi^2\nu}{g^2} \tag{1.40}$$

Only for the value of $\nu = 0$, the usual disregarding of the boundary term is justifiable. But what is the physical meaning of these gauge configurations $\bar{A}_\mu$ with non-trivial boundary conditions (i.e for which $\nu \neq 0$)?

### 1.2.2 Vacuum configurations

As well explained in [9], the interesting topological structure of non-abelian field theories has further deep consequences. Let us consider two four-potentials obtained through two different gauge transformations of the vacuum, e.g $A_\mu = \frac{i}{g} U\partial_\mu U^\dagger$ and $\tilde{A}_\mu = \frac{i}{g}\tilde{U}\partial_\mu \tilde{U}^\dagger$. Clearly, for both of these four-potentials, the field strength tensor equals zero all over the space. But if $U$ and $\tilde{U}$ are not smoothly deformable into each other, we will pass from $A_\mu$ to $\tilde{A}_\mu$ crossing vector potential configurations that are not gauge transformations of zero: they will not annul $\mathcal{F}_{\mu\nu}$ and there will be non-zero energy values associated to them. That implies $A_\mu$ and $\tilde{A}_\mu$ represent two different vacuum states separated from an energy barrier, that is a physical potential.

However, the previous argument relies on the existence of field configurations which are not equivalent under a smooth gauge transformation. Is it true that two $SU(m)$ elements $U$ and $\tilde{U}$ can not be in general deformed into each other? Using our freedom of fixing the gauge, we can carry out this brief analysis considering the temporal gauge in its euclidean version $\bar{A}_4(x) = 0\ \forall x \in \mathbb{R}^4$. We can readily write down

$$\bar{A}_4(x) = \frac{i}{g} U(x)\partial_4 U(x)^{-1} = 0 \qquad \bar{A}_i(x) = \frac{i}{g} U(x)\partial_i U(x)^{-1} \tag{1.41}$$

The first equality implies that, for this gauge choice, $U(x)$ is just a function of $\vec{x} \in \mathbb{R}^3$. It can be further proved that any function $U(\vec{x})$ can be smoothly deformed so as to map the spatial infinity $r \to +\infty$ to a single constant value: topologically speaking, the infinity is identified with a point. With this extra arrangement, $U(\vec{x})$ will be defined over $\mathbb{R}^3$ plus a point, which has the topology of $S^3$. Eventually, it emerges that $U(\vec{x})$ can always be thought as a map $S^3 \to SU(m)$ and we know from the previous paragraph that this lets us classify $U(x)$ functions into homotopy classes with different winding numbers (for a more detailed description on the topic look at [10]).

Consequently, there will be as many vacua as the number of non-equivalent gauge transformations, which means that QCD vacua possess the cardinality of $\mathbb{Z}$. They can again be labelled by a winding number $n$, defined by (1.34). For sake of clarity, we point out how, in the preceding paragraph, $U$ just refers to a *small gauge transformation*, connected to the identity operator and well-defined at each point. Here we are introducing some fictitious gauge transformations, belonging to non-trivial homotopy classes: as stressed by [11], these objects will unavoidably show a singularity somewhere on the compactified space $S^3$.

In principle, the presence of multiple vacuum states should not be all that novelty. A simple spontaneously broken scalar field with a $Z_2$ symmetry will present a couple of vacua. It is very well known, anyway, that the probability of transition between each other is zero in practise. This is clear by looking at the transistion probability amplitude between an in and out state separated by a potential barrier $V$ (see [12])

$$\langle n_{out}|V|n_{in}\rangle \sim e^{-S_E} \tag{1.42}$$

Of course, in the limit of infinite volume, the euclidean action increases and this matrix element goes rapidly to zero (which means the potential barrier is raised more and more). But the situation is now different for QCD. We can consider an inner state $n_{-\infty}$, coming from $t \to -\infty$, and an outer state $n_{+\infty}$, going to $t \to +\infty$. The transition probability between these two vacuum states of different winding numbers is not zero, because the Euclidean action admits some classical solutions which can mediate between them, allowing a tunneling even in the limit of large volume space.

To see that, we can start considering the following object

$$\frac{1}{2} tr\left[\int d^4x_E (\bar{\mathcal{F}}_{\mu\nu} \pm \tilde{\bar{\mathcal{F}}}_{\mu\nu})^2\right] = \int d^4x_E \big(tr[\bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}] \pm tr[\tilde{\bar{\mathcal{F}}}_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}]\big) \geq 0 \tag{1.43}$$

where the second equality follows from the fact that $\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta} = 2(\delta_{\rho\alpha}\delta_{\sigma\beta} - \delta_{\rho\beta}\delta_{\sigma\alpha})$, which implies $\tilde{\bar{\mathcal{F}}}_{\mu\nu}\tilde{\bar{\mathcal{F}}}_{\mu\nu} = \bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}$. So, one gets

$$-\int d^4x_E tr[\bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}] \geq \left| \int d^4x_E tr[\tilde{\bar{\mathcal{F}}}_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}] \right| \tag{1.44}$$

Just because the left-hand side is twice the euclidean action and the right-hand side can be rewritten using (1.40), we obtain the following condition:

$$S_E \geq \frac{8\pi^2|n'|}{g^2} \tag{1.45}$$

which tells us that in any homotopy class we have a field configuration minimizing the euclidean action, that is a corresponding solution of the classical field equations.

If we now plug into equation (1.42) the expression of $S_E$ saturating the bound (1.45), we will attain

$$\langle n_{out}|V|n_{in}\rangle \sim exp\left\{ -\frac{8\pi^2|n'|}{g^2} \right\} \tag{1.46}$$

where $n' = n_{out} - n_{in}$ and the tunneling probability does not depend on volume any more. These field configurations which mediate between different vacua are called *instantons*. However, one can immediately realize that the role of instantons can be highly ruled by the gauge coupling $g$. Concerning QCD, we know that at high energy $g \ll 1$, which means that we are in the perturbative regime. Here the previous exponential is suppressed by a large factor $1/g^2$. The tunneling processes will start being relevant in the non-perturbative regime.

The presence of instantons has an immediate consequence on the construction of our field theory. Indeed, a quantum theory should be able to rely on a vacuum labelled by a time indepedent quantity. Therefore, it is clear that the use of winding numbers is not a good choise, because of these possible vacuum transitions. Moreover, we would like that a gauge transformation did not change the physical vacuum state, because we know that acting with the gauge group can not have any physical effect. We could be left, at most, with a change of phase $|\text{vacuum}'\rangle = e^{i\theta}|\text{vacuum}\rangle$, which is clearly not a problem, if we consider that a quantum state is defined as a ray in the Hilbert space. But if we act with a *large gauge transformation* $\Omega_m$ (a trasformation which connects states which differ by a winding number $m$) on a vacuum state $|n\rangle$, we will obviously obtain $\Omega_m|n\rangle = |n+m\rangle$ by very definition. A good vacuum state for QCD can be designed as

$$|\theta\rangle = \sum_n e^{in\theta}|n\rangle \tag{1.47}$$

in which $\theta$ is a real parameter. This state is properly constructed in order to be gauge invariant, as it is evident from

$$\Omega_m|\theta\rangle = \sum_n e^{in\theta}\Omega_m|n\rangle = \sum_n e^{in\theta}|n+m\rangle = e^{-im\theta}\sum_{n'} e^{in'\theta}|n'\rangle = e^{-im\theta}|\theta\rangle \tag{1.48}$$

where we just have a harmless change of phase.

In addition, here there is no overlap between different $\theta$-vacua

$$\begin{aligned} Z_{\theta\to\theta'}[0] = \langle\theta'_{out}|\theta_{in}\rangle = \sum_{n,m} e^{im_o\theta'_{out}}e^{-in_i\theta_{in}}\langle m_o|n_i\rangle = \\ = \sum_{n,\nu} e^{i(n_i+\nu)\theta'_{out}}e^{-in_i\theta_{in}}\langle n_i+\nu|n_i\rangle = \delta(\theta'_{out}-\theta_{in})\sum_\nu e^{i\nu\theta_{in}}\langle\nu|0\rangle \end{aligned} \tag{1.49}$$

In the last passage we used the presence of the Dirac delta function to set $\theta_{out} = \theta_{in}$. In order to get our final result, we have put $n_i$ to zero in $\langle n_i + \nu | n_i \rangle$, because the transition probability will depend only on the winding number difference between two states. The $\delta(\theta'_{out} - \theta_{in})$ simply tells us that the parameter $\theta$ does not depend on time. This latter is a very convinient aspect to define a quantum vacuum, but it will prove to be a further problem later on.

If we now drop the delta function, that just ensures the conservation of $\theta$, we can define

$$\begin{aligned} Z_\theta[0] &= \sum_\nu e^{i\nu\theta} \langle \nu | 0 \rangle = \sum_\nu \int \mathcal{D}\bar{A}_{\mu,\nu} e^{i\nu\theta} e^{-S_E} = \\ &= \sum_\nu \int \mathcal{D}\bar{A}_{\mu,\nu} exp\Big\{ - \int d^4 x_E \Big( \mathcal{L}_E + \frac{i\theta g^2}{32\pi^2} tr[\epsilon_{\mu\alpha\rho\sigma} \bar{\mathcal{F}}_{\mu\alpha} \bar{\mathcal{F}}_{\rho\sigma}] \Big) \Big\} \end{aligned} \tag{1.50}$$

where the definition of winding number for $\nu$ has been employed. The notation $\int \mathcal{D}\bar{A}_{\mu,\nu}$ stands for an integration over all euclidean field configurations with a definite winding number (we should clearly solely integrate over inequivalent gauge configurations, but this is another story). Now we just need to move back to Minkowski space through an anti-Wick rotation. To do that, we have to proceed carefully and to consider that we will get a factor $i$ from the four space measure, an extra $-i$ from the singled-out time-like derivative that appears in $tr[\mathcal{F}_{\mu\alpha}\tilde{\mathcal{F}}^{\mu\alpha}]$ because of the Levi-Civita symbol, which, in its turn, contributes with an extra minus sign, due to the conversion $\epsilon^{4123} = -1$ but $\epsilon^{0123} = +1$.

The take-home message of this discussion is that the contribution of instantons to QCD theory appears in an extra term in the Lagrangian, which comes as

$$\mathcal{L}_\theta = \frac{\theta g^2}{16\pi^2} tr[\mathcal{F}_{\mu\alpha}\tilde{\mathcal{F}}^{\mu\alpha}]. \tag{1.51}$$

Thereby, we can either consider the correct quantum vacuum of QCD or go on using a simple vacuum (possibly associated with $\nu = 0$), but in the latter case we will have to introduce an extra term in our description which accounts for all possible vacuum transitions. $\theta$ is a new parameter of the theory, describing a QCD vacuum property. The field configurations $A_\mu$ that contribute to (1.51) and impede simply to neglect it as a total derivative are the instantons: that answers the question we left open at the end of the previous paragraph.

## 1.3 Chiral anomalies

The $\theta$-term that we derived in the previous section sums up in a powerful way the non-trivial properties of the QCD vacuum. What we want to show now is the profound connection between this term and the problem of quantum anomalies.

It is well-known that at classical level, the symmetries of a theory are those of the Lagrangian and to each of them it will be associated a conserved current, according to Noether theorem. Then, conservation laws will be modified into Ward-Takahashi identities at quantum level. But nothing assures us that a classical symmetry will still survive in a quantum framework. In a quantum field theory approach, the really fundamental object is not the Lagrangian, but the generating functional of correlation functions $Z[J]$. For a theory with one Dirac field and an abelian four-potential, upon setting sources to zero, we will have

$$Z[0] = \mathcal{N} \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS} \tag{1.52}$$

As a consequence, for $Z[0]$ to be invariant, we do not just need the invariance of $S$, but of the functional measure, too.

Let us consider the interesting example of an axial abelian transformation $U(1)_A$, acting as

$$\psi(x) \longmapsto e^{i\alpha\gamma_5}\psi(x) \qquad \bar{\psi}(x) \longmapsto \bar{\psi}(x)e^{i\alpha\gamma_5} \tag{1.53}$$

By taking the $\gamma_5$ matrix in the Weyl representation, where $\gamma^5 = diag(-\mathbb{1}, \mathbb{1})$, it is easy to appreciate how Weyl fermions of opposite chirality rotate in reverse directions. As already seen, without a mass term in the Lagrangian mixing up the left and right parts of the spinor, we have a conserved chiral current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ and an associated charge $Q^5$, which encodes the difference between left- and right-handed particles. By adding a mass, the conservation law is modified into

$$\partial_\mu j_5^\mu = 2im\bar{\psi}\gamma_5\psi, \tag{1.54}$$

Therefore, a particle can change chirality during its time evolution, which is responsible for a $\Delta Q^5 \neq 0$. So much for what we have in a classical approach. We can now set to zero the mass term again and consider how $Z[0]$ changes under a chiral transformation. To derive our result, we will follow the customary trick of generalizing $\alpha$ to a function of space-time coordinates.

We know that, when dealing with path integrals, the properties of convergence are better studied by Wick rotating our variables. In the Euclidean formulation, we will have $\bar{A}_\mu = (A_4 = -iA_0, \vec{A})$, $\psi_E$ and $\bar{\psi}_E$ will be treated as indipendent fields and for the $\gamma$-algebra it will hold that $\bar{\gamma}_\mu = \bar{\gamma}_\mu^\dagger$ (for $\bar{\gamma}_4 = \gamma^0$ and $\bar{\gamma}_k = -i\gamma^k$). Moreover, to handle adimensional differential operator, it will prove to be convenient the rescaling $\psi_E' = i\mu\psi_E$, where $\mu$ is an arbitrary constant with mass dimensions. If we consider all that, expression (1.52) can be recast as

$$\begin{aligned} Z_E[0] &= \mathcal{N}\int \mathcal{D}\bar{A}_\mu \mathcal{D}\psi_E \mathcal{D}\bar{\psi}_E e^{\int d^4x_E(\bar{\psi}_E(\frac{i\bar{\not{D}}}{\mu})\psi_E - \frac{1}{4}\bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu})} = \\ &= \mathcal{N}_A \int \mathcal{D}\bar{A}_\mu det\left|\left|\frac{i\bar{\not{D}}}{\mu}\right|\right| e^{-\frac{1}{4}\int d^4x_E \bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}} \end{aligned} \tag{1.55}$$

We can focus on $Z_A[0] = det||i\bar{\not{D}}/\mu||$, where $\bar{A}_\mu$ is treated as a classical field, not to deal with gauge fixing complications. We should take into account that a continuous Wick rotation over fermion fields will not affect spinor indeces:

$$\psi_E(\tau, \vec{x}) = \psi(e^{i\theta}t, \vec{x})|_{\theta=\pi/2} \qquad \bar{\psi}_E(\tau, \vec{x}) = \bar{\psi}(e^{i\theta}t, \vec{x})|_{\theta=\pi/2} \tag{1.56}$$

but it will just change the phase of spatial gamma matrices, as mentioned above. As a consequence, we can mimic the axial phase transformation (1.53) for minkowski fermions and translate it in euclidean formalism in a direct way:

$$\psi_E' = e^{i\alpha(x)\bar{\gamma}_5}\psi_E \qquad \bar{\psi}_E' = \bar{\psi}_E e^{i\alpha(x)\bar{\gamma}_5} \tag{1.57}$$

where $\bar{\gamma}_5 = \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\bar{\gamma}_4 = -\gamma^5$.

Therefore, it can be straightforwardly fathomed that, after this chiral change of variables, (1.55) becomes:

$$\begin{aligned} Z_A'[0] &= det||i\bar{\not{D}}'/\mu|| = \mathcal{N}_\psi' \int \mathcal{D}\psi_E'\mathcal{D}\bar{\psi}_E' e^{-S_E'} = \mathcal{J}^{-1}\mathcal{N}_\psi' \int \mathcal{D}\psi_E \mathcal{D}\bar{\psi}_E e^{-S_E + \int d^4x_E \partial_\mu\alpha(x) j_\mu^5} = \\ &= \mathcal{J}^{-1} det||e^{i\bar{\gamma}_5\alpha(x)}(i\bar{\not{D}}/\mu)e^{i\bar{\gamma}_5\alpha(x)}|| \end{aligned} \tag{1.58}$$

where, in the previous expression, $\mathcal{J}$ is the jacobian factor arising from the redefinition of spinor fields. The power of minus one accounts for the grassmann nature of the variables we are varying. The crux of the problem is just the evalutation of $\mathcal{J}$. In contrast to what we are accustomed, the chirality of the transformation does not lead to a compensation between $\psi$'s and $\bar{\psi}$'s variation in the fermionic measure and, hence, to a trivial $\mathcal{J} = 1$. Therefore, a classical symmetry will be preserved in a quantum field theory, if the functional measure will be invariant. If not, we speak of an *anomalous symmetry*, whose conservation law gets quantum corrections.

Nevertheless, the problem of anomaly is even subtler. Speaking of an anomalous contribution is equivalent to say that the determinant of (1.55) is not invariant under a change of basis. That sounds a bit weird in terms of customary algebra. But we know that these properties hold only if our formulae are well-defined and convergent: the anomaly simply tells us that our initial expression was not so and that we will need a regulator, in order for our construction to make sense. These underlying divergences emerge from the fact that the product of eigenvalues $\Pi_j \lambda_j$ increases without bound. The regularization procedure to extract a finite value from it is not unique, but the achieved result can not depend on the way we get it.

Historically, the methodology developed by Fujikawa [13] to handle quadratic path integrals played an important role in the process of understanding of chiral anomalies. Despite that, it was realized later on that Fujikawa method was not reliable to deal with all kind of anomalies. A more general technique, which has proved to be very useful, is the $\zeta$-function method, devised by Hawking [14] to regulate divergent jacobians of the path integral realm.

In the $\zeta$-function regularization, the possibility of rendering divergent quantities finite is based on the formalism of analytic continuation of $\zeta$ on the complex plane, which is an entrenched sector of mathematics. Once we defined the $\zeta$ function associate to an hermitian or normal operator $A$ as

$$\zeta(s, A) = \sum_j \lambda_j^{-1} \tag{1.59}$$

where $\{\lambda_j,\ j \in \mathbb{N}\}$ is the discrete or countable set of eigenvalues of $A$, the key identity of this procedure is

$$det(A) = exp\left\{-\frac{d\zeta}{ds}(s, A)\right\}_{s=0} \tag{1.60}$$

That gives us the possibility of assessing the jacobian factor of our problem as a difference of two determinants, already regularized from the outset [15]:

$$log\mathcal{J}^{-1} = \frac{d\zeta}{ds}\left(0, e^{i\bar{\gamma}_5\delta\alpha(x)}\frac{i\bar{\not{\mathcal{D}}}}{\mu}e^{i\bar{\gamma}_5\delta\alpha(x)}\right) - \frac{d\zeta}{ds}\left(0, \frac{i\bar{\not{\mathcal{D}}}}{\mu}\right) \tag{1.61}$$

A brief summary of the $\zeta$-function technique, together with a thorough evalutation of the previous expression, is reported in Appendix A.2. Here, we just need to quote that, after a pretty lengthy calculation, one can obtain the formula encoding the anomalous contribution:

$$\mathcal{J}^{-1} = exp\left\{-\frac{ig^2}{16\pi^2}\int d^4x\alpha(x)\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\right\} \tag{1.62}$$

In light of this result, the effect of a chiral transformation on the generating functional will be

$$Z'_A[0] = \mathcal{N}'_\psi \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS-i\int d^4x\alpha(x)[\partial_\mu j^{5\mu}(x)+\frac{g^2}{16\pi^2}\int d^4x\alpha(x)\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}]} \tag{1.63}$$

But, after all, we act on $Z[0]$ through nothing more than a change of variables, so that the generating functional is expected not to depend on the arbitrary parameter $\alpha$ of this chiral rotation.

Therefore, it is straightforward to see that this fact entails

$$\left.\frac{\delta Z_A[0]}{\delta\alpha}\right|_{\alpha=0} = 0 \qquad \text{and so}$$

$$\mathcal{N}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left(\partial_\mu j^{5\mu} + \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\right)e^{iS-i\int d^4x\alpha(x)[\partial_\mu j^{5\mu}+\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}]}\Bigg|_{\alpha=0} = 0 \tag{1.64}$$

$$\langle 0|\partial_\mu j^{5\mu}(z)|0\rangle = -\frac{g^2}{16\pi^2}\langle 0|\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}(z)\mathcal{F}_{\rho\sigma}(z)|0\rangle$$

which shows that, even in absence of mass terms, the 4-divergence of the chiral current gets quantum corrections. This results is known as *Adler-Bardeen theorem*, after the names of the first ones who derive this formula by an attentive inspection of Feynman diagrams. It is noteworthy that this result is exact, because equation (1.62) was derived by Fujikawa and by us in Appendix A.2 without any use of a perturbative expansion in the parameter $g$. Hence, there are no additional higher order contributions and there is no need to restrict the validity of (1.64) to a perturbative regime. The method that we employed is based on a direct handling of path integrals, showing the exactness of the formula (something that a diagram approach can not tell, because of its perturbative nature).

If we now consider the corresponding charge associated to $j^{5\mu}(x)$, it is easy to notice from the first expression that, on account of (1.40)

$$\int d^4x j^{50}(x) = Q^5_f - Q^5_i = \int d^4x_E \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\rho\sigma} = -2\nu \tag{1.65}$$

The amount of change of chirality is related to the winding number: even without a mass, particles change their chirality by interacting with instantons, e.g field configurations which mediate transitions between different quantum vacua. It is fundamental to notice that the term emerging from a chiral transformation is of the same form of that we added to the Lagrangian, to account for the complicated QCD vacuum structure: this point will prove to be crucial later.

Up to now, we have just considered a simple abelian axial transformation. A non-abelian generalization it is quite straightforward, but, first of all, it is worth noticing another detail. We emphasized that, in order to have a chiral transformation, we just need to transform differently fermions of opposite chirality. So, it is convenient to highlight the two independent roles of right and left fermions in (1.62). This can be unambiguously done remembering that $j^\mu = \bar{\psi}\gamma^\mu\psi$ and $j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$, from which one can construct $j^{5\mu}_L = \bar{\psi}_L\gamma^\mu\psi_L = (j^\mu - j^{5\mu})/2$ and $j^{5\mu}_R = \bar{\psi}_R\gamma^\mu\psi_R = (j^\mu + j^{5\mu})/2$. Taking into account that $\langle 0|\partial_\mu j^\mu(x)|0\rangle = 0$ even at quantum level, we can write from (1.64)

$$\begin{aligned}\langle 0|\partial_\mu j^{5\mu}_L(x)|0\rangle &= +\frac{g^2}{32\pi^2}\langle 0|\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}|0\rangle \\ \langle 0|\partial_\mu j^{5\mu}_R(x)|0\rangle &= -\frac{g^2}{32\pi^2}\langle 0|\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}|0\rangle\end{aligned} \tag{1.66}$$

Therefore, we can rearrange (1.62) as

$$\mathcal{J}^{-1}_{L/R} = exp\left\{\pm\frac{ig^2}{32\pi^2}\int d^4x\alpha(x)\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\right\} \tag{1.67}$$

where $\pm$ refers to left or right respectively (with half of the whole contribution presented in the original expression). Actually, as it will be clearer in the following chapters, only the relative minus sign between left and right particles will be relevant.

The remaining part of the job will just be a matter of generalizing abelian to non-abelian case. This will be done on two separate levels: both considering a non-abelian group symmetry acting on spinors and by coupling them to non-abelian gauge fields. The new chiral transformation laws for Weyl spinors will be:

$$\psi'_L = exp\{i\omega^a_L(x)\tau^a\}\psi_L \qquad \psi'_R = exp\{i\omega^a_R(x)\tau^a\}\psi_R \tag{1.68}$$

where $\tau^a$ are the generators of some group (an abelian or non-abelian one). All we have to include is an extra factor, comprising a trace over internal symmetry indeces. In our Appendix A.2, we cover this simple problem, too, obtaining the result:

$$\mathcal{J}^{-1}_{L/R} = exp\bigg\{ \mp \frac{ig^2}{32\pi^2} \int d^4x\omega_a(x)\epsilon^{\mu\nu\rho\sigma} tr[\tau^a \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}]\bigg\}. \tag{1.69}$$

Now, on account of the fact that $\mathcal{F}_{\mu\nu} = T^c\mathcal{F}^c_{\mu\nu}$, where $T^c$ are the generators associated to gauge symmetry, we can rewrite the trace operation in the exponent in a very interesting way

$$\epsilon^{\mu\nu\rho\sigma} tr[\tau^a \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}] = \epsilon^{\mu\nu\rho\sigma} tr[\tau^a T^b T^c]\mathcal{F}^b_{\mu\nu}\mathcal{F}^c_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} tr[\tau^a\{T^b, T^c\}]\mathcal{F}^b_{\mu\nu}\mathcal{F}^c_{\rho\sigma} \tag{1.70}$$

where we have highlighted the expression

$$d^{abc} = tr[\tau^a\{T^b, T^c\}] \tag{1.71}$$

This object is the really elementary quantity that tells us if a symmetry transformation, represented by the abelian or non-abelian generators $\tau^a$ and which couples to spinor fields, gets quantum corrections. The latters stem from the presence of gauge symmetries to which fermions are sensitive and whose infinitesimal generators appear inside the anticommutator (1.71). If $d^{abc} = 0$, there are no anomalies and a classical symmetry is still valid at quantum level; if $d^{abc} \neq 0$, the current associated to the symmetry involving $\tau^a$ will not be quantum conserved. It is just calculating objects of this kind that we are able to claim that the gauge symmetry $SU(3)_c \times SU(2)_L \times U(1)_Y$ of the standard model is anomaly free: charges and number of fermions are designed so that all possible $d^{abc}$ coefficients disappear. Therefore, we are left with the remarkable result that gauge symmetry is a real symmetry of the standard model, even at quantum level: that is a real luck, considering how gauge invariance is a foundational principle of it!

As already said, this general expression for quantum anomalies was originally obtained by Feynman diagram calculations. We are not going to tackle this kind of problem in this context, but we refer to our Appendix A.1 for a direct calculation of the anomaly with a perturbative approach. Nonetheless, it will be useful for the following development of this work to understand the underlying diagram structure behind the anomaly term by means of a heuristic reasoning.

As a matter of fact, we know that any term in the Lagrangian must correspond to a particular diagrammatic rule. If we rescale the action by a constant $\lambda$ with dimensions of $[\hbar]$, the exponential weight in $Z[J]$ will turn into

$$e^{\frac{i}{\hbar}S} \longmapsto e^{\frac{i}{\hbar/\lambda}\frac{S}{\lambda}} \tag{1.72}$$

Thereby, we can spot the adimensional parameter $\tilde{\lambda} = \hbar/\lambda$. We see that vertices will present powers of this parameter in the form $\tilde{\lambda}^{-1}$, while propagators (being related to the opposite of the inverse of the Fourier transform of the kinetic term) will acquire a $\tilde{\lambda}$. So, every Feynman diagram will come with a factor $\tilde{\lambda}^{I-V}$, with $I$ the number of internal lines and $V$ that of vertices. Then, if we take into account the topological relation $I - V = L - 1$, it emerges how a Feynman diagram will be always followed by a $\tilde{\lambda}^{L-1}$ factor, that turns out to be a loop counter.

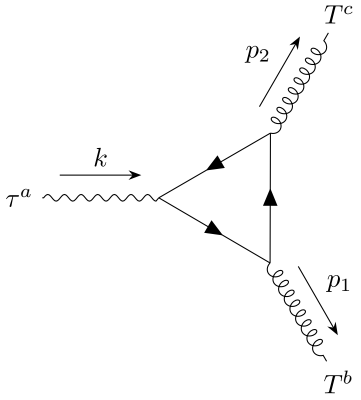



Figure 1.2: A triangle diagram related to the Lagrangian term $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}tr[\tau^a\{T^b,T^c\}]\omega^a\mathcal{F}^b_{\mu\nu}\mathcal{F}^c_{\rho\sigma}$

A simple tree level diagram will have a factor $\tilde{\lambda}^{-1}$, directly linked to the contribution of propagators and vertices: so $L=0$. The determinant giving rise to the anomaly term will have no factor $\tilde{\lambda}$ at all, because generated by the functional measure: therefore, $L=1$. Thus, the anomaly contribution introduced in the Lagrangian is associated to a loop diagram. Moreover, by inspection of formula (1.69), we have three fields appearing in the new term, as it happens for a three-external-legs diagram: a quadratic combination of the field strength tensor and $\omega(x)$, that here clearly plays the role of a field. What about the number of internal verteces? Equation (1.69) has been written down in the most general form and it can be applied to any sort of transformation acting on fermions: of course, there will always be a coupling constant parametrizing that. Just as an example, we could have inserted an explicit factor $g$ in (1.68), in case of a gauge transformation of the same kind: this would have generated a power of three in (1.69). All that suggests how the diagram related to $\mathcal{J}^{-1}$ should be a loop with three external legs and three internal verteces, that is to say a *triangle diagram* (see figure 1.2).

The loop expression associated to this diagram can be readily written down, considering again $\omega(x)$ as a field related to a local chiral transformation $\psi' = exp\{i\omega^a(x)\tau^a\gamma^5\}\psi$. Just because it is a phase field, its interaction with quarks will be, up to a coupling constant $y$, of the form $iy_j\partial_\mu\omega\bar{q}_j\gamma^\mu q_j$. Consequently, the integral expression of our triangle diagram will be:

$$-\frac{1}{2}d^{abc}\sum_j\int\frac{d^4l}{(2\pi)^4}tr\left[\frac{i}{\not{l}-m_j}(-iy_j\not{k}\gamma^5)\frac{i}{\not{l}-\not{p}_1-m_j}(ig\gamma^\mu)\frac{i}{\not{l}-\not{p}_1-\not{p}_2-m_j}(ig\gamma^\nu)\right] \quad (1.73)$$

where $j$ is a quark flavor index and $g$ the gauge coupling. We can easily notice that the integral is linearly divergent, as suggested by our starting reasoning on the algebraic properties of the fermion determinant: just this ill-defined integral is at the origin of the anomalous behavior of chiral transformations. In the path integral procedure, the divergence shows up as bad behaving determinant, which demanded a suitable regularization scheme. Here, we have customary indefinite integrals of perturbation theory, but with a further peculiar difficulty: the presence of one $\gamma^5$ matrix. If we insist on making use of the powerful instrument of dimensional regularization to counter this calculation, as we did in Appendix A.2, we will have to face the non-trivial issue of extending $\gamma^5$ to an arbitrary number of dimensions!

## 1.4 The $\theta$-term consequences

We have now all the tools together to understand why the $U(1)_A$ problem is not a problem at all. We just need to consider if this global symmetry, that we examined in the first paragraph, survives quantum corrections or not. This can be straightforwardly calculated using the expression of $d^{abc}$ we derived earlier.

We can easily realize that the conservation law for the axial triplet current is unaffected by anomalous terms. Indeed, on account of (1.64) (properly generalized), one gets

$$\partial_\mu j^{5\mu a}(x) = -\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} tr[\tau^a\{T^b, T^c\}]\mathcal{G}^b_{\mu\nu}\mathcal{G}^c_{\rho\sigma} \tag{1.74}$$

where $\tau^a$ are now axial isospin generators and $T^b$ color generators, which can be expressed through the Gell-Mann matrices. The trace is taken both over color and flavor. Using a normalization for Gell-Mann matrices which is typical of non-abelian gauge theory (i.e $tr(\lambda_a\lambda_b) = \delta_{ab}/2$) and on account of $\{\lambda_a, \lambda_b\} = \delta_{ab}/3 + d_{abc}\lambda_c$ (where $d_{abc} = 2tr[\{\lambda_a, \lambda_b\}\lambda_c]$ coincides with our previous definition up to a factor two, which depends on matrix normalization), one gets

$$tr[\tau^a\{T^b, T^c\}] = \frac{1}{3}tr[\tau^a\delta_{bc}] + tr[\tau^a d_{bce}\lambda_e] = 0 \quad \Rightarrow \quad \partial_\mu j^{5\mu a}(x) = 0 \tag{1.75}$$

in which we fixed $T^a = \lambda^a$ and we used the property of null trace of $SU(2)$ and $SU(3)$ generators. So, we obtained the anticipated result: the axial non-singlet current is only broken by the explicit quark mass term, which is responsible for pion masses.

For the singlet current associated to $U(1)_A$, things are different. As a matter of fact, we see that

$$\begin{aligned}\partial_\mu j^{5\mu}(x) &= -\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} tr[\mathbb{1}_{2\times 2}\{T^b, T^c\}]\mathcal{G}^b_{\mu\nu}\mathcal{G}^c_{\rho\sigma} = \\ &= -\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} tr[\mathbb{1}_{2\times 2}]tr\left[\frac{\delta_{bc}}{3}\right]\mathcal{G}^b_{\mu\nu}\mathcal{G}^c_{\rho\sigma} = -\frac{2g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathcal{G}^b_{\mu\nu}\mathcal{G}^c_{\rho\sigma}\end{aligned} \tag{1.76}$$

where $\mathbb{1}_{2\times 2}$ is the infinitesimal generator of $U(1)_A$, acting on the two flavours of our model. It is noteworthy that the current conservation law gets a quantum correction, which directly comes from QCD and its vacuum properties. It is just the quite tricky structure of QCD vacuum that makes the $U(1)_A$ group not a real symmetry of the theory: after all, $tr[\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma}]$ is nothing but the instanton topological term. Actually, we know that the same holds for $SU(2)_A$, once we introduce a mass term, but, nevertheless, we can still speak of pseudo-goldstone bosons. Anyway, if we could have treated $SU(2)_A$ and $U(1)_A$ on the same footing, as symmetries violated at tree level by mass terms, we would have expected to find an extra goldstone boson with a mass comparable to that of pions. However, we know that this is not the case. The same problem arises while considering the less precise $SU(3)$ flavour symmetry, where this singlet state is still missing. Therefore, the mass of $\eta'$ requires a different explanation, which just stems from the anomaly term. Indeed, we can envisage that the anomalous triangle diagram, which couples to $j^5_\mu$, will directly contribute to the associated "goldstone boson" mass with some loop corrections to its propagator (remember that $j^\mu_5 \sim \partial_\mu\eta^0$). The $\eta^0$ mass will be now lifted proportionally to the strong coupling constant, resulting in a much higher contribution than that coming from an explicit symmetry breaking mass term. After all, we have to take into account that, when we consider QCD bound states, we are in a non-perturbative regime. After the rotation (1.28) to physical states, we understand how the anomaly will affect both $\eta$ and $\eta'$, according to the amount of $\eta^0$ they contain. For some results on theoretical predictions of the $\eta$ and $\eta'$ masses, one can look for example at [16] (where they are evalutated using lattice QCD calculations).

But if we have sorted out the $U(1)_A$ conundrum, on the other hand a new problem appears upon introducing this $\theta$-term in the Lagrangian. First of all we notice that, just due to the presence of the Levi-Civita symbol, our new contribution will not be invariant under parity and, therefore, under CP (or equivalently time-reversal). So, we are introducing in the theory a source of CP violation. But this is not in principle a problem: only CPT transformations are expected to be real symmetry of the standard model. As a matter of fact, the complex phase $\delta$ of the CKM matrix in Wolfenstein parametrization already violates the CP symmetry. Anyway, it is better to highlight a difference: the $\theta$-term is a contribution which directly generates parity violation at tree level in the strong sector of the SM.

With a better inspection of the Lagrangian at energy lower than the electroweak symmetry breaking scale, we also observe that there is another non CP-preserving term

$$\mathcal{L}_{\not{CP}} = -(\bar{q}_L^i M_i{}^j e^{i\theta_Y} q_{Rj} + h.c.) + \frac{\alpha_s \theta_{QCD}}{4\pi} tr[\mathcal{G}_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}] \tag{1.77}$$

where $q^i_{R/L}$ is a Weyl spinor, whose index $i$ runs over quark flavours, and $\alpha_s = g^2_{strong}/4\pi$. In the first term, we singled out a complex phase connected to the possibility of a complex mass matrix. Of course, we can always perform a chiral transformation over spinors

$$q_L \longmapsto e^{-i\beta} q_L \qquad\qquad q_R \longmapsto e^{i\beta} q_R \tag{1.78}$$

leading to a quantum anomaly contribution of the form

$$\mathcal{L}_{anomaly} = +N_g \beta \frac{\alpha_s}{4\pi} tr[\mathcal{G}_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}] \tag{1.79}$$

where $N_g$ refers to the number of quarks we rotate. In so doing, $\theta_{QCD}$ transforms as $\theta_{QCD} \longmapsto \theta_{QCD} + N_g\beta$. If $\beta = -\theta_Y$, we could render the mass matrix real, but we would have shifted of a corresponding angle the phase of the QCD vacuum. On the other hand, we could have rotated fermions of a phase $-\theta_{QCD}/N_g$, so as to annul the $\theta$-term, but just leaving a complex quark matrix. Thus, the two parity violating phases are not unrelated and the Lagrangian essentially depends on a combination of them. But as already remarked earlier, this term induces a CP violation in strong interactions: this is at the origin of an issue dubbed *strong CP problem.* This latter arises from the fact that experiments suggest, with a great level of accuracy, the preservation of parity by strong interactions. Consequently, all bound states of QCD will be eigenstates of parity in their turn. If this is true, the neutron will not be the exception and it will be invariant under parity, too. If we consider its electric dipole momentum, using the property of parity operator $\mathbb{P} = \mathbb{P}^\dagger = \mathbb{P}^{-1}$ and defining $|N\rangle$ the neutron state, we will get

$$\langle N|\vec{E}|N\rangle = \langle N|\mathbb{P}^\dagger(\mathbb{P}\vec{E}\mathbb{P}^\dagger)\mathbb{P}|N\rangle = \langle N|(\mathbb{P}\vec{E}\mathbb{P}^\dagger)|N\rangle = -\langle N|\vec{E}|N\rangle \quad \Rightarrow \langle N|\vec{E}|N\rangle = 0 \tag{1.80}$$

where $\vec{E}$ is clearly the electric vector field, which gets a minus sign under parity. Thereby, we should have a neutron electric dipole momentum equal to zero, as a direct consequence of parity invariance. But if we now perturbe our theory with this new $\theta$ parameter, which must appear in the Lagrangian for all the reasons discussed previously, parity will not be a symmetry of the strong theory any more: we expect to obtain a contribution to the electric dipole momentum of neutrons. This quantity can be computed, by means of chirality techniques or in a OPE formalism [17]

$$d_n = 2.4 \times 10^{-3}\theta efm \tag{1.81}$$

The outcome must be compared with the experimental result of $|d_n| < 3.0 \times 10^{-13} efm$, fixing an upper bound on the new parameter of our theory of $|\theta| < 1.3 \times 10^{-10}$. This is clearly an incredibly tiny value that calls for an explanation; all the more so if we consider that this angle arises from the addition of two different and independent phases (i.e $\theta_Y$ and $\theta_{QCD}$). $\theta$ can freely vary inside the interval $[0, 2\pi[$: so, why should it assume this unnatural value compatible with zero? Experiments seem to suggest that this parameter is actually not physical or, anyway, it does not produce any observable result. But its introduction in the SM theory is highly justified: hence, further justifications must be provided.

## 1.5 The Peccei-Quinn mechanism

Actually, a conceptually easy solution could be given just by supposing that one, among the six quark masses, is zero. If that were the case, we could always act with a $U(1)_A$ transformation on this fermion field: by doing so, we will be able to erase the $\theta$-term by means of the color anomaly developed by this operation. Of course, this will not redefine the phase of any mass term any more. Consequently, because of the freedom of changing this fermion field, the extra term that we introduced earlier will not be physical. Unfortunately, so long as we remain inside the SM scenary, the hypothesis of a massless quark has been highly discouraged by experimental data: the only reasonable candidate could be the up quark, but experiments furnished $m_d/m_u = 1.76 \pm 0.13$, which obviously makes the idea of massless up quite difficult to be supported. But the existence of a new massless quark is hard to back up, too, because of a missing phenomenology of hadron states involving this alleged species.

A much more elegant way to escape from the strong CP problem was suggested by R.P. Peccei and H.R.Quinn in 1977 in an important article [18]. But before exploring this solution, one more comment regarding this $\theta$ parameter should be done. Indeed, let us consider expression (1.50). Here we are given a formula for $Z_\theta[0]$ in the euclidean formalism. On account of the deep connection between quantum field theory in euclidean space and statistical mechanics, we know that the generating functional $Z$ is related to the partition function $Z_s$, which can be written as $Z_s = exp\{-\beta F\}$. $F$ is clearly the free energy, whose minimum gives us the stability conditions of the system. If, in this context, we refer to $\mathcal{E}$ as the free energy density, we see from (1.50) that

$$e^{-V_4\mathcal{E}(\theta)} = \sum_\nu \int \mathcal{D}\bar{A}_{\mu,\nu} exp\left\{ - \int d^4x_E \left( \frac{1}{2} tr[\bar{\mathcal{F}}_{\mu\alpha}\bar{\mathcal{F}}_{\mu\alpha}] + \frac{i\theta g^2}{32\pi^2} tr[\epsilon_{\mu\alpha\rho\sigma}\bar{\mathcal{F}}_{\mu\alpha}\bar{\mathcal{F}}_{\rho\sigma}] \right) \right\} \tag{1.82}$$

where $V_4$ is a 4-dimensional euclidean volume factor. In euclidean notation, the first term in the exponential is clearly positive definite, while the second one is a simple phase: any value of $\theta$ different from zero will just lower the result of the integration. Thereby, we will obtain: $\mathcal{E}(\theta) \geq \mathcal{E}(0)$, i.e the minimum of the energy is reached when the value of $\theta$ equals zero. But we have to point out that $\theta$ is here a parameter of the theory and not a dynamical variable: its value can not evolve over time.

That previous disequality is a direct consequence of *Vafa-Witten theorem* [19], stating that in any theory devoid of sources of CP violation, parity can not be spontaneously broken and so we must have a minimum for the QCD vacuum energy at $\theta = 0$. Up to now, anyway, the $\theta$-term is a form of explicit CP-violation: hence, we can not apply this theorem to our situation. However, we can start envisaging that, if our $\theta$ was made a dynamical field, this latter could relax itself until reaching the minimum value of energy, corresponding to a null $\theta$. Actually, this idea is the cornerstone of the Peccei-Quinn mechanism.

In order to restore CP invariance, Peccei and Quinn required the existence of an extra global $U(1)_{PQ}$ symmetry, dubbed *Peccei-Quinn symmetry.* The particular realization of this symmetry is clearly model dependent, but what is really fundamental is that this transformation has to be anomalous: only this way we can cancel the $\theta$-term through a colour anomaly produced by a

$U(1)_{PQ}$ rotation of fields. So, in much the same way as for the massless fermion case, we can claim that the $\theta$ parameter is unphysical. But this is not the end of the story. Indeed, this new global symmetry could be spontaneously broken at some energy scale: this will result in the appearence of an extra degree of freedom in the model, i.e a goldstone boson associated with the broken symmetry. Actually, this latter will be a pseudo-goldstone boson, being the $U(1)_{PQ}$ symmetry anomalous at quantum level. But, whichever way you look at it, the Peccei-Quinn mechanism has, as a side-effect, the prediction of a new particle, that Wilczek called *axion*, after a famous detergent for its virtue of cleansing the SM out of its strong CP stain.

As it will be extensively discussed later, below the low energy breaking scale, we can write an effective Lagrangian, including the axion field $a(x)$ in this form

$$\mathcal{L}_a = \frac{1}{2}\partial_\mu a \partial^\mu a + \left(\theta_{QCD} + \frac{a}{f_a}\right)\frac{\alpha_s}{8\pi}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_a \tag{1.83}$$

where $f_a$ is an energy scale, called *axion decay constant.* We can appreciate how this new pseudo-goldstone boson plays the role of the $\theta$-parameter: we can actually incorporate this one in the definition of the axion field after a phase shift, which is a symmetry of all terms of the Lagrangian, apart from this anomalous coupling to gluons. Thus, it seems that a spontaneous breaking of $U(1)_{PQ}$ could threat our construction, because, after all, it introduces back in the Lagrangian something similar to a $\theta$ contribution. However, we have to consider that now $a(x)$ is a dynamical field. Its VEV has exactly the some power to induce a parity violation as the original parameter $\theta$, but the Vafa-Witten theorem ensures us that this should not happen, because the axion field will evolve towards the value $\langle a \rangle = 0$. The QCD dynamic itself, eventually, provides us with the solution to the strong CP problem and with the restoration of parity and time-reversal invariance!

The previous reasoning holds as long as the Vafa-Witten theorem can be applied. In presence of explicit CP violation terms, the axion potential will be modified so that $\langle a \rangle \neq 0$, potentially threatening the PQ solution. These sources of CP violation can arise from physics beyond the SM (to explain for example the baryon-antibaryon asymmetry of the universe). However, the SM itself presents, as already mentioned, a complex phase in the CKM matrix. Even though some new CP violating contributions $\theta_k$ will prevent a perfect cancellation of $d_n$, they should not be so large to the point of evading the experimental bound $\sum_k |\theta_k| < 1.3 \times 10^{-10}$.

We want to conclude this paragraph stressing how, from a theoretical point of view, the role of axion is twofold. Indeed, it does not only solve a highly disturbing SM problem, but because of its properties of interacting very weakly (as it emerges from different models that will be discussed later), it is also a lawful dark matter candidate, with all cosmological and astrophysical consequences of that. In this framework (that we will be explored better in chapter 3), after that the axion field has roled down towards the minimum of the potential, it will go on oscillating around it, which means $a = \langle a \rangle + \delta a$. Just considering that $d_n = d_n(\theta_{QCD})$ and that, in the PQ solution, we can substitute $\theta_{QCD} \longmapsto a/f_a$, we will have $d_n = d_n(a/f_a) = d_n(\delta a/f_a)$ (where we used $\langle a \rangle \approx 0$). The axion is a pseudo-scalar field, so that there are no CP violation problems in the Lagrangian (1.83), but time oscillations of $a$ will give rise to a time-varying neutron electric dipole momentum. That does not clash with (1.80), which refers to the mean value of $d_n$. This simple observation offered, pretty recently, a new way of detecting the effects of axions, in addition to a plethora of designed experiments [20].

## 1.6 The need for an extension of the SM

Before analysing a concrete example of axion model, some general aspects can be pointed out. A first thing that must be immediately clarified is the impossibility of implementing a PQ mechanism in a standard model scenary. That can be noticed by looking at the well-known Lagrangian, which presents the Yukawa terms for the first quark generation

$$\mathcal{L}_Y = -y_d \bar{q}_L \phi d_R - y_u \bar{q}_L \tilde{\phi} u_R \tag{1.84}$$

where $q_L$ (the left-handed quarks) and $\phi$ (the Higgs field) are $SU(2)_L$ doublets, whereas $d_R$ and $u_R$ are the right-handed quarks (which do not couple to isospin symmetry). Then, $y_u$ and $y_d$ denote the Yukawa couplings and, finally, $\tilde{\phi} = i\tau_2\phi^*$, employed to give mass to the second component of $\bar{q}_L$, once $SU(2)_L \times U(1)_Y$ is spontaneously broken. But before this threshold, the PQ symmetry will be satisfied by all terms in the Lagrangian. We remind how it is necessary to couple differently left and right quarks, in order to get an anomalous contribution. If, under a PQ transformation, we have

$$\phi \longmapsto e^{iX_h\alpha}\phi \qquad u_R \longmapsto e^{iX_u\alpha}u_R \qquad d_R \longmapsto e^{iX_d\alpha}d_R \tag{1.85}$$

and we suppose that left quarks have no PQ charge (a choice that can always be done, as it will be justified later on), it is straightforward to see that Yukawa terms will respect the PQ symmetry if and only if $X_u = X_h$ and $X_d = -X_h$. Therefore, the up and down PQ charges are the same up to a sign and, upon doing a PQ transformation and trying to evalutate the anomalous term, we realize there will be no change for the $\theta$-term

$$\frac{\alpha_s\theta_{QCD}}{8\pi}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_a \longmapsto \frac{\alpha_s[\theta_{QCD} - \alpha(X_u + X_d)/2]}{8\pi}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_a \tag{1.86}$$

because $X_u + X_d = X_h - X_h = 0$. That means we can not reabsorb the $\theta$ parameter and so we have no chance to pursue a PQ construction. Moreover, we do not have enough freedom to introduce an axion in our description: indeed, all phase degrees of freedom of the Higgs field around its VEV are used to give mass to gauge bosons and there is no room to accomodate an extra axion field. Eventually, we will have to enlarge the SM with some extra fields associated to particles which have not been observed yet.

There is probably no need to say how our extensions should be compatible with current phenomenology, which can highly limit and restrict possible developments beyond the SM. A very important restriction on the scalar structure of the Lagrangian comes from the experimental value of $\rho = 1.00037 \pm 0.00023$ (see [8]), which is defined as

$$\rho \equiv \frac{M_W^2}{M_Z^2\cos^2\theta_W} \tag{1.87}$$

where $\theta_W$ is the Weinberg angle and $M_W$, $M_Z$ the electroweak gauge boson masses. The experimental value of $\rho$ is increadibly close to 1: that is justified in the SM by the presence of the so-called custodial symmetry, an accidental symmetry of the theory. This latter protects the value of this parameter against perturbative corrections. Possible deviations from the exact value of 1 will emerge by including some non-custodial preserving terms. But let us clarify this idea a bit better.

### 1.6.1 The custodial symmetry

First of all, let us write the most general mass matrix for gauge bosons, so as to be consistent with the symmetry-breaking pattern of the SM: $SU(2)_L \times U(1)_Y \to U(1)_{em}$ (as described in [21]). It turns out to be:

$$\mathcal{M} = \begin{pmatrix} M_W^2 & 0 & 0 & 0 \\ 0 & M_W^2 & 0 & 0 \\ 0 & 0 & M_W^2 & m^2 \\ 0 & 0 & m^2 & M_0^2 \end{pmatrix} \tag{1.88}$$

where the first two entries refer to $W_\mu^\pm$ and the other ones to $W_3^\mu$ and $B_0^\mu$ (these latters correspond respectively to the third gauge field of the $SU(2)_L$ symmetry and the hypercharge gauge field). The first block is clearly diagonal and endowed with two equal entries, as ensured by the residual $U(1)_{em}$ symmetry, which forces the masses of the two charged $W_\mu$ to be the same. On the other hand, in the second block we have a mixing between the two neutral fields, as very well known. It is also clear that, if $U(1)_{em}$ is preserved, the neutral $2\times 2$ mass matrix will have a null eigenvalue and so

$$det\mathcal{M} = 0 = M_W^2 M_0^2 - m^4 \tag{1.89}$$

with the corresponding eigenvector $\begin{pmatrix} -\sin\theta_W \\ \cos\theta_W \end{pmatrix}$, where $\theta_W$ is the *Weinberg angle*, defined by $cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$. Employing the algebraic definition of eigenvector, the following relation will hold true

$$\tan\theta_W = \frac{m^2}{M_W^2} = \left|\frac{M_0}{M_3}\right| \tag{1.90}$$

where (1.89) has been utilised. In the end, being the trace defined as a sum of eigenvalues, we can express the non-zero eigenvalue this way:

$$M_Z^2 = tr\mathcal{M} = M_0^2 + M_W^2 = M_W^2 \sec^2\theta_W \tag{1.91}$$

Thus, at tree-level order we are left with:

$$\frac{M_W^2}{M_Z^2\cos^2\theta_W} = 1 \tag{1.92}$$

In the limit $g' \to 0$, we obtain $M_W = M_Z$. That tells us the three gauge fields $W_\mu^\pm$ and $Z_\mu$ transform as a triplet of $SO(3) \sim SU(2)$, which turns out to be an extra symmetry of the theory. Nevertheless, the hypercharge sector slightly breaks it with a tiny $g'$, so that we can only speak of an approximate global symmetry, whose effect in (1.92) is enclosed in $\cos^2\theta_W$. For $W_\mu^\pm$ and $Z_\mu$ to get masses in an almost symmetric way, an *accidental* symmetry must reside in the scalar sector of the SM Lagrangian, which is responsible for gauge boson masses through the Higgs mechanism.

In order to make this symmetry more manifest, it will be wiser to rewrite the Higgs sector using the *two doublets formalism.* We can introduce

$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi_0 \end{pmatrix} \quad \tilde{\phi} = i\tau_2\phi^* = \begin{pmatrix} \varphi_0^* \\ -\varphi^- \end{pmatrix} \quad \Longrightarrow \quad \Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{\phi} & \phi \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_0^* & \varphi^+ \\ -\varphi^- & \varphi_0 \end{pmatrix} \tag{1.93}$$

with isospin $I_\phi = I_{\tilde{\phi}} = 1/2$ and hypercharge $Y_\phi = -Y_{\tilde{\phi}} = 1$. This way we can rewrite the Higgs Lagrangian

$$\mathcal{L}_{Higgs} = (\mathcal{D}_\mu\phi)^\dagger\mathcal{D}^\mu\phi + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \quad \text{with} \quad \mathcal{D}_\mu\phi = \partial_\mu\phi - ig\vec{\tau}\vec{W}_\mu\phi - ig'\frac{Y_\phi}{2}B_\mu\phi \tag{1.94}$$

in terms of the new parametrization $\Phi$ as

$$\mathcal{L}'_{Higgs} = tr[(\mathcal{D}_\mu\Phi)^\dagger\mathcal{D}^\mu\Phi + \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2] \quad \text{with} \quad \mathcal{D}_\mu\Phi = \partial_\mu\Phi - ig\vec{\tau}\vec{W}_\mu\Phi + ig'B_\mu\Phi\tau_3 \tag{1.95}$$

Of course, $\tau^i = \sigma^i/2$. It is a straightforward calculation to verify that $\mathcal{L}_{Higgs} = \mathcal{L}'_{Higgs}$. Under gauge transformations, we know that

$$\begin{cases} \phi' = U_L\phi \quad \tilde{\phi}' = U_L\tilde{\phi} & \text{for} \quad U_L \in SU(2)_L \\ \phi' = e^{i\alpha/2}\phi \quad \tilde{\phi}' = e^{-i\alpha/2}\tilde{\phi} & \text{for} \quad e^{i\alpha/2} \in U(1)_Y \end{cases} \tag{1.96}$$

which can be recast for the new field $\Phi$ as

$$\begin{cases} \Phi' = U_L \Phi = U_L \tilde{\phi} & \text{for} \quad U_L \in SU(2)_L \\ \Phi' = \Phi e^{-i\alpha\tau_3/2} & \text{for} \quad U_{R3}^\dagger = e^{-i\alpha\tau_3/2} \in SU(2)_R \end{cases} \tag{1.97}$$

Then, we can try to extend the gauge group $SU(2)_L \times U(1)_Y$ to a larger (but global) one, i.e $SU(2)_L \times SU(2)_R \sim SO(4)$, under which $\Phi' = U_L \Phi U_R^\dagger$. The potential sector of the Lagrangian (1.95) manifestly respects this extended symmetry, which, in the old formalism, simply mixes the four degrees of freedom of $\phi$. On the other hand, the kinetic term partially breaks it. As a matter of fact, $(\mathcal{D}_\mu \Phi)' = U_L \mathcal{D}_\mu \Phi$ under $SU(2)_L$, but it is not true in general that for a $SU(2)_R$ transformation a similar relation holds, because $(\mathcal{D}_\mu \Phi)' \neq \mathcal{D}_\mu \Phi U_R^\dagger$. It is again the hypercharge contribution that violates it, due to the fact that $[U_{R3}^\dagger, \tau_i] = 0$ if and only if $i = 3$. Therefore, $(\mathcal{D}_\mu \Phi)' = U_L \mathcal{D}_\mu \Phi U_R^\dagger$ under $SU(2)_L \times SU(2)_R$ just in the limit $g' \to 0$.

Below the electroweak scale, if $\langle \phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$, we will obviously have $\langle \Phi \rangle = (v/2) \mathbb{1}_{2\times2}$. The VEV of $\Phi$ breaks the global $SU(2)_L \times SU(2)_R$ to its subset $SU(2)_V$, for which $U_L = U_R$. This remaining symmetry group is dubbed *custodial symmetry*, which is an accidental one, because the theory has not been explicitly required to satisfy it: it just stems from the structure of the scalar sector. Despite that, there are phenomenological reasons to claim it is an approximate symmetry of nature.

We said it is an approximate one, because, again, the term proportional to $g'$ in the covariant derivative violates the $SU(2)_V$ symmetry. In the limit $g' \to 0$, the covariant derivative $(\mathcal{D}_\mu \Phi)' = U_L \mathcal{D}_\mu \Phi U_L^\dagger$ will transform as a $SU(2)_V$ triplet: consequently, from $tr[(\mathcal{D}_\mu \Phi)^\dagger \mathcal{D}^\mu \Phi]$ the $W_\mu^\pm$ and $Z_\mu$ bosons will receive the same mass. It is just the presence of this residual $SU(2)_V$ symmetry that enforces the value of $\rho$ to be so close to one: departures from the tree level behaviour will arise from perturbative corrections involving explicit custodial breaking terms.

The presence of custodial violation will be responsible for the experimental $\Delta\rho = \rho_{exp} - 1 = 3.7 \times 10^{-4}$. The hypercharge sector will contribute through the non-null value of $g'$ inside loop corrections to $M_W$ and $M_Z$. However, a larger contribution will come from the quark part of the Lagrangian. Indeed, we can write down the Yukawa sector of the SM in the form

$$\mathcal{L}_{yuk} = \sum_{i=1}^{3} \begin{pmatrix} \bar{u}_{Li} & \bar{d}_{Li} \end{pmatrix} \Phi \begin{pmatrix} y_{ui} u_{Ri} \\ y_{di} d_{Ri} \end{pmatrix} \tag{1.98}$$

where the custodial symmetry is restored just if $y_{ui} = y_{di}$: the mass difference between quarks inside the same isospin doublet is another explicit $SU(2)_V$ violating source. The greatest correction will be produced by the third generation, owning to the huge top quark mass.

Now it is clear that all possible beyond the standard model (BSM) extensions should not clash with the phenomenological evidence of an approximate custodial symmetry: non-custodial preserving terms can not affect $\Delta\rho$ more than $\Delta\rho - \Delta\rho_{SM}$. Moreover, the scalar structure of these BSM theories should be designed in order to give equal masses to gauge bosons $W_\mu^\pm$ and $Z_\mu$ at tree level (up to $g'$ corrections). Let us analyse how the second requirement can costrain a model, by considering an undefined number of scalar multiplets $\phi_i$ of weak isospin $I_i$, hypercharge $Y_i$ and with a VEV for the neutral component of $\langle \varphi_{0i} \rangle = v_i/\sqrt{2}$. Then, upon defining the covariant derivative acting on the $\phi_i$ fields as

$$\mathcal{D}_\mu \phi_i \equiv \partial_\mu \phi_i - igW_\mu^a \tau^a \phi_i - ig' \frac{Y_i}{2} B_\mu \phi_i \tag{1.99}$$

we know that we can read off from the kinetic term $(\mathcal{D}_\mu \phi_i)^\dagger \mathcal{D}^\mu \phi_i$ the masses for gauge bosons, when the $\phi_i$ fields are evalutated at their own VEVs $\langle \phi_i \rangle$. If we concentrate only on mass contributions, we will get

$$[(\mathcal{D}_\mu\phi_i)^\dagger\mathcal{D}^\mu\phi_i]_{\text{gauge masses}} = \\ = -i^2\langle\phi_i\rangle^T\left(g[W^1_\mu\tau^1_i + W^2_\mu\tau^2_i] + gW^3_\mu\tau^3_i + g'\frac{Y_i}{2}B_\mu\right)\left(g[W^\mu_1\tau^1_i + W^\mu_2\tau^2_i] + gW^\mu_3\tau^3_i + g'\frac{Y_i}{2}B^\mu\right)\langle\phi_i\rangle \tag{1.100}$$

Then, we can make use of the expressions for the charge fields $W^\pm_\mu = \frac{W^2_\mu \pm iW^1_\mu}{\sqrt{2}}$ and introduce the ladder operators $\tau^\pm = \tau_1 \pm i\tau_2$. If we furthermore consider the existence of the unbroken electric charge generator $Q_{em}\langle\phi_i\rangle = (\tau^3 + Y_i\mathbb{1}/2)\langle\phi_i\rangle = 0$, we can write down

$$[(\mathcal{D}_\mu\phi_i)^\dagger\mathcal{D}^\mu\phi_i]_{\text{gauge masses}} = \\ = \langle\phi_i\rangle^T\left(g\frac{W^+_\mu\tau^-_i + W^-_\mu\tau^+_i}{\sqrt{2}} + gW^3_\mu\tau^3_i + g'\frac{Y_i}{2}B_\mu\right)\left(g\frac{W^{-\mu}\tau^+_i + W^{+\mu}\tau^-_i}{\sqrt{2}} + gW^\mu_3\tau^3_i + g'\frac{Y_i}{2}B^\mu\right)\langle\phi_i\rangle \\ = \langle\phi_i\rangle^T\left(\frac{g^2}{2}W^+_\mu W^\mu_-\{\tau^-_i,\tau^+_i\} + (gW^3_\mu - g'B_\mu)^2(\tau^3_i)^2\right)\langle\phi_i\rangle + \frac{g^2}{2}\langle\phi_i\rangle^T\underbrace{[(W^+_\mu\tau^-_i)^2 + (W^-_\mu\tau^+_i)^2]}_{=0}\langle\phi_i\rangle + \\ + \langle\phi_i\rangle^T\left[\underbrace{\left(\frac{g^2}{\sqrt{2}}W^\mu_3 - gg'\frac{B^\mu}{\sqrt{2}}\right)W^+_\mu\{\tau^-_i,\tau^3_i\}}_{=0} + \underbrace{\left(\frac{g^2}{\sqrt{2}}W^\mu_3 - gg'\frac{B^\mu}{\sqrt{2}}\right)W^-_\mu\{\tau^+_i,\tau^3_i\}}_{=0}\right]\langle\phi_i\rangle \tag{1.101}$$

The last three terms we have highlighted equal zero, because of the Lie algebra of $SU(2)$, whose properties are independent of the representation $i$. Indeed, when $\phi_i$ is evalutated at the minimum of the potential, it is just a constant vector in the $i$-dimensional representation of the group. Its third component of isospin $I_3$, given by $\tau_3\langle\phi_i\rangle_{i,m} = m\langle\phi_i\rangle_{i,m}$, will depend on the particular choise of the VEV. It is known from the angular momentum theory that:

$$\tau^\pm\langle\phi_i\rangle_{i,m} = \langle\phi_i\rangle_{i,m\pm1}\sqrt{i(i+1) - m(m\pm1)} \\ \langle\phi_i\rangle^T_{i,m'}\langle\phi_i\rangle_{i,m} = \delta_{m,m'} \tag{1.102}$$

Consequently, the last three addends of the expression (1.101) vanish, because they involve scalar products of orthogonal states. We can employ once more a relation derived from $SU(2)$ Lie algebra

$$\tau^\pm_i\tau^\mp_i = (\vec{\tau}_i)^2 - (\tau^3_i)^2 \pm \tau^3_i \tag{1.103}$$

to eventually get

$$[(\mathcal{D}_\mu\phi_i)^\dagger\mathcal{D}^\mu\phi_i]_{\text{gauge masses}} = \langle\phi_i\rangle^T\left(g^2W^+_\mu W^\mu_-[(\vec{\tau}_i)^2 - (\tau^3_i)^2] + (gW^3_\mu - g'B_\mu)^2(\tau^3_i)^2\right)\langle\phi_i\rangle = \\ = \left(\frac{g^2v_i^2}{2}W^+_\mu W^\mu_-[I_i(I_i+1) - I^2_{3i}] + \frac{v_i^2 I^2_{3i}}{2}(gW^3_\mu - g'B_\mu)^2\right) \tag{1.104}$$

where in the last passage we substitute the eigenvalues of isospin operators on the vacuum. If we now add up all of the Higgs contributions, we can identify

$$M^2_W = g^2\sum_i(I_i(I_i+1) - I^2_{3i})v_i^2/2 \qquad M^2_Z = \sum_i I^2_{i3}v_i^2(g^2 + g'^2), \tag{1.105}$$

Here we have reached the crucial point of our reasoning. In order for the $\rho$ parameter to equal one at tree level, we just need to impose

$$M_W^2 = M_Z^2 cos^2\theta_W \Rightarrow (I_i(I_i+1) - I_{3i}^2) = 2I_{i3}^2 \quad \forall i \tag{1.106}$$

Therefore, equation (1.106) tells us something fundamental: if we want to preserve the experimental value of the $\rho$ parameter, the only way is to extend the scalar sector of the SM model with Higgs fields whose isospin satisfies

$$I_i = \frac{1}{2}(-1 + \sqrt{1 + 12I_{i3}^2}) = \ 0, \ \frac{1}{2}, \ 3, \ ... \tag{1.107}$$

That means singlets, doublets, septuplets ... of scalar fields are allowed, but not triplets, for instance. Then, using again the property $\tau^3\langle\phi_i\rangle = -\frac{Y_i}{2}\mathbb{1}\langle\phi_i\rangle$, we can eventually claim [22]

$$\rho = \frac{\sum_i (I_i(I_i+1) - Y_i^2/4)v_i^2}{\sum_i Y_i^2 v_i^2/2} \tag{1.108}$$

Considering the values of $I_i = 0, \ \frac{1}{2}, \ 3, \ ...$ , we obtain the corresponding custodial preserving hypercharges $Y_i = 0, \ 1, \ 4, \ ...$ , stemming from the condition $I_i(I_i+1) = \frac{3}{4}Y_i^2$ imposed for all $i$. All these considerations have provided us with an interesting contraint on model building.

## 1.7 Possible implementations

One of the first model presented to host a PQ mechanism in the SM was the Peccei-Quinn-Weinberg-Wilczek (PQWW) model [23], which adds to the original field content of the theory an extra Higgs doublet. The Yukawa terms were modified as

$$\mathcal{L}_Y = -y_d \bar{q}_L \phi_2 d_R - y_u \bar{q}_L \tilde{\phi}_1 u_R \tag{1.109}$$

where $\phi_i$ develops a corresponding vacuum $v_i$. The two vacua should reasonably satisfy the relation $\sqrt{v_1^2 + v_2^2} = v = 246GeV$, in order to reproduce the electroweak scale. In so doing, we have eight degrees of freedom shared by two Higgs doublets and, consequently, we can introduce a non-trivial PQ charges assignment as

$$\phi_i \longmapsto e^{iX_i\alpha}\phi_i \qquad u_R \longmapsto e^{iX_u\alpha}u_R \qquad d_R \longmapsto e^{iX_d\alpha}d_R \tag{1.110}$$

in which $X_1 = X_u$ and $X_2 = -X_d$. Problems with this model appear when it comes to dealing with phenomenology. As it will be shown in the next chapter, the axion mass turns out to be related to a special quantity: $m_a \approx f_a^{-1}$, where $f_a$ is a constant with energy dimensions named *axion decay constant*. In this simple model, where there is only one energy scale (i.e $v$), $f_a \approx v$ and consequently $m_a \approx v^{-1}$. Moreover, the constant $f_a$ controls all axion couplings with SM particles, which turn out to be proportional to $1/f_a$. An axion of this kind interacts too strongly with matter if compared to current experimental results, which undoubtedly exclude it. A famous example is the experimental bound on the branching ratio $\mathcal{B}(K^+ \to \pi^+ + nothing) < 7.3 \times 10^{-11}$, for which this so-called *visible axion* model predicts a significantly larger value. Even more directly, astrophysical considerations impose $10^9 GeV \leq f_a \leq 10^{17} GeV$ (as we will contextualize better later on): an $f_a \approx 246GeV$ grossly falls out of this range. Therefore, just because the PQ mechanism seems to be well justified, in order to reconcile its side-effect with experiments we need to heavily decrease the axion mass by introducing a new energy scale, that means other degrees of freedom.

This way has been pursed by different models that go under the name of *invisible axion models*. One of those is the Dine-Fischler-Srednicki-Zhitnisky (DFSZ) model [24, 25], that is one of the main topic of this Thesis and it will be extensively described later. The essential idea of this construction is to extend solely the scalar sector with an extra complex singlet field with respect to the PQWW model and to require that all fields appearing in the theory enjoy a Peccei-Quinn symmetry (excluding gauge bosons). A different approach is that followed by the Kim-Shifman-Vainshtein-Zakharov (KSVZ) model [26, 27], where a complex singlet scalar that couples to a new heavy quark is introduced in the SM. Here only these two extra fields are endowed with a PQ charge and, thereby, they alone are responsible for the axion degree of freedom: the rest of the standard model is PQ neutral.

Of course, these are only two exemplary models, but it is easy to realise that plenty of possibilities to tweak the SM can be suggested. Anyway, all of them should take into account the experimental restrictions coming from particle physics experiments and, being the axion a dark matter candidate, constraints from astrophysical and cosmological data.

# Chapter 2

# The mass spectrum of a DFSZ model

In this chapter we are going to study in depth the degrees of freedom of a DFSZ theory and the problem of identifying the different fields of the standard model arising from its scalar sector, whose structure is here much more convoluted. In particular, we will concentrate on the kinetic and the potential part of the Lagrangian, leaving momentarily aside the Yukawa terms. Following the general procedure applicable for the calculation of the mass spectrum of a typical DFSZ model, we will analyse a slightly different case, where the quartic c-term of the potential, that mixes the three different Higgs fields, is replaced by a cubic one.

## 2.1 The DFSZ potential

As we have already anticipated in the previous chapter, in order to implement a model enjoying a PQ symmetry, we have to enlarge the field content of the theory. To do that, a viable possibility is that presented by the DFSZ paradigm, which leaves untouched the fermionic sector of the standard model, but enlarges the scalar one firstly by means of a second Higgs doublet. But we have already pointed out how this solution possesses great phenomenological problems: its prediction of axion properties is ruled out by experiments. Thus, the DFSZ model adds an extra Higgs singlet, for reasons that we are now going to discuss.

Indeed, we can define the two Higgs fields as

$$\phi_1 = \begin{pmatrix} \alpha_+ \\ \alpha_0 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} \beta_+ \\ \beta_0 \end{pmatrix} \tag{2.1}$$

which are simply copies of the SM Higgs field: two $SU(2)_L$ doublets (with isospin $I = 1/2$) with hypercharge $Y = 1$ and, consequently, with a upper charged component of $Q = 1$ and a lower discharged one (according to the Weinberg relation $Q = I^{(3)} + Y/2$). These two fields are responsible for quark and lepton masses through Yukawa couplings similar to those of (1.109). Therefore, it is clear that their VEVs, given by $\langle\alpha_+\rangle = \langle\beta_+\rangle = 0$, $\langle\alpha_0\rangle = v_1/\sqrt{2}$ and $\langle\beta_0\rangle = v_2/\sqrt{2}$, should be related to the usual electroweak vacuum expectation value as $v^2 = v_1^2 + v_2^2$ (with $v = 246GeV$), where we define $\tan\beta = v_2/v_1$.

If we now enrich the model with the extra Higgs $\phi$, requiring it to be a $SU(2)_L$ singlet with $Y = 0$, it is immediately evident that it will not take part in the gauge symmetry of the Lagrangian. Moreover, if $\langle\phi\rangle = v_\phi$, we are introducing by hand a new energy scale in the theory, which has nothing to do with the electroweak one. Actually, it will be ultimately related to the axion decay constant $f_a$. Just thanks to this new scale, we will have enough freedom to make axions *invisible.*

The kinetic terms of these Higgs fields can be fixed, accordingly to their definitions, as

$$\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi + \mathcal{D}_\mu \phi_1^\dagger \mathcal{D}^\mu \phi_1 + \mathcal{D}_\mu \phi_2^\dagger \mathcal{D}^\mu \phi_2 \tag{2.2}$$

where the covariant derivative is obviously defined this way:

$$\mathcal{D}_\mu \phi_i \equiv \partial_\mu \phi_i - i\frac{g}{2}\sigma_a W^a_\mu \phi_i - i\frac{g'}{2} Y \phi_i. \tag{2.3}$$

Now, if we consider how these Higgs fields transfom under the PQ symmetry, we are free to set

$$\phi_1 \longmapsto e^{iX_1\theta}\phi_1 \qquad \phi_2 \longmapsto e^{iX_2\theta}\phi_2 \qquad \phi \longmapsto e^{iX_\phi\theta}\phi \tag{2.4}$$

where, up to now, the different PQ charges are unrelated. Putting aside the Yukawa terms, which play no role in the determination of the scalar and gauge mass spectrum, we can write down the most general potential respecting CP invariance, the gauge symmetry of the standard model $SU(3)_c \times SU(2)_L \times U(1)_Y$ and the additional anomalous $U(1)_{PQ}$ transformation. It can be readily realised that the PQ invariance requirement imposes severe constraints on the possible terms entering the potential, that can be expressed in the following form:

$$\begin{aligned} V(\phi,\phi_1,\phi_2) =& \lambda_\phi(\phi^*\phi - V_\phi^2)^2 + \lambda_1(\phi_1^\dagger\phi_1 - V_1^2)^2 + \lambda_2(\phi_2^\dagger\phi_2 - V_2^2)^2 + \\ &+ \lambda_3(\phi_1^\dagger\phi_1 - V_1^2 + \phi_2^\dagger\phi_2 - V_2^2)^2 + \lambda_4\big[(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \\ &- (\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1)\big] + (a\phi_1^\dagger\phi_1 + b\phi_2^\dagger\phi_2)\phi^*\phi + cV_{c-term} \end{aligned} \tag{2.5}$$

In the previous expression, clearly, all of the parameters are assumed to be real. To start with, $V_1$, $V_2$ and $V_\phi$ individuate the unperturbed VEVs of the theory, that is to say the individual VEVs of the Higgs fields when all interactions have been switched off. That can be easily read off from the first three terms. Then, we have two independent contributions parametrized by $\lambda_3$ and $\lambda_4$, which display a mixing between the two Higgs doublets and, eventually, two terms which relate all of the Higgs fields. The second-to-last coupling, just as the previous ones, does not require further conditions on PQ charges in order for $U(1)_{PQ}$ to be a symmetry.

The $V_{c-term}$, which has not been made explicit yet, deserves a different kind of discussion. As a matter of fact, it can come in two distinct versions and, thereby, it can individuate two slightly different models:

$$V_{c-term} = \begin{cases} \phi_1^\dagger\phi_2\phi^2 + \phi_2^\dagger\phi_1\phi^{*2} & X_2 - X_1 = -2X_\phi \\ \phi_1^\dagger\phi_2\phi + \phi_2^\dagger\phi_1\phi^* & X_2 - X_1 = -X_\phi \end{cases} \tag{2.6}$$

with a quartic or cubic interaction term. Next to each version, we have indicated the required charge relation in order to make this extra contribution PQ invariant. The quartic model has already been studied quite deeply in other works (see for example [28]). What we are going to tackle here is to develop the consequences of introducing a cubic term in our Lagrangian in a very similar fashion to previous approaches. Customarily, the $X_\phi$ is fixed in literature to a precise constant value of $-1/2$, for reasons that will be clear later. To make use of the advantages of this choice, we will select a value of $X_\phi = -1$ for the cubic model, in order to obtain a completely equivalent relation among charges.

### 2.1.1 Realization of the custodial symmetry

A non-trivial issue will be to understand the roles of the different Higgs degrees of freedom in our model, but we will take it up in following sections. Now it is important to explore the global symmetries of this potential. As we have already motivated quite at length, there are good reasons to think that the custodial symmetry is an approximate symmetry of the standard model. It is obviously approximate, because it is not a symmetry of the whole Lagrangian: the Yukawa and hypercharge sector violate it, as we have already shown. Nevertheless, we know how crucial its presence is to account for the close values of the $W^{\pm}$ and $Z$ boson masses. However, it is not self-evident at all how this $SU(2)_L \times SU(2)_R$ symmetry (then broken to $SU(2)_V$ by the vacuum) could be achieved in a DFSZ model.

As a first step, we need to reorganize the Higgs doublets degrees of freedom using the two doublets formalism, to make more manifest our problem:

$$\Phi_{12} = \begin{pmatrix} \tilde{\phi}_1 & \phi_2 \end{pmatrix} = \begin{pmatrix} \alpha_0^* & \beta_+ \\ -\alpha_- & \beta_0 \end{pmatrix} \qquad \Phi_{21} = \begin{pmatrix} \tilde{\phi}_2 & \phi_1 \end{pmatrix} = \sigma_2 \Phi_{12}^* \sigma_2 = \begin{pmatrix} \beta_0^* & \alpha_+ \\ -\beta_- & \alpha_0 \end{pmatrix} \tag{2.7}$$

It will also prove useful to introduce the following matrices

$$\begin{aligned} I &= \Phi_{12}^\dagger \Phi_{12} = \begin{pmatrix} \phi_1^\dagger \phi_1 & \tilde{\phi}_1^\dagger \phi_2 \\ -\phi_1^\dagger \tilde{\phi}_2 & \phi_2^\dagger \phi_2 \end{pmatrix} \\ J &= \Phi_{12}^\dagger \Phi_{21} = \begin{pmatrix} \phi_2^\dagger \phi_1 & 0 \\ 0 & \phi_2^\dagger \phi_1 \end{pmatrix} \\ W &= (V_1^2 + V_2^2)\frac{\mathbb{1}}{2} + (V_1^2 - V_2^2)\frac{\sigma_3}{2} = \begin{pmatrix} V_1^2 & 0 \\ 0 & V_2^2 \end{pmatrix} \end{aligned} \tag{2.8}$$

The last one is simply a constant matrix, while the remaining ones are particular combinations of the original fields, whose VEVs take on the form

$$\langle I \rangle = \begin{pmatrix} v_1^2 & 0 \\ 0 & v_2^2 \end{pmatrix} \qquad \langle J \rangle = v_1 v_2 \mathbb{1} \tag{2.9}$$

It is noteworthy that these vacua break the $SU(2)_L \times SU(2)_R$, as expected. But if we want a residual $SU(2)_V$ symmetry, we have to impose the condition $v_1 = v_2$, rendering $I$ proportinal to an identity matrix, too.

If we make use of all the aforementioned notations, we can rewrite the potential as

$$\begin{aligned} V(\phi, \phi_1, \phi_2) =& \lambda_\phi (\phi^* \phi - V_\phi^2)^2 + \frac{\lambda_1}{4} \{tr[(I - W)(1 + \sigma_3)]\}^2 + \frac{\lambda_2}{4} \{tr[(I - W)(1 - \sigma_3)]\}^2 + \\ &+ \lambda_3 [tr(I - W)]^2 + \frac{\lambda_4}{4} tr[I^2 - (I\sigma_3)^2] + \frac{1}{2} tr[(a + b)I + (a - b)I\sigma_3]\phi^*\phi + \\ &+ \frac{c}{2} tr(J\phi + J^\dagger \phi^*) \end{aligned} \tag{2.10}$$

As usual, under a $SU(2)_L \times SU(2)_R$ symmetry, we will have $\Phi_{ij} \longmapsto L\Phi_{ij}R^\dagger$ and, thus, starting from their definitions, we can deduce

$$I \longmapsto RIR^\dagger \qquad J \longmapsto J \tag{2.11}$$

where the first relation can be straightforwardly obtained, while the second can be simply computed remembering that $U\sigma_2 = \sigma_2 U^*$, with $U$ unitary. Now that the transformation of the potential under the desired symmetry has been highlighted, the constraints on our parameters in order to satisfy the custodial symmetry can be calculated.

*Proof.* It is pretty easy to notice that $\lambda_\phi$ and $V_\phi$ will not be involved in the transformation and they will be automatically custodial preserving. Using the cyclicity property of the trace, it can be also claimed the same thing for $\lambda_3$ and $c$. The remaining terms deserve a bit more of caution. We can start developing the $\lambda_1$, $\lambda_2$ and $\lambda_4$ contributions:

$$\begin{aligned}&\frac{\lambda_1}{4}\{tr[I]-tr[W]+tr[I\sigma_3]-tr[W\sigma_3]\}^2+\frac{\lambda_2}{4}\{tr[I]-tr[W]+\\&-tr[I\sigma_3]+tr[W\sigma_3]\}^2+\frac{\lambda_4}{4}tr[I^2-(I\sigma_3)^2]\end{aligned} \tag{2.12}$$

We see that the only troublesome term which violates our global symmetry is $tr[I\sigma_3]$, owing to its transformation properties: $tr[I'\sigma_3]=tr[RIR^\dagger\sigma_3]\neq tr[I\sigma_3]$. The idea to pursue is trying to simplify terms of this sort, appearing in the $\lambda_1$ and $\lambda_2$ couplings, with those in the $\lambda_4$ parenthesis. To achieve that, we can just expand a bit more our expression as

$$\begin{aligned}&\frac{\lambda_1}{4}\{(tr[I]+tr[I\sigma_3])^2+tr[W]^2+tr[W\sigma_3]^2-2(tr[I]+tr[I\sigma_3])tr[W]-2(tr[I]+tr[I\sigma_3])tr[W\sigma_3]+\\&+2tr[W]tr[W\sigma_3]\}+\frac{\lambda_2}{4}\{(tr[I]-tr[I\sigma_3])^2+tr[W]^2+tr[W\sigma_3]^2-2(tr[I]-tr[I\sigma_3])tr[W]+\\&+2(tr[I]-tr[I\sigma_3])tr[W\sigma_3]-2tr[W]tr[W\sigma_3]\}+\frac{\lambda_4}{4}tr[I^2-(I\sigma_3)^2]\end{aligned} \tag{2.13}$$

In order to reach the compensation with the term showing up in the last bracket, we have to start making a set of assumptions that will ultimately lead us to the identification of the custodial preserving conditions. First of all, we have to require $\lambda_1=\lambda_2=\lambda$, so that we can write

$$\begin{aligned}&\frac{\lambda}{4}\{(tr[I]+tr[I\sigma_3])^2+(tr[I]-tr[I\sigma_3])^2+2tr[W]^2+2tr[W\sigma_3]^2-4tr[I]tr[W]+\\&-4tr[I\sigma_3]tr[W\sigma_3]\}+\frac{\lambda_4}{4}tr[I^2-(I\sigma_3)^2]\end{aligned} \tag{2.14}$$

It is easy to see that there is no hope of simplifying a term like $-4tr[I\sigma_3]tr[W\sigma_3]$, because the $\lambda_4$ contribution does not depend on $W$. So, we can only require $tr[W\sigma_3]=0$, that is achieved by setting $V_1=V_2$. Therefore, one gets

$$\frac{\lambda}{4}\{2tr[I]^2+2tr[I\sigma_3]^2+2tr[W]^2-4tr[I]tr[W]\}+\frac{\lambda_4}{4}tr[I^2-(I\sigma_3)^2] \tag{2.15}$$

The last two terms in the first parenthesis are harmless. Thereby, we just need to impose $\lambda_4=2\lambda$ to eventually obtain

$$\frac{\lambda}{2}\{2tr[I]^2+2tr[W]^2-4tr[I]tr[W]\} \tag{2.16}$$

which is now custodial preserving.

Regarding the term containing the parameters $a$ and $b$, we can simply get rid of the annoying $tr[I\sigma_3]$ contribution by requiring $a=b$.

Finally, we can sum up the custodial preserving conditions obtained in this proof:

$$\begin{cases}\lambda_1=\lambda_2=\lambda\\ \lambda_4=2\lambda\\ V_1=V_2=V\\ a=b\end{cases} \tag{2.17}$$

□

## 2.2 The field content

In this section we will go through the tricky issue of identifying the physical degrees of freedom of the scalar sector of the model. We know that each Higgs doublet possesses four components (a radial and three angular fluctuations), while the Higgs singlet has two of them: therefore, we have to end up with ten possible fields. We can envisage, just from theoretical considerations, that three of them will be eaten by the gauge bosons, contributing to their masses, exactly as in the standard model. Nevertheless, we also expect to find a goldstone boson (i.e the axion), because the global PQ symmetry is broken by the VEV of $\phi$ at a very high energy scale.

### 2.2.1 The gauge bosons

To start with, it will be extremely useful to spot the degrees of freedom eaten by the gauge bosons of the standard model. It is immediately clear that a complication arises from the fact that now the gauge symmetry is shared by two Higgs fields.

We know that we can write our fluctuations around the vacuum for the two doublets in a similar fashion as

$$\begin{aligned}\phi_1 &= \frac{1}{\sqrt{2}} exp\left\{\frac{i}{2}[\zeta_1\sigma_1 + \zeta_2\sigma_2 + \zeta_3(\sigma_3 - \mathbb{1})]\right\}\begin{pmatrix}0\\ v_1 + h_1\end{pmatrix} = \frac{1}{\sqrt{2}}E_1(x)\begin{pmatrix}0\\ v_1+h_1\end{pmatrix}\\ \phi_2 &= \frac{1}{\sqrt{2}} exp\left\{\frac{i}{2}[\eta_1\sigma_1 + \eta_2\sigma_2 + \eta_3(\sigma_3 - \mathbb{1})]\right\}\begin{pmatrix}0\\ v_2 + h_2\end{pmatrix} = \frac{1}{\sqrt{2}}E_2(x)\begin{pmatrix}0\\ v_2+h_2\end{pmatrix}\end{aligned} \tag{2.18}$$

where we have brought to light that $\sigma_3 + \mathbb{1}$ is the unbroken generator. Here, to avoid a cumbersome notation, we will consider $\zeta_i = \tilde{\zeta}_i/v_1$ and $\eta_i = \tilde{\eta}_i/v_2$, where $\tilde{\zeta}_i$ and $\tilde{\eta}_i$ are fields with mass dimensions. To see which particular combination of the phases $\zeta$ and $\eta$ will be absorbed in the gauge boson fields, we have to look at the kinetic term of the Lagrangian. Of course, we are going to individuate the degrees of freedom gobbled up by a linearized gauge transformation: thereby, we will be interested only in quadratic terms in gauge and goldstone fields. Using the Higgs $\phi_1$ as an example, we could write its partial derivative explicitly as

$$\begin{aligned}\partial_\mu\phi_1 =& \frac{1}{\sqrt{2}}\left\{E_1(x)\begin{pmatrix}0\\ \partial_\mu h_1\end{pmatrix} + \frac{i}{2}[\partial_\mu\zeta_1\sigma_1 + \partial_\mu\zeta_2\sigma_2 + \partial_\mu\zeta_3(\sigma_3 - \mathbb{1})]E_1(x)\begin{pmatrix}0\\ v_1+h_1\end{pmatrix}\right\} =\\ &\approx \frac{1}{\sqrt{2}}\left\{\begin{pmatrix}0\\ \partial_\mu h_1\end{pmatrix} + \frac{i}{2}\begin{pmatrix}0 & \partial_\mu\zeta_1 - i\partial_\mu\zeta_2\\ \partial_\mu\zeta_1 + i\partial_\mu\zeta_2 & -2\partial_\mu\zeta_3\end{pmatrix}\begin{pmatrix}0\\ v_1+h_1\end{pmatrix}\right\} =\\ &= \frac{1}{\sqrt{2}}\left\{\begin{pmatrix}0\\ \partial_\mu h_1\end{pmatrix} + \frac{i}{2}(v_1+h_1)\begin{pmatrix}\partial_\mu\zeta_1 - i\partial_\mu\zeta_2\\ -2\partial_\mu\zeta_3\end{pmatrix}\right\}\end{aligned} \tag{2.19}$$

where in the second-to-last passage we approximate $E_1(x) \approx \mathbb{1}$, because we will ultimately need the square of this object. As a consequence, quadratic terms in goldstone bosons can be neglected, together with all higher contributions.

Now, we can express our covariant derivative in a more explicit form as

$$\begin{aligned}\mathcal{D}_\mu\phi_1 =& \partial_\mu\phi_1 - \frac{1}{\sqrt{2}}\frac{i}{2}\begin{pmatrix}gW^3_\mu + g'B_\mu & gW^1_\mu - igW^2_\mu\\ gW^1_\mu + igW^2_\mu & -gW^3_\mu + g'B_\mu\end{pmatrix}E_1(x)\begin{pmatrix}0\\ v_1+h_1\end{pmatrix} =\\ &\approx \frac{1}{\sqrt{2}}\left\{\begin{pmatrix}0\\ \partial_\mu h_1\end{pmatrix} + \frac{i}{2}(v_1+h_1)\begin{pmatrix}\partial_\mu\zeta_1 - i\partial_\mu\zeta_2\\ -2\partial_\mu\zeta_3\end{pmatrix} - \frac{i}{2}(v_1+h_1)\begin{pmatrix}gW^1_\mu - igW^2_\mu\\ -gW^3_\mu + g'B_\mu\end{pmatrix}\right\}\end{aligned} \tag{2.20}$$

where we used $Y = 1$ and again the approximation $E_1(x) \approx \mathbb{1}$. If we now consider the term $\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_1$, products generating mixed interactions of gauge and goldstone fields with the Higgs field automatically simplify out. Consequently, we could claim

$$\begin{aligned}\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_1 =&\partial_\mu\phi_1^\dagger\partial^\mu\phi_1 + \frac{(v_1+h_1)^2}{8}[W_1^\mu W_\mu^1 + W_2^\mu W_\mu^2] + \frac{(v_1+h_1)^2}{8}[gW_3^\mu - g'B^\mu][gW_\mu^3 - g'B_\mu] + \\ &+ \frac{1}{8}(v_1+h_1)^2[4\partial_\mu\zeta_3(-gW_3^\mu + g'B^\mu) - 2\partial_\mu\zeta_1 W_1^\mu - 2\partial_\mu\zeta_2 W_2^\mu] + \dots\end{aligned} \tag{2.21}$$

The remaining term can be easily computed using (2.19):

$$\partial_\mu\phi_1^\dagger\partial^\mu\phi_1 = \frac{1}{2}\partial_\mu h_1\partial^\mu h_1 + \frac{(v_1+h_1)^2}{2}\left[\frac{(\partial_\mu\zeta_1)^2 + (\partial_\mu\zeta_2)^2}{4} + (\partial_\mu\zeta_3)^2\right] \tag{2.22}$$

so that, substituing in (2.21), adding also the contribution of the second Higgs doublet, we get

$$\begin{aligned}&\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_1 + \mathcal{D}_\mu\phi_2^\dagger\mathcal{D}^\mu\phi_2 = \frac{\partial_\mu h_1^\dagger\partial^\mu h_1}{2} + \frac{\partial_\mu h_2^\dagger\partial^\mu h_2}{2} + \frac{(v_1+h_1)^2}{2}\left[\frac{(\partial_\mu\zeta_1)^2 + (\partial_\mu\zeta_2)^2}{4} + (\partial_\mu\zeta_3)^2\right] + \\ &+ \frac{(v_2+h_2)^2}{2}\left[\frac{(\partial_\mu\eta_1)^2 + (\partial_\mu\eta_2)^2}{4} + (\partial_\mu\eta_3)^2\right] + \frac{g^2}{8}[(v_1+h_1)^2 + (v_2+h_1)^2][W_1^\mu W_\mu^1 + W_2^\mu W_\mu^2] + \\ &+ \frac{1}{8}[(v_1+h_1)^2 + (v_2+h_2)^2][gW_3^\mu - g'B^\mu][gW_\mu^3 - g'B_\mu] + \frac{1}{8}(v_1+h_1)^2[4\partial_\mu\zeta_3(-gW_3^\mu + g'B^\mu) + \\ &- 2\partial_\mu\zeta_1 W_1^\mu - 2\partial_\mu\zeta_2 W_2^\mu] + \frac{1}{8}(v_2+h_2)^2[4\partial_\mu\eta_3(-gW_3^\mu + g'B^\mu) - 2\partial_\mu\eta_1 W_1^\mu - 2\partial_\mu\eta_2 W_2^\mu] + \dots\end{aligned} \tag{2.23}$$

In order to obtain our result, we do not need to take into account the radial fluctuations around the VEVs ($h_1$ and $h_2$): hence, we will ignore them in the following considerations. But written in this form, equation (2.23) does not unveil the combinations of phases that can be absorbed through a gauge trasformation of $W_i^\mu$. It will prove to be convinient to add and subtract the expression $c_\beta s_\beta(\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2 + \mathcal{D}_\mu\phi_2^\dagger\mathcal{D}^\mu\phi_1)$ (where we use the shorthand notation $c_\beta$ and $s_\beta$, which stand respectively for $\cos\beta$ and $\sin\beta$):

$$\begin{aligned}&\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2 + \text{h.c.} = \frac{\partial_\mu h_1\partial^\mu h_2}{2} + \frac{1}{8}(v_1+h_1)(v_2+h_2)\Big[\partial_\mu\zeta_1\partial^\mu\eta_1 + \partial_\mu\zeta_2\partial^\mu\eta_2 + 4\partial_\mu\zeta_3\partial^\mu\eta_3 + \\ &- i\partial_\mu\zeta_1\partial^\mu\eta_2 + i\partial_\mu\zeta_2\partial^\mu\eta_1\Big] + \frac{1}{8}(v_1+h_1)(v_2+h_2)\Big[g^2(W_1^\mu)^2 + g^2(W_2^\mu)^2 + (gW_3^\mu - g'B^\mu)^2\Big] + \\ &- \frac{1}{8}(v_1+h_1)(v_2+h_2)\Big[g(W_\mu^1 - iW_\mu^2)(\partial^\mu\zeta_1 + i\partial^\mu\zeta_2) + (-gW_\mu^3 + g'B_\mu)(-2\partial^\mu\zeta_3)\Big] + \\ &- \frac{1}{8}(v_1+h_1)(v_2+h_2)\Big[g(W_\mu^1 + iW_\mu^2)(\partial^\mu\eta_1 - i\partial^\mu\eta_2) + (-gW_\mu^3 + g'B_\mu)(-2\partial^\mu\eta_3)\Big] + \text{h.c.} + \dots\end{aligned} \tag{2.24}$$

and neglecting terms involving the radial Higgs fluctuations once and for all, as already suggested, we are left with

$$\begin{aligned}\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2 + \text{h.c.} =&\frac{1}{4}v_1v_2\Big[\partial_\mu\zeta_1\partial^\mu\eta_1 + \partial_\mu\zeta_2\partial^\mu\eta_2 + 4\partial_\mu\zeta_3\partial^\mu\eta_3\Big] + \frac{1}{4}v_1v_2\Big[g^2(W_1^\mu)^2 + g^2(W_2^\mu)^2 + \\ &+ (gW_3^\mu - g'B^\mu)^2\Big] - \frac{1}{4}v_1v_2\Big[gW_\mu^1(\partial^\mu\zeta_1 + \partial^\mu\eta_1) + gW_\mu^2(\partial^\mu\zeta_2 + \partial^\mu\eta_2) + \\ &+ (-gW_\mu^3 + g'B_\mu)(-2\partial^\mu\zeta_3 - 2\partial^\mu\eta_3)\Big] + \dots\end{aligned} \tag{2.25}$$

Now, going on with this little trick, we can rewrite the kinetic term for the Higgs doublets in a clever manner

$$\underbrace{(c_\beta^2+s_\beta^2)}_{=1}\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_1+\underbrace{(c_\beta^2+s_\beta^2)}_{=1}\mathcal{D}_\mu\phi_2^\dagger\mathcal{D}^\mu\phi_2+\underbrace{c_\beta s_\beta(\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2+h.c)-c_\beta s_\beta(\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2+h.c)}_{=0}=$$
$$=\Big[c_\beta^2\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_1+s_\beta^2\mathcal{D}_\mu\phi_2^\dagger\mathcal{D}^\mu\phi_2+c_\beta s_\beta(\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2+h.c.)\Big]+\Big[s_\beta^2\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_1+c_\beta^2\mathcal{D}_\mu\phi_2^\dagger\mathcal{D}^\mu\phi_2+$$
$$-c_\beta s_\beta(\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2+h.c.)\Big] \tag{2.26}$$

If we consider the first term in square brackets, plugging in all the aforementioned expressions, we can obtain the following result:

$$c_\beta^2\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_1+s_\beta^2\mathcal{D}_\mu\phi_2^\dagger\mathcal{D}^\mu\phi_2+c_\beta s_\beta(\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2+h.c.)=\frac{c_\beta^2v_1^2}{2}\left[\frac{(\partial_\mu\zeta_1)^2+(\partial_\mu\zeta_2)^2}{4}+(\partial_\mu\zeta_3)^2\right]+$$
$$+\frac{s_\beta^2v_2^2}{2}\left[\frac{(\partial_\mu\eta_1)^2+(\partial_\mu\eta_2)^2}{4}+(\partial_\mu\eta_3)^2\right]+\frac{g^2}{8}[c_\beta^2v_1^2+s_\beta^2v_2^2][(W_1^\mu)^2+(W_2^\mu)^2]+$$
$$+\frac{[c_\beta^2v_1^2+s_\beta^2v_2^2]}{8}(gW_3^\mu-g'B^\mu)^2+\frac{c_\beta^2v_1^2}{8}[4\partial_\mu\zeta_3(-gW_3^\mu+g'B^\mu)-2g\partial_\mu\zeta_1W_1^\mu-2g\partial_\mu\zeta_2W_2^\mu]+$$
$$+\frac{s_\beta^2v_2^2}{8}[4\partial_\mu\eta_3(-gW_3^\mu+g'B^\mu)-2g\partial_\mu\eta_1W_1^\mu-2g\partial_\mu\eta_2W_2^\mu]+$$
$$+\frac{c_\beta s_\beta v_1v_2}{4}\Big[\partial_\mu\zeta_1\partial^\mu\eta_1+\partial_\mu\zeta_2\partial^\mu\eta_2+4\partial_\mu\zeta_3\partial^\mu\eta_3+g^2(W_1^\mu)^2+g^2(W_2^\mu)^2+(gW_3^\mu-g'B^\mu)^2\Big]+$$
$$-\frac{c_\beta s_\beta v_1v_2}{4}\Big[gW_\mu^1(\partial^\mu\zeta_1+\partial^\mu\eta_1)+gW_\mu^2(\partial^\mu\zeta_2+\partial^\mu\eta_2)+(-gW_\mu^3+g'B_\mu)(-2\partial^\mu\zeta_3-2\partial^\mu\eta_3)\Big]+\dots \tag{2.27}$$

Going on with some cumbersome algebraic passages, we can manipulate another bit the previous lines, in order to obtain something neater and more readable:

$$c_\beta^2\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_1+s_\beta^2\mathcal{D}_\mu\phi_2^\dagger\mathcal{D}^\mu\phi_2+c_\beta s_\beta(\mathcal{D}_\mu\phi_1^\dagger\mathcal{D}^\mu\phi_2+h.c.)=$$
$$=\frac{(c_\beta v_1)^2}{8}\Big[g^2(W_1^\mu)^2+g^2(W_2^\mu)^2+(\partial_\mu\zeta_1)^2+(\partial_\mu\zeta_2)^2-2g\partial_\mu\zeta_1W_1^\mu-2g\partial_\mu\zeta_2W_2^\mu\Big]+$$
$$+\frac{(s_\beta v_2)^2}{8}\Big[g^2(W_1^\mu)^2+g^2(W_2^\mu)^2+(\partial_\mu\eta_1)^2+(\partial_\mu\eta_2)^2-2g\partial_\mu\eta_1W_1^\mu-2g\partial_\mu\eta_2W_2^\mu\Big]+$$
$$+\frac{c_\beta s_\beta v_1v_2}{4}\Big[g^2(W_1^\mu)^2+g^2(W_2^\mu)^2+\partial_\mu\zeta_1\partial^\mu\eta_1+\partial_\mu\zeta_2\partial^\mu\eta_2-gW_\mu^1(\partial^\mu\zeta_1+\partial^\mu\eta_1)+$$
$$-gW_\mu^2(\partial^\mu\zeta_2+\partial^\mu\eta_2)\Big]+\frac{(c_\beta v_1)^2}{8}\Big[(gW_3^\mu-g'B^\mu)^2+4(\partial_\mu\zeta_3)^2-4(\partial_\mu\zeta_3)(gW_3^\mu-g'B^\mu)\Big]+$$
$$+\frac{(s_\beta v_2)^2}{8}\Big[(gW_3^\mu-g'B^\mu)^2+4(\partial_\mu\eta_3)^2-4(\partial_\mu\eta_3)(gW_3^\mu-g'B^\mu)\Big]+$$
$$+\frac{c_\beta s_\beta v_1v_2}{4}\Big[(gW_3^\mu-g'B^\mu)^2+4\partial_\mu\zeta_3\partial^\mu\eta_3-2(\partial_\mu\zeta_3+\partial_\mu\eta_3)(gW_3^\mu-g'B^\mu)\Big] \tag{2.28}$$

Thus, when the smoke clears, we can eventually write down a compact and useful expression:

$$\begin{aligned}&\frac{1}{8}\sum_{i=1}^{2}\Big[g\underbrace{(c_\beta v_1+s_\beta v_2)}_{=1}W_i^\mu-c_\beta v_1\partial^\mu\zeta_i-s_\beta v_2\partial^\mu\eta_i\Big]^2+\\&+\frac{1}{8}\Big[\underbrace{(c_\beta v_1+s_\beta v_2)}_{=1}(gW_3^\mu-g'B^\mu)-2(c_\beta v_1\partial^\mu\zeta_3+s_\beta v_2\partial^\mu\eta_3)\Big]^2\end{aligned}\tag{2.29}$$

Noticing that $c_\beta = v_1/v$ and $s_\beta = v_2/v$, we can easily see that the charged $W^\pm$ bosons get the usual mass term $m_W = gv/2$, where $v$ is the electroweak scale; the same is true for the combination that defines the $Z$ boson in the SM, which has $m_Z = \sqrt{g^2+g'^2}v/2$. But now we can clearly observe that the gauge symmetry is shared by the two doublets. As a matter of fact, the gauge transformation to apply to our bosons in order to jump into a unitary gauge must involve both of the two Higgs phases in a non-trivial arrangement. Indeed, considering that $\zeta_i = \tilde{\zeta}_i/v_1$ and $\eta_i = \tilde{\eta}_i/v_2$, it is evident for example that the combination $\theta_Z$ eaten by the zeta boson at infinitesimal order is:

$$Z_\mu \longmapsto Z_\mu - 2\partial_\mu\left(\frac{c_\beta\tilde{\zeta}_3+s_\beta\tilde{\eta}_3}{v\sqrt{g^2+g'^2}}\right)\tag{2.30}$$

and, so, $\theta_Z = 2(c_\beta\tilde{\zeta}_3 + s_\beta\tilde{\eta}_3)$. In a similar fashion, for the charged gauge fields $W_\mu^\pm = (W_\mu^2 \pm iW_\mu^1)/\sqrt{2}$ one gets $\theta^\pm = [c_\beta\tilde{\zeta}_2 + s_\beta\tilde{\eta}_2 \pm i(c_\beta\tilde{\zeta}_1 + s_\beta\tilde{\eta}_1)]/\sqrt{2}$.

If we now consider the second part of equation (2.26), working out the various terms in a completely similar way, one finally attains

$$\begin{aligned}&\frac{1}{8}\sum_{i=1}^{2}\Big[g\underbrace{(-s_\beta v_1+c_\beta v_2)}_{=0}W_i^\mu-(-s_\beta v_1\partial^\mu\zeta_i+c_\beta v_2\partial^\mu\eta_i)\Big]^2+\\&+\frac{1}{8}\Big[\underbrace{(-s_\beta v_1+c_\beta v_2)}_{=0}(gW_3^\mu-g'B^\mu)-2(-s_\beta v_1\partial^\mu\zeta_3+c_\beta v_2\partial^\mu\eta_3)\Big]^2\end{aligned}\tag{2.31}$$

where we see that in each couple of square brackets addends proportinal to the gauge fields disappear, because of $-s_\beta v_1 + c_\beta v_2 = 0$. As a matter of fact, these last phase combinations are just those orthogonal to the previous ones. Therefore, there are some phase fields which survive a gauge redefinition of vector bosons. For the neutral sector, the field orthogonal to the Z-goldstone boson phase (2.30) is:

$$\tilde{A}_0 = s_\beta\tilde{\zeta}_3 - c_\beta\tilde{\eta}_3\tag{2.32}$$

Nevertheless, the fact that adding and subtracting a quite convoluted term lets us achieve the correct result means something more profound. If we look again at (2.26), it is immediate to see that this expression can be written down as

$$\mathcal{D}_\mu(c_\beta\phi_1+s_\beta\phi_2)^\dagger\mathcal{D}_\mu(c_\beta\phi_1+s_\beta\phi_2)+\mathcal{D}_\mu(-s_\beta\phi_1+c_\beta\phi_2)^\dagger\mathcal{D}_\mu(-s_\beta\phi_1+c_\beta\phi_2)\tag{2.33}$$

which tells us that the real degrees of freedom can be obtained by rotating the original Higgs doublets: only this way one can correctly spot the direction giving mass to gauge bosons. Indeed, if we start with our doublets written in the form (2.1), then we will have

$$\phi_1' = c_\beta\phi_1 + s_\beta\phi_2 = \begin{pmatrix} c_\beta\alpha^+ + s_\beta\beta^+ \\ c_\beta Re[\alpha_0] + s_\beta Re[\beta_0] + i(c_\beta Im[\alpha_0] + s_\beta Im[\beta_0]) \end{pmatrix}\tag{2.34}$$

where the upper component is given by the sum of the two uppers components of the original fields, having charges $Q = +1$: that suggests how this phase is related to the $W^{\pm}$-goldstone bosons; the neutral imaginary part of the lower component, instead, exactly resembles (2.30). On the other hand, the real part is the only one with a non-null VEV. This one can be defined as the sum of the right VEV and a fluctuation $H$, because, consistently

$$\langle\frac{v+H}{\sqrt{2}}\rangle = \frac{v}{\sqrt{2}} + \frac{1}{\sqrt{2}}\underbrace{\langle H\rangle}_{=0} \quad \text{and} \quad \langle c_\beta Re[\alpha_0] + s_\beta Re[\beta_0]\rangle = \frac{c_\beta}{\sqrt{2}}v_1 + \frac{s_\beta}{\sqrt{2}}v_2 = \frac{v}{\sqrt{2}} \tag{2.35}$$

So, upon carrying out the rotation, this field combination $\phi_1'$ has exactly the standard form of the SM Higgs. At the same time, the second orthogonal part comes as:

$$\phi_2' = -s_\beta\phi_1 + c_\beta\phi_2 = \begin{pmatrix} -s_\beta\alpha^+ + c_\beta\beta^+ \\ -s_\beta Re[\alpha_0] + c_\beta Re[\beta_0] + i(-s_\beta Im[\alpha_0] + c_\beta Im[\beta_0]) \end{pmatrix} \tag{2.36}$$

Here, the charged and neutral phases just look like the fields which are not gobbled up by the gauge fields (that we have already discussed above): the neutral one will be analysed more deeply in the next paragraph, while the charged ones are the so-called *charged Higgs* $H^{\pm}$. The real neutral component is a scalar field $S/\sqrt{2}$ with a null VEV: $\langle S/\sqrt{2}\rangle = \langle -s_\beta Re[\alpha_0] + c_\beta Re[\beta_0]\rangle = 0$.

### 2.2.2 The Z-boson and the PQ symmetry

As already stated, the only PQ charged objects in our model are the Higgs fields and fermions: no direct PQ coupling is assumed to exist for gauge bosons. To ensure this fact, we have to guarantee that the phases eaten by these bosons are PQ neutral.

Restricting as usual our discussion to the scalar sector, we know that, when it comes to speaking of PQ symmetry, also the singlet field plays the game. There will be a degree of freedom associated to the PQ transformation (that in the end will identify the axion) that we will name $\omega$, shared by all the Higgs fields and, hence, given by a combination of their phases. As already said, under a PQ transformation, we have

$$\phi_1 \longmapsto e^{iX_1\omega(x)/v}\phi_1 \qquad \phi_2 \longmapsto e^{iX_2\omega(x)/v}\phi_2 \qquad \phi \longmapsto e^{iX_\phi\omega(x)/v}\phi \tag{2.37}$$

where we have made the substitution $\theta \longmapsto \omega/v$. It is easy to realize that, starting from (2.18), the first two fields can be written as

$$\phi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} (\tilde{\zeta}_2 + i\tilde{\zeta}_1)/2 \\ v_1 + h_1 + i(-\tilde{\zeta}_3 + \sqrt{2}\omega X_1 v_1/v) \end{pmatrix} \quad \phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} (\tilde{\eta}_2 + i\tilde{\eta}_1)/2 \\ v_2 + h_2 + i(-\tilde{\eta}_3 + \sqrt{2}\omega X_2 v_2/v) \end{pmatrix} \tag{2.38}$$

For the Higgs singlet, we can instead simply claim

$$\phi = v_\phi + \rho + i(\tilde{G}_\phi + X_\phi v_\phi \omega/v) \tag{2.39}$$

So, we have the following transformation laws under a PQ symmetry for the neutral phases (we see that the charged ones are untouched, as expected):

$$\begin{cases} \tilde{\zeta}_3' = \tilde{\zeta}_3 - \sqrt{2}\omega X_1(v_1/v) \\ \tilde{\eta}_3' = \tilde{\eta}_3 - \sqrt{2}\omega X_2(v_2/v) \\ \tilde{G}_\phi' = \tilde{G}_\phi + X_\phi\omega(v_\phi/v) \end{cases} \tag{2.40}$$

We can use now the relation achieved in (2.30), which identifies the eaten Z-gauge boson phase $\theta_Z$, in order to realize that it changes as

$$\theta'_Z = \theta_Z - 2\sqrt{2}\omega(c^2_\beta X_1 + s^2_\beta X_2) \tag{2.41}$$

Therefore, all we have to do is just setting $c^2_\beta X_1 + s^2_\beta X_2 = 0$ to endowed $\theta_Z$ with the property of being PQ neutral. This latter is our second relation among PQ charges. If we consider that $-X_1 + X_2 = -X_\phi = 1$, we readily get $X_1 = -s^2_\beta$ and $X_2 = c^2_\beta$. Actually, this constraint on $\theta_Z$ can also be considered necessary so as to diagonalize the kinetic term, avoiding expressions as $\partial_\mu\theta_Z(x)\partial^\mu\omega(x)$ in the Lagrangian: if the kinetic part is not diagonal, we can not extract a propagator, because we do not know which ones are the truly physical degrees of freedom that propagate.

From this point of view, the problem is still open for the fields $\tilde{A}_0$ and $\tilde{G}_\phi$ which transform as

$$\begin{cases} \tilde{A}'_0 = \tilde{A}_0 + \sqrt{2}\omega[X_2 - X_1]c_\beta s_\beta = \tilde{A}_0 + \sqrt{2}\omega c_\beta s_\beta \\ \tilde{G}'_\phi = \tilde{G}_\phi + \omega X_\phi(v_\phi/v) \end{cases} \tag{2.42}$$

It is clear that this $\omega$-degree will be a suitable combination of these two. If we now consider the original Higgs doublets written in terms of the rotated ones (which highlight the physical fields of the problem) and if we keep in mind the relations $\alpha^+ = \frac{\tilde{\zeta}_2 + i\tilde{\zeta}_1}{2\sqrt{2}}$, $\beta^+ = \frac{\tilde{\eta}_2 + i\tilde{\eta}_1}{2\sqrt{2}}$, $Im[\alpha_0] = -\tilde{\zeta}_3/\sqrt{2}$ and $Im[\beta_0] = -\tilde{\eta}_3/\sqrt{2}$ to move from one parametrization to the other (compare (2.1) and (2.18)), we will reach these expressions, until now:

$$\begin{aligned} \phi_1 &= c_\beta\phi'_1 - s_\beta\phi'_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} c_\beta\frac{\theta^+}{\sqrt{2}} - \sqrt{2}s_\beta H^+ \\ (v + H - i\theta_Z/2)c_\beta - s_\beta(S + i\tilde{A}_0) \end{pmatrix} \\ \phi_2 &= s_\beta\phi'_1 + c_\beta\phi'_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} s_\beta\frac{\theta^+}{\sqrt{2}} + \sqrt{2}c_\beta H^+ \\ (v + H - i\theta_Z/2)s_\beta + c_\beta(S + i\tilde{A}_0) \end{pmatrix} \\ \phi &= v_\phi + \rho + i\tilde{G}_\phi \end{aligned} \tag{2.43}$$

As already mentioned, $\tilde{A}_0$ and $\tilde{G}_\phi$ are not physical fields yet. We have also set $H_{1/2} = (-s_\beta\tilde{\zeta}_{1/2} + c_\beta\tilde{\eta}_{1/2})/2$, $H^\pm = (H_2 \pm iH_1)/\sqrt{2}$ and, in a similar fashion, $\theta^\pm = (\theta_2 \pm i\theta_1)/\sqrt{2}$.

### 2.2.3 The axion emergence

We have already claimed how the axion component should be associated with the freedom of carrying out PQ transformations. However, we should still individuate this massless field. There is more than one field possessing a PQ charge and, thus, in a reasoning similar to that done for the gauge symmetry, we need a right combinantion of phases. The remark that the axion field can not be completely identified with the phase of $\phi$ is self-evident just by looking at our cubic term $V_{c-term}$: here the phase of $\phi$ does not disappear from the potential and, thereby, it can not be associated to a massless field. Indeed, any field which has place in the potential contribution of the Lagrangian will eventually get a mass through the spontaneous symmetry breaking mechanism.

A powerful instrument to cope with this problem is the *currents algebra* of the theory. As a matter of fact, if we have a symmetry associated to an internal transformation, we can define a conserved current, whose expression will get a contribution (according to Noether theorem) solely from the kinetic part, being the potential independent of the derivatives of fields. Employing this definition, we can obtain the PQ conserved current associated to the scalar part of the Lagrangian

$$\begin{aligned} J^{PQ}_\mu &= \sum_{k=1}^{2}\left(\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi_k}\delta\phi_k + \delta\phi_k^\dagger\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi_k^\dagger}\right) + \left(\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi}\delta\phi + \delta\phi_k^*\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi^*}\right) = \\ &= \frac{1}{2}X_\phi\phi^* i\overleftrightarrow{\partial}_\mu\phi + X_1\phi_1^\dagger i\overleftrightarrow{\partial}_\mu\phi_1 + X_2\phi_2^\dagger i\overleftrightarrow{\partial}_\mu\phi_2 + \dots \end{aligned} \tag{2.44}$$

where dots stand for the additional part to which fermions could contribute through their kinetic terms, if PQ charged. As we will see, this part is extremely model-dependent. However, using our freedom of choosing the definition of a field, it will not be important in order to fix the axion component. Anyway, this non-zero projection of the PQ current on fermions will have a crucial role while considering axion mass and its couplings to matter.

If we now use the usual parametrization (2.18) for our fields, together with (2.43) for $\phi$, we notice that the piece proportional to $h_1$, $h_2$ and $\rho$ simplifies out, so that we are left with:

$$J^{PQ}_\mu \approx -X_\phi v_\phi\partial_\mu\tilde{G}_\phi + v(X_1c_\beta\partial_\mu\zeta_3 + X_2s_\beta\partial_\mu\eta_3) \tag{2.45}$$

In the space of fields, this current is a field itself, individuating the direction associated to the PQ transformation. If we want for a field not to couple to PQ symmetry, we just need to impose an orthogonality condition or, in operatorial language, that the commutator between the field and the current disappears. As a matter of fact, our previous condition $c_\beta^2X_1 + s_\beta^2X_2 = 0$ could have also been found by requiring the two vectors $\hat{J}^{PQ}_\mu|0\rangle$ and $\hat{\theta}_Z|0\rangle$ to satisfy

$$\langle 0|\hat{J}^{PQ}_\mu\hat{\theta}_Z|0\rangle = 0 \quad \text{or} \quad \langle 0|[\hat{J}^{PQ}_\mu,\hat{\theta}_Z]|0\rangle = 0 \tag{2.46}$$

Indeed, one has:

$$\begin{aligned} &\langle\hat{J}^{PQ}_\mu(x)|\hat{\theta}_Z(y)\rangle = \\ &= \Big(-X_\phi v_\phi\langle\partial_\mu\tilde{G}_\phi(x)| + vX_1c_\beta\langle\partial_\mu\zeta_3(x)| + vX_2s_\beta\langle\partial_\mu\eta_3(x)|\Big)\Big(c_\beta|\zeta_3(y)\rangle + s_\beta|\eta_3(y)\rangle\Big) = \\ &= vX_1c_\beta^2\langle\partial_\mu\zeta_3(x)|\zeta_3(y)\rangle + vX_2s_\beta^2\langle\partial_\mu\eta_3(x)|\eta_3(y)\rangle = -iv\hbar c(X_1c_\beta^2 + X_2s_\beta^2)\partial_\mu D(x-y) \end{aligned} \tag{2.47}$$

which readily replicates our initial condition, provided that the orthogonality relations $\langle\partial_\mu\tilde{G}_\phi|\zeta_3\rangle = \langle\partial_\mu\tilde{G}_\phi|\eta_3\rangle = \langle\partial_\mu\zeta_3|\eta_3\rangle = \langle\partial_\mu\eta_3|\zeta_3\rangle = 0$ hold (being $\zeta_3$ and $\eta_3$ independent fields by very definition). Moreover, the commutation formulae $[\eta_3(x),\eta_3(y)] = [\zeta_3(x),\zeta_3(y)] = -i\hbar cD(x-y)$ (where $D(x-y)$ is the Pauli-Jordan distribution) will be satisfied, too, in the limit of a free field theory.

Once observed that, it should not be so difficult to be convinced that the axion direction can be directly extracted from the PQ current itself. Indeed, we can easily see that

$$\begin{aligned} J^{PQ}_\mu &\approx -\frac{1}{2}\partial_\mu[2v_\phi\tilde{G}_\phi - 2vc_\beta s_\beta(-s_\beta\zeta_3 + c_\beta\eta_3)] = \\ &= -\frac{1}{2}\partial_\mu[2v_\phi\tilde{G}_\phi + 2vc_\beta s_\beta\tilde{A}_0] \end{aligned} \tag{2.48}$$

where we used the definition of $\tilde{A}_0$ (2.32) and, so, we can finally claim

$$J^{PQ}_\mu \approx -\frac{\Delta}{2}\partial_\mu a'_\phi \quad \Rightarrow \quad a'_\phi = \frac{2v_\phi\tilde{G}_\phi + 2vc_\beta s_\beta\tilde{A}_0}{\Delta} \tag{2.49}$$

in which $\Delta$ is a dimensional constant we need to adjust dimensions. If we want to recover the standard normalization $[a_\phi(x), a_\phi(y)] = -i\hbar c D(x-y)$ for the axion field in a free-fields limit, it is simple to show that the correct definition endowing our model with this consistency property is:

$$a_\phi = \frac{2v_\phi \tilde{G}_\phi + v\sin(2\beta)\tilde{A}_0}{\sqrt{4v_\phi^2 + v^2\sin(2\beta)}} \tag{2.50}$$

It is noteworthy that, because of different choices of the Peccei-Quinn charges, the definition of the axion for a cubic $V_{c-term}$ theory slightly departs from the usual one, where a quartic term is adopted: this very tiny change will have, anyway, visible effects on the mass spectrum formulae. Moreover, we want to point out a typical aspect of these DFSZ models: even if the phase of $\phi$ does not contribute by itself to the axion, as already enhanced, the axion field receives a substatial part of its definition from it, because of the relation $v_\phi \gg v$. Until now, we did not say anything about a numerical value of $v_\phi$, but we just required it to be a great number in order to make the axion invisible: we will come back on this point later on.

Actually, our axion definition (2.50) individuates the rotation matrix between the last two non-physical degrees of freedom that were left aside, i.e $\tilde{A}_0$ and $\tilde{G}_\phi$. Hence, we can define the additional field orthogonal to the axion simply as

$$A_0 = \frac{2v_\phi \tilde{G}_\phi - v\sin(2\beta)\tilde{A}_0}{\sqrt{4v_\phi^2 + v^2\sin(2\beta)}} \tag{2.51}$$

and we can eventually pin down the complete field parametrization in a compact form, employing $\Phi_{12}$ and $\phi$. The kinetic term can be written down as

$$\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi + tr[\mathcal{D}_\mu \Phi_{12}^\dagger \mathcal{D}^\mu \Phi_{12}] \tag{2.52}$$

Here, we have $\mathcal{D}^\mu \Phi_{12} = \partial^\mu \Phi_{12} - ig\vec{W}^\mu \Phi_{12}\vec{\sigma}/2 + ig' B^\mu \Phi_{12}\sigma_3/2$, in which the presence of $\sigma_3$ in the last term is justified by considering the definition $\Phi_{12} = \begin{pmatrix}\tilde{\phi}_1 & \phi_2\end{pmatrix}$ (the first component has opposite hypercharge). Therefore, we will end up with

$$\begin{aligned}
\Phi_{12} &= \frac{1}{\sqrt{2}} U \begin{pmatrix} (v+H)c_\beta - s_\beta\left(S - i\frac{2v_\phi A_0}{\sqrt{4v_\phi^2 + v^2 s_{2\beta}}}\right) & \sqrt{2}c_\beta H^+ \\ \sqrt{2}s_\beta H^- & (v+H)s_\beta + c_\beta\left(S + i\frac{2v_\phi A_0}{\sqrt{4v_\phi^2 + v^2 s_{2\beta}}}\right)\end{pmatrix} U_a \\
\phi &= \left(v_\phi + \rho - i\frac{v s_{2\beta} A_0}{\sqrt{4v_\phi^2 + v^2 s_{2\beta}}}\right) exp\left\{\frac{2ia_\phi}{\sqrt{4v_\phi^2 + v^2 s_{2\beta}}}\right\}
\end{aligned} \tag{2.53}$$

with $U$ the matrix containing the goldstone bosons and $U_a$ the axion field:

$$U = exp\left\{\frac{i\theta_k \sigma_k}{2v}\right\} \qquad U_a = exp\left\{\frac{2a_\phi X}{\sqrt{4v_\phi^2 + v^2 s_{2\beta}}}\right\} \quad \text{with} \quad X = \begin{pmatrix}\sin^2\beta & 0 \\ 0 & \cos^2\beta\end{pmatrix} \tag{2.54}$$

where the factor two multiplying $a_\phi$ is used to recover the $s_{2\beta}$ of the inverse transformation of (2.50) and (2.51). By substituing this explicit expression inside the kinetic Lagrangian, one can directly verify that all of the fields show up with a diagonal and standardly normalized kinetic term: that means we are on the right track to identify the physical degrees of freedom of the theory.

Just to give a flavour of how things work and that everything has been correctly done until now, we will sketch in what follows a schematic calculation of the diagolalization of the mass term for $\tilde{A}_0$ and $\tilde{G}_\phi$.

*Proof.* By using our initial parametrization (2.43), we can understand that the sole contributions containing quadratic terms in $\tilde{A}_0$ and $\tilde{G}_\phi$ could come out from $V_{c-term}$: indeed, these fields are phase degrees of freedom that disappear while multiplying a Higgs by its complex conjugate. So, we can concentrate solely on this part of the potential. We can start by working out $\phi_1^\dagger\phi_2$:

$$\phi_1^\dagger\phi_2 \approx \frac{1}{2}(v^2 c_\beta s_\beta + \ldots)exp\left\{i\left(\frac{s_\beta}{c_\beta}+\frac{c_\beta}{s_\beta}\right)\frac{\tilde{A}_0}{v}\right\} \tag{2.55}$$

and then

$$\phi \approx (v_\phi + \ldots)exp\left\{i\frac{\tilde{G}_\phi}{v_\phi}\right\} \tag{2.56}$$

Consequently, we can write the expansion of the $V_{c-term}$ as:

$$\begin{aligned}\mathcal{L}_{\text{c-term}} &= c\phi_1^\dagger\phi_2\phi + h.c. = \frac{c}{2}[v^2 c_\beta s_\beta + \ldots][v_\phi + \ldots]exp\left\{i\left[\left(\frac{s_\beta}{c_\beta}+\frac{c_\beta}{s_\beta}\right)\frac{\tilde{A}_0}{v}+\frac{\tilde{G}_\phi}{v_\phi}\right]\right\} + h.c. = \\ &= \frac{c}{2}[v^2 c_\beta s_\beta + \ldots][v_\phi + \ldots]\left[1 + i\left(\frac{s_\beta}{c_\beta}+\frac{c_\beta}{s_\beta}\right)\frac{\tilde{A}_0}{v} + i\frac{\tilde{G}_\phi}{v_\phi} - \frac{1}{2}\left(\frac{s_\beta}{c_\beta}+\frac{c_\beta}{s_\beta}\right)^2\frac{\tilde{A}_0^2}{v^2} - \frac{1}{2}\frac{\tilde{G}_\phi^2}{v_\phi^2} + \right. \\ &\left. - \left(\frac{s_\beta}{c_\beta}+\frac{c_\beta}{s_\beta}\right)\frac{\tilde{A}_0\tilde{G}_\phi}{vv_\phi}\right] + h.c.\end{aligned} \tag{2.57}$$

If we retain again only the quadratic terms which enter the mass matrix, the preceding formula will yield

$$\begin{aligned}\mathcal{L}_{\text{c-term}} &= -\frac{c}{2}[v^2 c_\beta s_\beta v_\phi]\left[\left(\frac{s_\beta}{c_\beta}+\frac{c_\beta}{s_\beta}\right)^2\frac{\tilde{A}_0^2}{v^2} + \frac{\tilde{G}_\phi^2}{v_\phi^2} + 2\left(\frac{s_\beta}{c_\beta}+\frac{c_\beta}{s_\beta}\right)\frac{\tilde{A}_0\tilde{G}_\phi}{vv_\phi}\right] = \\ &= -\frac{c}{2}\left[\frac{v_\phi}{c_\beta s_\beta}\tilde{A}_0^2 + \frac{v^2 c_\beta s_\beta}{v_\phi}\tilde{G}_\phi^2 + 2v\tilde{A}_0\tilde{G}_\phi\right]\end{aligned} \tag{2.58}$$

This gives rise to a two-by-two mass matrix, whose eigenvalues and eigenstates are our next goal. The minus sign in front of this mass term enables us to understand that, in order not to deal with tachyon particles, c has to be negative: $c < 0$. The associated secular equation can be immediately read off from the previous expression as

$$det\begin{pmatrix} m^2 - |c|\frac{v_\phi}{c_\beta s_\beta} & |c|v \\ |c|v & m^2 - |c|\frac{v^2 c_\beta s_\beta}{v_\phi}\end{pmatrix} = 0 \quad\Rightarrow\quad m\left(m - \frac{|c|}{2v_\phi}\left[\frac{4v_\phi^2}{s_{2\beta}} + v^2 s_{2\beta}\right]\right) = 0 \tag{2.59}$$

which gives a massless solution, as expected, and a massive eigenvalue

$$m_{A_0}^2 = \frac{|c|}{2v_\phi}\left(\frac{4v_\phi^2}{s_{2\beta}} + v^2 s_{2\beta}\right) \tag{2.60}$$

Now the eigenvectors can be found very easily by identifying the correctly normalized vectors solving the eigenvalues equation. This leads us to the orthogonal rotation matrices

$$O = \begin{pmatrix} 2v_\phi & -vs_{2\beta} \\ vs_{2\beta} & 2v_\phi \end{pmatrix} \tag{2.61}$$

which eventually generates the axion and $A_0$ fields of expressions (2.50) and (2.51) respectively. □

## 2.3 The mass spectrum

Once the theory has been described in details for what concerns its field content, we have to move onto something which lets us compare our theoretical construction with experimental results. A first decisive step will be the calculation of the mass spectrum. Actually, this latter can tell us if our theory is consistent with the current knowledge of nature, coming from collected data, and if there is some hope to verify it with some future experiments.

Before proceeding, we will tweak the value of some parameters in the Lagrangian, in order to better compare our results with the existent literature. We will rescale some of them as follows: $\lambda_1 \to 16\lambda_1$, $\lambda_2 \to 16\lambda_2$, $\lambda_3 \to 16\lambda_3$, $\lambda_4 \to 16\lambda_4$ , $a \to 4a$, $b \to 4b$ and finally $c \to 4c$. This choise arises from the fact that, in general, a different parametrization of the Higgs fields $\phi_1$ and $\phi_2$ can be done, which departs from ours by a factor two.

We have already calculated the gauge boson masses: we saw how they remain just the same as in the standard model at tree level (something more about what happens at higher order will be shown later on). Then, with an explicit proof, we derived the mass of the pseudo-scalar $A_0$: by comparing this result with the one coming out from a quartic c-term [28]:

$$m^2_{A_0(\text{quartic})} = 8|c|\left(\frac{v_\phi^2}{s_{2\beta}} + v^2 s_{2\beta}\right) \tag{2.62}$$

we can notice two things. First of all, the mass term for $A_0$ remains proportional to the parameter $c$: as stated for example in [29], when $c = 0$ the PQ charges of the fields $\phi$ and $\phi_{1,2}$ are naturally decoupled and so there is an additional global symmetry or, to put it another way, an enlarged PQ symmetry $U(1)_{PQ(1/2)} \times U(1)_{PQ(\phi)}$ (where the notation adopted is self-evident). Below the $v_\phi$ energy scale, the vacuum breaks this symmetry and we will have two goldstone bosons: the axion and, now, $A_0$, whose mass goes to zero with $c = 0$. Just to introduce some terminology, these two neutral pseudoscalar fields comprise the $0^-$ *sector*.

A second remarkable aspect is that the original numerical factor $|c|$ in (2.62) is replaced in (2.60) by the adimensional combination $|c|/v_\phi$, because here, in order for the Lagrangian to have the right physical dimensions, $c$ has to be a massive parameter. So, it seems that, from the point of view of the mass spectrum, nothing should be substantially modified. Indeed, the role of the missing factor $\phi$ in the $V_{c-term}$, who generally increases the spectrum of a factor $v_\phi$ in the original quartic model, is performed by the constant $c$.

### 2.3.1 The axion mass

In the classical approach developed up to now, the axion emerges as a goldstone boson with zero mass: this is due to the fact that, classically, the PQ transformation realizes a genuine symmetry of the DFSZ Lagrangian. Nevertheless, we already discussed at leght that, at quantum level, there is an additional term which breaks this symmetry explicitly. In this new framework, the axion will be a pseudo-goldstone boson and, so, it will be endowed with a mass. Being the axion, anyway, a very light particle, its mass should arise from a theory of low energy, where all the heavier degrees of freedom are frozen: being quarks hidden by confinement, the only possible particles contributing to axion mass at this energy scale will be the pseudo-goldstone bosons of QCD. The major contribution will come from the lightest of them, i.e the pions. It is quite easy to get conviced that, under these circumstances, the formula giving the axion mass will be model independent: all particles whose masses could be different for our cubic model do not take part in the effective potential which provides the axion with a mass.

After this necessary introduction, we can start going through this issue. As already shown in (1.83), the effective Lagrangian related to axions can be expressed as

$$\mathcal{L}_a = \frac{1}{2}\partial_\mu a \partial^\mu a + \frac{a}{f_a}\frac{\alpha_s}{8\pi}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_a - \bar{q}_L M_q q_R - \bar{q}_R M_q^\dagger q_L + \dots \tag{2.63}$$

where we have written down the correctly normalized kinetic contribution for the axion, the anomalous axion-gluon coupling (from color anomaly) and the mass term for quarks. This latter will be generated by a suitable Higgs mechanism through Yukawa terms (that we will describe later on). We notice how the effective axion-gluon interaction has been written using the standard form, which includes the more physical $f_a$ constant instead of $v_f = \sqrt{4v_\phi^2 + v^2 s_{2\beta}}$, explicitly appearing in the axion phase definition: we will better analyse their relation in the next chapter. Just because we are dealing with a low energy Lagrangian, we can restrict our analysis to the up and down quarks, considering all the heavier particles as integrated out. Incidentally, we should stress how the color anomaly is not the only one which should appear here: we should have added additional anomalous contributions that, anyway, are not important for the current discussion and that will be presented in what follows.

We are now going to prove that the anomaly term is the real responsible for the axion mass: it generates an effective potential which turns the axion into a pseudo-goldstone boson. Acting with a well-known chiral trasformation, we can move the anomaly contribution from the gluon term to the quark masses, getting

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \longmapsto e^{i\gamma_5 \frac{a}{4f_a}\mathbb{1}_{2\times 2}} \begin{pmatrix} u \\ d \end{pmatrix} \quad \text{and so} \quad \mathcal{L}_a = \frac{1}{2}\partial_\mu a \partial^\mu a - \bar{q}_L M_a q_R - \bar{q}_R M_a^\dagger q_L \tag{2.64}$$

where the extra factor one-half in the exponential has been used to compensate for the fact that we are transforming two quarks and where

$$M_a = e^{i\frac{a}{4f_a}\mathbb{1}_{2\times 2}} M_q e^{i\frac{a}{4f_a}\mathbb{1}_{2\times 2}}, \qquad M_q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \tag{2.65}$$

Now that we have redefined through a chiral transformation the two-by-two quark matrix, we can tweak the low energy Lagrangian that we presented in (1.24) simply as

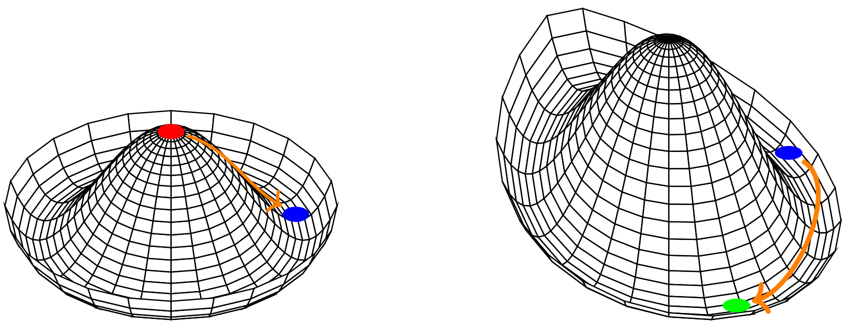
$$\mathcal{L}_{\text{mass}} = v_\psi^3 tr[\Sigma M_a^\dagger + M_a \Sigma^\dagger] \tag{2.66}$$

in which we can not overlook the possible complex nature of the quark mass matrix any more. We remind that

$$\Sigma(x) = \exp\left\{i\sigma^a \frac{\pi^a(x)}{f_\pi}\right\} = exp\left\{\frac{i}{f_\pi}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\right\} \tag{2.67}$$

and that $v_\psi$ in (2.66) is an energy scale related to chiral condensates. Therefore, following [30], all we have to do is to expand (2.66), considering that

$$\begin{aligned} \Sigma M_a^\dagger + M_a \Sigma^\dagger &= exp\left\{\frac{i}{f_\pi}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\right\} e^{-i\frac{a}{2f_a}\mathbb{1}_{2\times 2}} M_q + h.c. = \\ &= exp\left\{ i \begin{pmatrix} \frac{\pi^0}{f_\pi} - \frac{a}{2f_a} & \frac{\sqrt{2}\pi^+}{f_\pi} \\ \frac{\sqrt{2}\pi^-}{f_\pi} & -\frac{\pi^0}{f_\pi} - \frac{a}{2f_a} \end{pmatrix}\right\} M_q + h.c. \end{aligned} \tag{2.68}$$



(a) The scalar potential before the introduction of quantum corrections: the angular fluctuations move along a flat direction of the potential, so that the goldstone boson remains massless.

(b) The quantum corrections embedded in $V(a)$ bend the originally flat direction of the scalar potential, giving a mass to axion oscillations.

Figure 2.1

Consequently, we could claim

$$\begin{aligned}\mathcal{L}_{\text{mass}} &= v_\psi^3 tr\left[exp\left\{i\begin{pmatrix}\frac{\pi^0}{f_\pi}-\frac{a}{2f_a} & \frac{\sqrt{2}\pi^+}{f_\pi}\\ \frac{\sqrt{2}\pi^-}{f_\pi} & -\frac{\pi^0}{f_\pi}-\frac{a}{2f_a}\end{pmatrix}\right\}M_q + h.c.\right] = \\ &= v_\psi^3\left(2m_u\cos\left(\frac{\pi^0}{f_\pi}-\frac{a}{2f_a}\right)+2m_d\cos\left(\frac{\pi^0}{f_\pi}+\frac{a}{2f_a}\right)\right) = \\ &= 2v_\psi^3\left(\left[(m_u+m_d)\cos\left(\frac{a}{2f_a}\right)\right]\cos\left(\frac{\pi^0}{f_\pi}\right)+\left[(m_u-m_d)\sin\left(\frac{a}{2f_a}\right)\right]\sin\left(\frac{\pi^0}{f_\pi}\right)\right)\end{aligned} \tag{2.69}$$

and making use of some standard trigonometric formulae, we eventually obtain

$$\mathcal{L}_{\text{mass}} = 2v_\psi^3(m_u+m_d)\sqrt{1-\frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\left(\frac{a}{2f_a}\right)}\cos\left(\frac{\pi^0}{f_\pi}-\phi_a\right) \tag{2.70}$$

where

$$tan\phi_a = \frac{m_u-m_d}{m_u+m_d}\tan\left(\frac{a}{2f_a}\right) \tag{2.71}$$

To get the effective potential acting on the axion, we can integrate out the pion field by simply noticing that the previous expression can be minimized with respect to pions by choosing $\langle\pi^0\rangle = \phi_a/f_\pi$. Moreover, if we keep in mind the formula of pion mass (1.25), we will finally get the form of the axion effective potential, whose effects have been plotted in figure 2.1:

$$V(a) = -f_\pi^2 m_\pi^2\sqrt{1-\frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\left(\frac{a}{2f_a}\right)} \tag{2.72}$$

The minus sign obviously comes out from the fact that the potential enters the Lagrangian with a change of sign. This important result deserves some remarks. First of all, it is noteworthy to see that the effective potential is minimized by $\langle a\rangle = 0$: this is in full agreement with the already stated Vafa-Witten theorem, which prohibits the spontaneuos violation of the discrete CP symmetry. The axion field will find rest in a null expectation value, without giving rise to a potential source of CP violation in QCD as the $\theta$ parameter itself.

Moreover, we know from Goldstone theorem that the directions of the potential associated to fluctuations of goldstone modes are flat, as it happens for the famous *mexican hat potential.* However, we see that, in our current, case quantum corrections slightly bend this direction, so that these fluctuations become now associated to a massive field. We can extract the axion mass from the previous formula very easily, by simply expanding the potetial $V(a)$ up to the quadratic order in $a$

$$V(a) \approx -f_\pi^2 m_\pi^2 \left(1 - \frac{1}{2}\frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\left(\frac{a}{2f_a}\right)\right) + \ldots \approx -f_\pi^2 m_\pi^2 \left(1 - \frac{1}{2}\frac{4m_u m_d}{(m_u+m_d)^2}\left(\frac{a}{2f_a} + \right.\right.$$
$$\left.\left. - \frac{1}{6}\frac{a^3}{8f_a^3} + \ldots\right)^2\right) + \ldots \approx -f_\pi^2 m_\pi^2 \left(1 - \frac{1}{2}\frac{4m_u m_d}{(m_u+m_d)^2}\frac{a^2}{4f_a^2}\right) + \ldots \tag{2.73}$$

so that we obtain the important model-independent axion mass formula

$$m_a^2 = \frac{m_u m_d}{(m_u+m_d)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2} = \frac{z}{(1+z)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2} \qquad \text{with} \quad z = \frac{m_u}{m_d} \tag{2.74}$$

In this result, we can appreciate that, up to this order of approximation, the axion mass is only related to that of the neutral pion by means of some QCD parameters. An explicit outcome involving loop contributions, computed in a chiral perturbation framework, can be found in [30]. Remarkably, even though we will discuss in the next chapter how the axion mass varies with energy scale, formula (2.74) tells us how its change is weakly dominated by the running of quark mass parameters. As a matter of fact, $z$ is essentially scale indipendent, unless tiny corrections in $\alpha_{em}$ and in Yukawa couplings. Substituing the values for the quarks mass ratio $z = m_u/m_d = 0.48(3)$ coming from lattice QCD simulations (evalutated at $2GeV$), for the pion mass $m_{\pi^0} = 135MeV$ and for the pion decay constant $f_\pi = 92MeV$, one gets

$$m_a = 5.70\mu eV \frac{10^{12}GeV}{f_a} \tag{2.75}$$

where, of course, the dependence on $f_a$ still remains. From here it clearly appears the essence of invisible axion models. We can appreciate that, in the Lagrangian, the adimensional combination $a/f_a$ necessarilly enters independently of the specific features of the model, being the axion associated to a phase transformation. Thus, we are in principle free to choose the dimensional constant $f_a$. By enlarging this scale, we can render the axion mass (and not just it) smaller and smaller. This is the underlying reason for setting $f_a \sim v_\phi$, with $v_\phi \gg v$. Of course, we will have to give a possible range of values for this crucial parameter in following chapters.

### 2.3.2 The scalar sector

The addition of extra Higgs fields to the theory necessarily introduces new parameters in the potential with respect to the SM case: the arbitrariness of their values manifests itself in a certain number of scenarios for the mass spectrum.

#### The extreme decoupling regime

To narrow down the number of variables at play, it is always possible to exploit the perturbative minimization conditions of the potential: we will use them to eliminate the dimensionful parameters $V_1$, $V_2$ and $V_\phi$. By calculating the partial derivatives of our potential, expressed in terms of the physical fields, and requiring just for zero order contributions to vanish, one finds trivial solutions while deriving with respect to the charged and pseudoscalar fields, whereas the conditions coming from $H$, $S$ and $\rho$ provide the following system of equations:

$$\begin{cases} (av_\phi^2 - 2(V_2\lambda_3 + V_1(\lambda_1+\lambda_3)))c_\beta^2 + 4v^2(\lambda_1+\lambda_3)c_\beta^4 + (bv_\phi^2 - 2(V_1\lambda_3 + \\ \quad +V_2(\lambda_2+\lambda_3)))s_\beta^2 + 4v^2(\lambda_2+\lambda_3)s_\beta^4 + s_{2\beta}(cv_\phi + 2v^2\lambda_3 s_{2\beta}) = 0 \\ cv_\phi c_{2\beta} - 4v^2\lambda_1 c_\beta^3 s_\beta + \left(\frac{(b-a)v_\phi^2 + 4v^2\lambda_2 s_\beta^2}{2} + V_1\lambda_1 - V_2\lambda_2\right)s_{2\beta} = 0 \\ \lambda_\phi(v_\phi^2 - V_\phi) + 2av^2c_\beta^2 + 2bv^2s_\beta^2 + \frac{cv^2 s_{2\beta}}{v_\phi} = 0 \end{cases} \tag{2.76}$$

The values of the parameters obtained by solving this system will be employed in all of the following calculations. Incidentally, one can also notice that, in the last equation, an expression for $c$ can be extracted, too, if $\lambda_\phi = 0$, which means there is no autointeraction term for $\phi$.

But now we are going to move onto the more difficult task of dealing with the scalar part of the mass spectrum. Here everything will be computed by using perturbative techniques. Nevertheless, still a quite simple result is that related to the calculation of the charged Higgs $H^\pm$: their masses can be directly derived from the Lagrangian, upon substituting our field parametrization in the potential:

$$m_{H^\pm}^2 = 8\left(\lambda_4 v^2 - \frac{cv_\phi}{s_{2\beta}}\right) \tag{2.77}$$

that reminds the quartic potential result. A discussion similar to that for $A_0$ could be carried out: again, the only modification with respect to the known case is the substitution of the second addend inside brackets, originally $cv_\phi^2/s_{2\beta}$, with $cv_\phi/s_{2\beta}$. For the situation we are going to analyse, with $|c| \sim v_\phi$, the $H^\pm$ fields lie in a very high energy region, because of the proportionality to $v_\phi^2$.

By looking at quadratic terms emerging from the potential after spontaneous symmetry breaking for the remaining three neutral fields $H$, $S$ and $\rho$ (the so-called $0^+$ *sector*, because neutral and parity even), it is immediately evident that these latter are not mass eigenstates. As a matter of fact, the mass matrix is

$$M_{HS\rho} = \begin{pmatrix} 4v^2(3(\lambda_1+\lambda_2) + 8\lambda_3 + 4(\lambda_1-\lambda_2)c_{2\beta} + (\lambda_1+\lambda_2)c_{4\beta}) & -8v^2 s_{2\beta}(\lambda_1 - \lambda_2 + (\lambda_1+\lambda_2)c_{2\beta}) & 8v(av_\phi c_\beta^2 + cc_\beta s_\beta + bv_\phi s_\beta^2) \\ -8v^2 s_{2\beta}(\lambda_1 - \lambda_2 + (\lambda_1+\lambda_2)c_{2\beta}) & -\frac{8cv_\phi}{s_{2\beta}} + 4v^2(\lambda_1+\lambda_2)s_{2\beta}^2 & 4v(cc_{2\beta} + (b-a)v_\phi s_{2\beta}) \\ 8v(av_\phi c_\beta^2 + cc_\beta s_\beta + bv_\phi s_\beta^2) & 4v(cc_{2\beta} + (b-a)v_\phi s_{2\beta}) & 4v_\phi^2\lambda_\phi - \frac{2cv^2 s_{2\beta}}{v_\phi} \end{pmatrix} \tag{2.78}$$

Thus, a rotation is required to find correct physical states. It is important to notice that this orthogonal transformation will not obviously modify the correctly normalized kinetic terms for these fields. Moreover, one can pretty quickly realize that the diagonalization procedure can only be carried out with a perturbative approach. Because $v/v_\phi \ll 1$, our calculations can be organized as an expansion in powers of this very tiny parameter. We will first of all consider the general case, where $a$, $b$ take on a natural value of order $\mathcal{O}(1)$. In a similar fashion, we will choose $|c| \sim v_\phi$, in such a way that the corresponding adimensional parameter $c/v_\phi$ (suggested for example by (2.60)) will be of order one, too.

The rotation matrix $R$ that will take us in the physical basis $H'$, $S'$ and $\rho'$, i.e

$$\begin{pmatrix} H' \\ S' \\ \rho' \end{pmatrix} = R^{-1} \begin{pmatrix} H \\ S \\ \rho \end{pmatrix} \tag{2.79}$$

can be chosen according to the following ansatz

$$R = exp\left\{\frac{v}{v_\phi}A + \frac{v^2}{v_\phi^2}B\right\} \tag{2.80}$$

up to second order in $v/v_\phi$. Of course, $A^T = -A$ and $B^T = -B$: hence, there are six unknown factors. To determine these ones, the perturbative diagonalization proceeds by calculating the matrix $D = R^T M_{HS\rho} R$, expanding it in powers of $v/v_\phi$ and imposing for the non-diagonal entries to be zero: that provides a set of conditions which define $R$ completely. Indeed, we obtain $A_{12} = B_{13} = B_{23} = 0$ and this way the rotation matrix turns out to be

$$R = \begin{pmatrix} 1 - \frac{v^2}{v_\phi^2}\frac{A_{13}^2}{2} & -\frac{v^2}{v_\phi^2}\frac{A_{13}A_{23}-2B_{12}}{2} & \frac{v}{v_\phi}A_{13} \\ -\frac{v^2}{v_\phi^2}\frac{A_{13}A_{23}+2B_{12}}{2} & 1 - \frac{v^2}{v_\phi^2}\frac{A_{23}^2}{2} & \frac{v}{v_\phi}A_{23} \\ -\frac{v}{v_\phi}A_{13} & -\frac{v}{v_\phi}A_{23} & 1 - \frac{v^2}{v_\phi^2}\frac{A_{13}^2+A_{23}^2}{2} \end{pmatrix} \tag{2.81}$$

where

$$\begin{aligned} A_{13} &= \frac{2av_\phi c_\beta^2 + cs_{2\beta} + 2bs_\beta^2}{v_\phi \lambda_\phi}, \quad A_{23} = \frac{cc_{2\beta} + (b-a)v_\phi s_{2\beta}}{v_\phi\lambda_\phi + \frac{2c}{s_{2\beta}}}, \\ B_{12} &= \frac{1}{2c}\Bigg(s_{2\beta}^2 + 2s_{2\beta}^2 v_\phi(\lambda_1 - \lambda_2) + s_{2\beta}^2 \frac{2a(b-a)v_\phi c_\beta^2 - c(a-b)s_{2\beta} + 2v_\phi(b-a)bs_\beta^2}{\lambda_\phi} + \\ &+ \frac{c}{2}\frac{(2av_\phi c_\beta^2 + cs_{2\beta} + 2bv_\phi s_\beta^2)}{v_\phi\lambda_\phi}\left(s_{4\beta} - 2\frac{cc_{2\beta} + (b-a)v_\phi s_{2\beta}}{v_\phi\lambda_\phi + \frac{2c}{s_{2\beta}}}\right) + 2v_\phi(\lambda_1+\lambda_2)s_{2\beta}^2 c_{2\beta}\Bigg) \end{aligned} \tag{2.82}$$

We want to stress once more that, for the cubic model, $c$ consistently has mass dimensions, so that all formulae make sense from a dimensional point of view.

Once the $R$ matrix is given, one can calculate the eigenvalues of the mass matrix. We provide the results in the limit of large $v_\phi$, not to deal with cumbersome expressions and to have a much clearer intuition of physical implications:

$$\begin{aligned} m_{H'}^2 &= 4v^2(3(\lambda_1+\lambda_2) + 8\lambda_3 + 4(\lambda_1-\lambda_2)c_{2\beta} + (\lambda_1+\lambda_2)c_{4\beta}) \\ m_{S'}^2 &= v_\phi\left(\frac{-4c}{s_\beta c_\beta}\right) + \frac{2v^2((a-b)^2 - 2(\lambda_1+\lambda_2)\lambda_\phi)(c_{4\beta}-1) - 16c^2/s_\beta^2}{\lambda_\phi} \\ m_{\rho'}^2 &= 4v_\phi^2\lambda_\phi + \frac{8cv_\phi}{s_{2\beta}} + \frac{8v^2(c_{2\beta}(a^2-b^2) + (a^2+b^2)/c_{2\beta})}{\lambda_\phi} \end{aligned} \tag{2.83}$$

We see that, again, as expected, a light mass in the electroweak spectrum arises: thereby, the $H'$ field can naturally be identified with the Higgs boson of mass $126GeV$ discovered at the LHC. Then, we have a very massive $\rho'$ state, whose mass is proportional to $v_\phi^2$, exactly as in the original quartic model. But the $S'$ state, now, thanks to the dimensionful parameter $c$, has a mass proportional to $v_\phi$ and not to its square. Nevertheless, in this approximation regime, we remind that $|c| \sim v_\phi$: the combination $v_\phi c$ plays in our model the role of $v_\phi^2$, so that nothing changes from a phenomenological viewpoint. Actually, this situation reproduces the *extreme decoupling regime* of the original model, where only one scalar field lives in the low energy spectrum.

**Case 2:** $a, b \sim v/v_\phi$ **and** $|c| \sim v$

The same diagonalization procedure, with due modifications, can be carried out for all of the remaining cases, ending up with mass expressions which are similar to the previous ones. A first possibility is to consider values for the coupling parameters as $a, b \sim v/v_\phi$ and $|c| \sim v$. It is important to observe that, in this scenario and in the other ones we are going to describe, the quantities $a, b$ and $c$ turn out to be small: therefore, we must carry out an expansion of the mass matrix not only in $v/v_\phi$, but also in $a$ and $b$. For $c$, the correct adimensional variable in which we have to expand is $c/v_\phi$.

However, the situation of interest for this subparagraph is fairly similar to the preceding one, from a phenomenological point of view, with only one light Higgs state. Despite that, the states $S'$, $H^\pm$ and $A_0$ have now masses proportional to $\sqrt{vv_\phi}$ and, consequently, their existences might be tested with some future particle accelerators.

**Case 3:** $a, b \sim v^2/v_\phi^2$ **and** $|c| \sim v^2/v_\phi$**, with** $|c| \gg \lambda_i v^2/v_\phi$

We can further decrease the values of the couplings and consider the case $a, b \sim v^2/v_\phi^2$ and $|c| \sim v^2/v_\phi$. Even if expressions (2.83) are still valid and the $\rho'$ state still lies in the high energy spectrum, the masses for $H'$ and $S'$ are both proportional to $v^2$, together with $m_{A_0}^2$ and $m_{H^\pm}^2$. Anyway, if $|c| \ll \lambda_i v^2/v_\phi$ (with $i \in \{1, 2, 3\}$), we will have $m_{S'}^2 \ll m_{H'}^2$, that means a situation which is excluded by experiments. Thus, the only viable possibilities are $|c| \gg \lambda_i v^2/v_\phi$ (where $S'$ is a light state, but more massive that the SM Higgs field) and $|c| \sim \lambda_i v^2/v_\phi$.

**Case 4:** $a, b \sim v^2/v_\phi^2$ **and** $|c| \sim v^2/v_\phi$**, with** $|c| \sim \lambda_i v^2/v_\phi$

This case is particularly interesting, even if not so different from the aforementioned ones, because the expressions for the two light $0^+$ states of case 3 can be written in a compact form. Indeed, the mass matrix, up to second order in $v/v_\phi$, happens to be:

$$M_{HS\rho} = \begin{pmatrix} 4v^2(3(\lambda_1+\lambda_2)+8\lambda_3+4(\lambda_1-\lambda_2)c_{2\beta}+(\lambda_1+\lambda_2)c_{4\beta}) & -8v^2 s_{2\beta}(\lambda_1-\lambda_2+(\lambda_1+\lambda_2)c_{2\beta}) & 0 \\ -8v^2 s_{2\beta}(\lambda_1-\lambda_2+(\lambda_1+\lambda_2)c_{2\beta}) & -\frac{8cv_\phi}{s_{2\beta}}+4v^2(\lambda_1+\lambda_2)s_{2\beta}^2 & 0 \\ 0 & 0 & 4v_\phi^2\lambda_\phi \end{pmatrix} \tag{2.84}$$

so that we just need to diagonalise the two-by-two upper block. It worth noticing that the condition $|c| \sim \lambda_i v^2/v_\phi$ just plays a role in the entry $-8cv_\phi/s_{2\beta}+4v^2(\lambda_1+\lambda_2)s_{2\beta}^2$, where the two addends are of the same order of magnitude. The diagonalization can be pursed with the customary procedure, getting a simplified version of the rotation matrix (2.81)

$$R = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad tan(2\theta) = -\frac{(\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2)s_{2\beta}}{(\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2)c_{2\beta} - \frac{cv_\phi}{4v^2 s_{2\beta}}} \tag{2.85}$$

The resulting spectrum will be

$$m_{H'/S'}^2 = 2v^2\Bigg(\frac{-2cv_\phi}{v^2 s_{2\beta}^2} + 8(\lambda_1 c_\beta^2 + \lambda_2 s_\beta^2 + \lambda_3) \mp \Bigg[\frac{4c^2 v_\phi^2}{v^4 s_{2\beta}^2} + \frac{8cv_\phi}{v^2 s_{2\beta}}(\lambda_1+\lambda_2+4\lambda_3+2(\lambda_1-\lambda_2)c_{2\beta} + \\ + (\lambda_1+\lambda_2)c_{4\beta}) + 64(\lambda_1 c_\beta^2+\lambda_2 s_\beta^2+\lambda_3)^2 - 64 s_{2\beta}^2(\lambda_2\lambda_3+\lambda_1\lambda_2+\lambda_1\lambda_3)\Bigg]^{1/2}\Bigg) \tag{2.86}$$

which allows us to write

$$m_{H'}^2 + m_{S'}^2 = 32v^2\left(\frac{-cv_\phi}{4v^2 s_{2\beta}^2} + \lambda_1 c_\beta^2 + \lambda_2 s_\beta^2 + \lambda_3\right) \tag{2.87}$$

This rule is important to highlight the difference of this latter case from all the others. Indeed, once the ratio between the two Higgs doublet VEVs (i.e $\tan\beta$) has been fixed, the mass of the lightest Higgs boson set to $126GeV$ and $v_\phi$ put equal to some high energy scale, the mass spectrum described in the previous subsections profoundly depends on three parameters. The singlet self-coupling $\lambda_\phi$ only affects the mass of the heavy Higgs $\rho'$, that is, nevertheless, out of reach of LHC experiments. Consequenty, a crucial role is that of $c$ and $\lambda_4$ (this latter entering the expressions of the charged Higgs masses). But in this last case, when $|c| \sim \lambda_i v^2/v_\phi$, if $m_{H'} = 126GeV$, it is clear from (2.87) that the mass of $S'$ depends on a slightly different combination of parameters, whose role, in the other situations above, is overshadowed by a large $c$.

Before moving onto another set of interesting cases, we would like to stress an important aspect. As opposed to the extreme decoupling regime, where the parameter $a$ and $b$ are assumed to be of order one and $c$ is identified with the new high energy scale, in case 2, 3 and 4 $a, b \sim 10^{-7} - 10^{-10}$ or $a, b \sim 10^{-14} - 10^{-20}$ and $c \sim v$ or $c \sim 10^{-10}v$. Nonetheless, these extremely small values of coupling constants do not turn out to be associated to an unnatural fine-tuning, but, as explained in [31], they are technically natural, according to the notion of naturalness introduced by 't Hooft in [32]. As a matter of fact, by performing the limit $a, b, c \to 0$, the Poincaré symmmetry of the Lagrangian is enhanced, because the scalar doublets and the singlet decoupled: thereby, they can be Poincaré transformed independently (there will be two separate energy-momentum tensors obeying separate conservation laws). By setting $a, b, c \neq 0$, if $\mathcal{P}$ denotes the Poincaré group, we are breaking an extended space-time symmetry as $\mathcal{P}^{\phi_1/\phi_2} \times \mathcal{P}^{\phi} \to \mathcal{P}^{\phi_1/\phi_2+\phi}$. Furthermore, the limit $c \to 0$ is also related to the possibility of enlarging the PQ phase symmetry, as already stated. Therefore, the cases explored earlier are perfectly reasonable and justified.

### The quasi-free singlet limit

Just because the mass of $\rho'$ is controlled by $\lambda_\phi$, one can try to perform the limit $\lambda_\phi \to 0$. In this situation, $c$ does not appear in the mass matrix any more, because, as already noticed, it can be removed starting from the minimization conditions (2.76). Working out the same calculation for a DFSZ potential with $\lambda_\phi = 0$, one gets for $a, b \sim \mathcal{O}(1)$:

$$m^2_{H^\pm} = 8v^2\lambda_4 + v_\phi^2\left(\frac{a}{c^2_\beta} + \frac{b}{s^2_\beta}\right) \qquad m^2_{A_0} = 2(a+b)v^2 + 2(a-b)v^2 c_{2\beta} + 4v_\phi^2\left(\frac{a}{c^2_\beta} + \frac{b}{s^2_\beta}\right) \tag{2.88}$$

which lie in the heavy sector and the following $0^+$ mass spectrum

$$\begin{aligned} m^2_{H'} &= 4v^2(3(\lambda_1+\lambda_2) + 8\lambda_3 + 4(\lambda_1-\lambda_2)c_{2\beta} + (\lambda_1+\lambda_2)c_{4\beta}) \\ m^2_{S'} &= 16v_\phi^2\left(\frac{ac^2_\beta + bs^2_\beta}{s^2_{2\beta}}\right) + \mathcal{O}(v^2) \qquad m^2_{\rho'} = \frac{4abv^2 s^2_{2\beta}}{ac^2_\beta + bs^2_\beta} \end{aligned} \tag{2.89}$$

where now $\rho'$ is proportional to the electroweak scale. Of course, one should be able to guarantee that $m^2_{H'} < m^2_{\rho'}$, in order for this case to be feasible. For the same reason, we can solely consider the situation $a, b \sim \mathcal{O}(1)$. It is worth noticing that this case is the only one which is substantially different from the quartic potential theory: in that scenario the quasi-free singlet limit was not viable, because it was not possible to make a choise of parameters where tachyon masses were absent. Here, instead, we just need to require $a, b > 0$. It is quite reasonable that the only phenomenological discrepancy between the two models appears when $\lambda_\phi = 0$: this parameter is directly related to the fourth power of the high scale $v_\phi$ through the operator $\lambda_\phi(\phi^*\phi)^2$, which is clearly the dominant contribution. When this latter is present, it obscures our tiny modification of $V_{\text{c-term}}$.

**The custodial limit:** $\tan\beta \neq 1$

We have already discussed the importance of custodial symmetry in the SM scenario and how a custodial symmetric situation can be achieved in a DFSZ model before spontaneous symmetry breaking: we just need to use the set of conditions (2.17). Below the electroweak scale, the VEV of the theory might or might not respect the global residual $SU(2)_V$ symmetry, depending on the relative value of $v_1$ and $v_2$, i.e on the value of $\tan\beta = v_2/v_1$. If $\tan\beta \neq 1$, the custodial symmetry is broken into a $U(1)$ symmetry. So, we expect to have two extra goldstone bosons, which can be identified with the charged Higgs fields, whose masses turn out to be identically zero. Moreover, the $A_0$ field has light quanta:

$$m_{A_0}^2 = -16\lambda v^2\left(1 + \frac{v^2}{4v_\phi^2}s_{2\beta}^2\right) \tag{2.90}$$

where we can compensate for the minus sign through $\lambda$ itself. Nevertheless, the scenario just depicted, comprising charged massless particles, is excluded by phenomenology and must be promptly ruled out.

**The custodial limit:** $\tan\beta = 1$

If the custodial symmetry is preserved by the VEV of the theory, that is to say $SU(2)_L \times SU(2)_R$ is broken to $SU(2)_V$ with $\tan\beta = 1$, we have

$$m_{H^\pm}^2 = 8(-cv_\phi + 2v^2\lambda) \qquad m_{A_0}^2 = \frac{-2c}{v_\phi}(v^2 + 4v_\phi^2) \tag{2.91}$$

which are heavy states, even if their values can be decreased by tuning the $c$ parameter. The mass matrix for the $0^+$ sector is instead given by

$$M_{HS\rho} = \begin{pmatrix} 16v^2(\lambda + 2\lambda_3) & 0 & 4v(c + 2av_\phi) \\ 0 & 8(-cv_\phi + 2v^2\lambda) & 0 \\ 4v(c + 2av_\phi) & 0 & \frac{-2cv^2}{v_\phi} + 4v_\phi^2\lambda_\phi \end{pmatrix} \tag{2.92}$$

so that it is clear how $S$ is already a mass eigenstate and in particular $m_{S'}^2 = m_{H^\pm}^2$: $S$ and the two charged Higgs bosons constitute a triplet of $SU(2)_V$. The remaining two-by-two matrix can be easily diagonalized, giving the following values for $H'$ and $\rho'$:

$$\begin{aligned} m_{H'}^2 &= 4v^2\left(4\lambda + 8\lambda_3 - \frac{(c + 2av_\phi)^2}{v_\phi^2\lambda_\phi}\right) \\ m_{\rho'}^2 &= 2\left(-\frac{cv^2}{v_\phi} + 2\lambda_\phi v_\phi^2 + \frac{2v^2(c + 2av_\phi)^2}{\lambda_\phi v_\phi^2}\right) \end{aligned} \tag{2.93}$$

We see that, within a scenary where $a, b \sim v^2/v_\phi^2$ and $|c| \sim v^2/v_\phi$ with $c \sim \lambda_i v^2/v_\phi$ (i.e case 4), the matrix (2.92) is diagonal, once we drop out terms of order $v^3/v_\phi^3$. Moreover, four fields enter the light spectrum: $H^\pm$, $S'$ and $A_0$, together with $H'$. By fixing as usual $m_{H'} = 126GeV$, the light mass spectrum mainly depends on $\lambda$ and $c$: in fact, the $\lambda_3$ dependence is entirely contained in the SM Higgs mass, while the $a = b$ presence in $m_{\rho'}$ is highly suppressed.

**The quasi-custodial limit**

A last interesting situation is the so-called *quasi-custodial limit*. Here, the custodial symmetry is fulfilled by all parameters, as in case $\tan\beta = 1$, less than by the coupling $\lambda_4$, that is not equal to

$2\lambda$ and hence breaks explicitly the symmetry. The mass spectrum is untouched, unless we consider the fields $H^\pm$ and $S'$, whose masses now do not satisfy the equality $m_S^2 = m_{H^\pm}^2$, but

$$m_{S'}^2 = m_{H^\pm}^2 - 8v^2\lambda_4 + 16v^2\lambda \tag{2.94}$$

The custodial condition $\lambda_4 = 2\lambda$ restores the previous result.

## 2.4 Vacuum stability conditions

In order for our model to make sense, we should be able to find some conditions on the parameters of our Lagrangian ensuring the vacuum stability and a bounded-by-below potential. Following [33], we can just consider the quartic and cubic terms of (2.5)

$$\begin{aligned} V_4(\phi, \phi_1, \phi_2) =& \lambda_\phi(\phi^*\phi)^2 + (\lambda_1+\lambda_3)(\phi_1^\dagger\phi_1)^2 + (\lambda_2+\lambda_3)(\phi_2^\dagger\phi_2)^2 + (2\lambda_3+\lambda_4)(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \\ & - \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + (a\phi_1^\dagger\phi_1 + b\phi_2^\dagger\phi_2)\phi^*\phi + c(\phi_1^\dagger\phi_2\phi + \phi_2^\dagger\phi_1\phi^*) \end{aligned} \tag{2.95}$$

Due to the tricky form of the potential, which is difficult to study analytically, we assume that the couplings $a$, $b$ and $c$ are small enough not to perturb the stability conditions obtained from the rest of the Lagrangian. This is an hypothesis on the values of these couplings in full agreement with our previous considerations. Thereby, the singlet field disappears from the potential almost everywhere, with the exception of the first term: here, the stability is achieved by simply requiring $\lambda_\phi > 0$. Moreover, we know that this last contribution $\lambda_\phi(\phi^*\phi)^2$ always dominates over the quadratic and linear terms in $\phi$, so that we should not worry too much about how they could affect stability.

The subtleties involved in case $\lambda_\phi = 0$ have already been discussed. In that situation the role of $a$, $b$ and $c$ can not be neglected any more. Nevertheless, we have seen that the requirement $a, b > 0$ must be imposed to avoid tachyon masses. Once these conditions are provided, we see that the term $(a\phi_1^\dagger\phi_1 + b\phi_2^\dagger\phi_2)\phi^*\phi$ is quadratic in all Higgs fields and surely dominates, asymptotically at infinity, over the $c$ contribution, which is linear in each of them. Independently of the way we approach infinity in field space, a non-bounded-by-below potential will not arise.

Moving back to the most general case, we can set $f = \phi_1^\dagger\phi_1$, $g = \phi_2^\dagger\phi_2$ and $e = Re(\phi_1^\dagger\phi_2)$, $d = Im(\phi_1^\dagger\phi_2)$: by doing so, we can immediately individuate a positive combinations of fields given by $fg - e^2 - d^2 \geq 0$. Taking this into account, it will prove to be convenient to rewrite (2.95) (putting aside the $\phi$-dependent parts) as

$$\begin{aligned} V_4(\phi_1, \phi_2) =& (\lambda_1+\lambda_3)f^2 + (\lambda_2+\lambda_3)g^2 + (2\lambda_3+\lambda_4)fg - \lambda_4(e^2+d^2) = \\ =& [(\lambda_1+\lambda_3)^{1/2}f - (\lambda_2+\lambda_3)^{1/2}g]^2 + [2\lambda_3+\lambda_4+2\sqrt{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}]\cdot \\ & \cdot(fg - e^2 - d^2) + [2\lambda_3 + 2\sqrt{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}](e^2+d^2) \end{aligned} \tag{2.96}$$

so that we get the following stability conditions:

$$\begin{cases} \lambda_1+\lambda_3 > 0 \qquad \lambda_2+\lambda_3 > 0 \\ 2\lambda_3+\lambda_4+2\sqrt{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} > 0 \\ \lambda_3 + \sqrt{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} > 0 \end{cases} \tag{2.97}$$

In the quasi-custodial limit, where $\lambda_1 = \lambda_2 = \lambda$, they are reduced to

$$\begin{cases} \lambda+\lambda_3 > 0 \qquad 4\lambda_3+\lambda_4+2\lambda > 0 \\ 2\lambda_3+\lambda > 0 \end{cases} \tag{2.98}$$

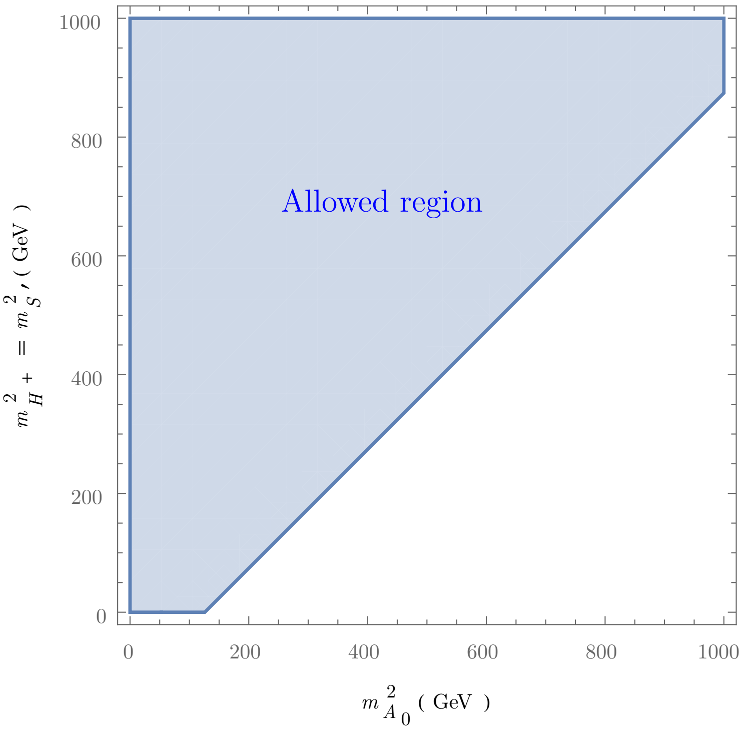



Figure 2.2: *Light spectrum of a cubic DFSZ model in the custodial limit. Once the mass of the light Higgs state $H'$ is fixed to the experimental value of $126GeV$ and $v_\phi$ to an arbitrary large scale (for example $10^{12}GeV$), the mass spectrum depends only on two parameters: $\lambda$, which is contained in $m^2_{S'}$, and $c$, which dictates the $m^2_{A_0}$ behaviour.*

Considering the formulae for the masses of $H^\pm$, $A_0$, $H'$ and $S'$ in the quasi-custodial limit (which can be read off from the custodial expressions, with the sole trick of substituing $2\lambda$ with $\lambda_4$ for $H^\pm$), we can solve the system

$$\begin{cases} m^2_{H^\pm} + 8(cv_\phi - \lambda_4 v^2) = 0 \\ m^2_{A_0} + 8cv_\phi = 0 \\ m^2_{H'} - 16v^2(\lambda + 2\lambda_3) = 0 \\ m^2_{S'} + 8(cv_\phi - 2\lambda v^2) = 0 \end{cases} \tag{2.99}$$

in terms of $c$, $\lambda_4$, $\lambda_3$ and $\lambda$. For $m^2_{A_0}$ and $m^2_{H'}$the limit $v_\phi \to \infty$ has been employed, so that terms respectively $\propto v^2 c/v_\phi$ and $\propto -4v^2(c+2av_\phi)^2/(v^2_\phi \lambda_\phi)$ have been neglected. The result can be used, together with the last set of disequalities, in order to derive an interesting couple of constraints on masses:

$$m^2_{H'} + m^2_{S'} - m^2_{A_0} > 0 \qquad m^2_{H'} + m^2_{H^\pm} - m^2_{A_0} > 0 \tag{2.100}$$

It is evident that these conditions will hold also for the less general situation of complete custodial symmetry, where they simply concide, because of $m^2_{S'} = m^2_{H^\pm}$. If $m_{H'} = 126GeV$ and $v = 246GeV$, we can utilize the approximate mass formulae (2.99) in (2.100) to obtain some rough bounds for $\lambda > -0.016$ and $\lambda_4 > -0.032$. Having imposed the experimental restriction $m^2_{H'} = (126GeV)^2 = 16v^2(\lambda + 2\lambda_3)$, if $\lambda_3$ grows up, its increase must be offset by a smaller value of $\lambda$. The latter can decrease up to the negative value of $-0.016$, so that $\lambda_3 < 0.016$.

As already noticed, the custodial symmetric spectrum of case 4 just depends on $\lambda$ and $c$: therefore, taking into account the mass disequalities for this situation, it is possible to plot the allowed region for these two parameters (look at figure 2.2). In a custodial symmetric scenario, there are no further constraints coming from the $\rho$ quantity, because $\Delta\rho_{BSM}$ is exactly zero: all custodial violating contributions will arise from the SM Lagrangian.

A custodial or quasi-custodial scenario can be pursed also in the quasi-free singlet limit: all of previous considerations are still lawful, if supported by the additional $a, b > 0$ requirements. In this simplified situation, (2.89) can be used to derive a bound on the $a$ parameter, in order to fulfil the hierarchy among the lighest masses of the spectrum: $m^2_{H'} < m^2_{\rho'}$ will imply $a > 0.066$. Moreover, it worth noticing how in (2.88) the roles of $\lambda$ and $\lambda_4$ are hidden by the large $v_\phi$ scale: even though custodial symmetry will be broken into a quasi-custodial case, the theory will not realise it. That means departures from the tree level value of $\rho$, owning to DFSZ loop corrections, will be negligible if $\lambda_\phi = 0$, at variance with the other cases, where setting $\lambda_4 \neq 2\lambda$ can have visible falls-out on $\rho$.

## 2.5 The electroweak precision test

For sake of completeness, we want to show in this paragraph how physics at higher energy scale can be powerfully constrained by the so-called *electroweak precision test*: of course, we are just going to give a flavour of the topic.

In order to incorporate the axion field in our theory, we have been compelled to enlarge the field content, including particles which generally lie in the high energy region. Even though they can not be directly produced in current particle accelerators, field fluctuations in form of virtual quanta can influence through loop corrections the value of some very precisely measured quantities. Comparing the predictions of the DFSZ theory with experiments can in principle give an indirect proof of axion existence.

An extremely important quantity that does this job is the already introduced $\rho$ parameter:

$$\rho = \frac{m_W^2}{m_Z^2 c_W^2} = 1 + \Delta\rho_{SM} + \Delta\rho_{BSM} \tag{2.101}$$

where $\Delta\rho_{SM}$ are the deviations from the custodial preserving scenario covered by the SM: this term includes the small corrections due to $g' \neq 0$ and the larger ones related to the top quark mass. The extra contribution $\Delta\rho_{BSM}$ will instead arise from custodial breaking effects coming from theories which extend the SM Lagrangian. One must have $\Delta\rho_{SM} + \Delta\rho_{BSM} = 3.7 \times 10^{-4}$.

All terms which violate custodial symmetry will affect $\rho$ through loop corrections to the values of the gauge boson masses: these quantum fluctuations will enter the perturbative series of their propagator diagrams. We will not work in the unitary gauge, where the goldstone bosons of the $H'$ field have been gobbled up by the gauge bosons, but these goldstone modes will be explicitly retained in loop calculations. Once defined the massive vector field and the scalar goldstone field propagators at zero order in perturbation theory respectively as

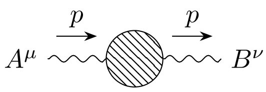
$$D_{\mu\nu}(k) = \frac{i}{k^2 - m_{Wi}^2 + i\epsilon}\left[-g_{\mu\nu} + \frac{(1-\xi)k_\mu k_\nu}{k^2 - \xi m_{Wi}^2 + i\epsilon}\right] \qquad D_{gold}(k) = \frac{i}{k^2 - \xi m_{Wi}^2 + i\epsilon} \tag{2.102}$$

where $\xi$ is the gauge fixing parameter, we will implement the Feynman-'t Hooft gauge setting $\xi = 1$

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 - m_{Wi}^2 + i\epsilon} \qquad D_{gold}(k) = \frac{i}{k^2 - m_{Wi}^2 + i\epsilon} \tag{2.103}$$

In a non-abelian gauge theory, the additional presence of some ghost fields underpins the unitarity of the whole construction. However, we will not need to explicitly consider them: they are allowed to interact solely with gauge fields and their associated goldstone bosons, without blending with the scalar sector through tree level verteces. Therefore, ghost degrees of freedom will simply enter $\Delta\rho_{SM}$ at 1-loop order.

With convention (2.103), we can write down the most general form for loop corrections to the quantum propagator diagram of a massive vector field as




$$= \Pi^{\mu\nu}_{AB}(p^2) = ig^{\mu\nu}\Pi_{AB}(p^2) + p^\mu p^\nu \Delta_{AB}(p^2) = \tag{2.104}$$
$$= ig^{\mu\nu}\left(\Pi_{AB}(0) + p^2\frac{d\Pi_{AB}}{dp^2}(0) + \mathcal{O}\left(\frac{p^2}{\Lambda_{NP}}\right)\right) + p^\mu p^\nu \Delta_{AB}(p^2) \qquad A, B = W_\mu, Z_\mu$$

where we have expanded for small energy $p^2 \approx 0$, owning to our interest in studying effects of the new physics energy scale $\Lambda_{NP} \sim v_\phi^2 \gg p^2$. The particle mass is defined as the energy $p^2$ at which the propagator $D_{\mu\nu}(p)$ exhibits a divergence: that implies quantum corrections to gauge boson masses will be given by $\Pi^{\mu\nu}_{AB}(p^2)$ evalutated at $p^2 = m^2_{W/Z}$. Consequently, referring to tree level masses as $m^2_{(tree)}$, we will have

$$\begin{aligned} m_W^2 &= m^2_{W(tree)} + \Pi_{WW}(p^2 = m_W^2) \approx \frac{g^2v^2}{2} + \Pi_{WW}(0) \\ m_Z^2 &= m^2_{Z(tree)} + \Pi_{ZZ}(p^2 = m_Z^2) \approx (g^2 + g'^2)\frac{v^2}{2} + \Pi_{ZZ}(0) \end{aligned} \tag{2.105}$$

that will affect $\rho$ as

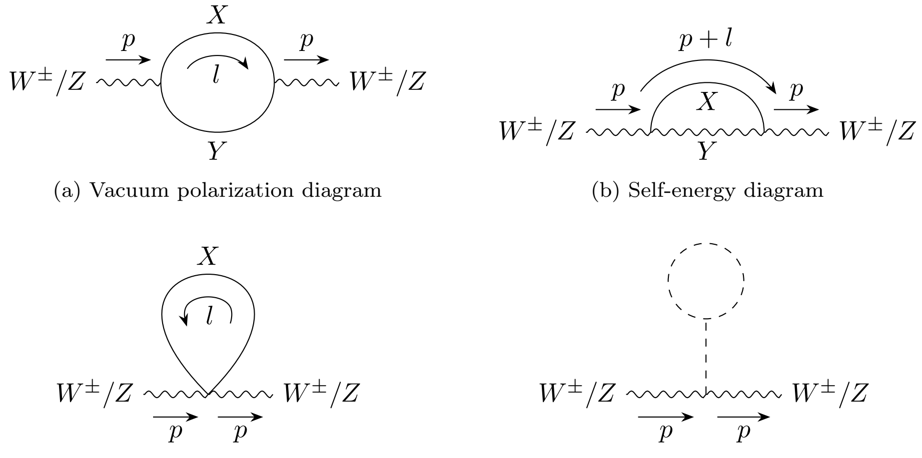
$$\begin{aligned} \rho = \frac{m_W^2}{m_Z^2 c_W^2} &= \frac{m^2_{W(tree)} + \Pi_{WW}(0)}{m^2_{Z(tree)}c_W^2 + \Pi_{ZZ}(0)c_W^2} = \frac{m^2_{W(tree)}\left(1 + \frac{\Pi_{WW}(0)}{m^2_{W(tree)}}\right)}{m^2_{Z(tree)}c_W^2\left(1 + \frac{\Pi_{ZZ}(0)}{m^2_{Z(tree)}}\right)} = \\ &\approx \frac{m^2_{W(tree)}}{m^2_{Z(tree)}c_W^2}\left(1 + \frac{\Pi_{WW}(0)}{m^2_{W(tree)}}\right)\left(1 - \frac{\Pi_{ZZ}(0)}{m^2_{Z(tree)}}\right) \approx \rho_{tree} + \underbrace{\frac{\Pi_{WW}(0)}{m^2_{W(tree)}} - \frac{\Pi_{ZZ}(0)}{m^2_{Z(tree)}}}_{\Delta\rho} \end{aligned} \tag{2.106}$$

Therefore, the effects of new physics will show up in

$$\Delta\rho_{BSM} = \frac{\Pi_{WW}(0)}{m^2_{W(tree)}} - \frac{\Pi_{ZZ}(0)}{m^2_{Z(tree)}} - \Delta\rho_{SM} \tag{2.107}$$

Thus, we just need to compute the quantities $\Pi_{WW}(0)$ and $\Pi_{ZZ}(0)$ starting from the general computation of $\Pi^{\mu\nu}_{XY}(p^2)$, where now subscripts refer to particles circulating inside loops. The diagrams contributing to it are (see figure 2.3):

- the scalar loop and tadpoles diagrams, which, nevertheless, will simplify in the difference appearing in (2.107) at the order of correction we are going to consider. Therefore, we will not explicitly evalutate them;
- the vacuum polaritazion diagrams;




(a) Vacuum polarization diagram

(b) Self-energy diagram

(c) Scalar loop diagram

(d) Tadpole diagram

Figure 2.3: Diagrams contributing to the calculation of $\Pi^{\mu\nu}_{XY}(p^2)$. The last diagram is a tadpole, which can anchor itself in the gauge propagator by means of different kinds of tails (a ghost-like, a scalar-like ...). Depending on that, the loop will host a suitable variety of virtual particles. This large number of options has been graphically depicted with a dashed line. Together with the scalar loop diagram, it will not directly enter our calculation of $\Delta\rho_{BSM}$.

- the self-energy diagrams.

In what follows, it will turn out to be unnecessary to consider the physical gauge boson basis, obtained through the Weinberg rotation: we will eventually work in the limit $g' \to 0$, where $c_W \to 1$ and $s_W \to 0$. Moreover, the Feynman rules that we are going to use for our calculations can be directly extracted expanding the kinetic term of the DFSZ Lagrangian: cubic verteces are summarized in tables 2.1$a$ and 2.1$b$, where their squares have already been taken into account for practical purposes. Just because we are not going to compute scalar loop diagrams, Feynman rules for quartic interactions have been left aside.

We can start evalutating the vacuum polarization loop integral $I^{XY}_{\mu\nu}$ using dimensional regularization, where space-time dimensions $d$ are extended from four to $2\omega$, $\omega \in \mathbb{C}$. Therefore, if $\mu$ is the ultraviolet regulator, we can write

$$
\begin{aligned}
I^{XY}_{\mu\nu}(p^2) &= -C_{XY}\mu^{4-2\omega}\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{l_\mu l_\nu}{[p^2 - m_X^2 + i\epsilon][(l+p)^2 - m_Y^2 + i\epsilon]} = \\
&= -C_{XY}\mu^{4-2\omega}\int_0^1 dxdy\delta(x+y-1)\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{l_\mu l_\nu}{\{x[l^2 - m_X^2 + i\epsilon] + y[(l+p)^2 - m_Y^2 + i\epsilon]\}^2} = \\
&= -C_{XY}\mu^{4-2\omega}\int_0^1 dx\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{(l-(1-x)p)_\mu(l-(1-x)p)_\nu}{[l^2 - \Delta(x,p^2) + i\epsilon]^2}
\end{aligned}
\tag{2.108}
$$

where we made use of the Feynman parameters $x$ and $y$ and we defined $\Delta(x, p^2) = x(x-1)p^2 + xm_X^2 + (1-x)m_Y^2$. Expanding the numerator of the integrand, terms proportional to $l^\mu$ will vanish for symmetry reasons, upon integrating, and hence

$$
\begin{aligned}
I^{XY}_{\mu\nu}(p^2) &= \int_0^1 dx \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{-C_{XY}\mu^{4-2\omega} l_\mu l_\nu}{[l^2 - \Delta(x,p^2) + i\epsilon]^2} + p_\mu p_\nu \underbrace{\int_0^1 dx \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{-\mu^{4-2\omega}(1-x)^2 C_{XY}}{[l^2 - \Delta(x,p^2) + i\epsilon]^2}}_{\to \Delta_{XY}(p^2)} = \\
&= -C_{XY}\mu^{4-2\omega}\frac{g_{\mu\nu}}{2\omega} \int_0^1 dx \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{l^2}{[l^2 - \Delta(x,p^2) + i\epsilon]^2} + p_\mu p_\nu K_{XY}(p^2)
\end{aligned}
\tag{2.109}
$$

Furthermore, terms containing the gauge boson momentum $p^\mu$ can be removed from the calculation, because they never contribute in practise. Indeed, all of these diagrams are generally sandwiched between fermion currents, so that the relation $p_\mu \bar{u}_r(q')\gamma^\mu u_s(q) = 0$ holds, where we required the vertex conservation law $p_\mu = q'_\mu - q_\mu$. Thereby, we are left with a scalar integral which contributes to $\Pi_{XY}(p^2)$ of expression (2.104), that is:

$$
I^{XY}(p^2) = i\frac{C_{XY}}{2\omega}\mu^{4-2\omega} \int_0^1 dx \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{l^2}{[l^2 - p^2\tilde{\Delta}(x) + i\epsilon]^2} \tag{2.110}
$$

with $\tilde{\Delta}(x) = x^2 - x + \frac{x m_X^2 + (1-x)m_Y^2}{p^2}$. Now, a Wick rotation can be performed without ado, provided that $p^2$ is kept below the threshold of physical production of the two virtual particles appearing in the loop. Indeed, we can check it by considering the parabola $\tilde{\Delta}(x)$, whose equation can be simply studied as:

$$
\frac{d\tilde{\Delta}(x)}{dx} = 0 \quad \Rightarrow \quad \bar{x} = \frac{1}{2}\left(\frac{m_Y^2}{p^2} - \frac{m_X^2}{p^2} + 1\right) \quad \text{and} \quad \frac{d^2\tilde{\Delta}(x)}{dx^2} = 2 \tag{2.111}
$$

so that

$$
\tilde{\Delta}(\bar{x}) \begin{cases} > 0 & (m_X - m_Y)^2 < p^2 < (m_X + m_Y)^2 \\ < 0 & p^2 < (m_X - m_Y)^2 \vee p^2 > (m_X + m_Y)^2 \end{cases} \tag{2.112}
$$

Nevertheless, there are no physical conditions providing $0 \leq \bar{x}(p^2) \leq 1$, which corresponds to our domain of interest, for $0 < p^2 < (m_X - m_Y)^2$: therefore, we can forget about the restrictions on $\tilde{\Delta}(\bar{x})$ in this interval. In addition, if we notice $\tilde{\Delta}(x=0) = (m_X^2/p^2) > 0$ and $\tilde{\Delta}(x=1) = (m_Y^2/p^2) > 0$, we will be able to state that

$$
p^2\tilde{\Delta}(x) > 0 \quad \Longleftrightarrow \quad 0 < p^2 < (m_X + m_Y)^2 \quad \forall x \in [0,1] \tag{2.113}
$$

which actually confirms we can perform a Wick rotation so long as the energy is not enough to produce resonance states.

By moving onto the Euclidean space and making use of the Mellin transform, we could write

$$
\begin{aligned}
I^{XY}(p^2) &= \frac{+C_{XY}\mu^{4-2\omega}}{2\omega} \int_0^1 dx \int \frac{d^{2\omega}l_E}{(2\pi)^{2\omega}} \frac{l_E^2}{[l_E^2 + p_E^2\tilde{\Delta}(x)]^2} = \\
&= \frac{C_{XY}\mu^{4-2\omega}}{2\omega} \int_0^1 dx \int \frac{d^{2\omega}l_E}{(2\pi)^{2\omega}} \frac{l_E^2}{\Gamma(2)} \int_0^{+\infty} dt t e^{-t[l_E^2 + \Delta]} = \\
&= \frac{C_{XY}\mu^{4-2\omega}}{2\omega\Gamma(2)} \int_0^1 dx \frac{1}{(2\pi)^{2\omega}} \int_0^{+\infty} dt t e^{-t\Delta}\left(-\frac{d}{dt}\right) \int d^{2\omega}l_E e^{-tl_E^2} = \\
&= \frac{C_{XY}\mu^{4-2\omega}}{2\Gamma(2)(4\pi)^\omega} \int_0^1 dx \int_0^{+\infty} dt t^{-\omega} e^{-t\Delta} = \frac{C_{XY}\mu^{4-2\omega}}{2\Gamma(2)(4\pi)^\omega} \int_0^1 dx \frac{\Gamma(1-\omega)}{\Delta^{1-\omega}}
\end{aligned}
\tag{2.114}
$$

| $X$ | $Y$ | Interaction term | Feynman rule for $\Pi^{\mu\nu}_{XY}(p^2)$: $C_{XY}$ | |
|---|---|---|---|---|
| $S$ | $H_{2/1}$ | $\mp\frac{g}{2}W^{\mu}_{1/2}S\overleftrightarrow{\partial}_{\mu}H_{2/1}$ | $-\frac{g^2}{4}(2l+p)_{\mu}(2l+p)_{\nu} \rightarrow -g^2 l_{\mu}l_{\nu}$ | |
| $A_0$ | $H_{1/2}$ | $g\frac{v_{\phi}}{v_f}W^{\mu}_{1/2}A_0\overleftrightarrow{\partial}_{\mu}H_{1/2}$ | $-g^2\frac{v_{\phi}^2}{v_f^2}(2l+p)_{\mu}(2l+p)_{\nu} \rightarrow -g^2\frac{4v_{\phi}^2}{v_f^2}l_{\mu}l_{\nu}$ | |
| $G_{1/2}$ | $H$ | $-gW^{\mu}_{1/2}H\partial_{\mu}G_{1/2}$ | $-g^2(p+l)_{\mu}(p+l)_{\nu} \rightarrow -g^2 l_{\mu}l_{\nu}$ | $\} \ C_{XY}l_{\mu}l_{\nu}$ |
| $G_0$ | $G_{2/1}$ | $\mp\frac{g}{2}W^{\mu}_{1/2}G_{2/1}\overleftrightarrow{\partial}_{\mu}G_0$ | $-\frac{g^2}{4}(2l+p)_{\mu}(2l+p)_{\nu} \rightarrow -g^2 l_{\mu}l_{\nu}$ | |
| $H_{1/2}$ | $a_{\phi}$ | $-g\frac{vs_{2\beta}}{v_f}H_{1/2}\partial_{\mu}a_{\phi}$ | $-g^2\frac{v^2s_{2\beta}^2}{v_f^2}(p+l)_{\mu}(p+l)_{\nu} \rightarrow -g^2\frac{v^2s_{2\beta}^2}{v_f^2}l_{\mu}l_{\nu}$ | |
| $G_{2/1}$ | $B_{\mu}$ | $\pm gg'\frac{v}{2}W^{\mu}_{1/2}B_{\mu}G_{2/1}$ | $g^2g'^2\frac{v^2}{4}g_{\mu\rho}g^{\rho\sigma}g_{\sigma\nu} \rightarrow g^2g'^2\frac{v^2}{4}g_{\mu\nu}$ | $\rightarrow C_{XY}g_{\mu\nu}$ |

(a)

| $X$ | $Y$ | Interaction term | Feynman rule for $\Pi^{\mu\nu}_{XY}(p^2)$: $C_{XY}$ | |
|---|---|---|---|---|
| $H_1$ | $H_2$ | $\frac{g}{2}W^{\mu}_{3}H_1\overleftrightarrow{\partial}_{\mu}H_2$ | $-\frac{g^2}{4}(2l+p)_{\mu}(2l+p)_{\nu} \rightarrow -g^2 l_{\mu}l_{\nu}$ | |
| $A_0$ | $S$ | $-g\frac{v_{\phi}}{v_f}W^{\mu}_{3}A_0\overleftrightarrow{\partial}_{\mu}S$ | $-g^2\frac{v_{\phi}^2}{v_f^2}(2l+p)_{\mu}(2l+p)_{\nu} \rightarrow -g^2\frac{4v_{\phi}^2}{v_f^2}l_{\mu}l_{\nu}$ | |
| $G_0$ | $H$ | $-gW^{\mu}_{3}H\partial_{\mu}G_0$ | $-g^2(p+l)_{\mu}(p+l)_{\nu} \rightarrow -g^2 l_{\mu}l_{\nu}$ | $\} \ C_{XY}l_{\mu}l_{\nu}$ |
| $G_1$ | $G_2$ | $\frac{g}{2}W^{\mu}_{3}G_1\overleftrightarrow{\partial}_{\mu}G_2$ | $-\frac{g^2}{4}(2p+l)_{\mu}(2l+p)_{\nu} \rightarrow -g^2 l_{\mu}l_{\nu}$ | |
| $S$ | $a_{\phi}$ | $g\frac{vs_{2\beta}}{v_f}H_{1/2}\partial_{\mu}a_{\phi}$ | $-g^2\frac{v^2s_{2\beta}^2}{v_f^2}(p+l)_{\mu}(p+l)_{\nu} \rightarrow -g^2\frac{v^2s_{2\beta}^2}{v_f^2}l_{\mu}l_{\nu}$ | |
| $H$ | $B_{\mu}$ | $gg'\frac{v}{2}W^{\mu}_{3}B_{\mu}H$ | $g^2g'^2\frac{v^2}{4}g_{\mu\rho}g^{\rho\sigma}g_{\sigma\nu} \rightarrow g^2g'^2\frac{v^2}{4}g_{\mu\nu}$ | $\rightarrow C_{XY}g_{\mu\nu}$ |

(b)

Table 2.1: The two tables show the (squared) Feynman rules, which enter the computation of $W^{1/2}_{\mu}$ (2.1a) and $W^{3}_{\mu}$ (2.1b) mass corrections. In both of them, the first five rules on the far right, used in the evalutation of the vacuum polarization diagram, have been stripped of linear terms in the loop momentum $l^{\mu}$ and of contributions proportional to the external gauge momentum $p^{\mu}$. In the last rows, the metric tensor appearing $g^{\rho\sigma}$ in the vector boson propagator has already been taken into account.

If we name $\epsilon = 2 - \omega$, expanding for small $\epsilon$ lets us claim

$$
\begin{aligned}
I^{XY}(p^2) &= \frac{C_{XY}}{32\pi^2}\int_0^1 dx\Gamma(-1+\epsilon)\Delta\left(\frac{4\pi\mu^2}{\Delta}\right)^{2-\omega} = \\
&\approx -\frac{C_{XY}}{32\pi^2}\int_0^1 dx\Delta\left[\frac{1}{\epsilon}+1-\gamma_E+...\right]\left[1+\epsilon ln\frac{4\pi\mu^2}{\Delta}+...\right] = \\
&\approx -\frac{C_{XY}}{32\pi^2}\int_0^1 dx\left[(1+\Delta_{\epsilon}+ln\mu^2)\Delta-\Delta ln\Delta+...\right]
\end{aligned}
\tag{2.115}
$$

with $\Delta_{\epsilon} = 1/\epsilon - \gamma_E + ln(4\pi)$ (where $\gamma_E = 0.5772...$ is the Euler-Mascheroni constant).

If we now extract the term which is constant with respect to $p^2$ by setting $p^2 = 0$, we can exactly compute the integration in $x$ through elementary integrals:

$$
\begin{aligned}
I^{XY}(0) = &-\frac{C_{XY}}{32\pi^2}(1+\Delta_{\epsilon}+ln\mu^2)\int_0^1 dx(xm_X^2+(1-x)m_Y^2)+ \\
&+\frac{C_{XY}}{32\pi^2}\int_0^1 dx(x(m_X^2-m_Y^2)+m_Y^2)ln(x(m_X^2-m_Y^2)+m_Y^2) =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{C_{XY}}{32\pi^2}(1+\Delta_\epsilon+ln\mu^2)\frac{m_X^2+m_Y^2}{2}+\frac{C_{XY}}{32\pi^2}\left[\frac{m_X^4 ln(m_X^2)-m_Y^4 ln(m_Y^2)}{2(m_X^2-m_Y^2)}-\frac{m_X^2+m_Y^2}{4}\right]= \\
&= -\frac{C_{XY}}{64\pi^2}\left\{\left(\frac{3}{2}+\Delta_\epsilon\right)(m_X^2+m_Y^2)-\left[\frac{m_X^4 ln(m_X^2/\mu^2)-m_Y^4 ln(m_Y^2/\mu^2)}{(m_X^2-m_Y^2)}\right]\right\}
\end{aligned}
\tag{2.116}
$$

Actually, it will prove useful to manipulate further the term in square brackets:

$$
\begin{aligned}
&\frac{m_X^4 ln(m_X^2/\mu^2)-m_Y^4 ln(m_Y^2/\mu^2)}{(m_X^2-m_Y^2)} = \frac{(m_X^2-m_Y^2)(m_X^2 ln(m_X^2/\mu^2)+m_Y^2 ln(m_Y^2/\mu^2))}{m_X^2-m_Y^2}+ \\
&-\frac{m_X^2 m_Y^2}{m_X^2-m_Y^2}(ln(m_Y^2/\mu^2)+ln(\mu^2/m_X^2)) = m_X^2 ln\left(\frac{m_X^2}{\mu^2}\right)+m_Y^2 ln\left(\frac{m_X^2}{\mu^2}\right)-\frac{m_X^2 m_Y^2}{m_X^2-m_Y^2}ln\left(\frac{m_Y^2}{m_X^2}\right)
\end{aligned}
\tag{2.117}
$$

so that we can eventually claim

$$
I^{XY}(0) = -\frac{C_{XY}}{64\pi^2}\left\{\left(\frac{3}{2}+\Delta_\epsilon\right)(m_X^2+m_Y^2)-m_X^2 ln\frac{m_X^2}{\mu^2}-m_Y^2 ln\frac{m_X^2}{\mu^2}+\frac{m_X^2 m_Y^2}{m_X^2-m_Y^2}ln\frac{m_Y^2}{m_X^2}\right\}
\tag{2.118}
$$

In a similar fashion, we can evalutate the contribution $J^{XY}_{\mu\nu}$ to $\Pi^{\mu\nu}_{XY}(p^2)$ coming from the self-energy diagram. According to the Feynman rules of our table, the only vector field entering this loop is the massless hypercharge field $B_\mu$ (that coincides with the photon in the limit $g' \to 0$). Therefore, we will have

$$
\begin{aligned}
J^{XY}_{\mu\nu}(p^2) &= C_{XY}\mu^{4-2\omega}\int\frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{g_{\mu\nu}}{[p^2-m_X^2+i\epsilon][(l+p)^2+i\epsilon]} = \\
&= iC_{XY}\mu^{4-2\omega}g_{\mu\nu}\int_0^1 dx\int\frac{d^{2\omega}l_E}{(2\pi)^{2\omega}}\frac{1}{[l_E^2+\Delta_p(x)]^2}
\end{aligned}
\tag{2.119}
$$

where $\Delta_p(x) = x(x-1)p^2+xm_X^2$. With exactly the same procedure, we can eventually obtain the scalar integral

$$
\begin{aligned}
J^{XY}(p^2) &= +\frac{C_{XY}\mu^{4-2\omega}}{(4\pi)^\omega}\int_0^1 dx\int_0^{+\infty}dtt^{-\omega}e^{-t\Delta_p(x)} = +\frac{C_{XY}\mu^{4-2\omega}}{(4\pi)^\omega}\int_0^1 dx\frac{\Gamma(1-\omega)}{\Delta_p(x)^{1-\omega}} = \\
&\approx -\frac{C_{XY}}{16\pi^2}\int_0^1 dx\left[\frac{1}{\epsilon}+1-\gamma_E+\cdots\right]\left[1+\epsilon ln\frac{4\pi\mu^2}{\Delta_p}+\cdots\right] = \\
&\approx -\frac{C_{XY}}{16\pi^2}\int_0^1 dx\left[1+\Delta_\epsilon+ln\mu^2-ln\Delta_p\right]\underset{p^2\to 0}{\longrightarrow}-\frac{C_{XY}}{16\pi^2}\left[1+\Delta_\epsilon-ln(m_X^2/\mu^2)\right]
\end{aligned}
\tag{2.120}
$$

Now that we have all the tools together to proceed, we can define the set of functions

$$
\begin{cases}
f(m_X^2,m_Y^2) = -\frac{C_{XY}}{64\pi^2}\left\{\left(\frac{3}{2}+\Delta_\epsilon\right)(m_X^2+m_Y^2)-m_X^2 ln\frac{m_X^2}{\mu^2}-m_Y^2 ln\frac{m_X^2}{\mu^2}+\frac{m_X^2 m_Y^2}{m_X^2-m_Y^2}ln\frac{m_Y^2}{m_X^2}\right\} \\
f(m_X^2,m_X^2) \overset{def}{=} \lim_{m_X^2\to m_Y^2} f(m_X^2,m_Y^2) = -\frac{C_{XY}}{32\pi^2}\left\{(1+\Delta_\epsilon)m_X^2-m_X^2 ln\frac{m_X^2}{\mu^2}\right\} \\
g(m_X^2) = -\frac{C_{XB_\mu}}{16\pi^2}\left[1+\Delta_\epsilon-ln(m_X^2/\mu^2)\right]
\end{cases}
\tag{2.121}
$$

where the limit, for the second definition, has been carried out using the simplified form (2.118) for $I^{XY}(0)$. If we further introduce the vector of scalar mass eigenstates $h_i = (H', S', \rho')$, we can simply compute $\Delta\rho_{BSM}$ using (2.107) and adding together all of the contributions given in table 2.1a and 2.1b. The result will be

$$\Delta\rho_{BSM} = \frac{1}{m_W^2}\Bigg[\sum_{i=1}^{3}(R_{Si}^2 f(m_{h_i}^2, m_{H^\pm}^2) + R_{Hi}^2 f(m_{G^\pm}^2, m_{h_i}^2)) + f(m_{A_0}^2, m_{H^\pm}^2) + f(m_{G^0}^2, m_{G^\pm}^2) + \\ + f(m_{H^\pm}^2, m_{a_\phi}^2) + g(m_{B_\mu}^2)\Bigg] - \frac{1}{m_Z^2}\Bigg[\sum_{i=1}^{3}(R_{Si}^2 f(m_{A_0}^2, m_{h_i}^2) + R_{Hi}^2 f(m_{G_0}^2, m_{h_i}^2) + \\ + R_{Si}^2 f(m_{h_i}^2, m_{a_\phi}^2) + R_{Hi}^2 g(m_{h_i}^2)) + f(m_{H^\pm}^2, m_{H^\pm}^2) + f(m_{G^\pm}^2, m_{G^\pm}^2)\Bigg] + \Delta\rho_{top} - \Delta\rho_{SM} \tag{2.122}$$

We can start neglecting the $g'$ corrections by approximating $m_Z^2 \approx m_W^2$. Moreover, we know that the top correction $\Delta\rho_{top}$, together with the contributions $f(m_{G^\pm}^2, m_{G^\pm}^2)$, $f(m_{G^0}^2, m_{G^\pm}^2)$ and $g(m_{B_\mu}^2)$ are already covered by the SM, so that they disappear upon subtracting $\Delta\rho_{SM}$. By direct inspection, one also notices that the two terms $R_{Hi} f(m_{G^\pm}^2, m_{h_i}^2)$ and $R_{Hi} f(m_{G_0}^2, m_{h_i}^2)$ exactly simplify in the limit $m_Z^2 \approx m_W^2$ (remembering that $m_{G^\pm}^2 = m_{W^\pm}^2$ and $m_{G_0}^2 = m_Z^2$). Using the property of rotation matrices $\sum_{i=1}^{3} R_{Hi}^2 = 1$, we can write

$$\Delta\rho_{BSM} = \frac{1}{m_W^2}\Bigg[\sum_{i=1}^{3} R_{Si}^2 (f(m_{h_i}^2, m_{H^\pm}^2) - f(m_{A_0}^2, m_{h_i}^2) - f(m_{h_i}^2, m_{a_\phi}^2)) + f(m_{A_0}^2, m_{H^\pm}^2) + \\ - f(m_{H^\pm}^2, m_{H^\pm}^2) + f(m_{H^\pm}^2, m_{a_\phi}^2)\Bigg] - \frac{1}{m_Z^2}\sum_{i=1}^{3} R_{Hi}^2 g(m_{h_i}^2) - \underbrace{\left(\sum_{i=1}^{3} R_{Hi}^2\right)}_{=1}\underbrace{\left(-\frac{1}{m_Z^2} g(m_{H_{SM}}^2)\right)}_{\to\Delta\rho_{SM}} \tag{2.123}$$

Just because $\Delta\rho_{BSM}$ is a measurable quantity, we expect the divergeces enclosed in $\Delta_\epsilon$ and the unphysical and arbitrary scale $\mu$ to disappear. This can be easily achieved between the last two addends of the previous expression, but it is by no means obvious how that can occur among terms in square brackets. Therefore, we can focus on them and go through some cumbersome calculations to gain a better insight of the procedure. In the interest of convenience, we further set $\tilde{f}(m_X^2, m_Y^2) = \frac{m_X^2 m_Y^2}{m_X^2 - m_Y^2} ln\left(\frac{m_X^2}{m_Y^2}\right)$. By employing our preceding definitions and the Feynman rules to make $C_{XY}$ explicit, we can start considering

$$\sum_{i=1}^{3} R_{Si}^2 (f(m_{h_i}^2, m_{H^\pm}^2) - f(m_{A_0}^2, m_{h_i}^2) - f(m_{h_i}^2, m_{a_\phi}^2)) + f(m_{A_0}^2, m_{H^\pm}^2) - f(m_{H^\pm}^2, m_{H^\pm}^2) + \\ + f(m_{H^\pm}^2, m_{a_\phi}^2) = \frac{g^2}{64\pi^2}\Bigg\{\left(\frac{3}{2} + \Delta_\epsilon\right)m_{H^\pm}^2 + \left(\frac{3}{2} + \Delta_\epsilon\right)\sum_{i=1}^{3} R_{Si}^2 m_{h_i}^2 - m_{H^\pm}^2 ln\frac{m_{H^\pm}^2}{\mu^2} + \\ - \sum_{i=1}^{3} R_{Si}^2 m_{h_i}^2 ln\frac{m_{h_i}^2}{\mu^2} - \sum_{i=1}^{3} R_{Si}^2 \tilde{f}(m_{H^\pm}^2, m_{h_i}^2) - \cancel{\frac{4v_\phi^2}{v_f^2}\left(\frac{3}{2} + \Delta_\epsilon\right)} m_{A_0}^2 - \frac{4v_\phi^2}{v_f^2}\left(\frac{3}{2} + \Delta_\epsilon\right)\sum_{i=1}^{3} R_{Si}^2 m_{h_i}^2 + \\ + \cancel{\frac{4v_\phi^2}{v_f^2} m_{A_0}^2 ln\frac{m_{A_0}^2}{\mu^2}} + \frac{4v_\phi^2}{v_f^2}\sum_{i=1}^{3} R_{Si}^2 m_{h_i}^2 ln\frac{m_{h_i}^2}{\mu^2} + \frac{4v_\phi^2}{v_f^2}\sum_{i=1}^{3} R_{Si}^2 \tilde{f}(m_{A_0}^2, m_{h_i}^2) - \underbrace{\frac{v^2 s_{s\beta}^2}{v_f^2}\left(\frac{3}{2} + \Delta_\epsilon\right)m_{a_\phi}^2}_{m_{a_\phi}^2 \to 0} +$$

$$
\begin{aligned}
&-\frac{v^2 s^2_{s\beta}}{v_f^2}\left(\frac{3}{2}+\Delta_\epsilon\right)\sum_{i=1}^{3}R^2_{Si}m^2_{h_i}+\underbrace{\frac{v^2 s^2_{s\beta}}{v_f^2}m^2_{a_\phi}ln\frac{m^2_{a_\phi}}{\mu^2}}_{m^2_{a_\phi}\to 0}+\frac{v^2 s^2_{s\beta}}{v_f^2}\sum_{i=1}^{3}R^2_{Si}m^2_{h_i}ln\frac{m^2_{h_i}}{\mu^2}+\\
&+\underbrace{\frac{v^2 s^2_{s\beta}}{v_f^2}\sum_{i=1}^{3}R^2_{Si}\tilde{f}(m^2_{a_\phi},m^2_{h_i})}_{m^2_{a_\phi}\to 0}+\frac{4v_\phi^2}{v_f^2}\left(\frac{3}{2}+\Delta_\epsilon\right)m^2_{A_0}+\frac{4v_\phi^2}{v_f^2}\left(\frac{3}{2}+\Delta_\epsilon\right)m^2_{H^\pm}-\frac{4v_\phi^2}{v_f^2}m^2_{H^\pm}ln\frac{m^2_{H^\pm}}{\mu^2}+\\
&-\frac{4v_\phi^2}{v_f^2}m^2_{A_0}ln\frac{m^2_{A_0}}{\mu^2}-\frac{4v_\phi^2}{v_f^2}\tilde{f}(m^2_{H^\pm},m^2_{A_0})\underbrace{-2(1+\Delta_\epsilon)m^2_{H^\pm}+2m^2_{H^\pm}ln\frac{m^2_{H^\pm}}{\mu^2}}_{-f(m^2_{H^\pm},m^2_{H^\pm})}+\frac{v^2 s^2_{s\beta}}{v_f^2}\left(\frac{3}{2}+\Delta_\epsilon\right)m^2_{H^\pm}+\\
&+\underbrace{\frac{v^2 s^2_{s\beta}}{v_f^2}\left(\frac{3}{2}+\Delta_\epsilon\right)m^2_{a_\phi}}_{m^2_{a_\phi}\to 0}-\frac{v^2 s^2_{s\beta}}{v_f^2}m^2_{H^\pm}ln\frac{m^2_{H^\pm}}{\mu^2}-\underbrace{\frac{v^2 s^2_{s\beta}}{v_f^2}m^2_{a_\phi}ln\frac{m^2_{a_\phi}}{\mu^2}}_{m^2_{a_\phi}\to 0}-\underbrace{\frac{v^2 s^2_{s\beta}}{v_f^2}\tilde{f}(m^2_{H^\pm},m^2_{a_\phi})}_{m^2_{a_\phi}\to 0}\Bigg\}
\end{aligned}
\tag{2.124}
$$

where we explicitly indicate where we are going to performe the invisible axion limit $m^2_{a_\phi}\to 0$. All terms proportional to the squared axion mass will vanish, together with $\tilde{f}(m^2_{a_\phi},m^2_X)\underset{m^2_{a_\phi}\to 0}{\longrightarrow}0$. Now, the use of the $v_f$ definition will yield

$$
\begin{aligned}
&\sum_{i=1}^{3}R^2_{Si}(f(m^2_{h_i},m^2_{H^\pm})-f(m^2_{A_0},m^2_{h_i})-f(m^2_{h_i},m^2_{a_\phi}))+f(m^2_{A_0},m^2_{H^\pm})-f(m^2_{H^\pm},m^2_{H^\pm})+\\
&+f(m^2_{H^\pm},m^2_{a_\phi})\underset{m^2_{a_\phi}\to 0}{\longrightarrow}\frac{g^2}{64\pi^2}\Bigg\{\underbrace{\left[\left(\frac{3}{2}+\Delta_\epsilon\right)\left(1+\frac{4v_\phi^2}{v_f^2}+\frac{v^2 s^2_{s\beta}}{v_f^2}\right)-2(1+\Delta_\epsilon)\right]}_{=1}m^2_{H^\pm}+\\
&+\left(\frac{3}{2}+\Delta_\epsilon\right)\left(1-\frac{4v_\phi^2}{v_f^2}-\frac{v^2 s^2_{s\beta}}{v_f^2}\right)\sum_{i=1}^{3}R^2_{Si}m^2_{h_i}+\left(2-\frac{v^2 s^2_{s\beta}}{v_f^2}-\frac{4v_\phi^2}{v_f^2}-1\right)m^2_{H^\pm}ln\frac{m^2_{H^\pm}}{\mu^2}+\\
&+\left(\frac{4v_\phi^2}{v_f^2}+\frac{v^2 s^2_{s\beta}}{v_f^2}-1\right)\sum_{i=1}^{3}R^2_{Si}m^2_{h_i}ln\frac{m^2_{h_i}}{\mu^2}-\sum_{i=1}^{3}R^2_{Si}\tilde{f}(m^2_{H^\pm},m^2_{h_i})+\frac{4v_\phi^2}{v_f^2}\sum_{i=1}^{3}R^2_{Si}\tilde{f}(m^2_{A_0},m^2_{h_i})+\\
&-\frac{4v_\phi^2}{v_f^2}\tilde{f}(m^2_{H^\pm},m^2_{A_0})\Bigg\}
\end{aligned}
\tag{2.125}
$$

We can finally plug this result in our initial expression (2.123), in order to get

$$
\begin{aligned}
\Delta\rho_{BSM}=\frac{g^2}{64\pi^2 m_W^2}&\left[\sum_{i=1}^{3}R^2_{Si}\left(\frac{4v_\phi^2}{4v_\phi^2+v^2 s^2_{2\beta}}\tilde{f}(m^2_{A_0},m^2_{h_i})-\tilde{f}(m^2_{h_i},m^2_{H^\pm})\right)+\right.\\
&\left.-\frac{4v_\phi^2}{4v_\phi^2+v^2 s^2_{2\beta}}\tilde{f}(m^2_{A_0},m^2_{H^\pm})+m^2_{H^\pm}\right]+\frac{g'^2}{16\pi^2}\sum_{i=1}^{3}R^2_{Hi}ln\left(\frac{m^2_{H_{SM}}}{m^2_{h_i}}\right)
\end{aligned}
\tag{2.126}
$$

The last addend, proportional to $g'^2$, that we are going to drop out, too, represents an overlap between the SM and BSM contributions. The dependence on scalar masses is only logarithmic.

On the other hand, the completely new contributions reveal a slightly stronger dependence on the mass variables. For example, we know that the mass splitting between $S'$ and $H^{\pm}$ states (that comprise a triplet in the custodial limit) violates custodial symmetry: consistently, we find a quadratic dependence on $m_{H^{\pm}}$ in the $\Delta\rho_{BSM}$ formula.

In order to compare the predictions of our DFSZ axion model with experimental data, we introduce one of the *Peskin-Takeuchi parameters* [34]

$$\begin{aligned}\Delta T = \frac{\Delta\rho_{BSM}}{\alpha_{em}} = \frac{1}{16\pi s_W^2 m_W^2}\Bigg[\sum_{i=1}^{3} R_{Si}^2 \left(\frac{4v_\phi^2}{4v_\phi^2 + v^2 s_{2\beta}^2}\tilde{f}(m_{A_0}^2, m_{h_i}^2) - \tilde{f}(m_{h_i}^2, m_{H^{\pm}}^2)\right) + \\ - \frac{4v_\phi^2}{4v_\phi^2 + v^2 s_{2\beta}^2}\tilde{f}(m_{A_0}^2, m_{H^{\pm}}^2) + m_{H^{\pm}}^2\Bigg]\end{aligned} \tag{2.127}$$

where we used $\alpha_{em} = e^2/4\pi = g^2 s_W^2/4\pi$.

We will just consider this parameter, among the possible others, because we are not going to develop a detailed analysis about electroweak precision test, which can be found in abundance in literature (see for example [35]). Furthermore, the $\Delta T$ value has been proved to be the one dictating the more severe bounds on BSM physics, if compared to the remaining ones.

We can realize that the only difference with respect to the $\Delta T$ formula obtained in a quartic DFSZ model resides in the four factor multiplying $v_\phi^2$ in the ratio $\frac{4v_\phi^2}{4v_\phi^2+v^2 s_{2\beta}^2}$: the ultimate reason for this tiny change is due to the modified axion definition given for the cubic theory. Nonetheless, being $v_\phi$ a large energy scale, the phenomenology will be again consistent with the old results, without great novelties. Consequently, we can claim that, from the point of view of the scalar spectrum, a cubic DFSZ model is practically indistinguishable from a quartic one: therefore, different subtlelties must be taken into account in order to discern between them.

By way of conclusion, we explicitly show the tight restrictions imposed by the experimental value of $\Delta T = 0.09 \pm 0.13$ (measurement done by [36]) on an exemplary spectrum, i.e that of case 4 ($a, b \sim v^2/v_\phi^2$ and $|c| \sim v^2/v_\phi$, with $|c| \sim \lambda_i v^2/v_\phi$). After all, this is the only one predicting a sizeable number of light scalars, in addition to the SM Higgs: the two charged Higgs $H^{\pm}$, the pseudo-scalar $A_0$ and the scalar $S'$. We will simply soften the custodial requirement up to the quasi-custodial scenario, where only $\lambda_4$ violates it: that will be enough to appreciate the strong constraints arising from electroweak measurements. We will fix again $m_{h_1} = 126GeV$, $v_\phi = 10^{12}GeV$ and the electroweak parameters will be chosen to the approximate values of $v \approx 250GeV$, $m_W \approx 80GeV$ and $s_{\theta_W}^2 \approx 0.22$. Now, using equations (2.100) to implement stability and noticing how in a quasi-custodial case 4 the rotation matrix simplifies as $R_{S1}^2 = R_{S3}^2 = 0$ and $R_{S2}^2 = 1$, the $\Delta T$ formula can be explicitly written in terms of the unknown factors $m_{H^{\pm}}$, $m_{A_0}$ and $\lambda_{4B} = \lambda_4 - 2\lambda$ (figure 2.4) or $m_{H^{\pm}}$, $m_{A_0}$ and $m_{h_2}$ (figure 2.5). Here, $\lambda_{4B}$ parametrizes the effect of custodial breaking. We can easily move from one set of variables to the other one using the relation $m_{h_2}^2 = m_{H^{\pm}}^2 - 8v^2\lambda_{4B}$.

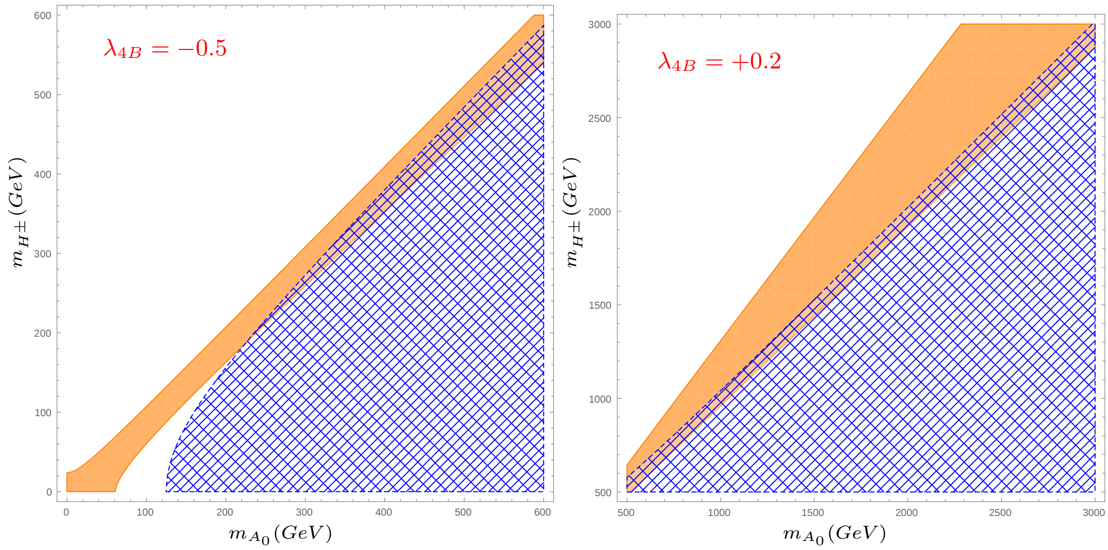



Figure 2.4: The two plots illustrate the light mass spectrum of a DFSZ model (quasi-custodial case 4) for two different values of the custodial breaking parameter $\lambda_{4B}$. The orange region is the viable range of masses for the couple $(m_{A_0}, m_{H^\pm})$ according to the $\Delta T = 0.09 \pm 0.13$ restriction; the blue grid area excludes points forbidden by the stability conditions of the potential.

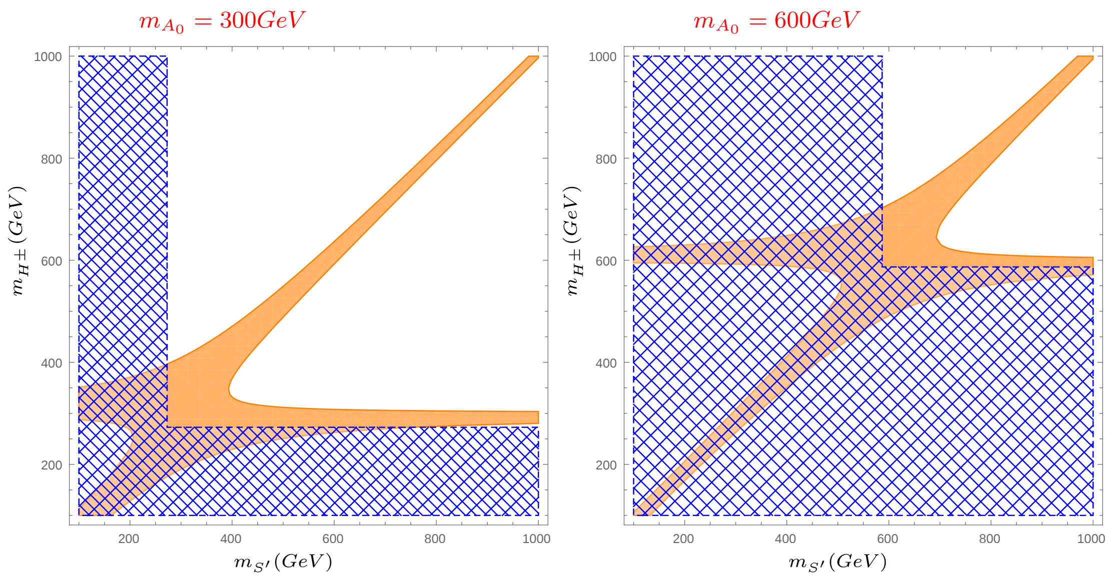



Figure 2.5: The two plots illustrate the light mass spectrum of a DFSZ model (quasi-custodial case 4) for two different values of the pseudoscalar mass $m_{A_0}$. The orange region is the viable range of masses for the couple $(m_{S'}, m_{H^\pm})$ according to the $\Delta T = 0.09 \pm 0.13$ restriction; the blue grid area excludes points forbidden by the stability stability of the potential.

# Chapter 3

# The domain wall issue

The axion concept is an extremely fruitful idea of theoretical physics with countless applications and consequences. Just that makes the completion of axion models an incredibly delicate task. Theories bearing the axion as a part of their spectrum should be compatible with particle physics: all of the fields we introduce in order to construct a consistent axion particle should deal with the current phenomenology at the LHC, for example. Nevertheless, this new pseudo-scalar field can potentially affect astrophysics and cosmology, too. So, there is another frontier that grossly limits our model building freedom. The domain wall puzzle is an exemplary problem that one has to take into account when coping with axions.

## 3.1 The $f_a$ and $v_\phi$ relation

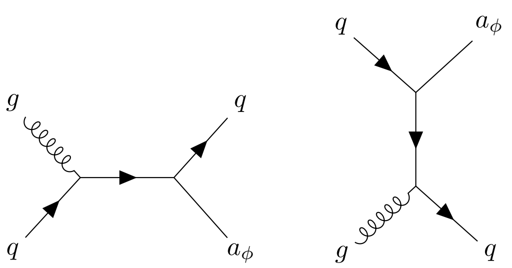
Until now, we have overlooked a subtle issue when dealing with the effective axion-gluon interaction. We discussed how the standard form to present it is (2.63), i.e in terms of the constant $f_a$:

$$\mathcal{L}_{anomaly} = \frac{a_\phi}{f_a}\frac{\alpha_s}{8\pi}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_a \tag{3.1}$$

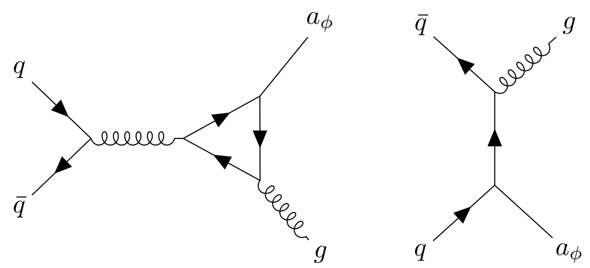
This is the axion decay constant, which directly parametrizes the strength of decay processes of axions into gluons through a triangle diagram. Despite that, we can also compute this term starting from our DFSZ model. In the next chapter, we will deeply study the possible Yukawa contributions to our Lagrangian; what is important to consider here is that, if we take into account a Yukawa term in the form $y\bar{Q}_L\Phi_{12}q_R$ (with $Q_L$ an isospin doublet and $q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}$ a vector of isospin singlets) and we make use of the expression (2.53) for $\Phi_{12}$, the phases associated to the matrix $U_a$ can be reabsorbed through a local chiral transformation. This procedure will generate an axion-gluon vertex of the form

$$\mathcal{L}_{anomaly} = \frac{2Na_\phi}{v_f}\frac{\alpha_s}{8\pi}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_a \tag{3.2}$$

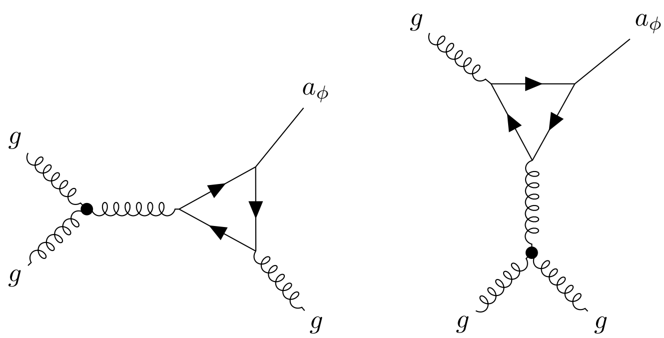
where we remind that $v_f = \sqrt{4v_\phi^2 + v^2 s_{2\beta}}$ for a DFSZ theory with a cubic potential or $v_f = \sqrt{v_\phi^2 + v^2 s_{2\beta}}$ in presence of a quartic one. $N$ is the color anomaly, which can be computed using (1.71), once the PQ charges for fermions have been assigned. In this last expression, the effective coupling constant of the interaction involves the high energy scale $v_f$ of the model, which is mainly




(a) $g + q(\bar{q}) \leftrightarrow a_\phi + q(\bar{q})$




(b) $q + \bar{q} \leftrightarrow a_\phi + g$




(c) $g + g \leftrightarrow g + a_\phi$

Figure 3.1: Thermal axion sources for $T > T_{QH}$ during the quark-gluon plasma phase.

related, thanks to the constraint $v_\phi \gg v$, to the VEV of the singlet field $\phi$. By requiring the matching of the two expressions (3.1) and (3.2), one gets

$$f_a = \frac{v_f}{2N} \underset{v_\phi \gg v}{\longrightarrow} f_a = -\frac{X_\phi v_\phi}{N} \tag{3.3}$$

where the last equality holds by considering the different $v_f$ definitions for the DFSZ models, together with the suitable $X_\phi$ value. Indeed, $X_\phi = -1$ for the cubic theory and $X_\phi = -1/2$ for the quartic one. The number linking $f_a$ and $v_\phi$ is called the *domain wall number*: $\mathcal{N}_{DW} = -N/X_\phi$. This latter is the leading character of this chapter.

## 3.2 Axion and cosmology

If axion existed, they would have highly influenced the past history of the universe. In particular, they would find a place in the cosmological scenario as dark matter components. As a matter of fact, they naturally possess a huge number of properties that a would-be dark matter candidate should have: a tiny mass, small couplings with SM particles (as we can already see from the suppression of the gluon vertex, due to $f_a$), a long life-time (greater that the age of the universe) and so on. All of these features derive solely from $f_a$, whose range of possible values is fixed as $10^9 GeV \leq f_a \leq 10^{17} GeV$ [37]: the lower bound comes from supernova cooling, while the upper limit from black hole superradiance. Through (2.75) one gets the axion mass range $5.70 \times 10^{-11} eV \leq m_a \leq 5.70 \times 10^{-3} eV$. Actually, the upper bound on axion mass is intimately related to the properties of this particle of interacting with nucleons. Despite that, the presented interval can be tweaked for the better, by opening the axion mass window: how to achieve that is the main topic of the next chapter.

Axions can take part in the history of cosmos both as hot and cold dark matter constituents. Hot axions can be efficiently produced through different processes during the universe thermal evolution. For temperature $T$ higher than the $T_{QH} \approx 200 - 300MeV$ of the quark-hadron transition, axions can be directly generated from gluon and quark reactions, as $g+q(\bar{q}) \leftrightarrow a_\phi+q(\bar{q})$, $q+\bar{q} \leftrightarrow a_\phi+g$ or $g+g \leftrightarrow g+a_\phi$ [38], whose Feynman diagrams are shown in figure 3.1. When the temperature drops below $T_{QH}$, we move from a quark-gluon plasma phase to a situation where gluons confine quarks into hadrons. Here, the most relevant axion thermalization processes are due to pions and nucleons interactions, such as $\pi + \pi \leftrightarrow \pi + a_\phi$ and $\pi + n \leftrightarrow n + a_\phi$, which can be described by an effective Lagrangian (presented in the following chapter). These reactions will be effective until the pion recombination temperature $T_\pi \approx 130MeV$. The presence of a new source of relativistic components in the universe, together with neutrinos and photons, can lead to important consequences. As said in [39], just as massive neutrinos, axions could leave an imprint on the CMB temperature anisotropies, could contribute to the effective number of relativistic degrees of freedom, could influence primordial nucleosynthesis and structure formation through the free-streaming scale.

In spite of the extreme similarity between axions and neutrinos, we claimed that axion can even constitute a cold dark matter component, which sounds pretty strange, if we consider its generally small mass. Recent experiments showed how some neutrino species must possess a mass. Nevertheless, this latter must be very tiny, so that, regarding the process of structure formation, neutrinos will be hot dark matter components: they will be produced by thermal fluctuations in the early universe and, even if they are cooled by the cosmos expansion, they will still be highly relativistic at that epoch of structure accretion. A similar reasoning applies to axions produced by the reactions we have previously presented. But it is now important to consider the differences between these two light particles just by remembering the peculiarities of the axion field.

The axion is a pseudo-goldstone boson: it arises as the phase of some complex scalar field. In one of the simplest invisible axion model, such as the KSVZ one, there is only one extra Higgs field $S$ enjoying a PQ simmetry, so that one can naturally set $S = (v_S + s)e^{ia_S/v_S}$. In our DFSZ theory, things are much more complicated, because we have a PQ simmetry shared by three Higgs fields. We have already studied the emergence of the axion phase, which is expressed by (2.50): by looking at the VEV $v_f$ which normalizes this axion phase, we can envisage that the field combination $\Upsilon$ acting as the KSVZ $S$, here, will be given by

$$\Upsilon = (v_f + \chi)e^{ia_\phi/v_f} \qquad \text{where} \qquad \Upsilon = \frac{4v_\phi\phi + vs_{2\beta}\tilde{H}}{v_f} \tag{3.4}$$

with $\tilde{H} = v + H$, whose VEV is $\langle\tilde{H}\rangle = v$. The $\Upsilon$ formula has been fixed in such a way that $\langle\Upsilon\rangle = v_f$. Of course, we know how $v_\phi \gg v$, so that $\Upsilon$ is essentially ruled by $\phi$. For the same reason, the major responsible for spontaneous symmetry breaking of PQ symmetry will be just the term proportional to $\lambda_\phi$ of (2.5). The non-perturbative vacuum $V_\phi$ itself, which can be obtained from the potential minimization conditions (for example (2.76)), turns out to be grossly $V_\phi \sim v_\phi^2$. Despite that, axions would emerge even without a $\phi$ field, because it is not the only PQ charged Higgs. In the same way, the symmetry breaking mechanism will occur in the quasi-free singlet limit ($\lambda_\phi = 0$), too, where the subleading terms of the potential related to $\tilde{H}$ will start contributing.

All that is of capital importance from a cosmological perspective: we will now consider the axion field evolution (and not just its perturbation, as we did up to now). If we are in the primordial universe, at energy higher than $f_a$, the PQ symmetry is preserved by $\langle\Upsilon\rangle = 0$, which minimizes the potential

$$V(\Upsilon) \approx \lambda_\phi(\Upsilon^*\Upsilon + v_\phi^2)^2 \tag{3.5}$$

and we have no axions. But when we drop below this scale, the PQ symmetry is broken: if our $\Upsilon$ field is subjected to the simplified mexican hat potential

$$V(\Upsilon) \approx \lambda_\phi(\Upsilon^*\Upsilon - v_\phi^2)^2 \tag{3.6}$$

it will roll down from the false vacuum state $\langle\Upsilon\rangle = 0$ to one arbitrary minimum between the infinite set of them, i.e $\langle\Upsilon\rangle = v_f e^{ia_\phi/v_\phi}$, where the angular variable $a_\phi/v_\phi$ just parametrizes each element. The sign flip between (3.5) and (3.6), which we modified by hand, shoud be in principle explained by the cosmological evolution of the parameters with temperature. Our energy is still greater than $\Lambda_{QCD} \sim 200MeV$, below which QCD can not be treated perturbatively any more and quarks are confined into hadrons: for the effects of asymptotic freedom, strong couplings are weak and, clearly, pions do not exist. At these scales there is no effective potential for the axion, which is practically massless. Nevertheless, this pretty simple mechanism must be explained a bit better when embedded in a cosmological background.

### 3.2.1 Axionic strings

When the PQ phase transition occurs, we know that it will take place simultaneously in different points of the universe: quantum fluctuations will move the $\Upsilon$ field to a completely casual value of the $U(1)$ symmetric set of vacua. Nevertheless, these values will not be the same everywhere, because the correlation length between various universe positions can not exceed the *casual particle horizon* $d_H$, defined as usual

$$d_H(t) = R(t)\int_0^t \frac{cdt'}{R(t')} \tag{3.7}$$

and associated to light propagation. $R$ is the scale factor which enters the Friedmann-Robertson-Walker metric

$$ds^2 = c^2dt^2 - R(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right) \tag{3.8}$$

and $k$ the curvature constant. The process, which leads to a universe comprised of distinct patches, is dubbed *Kibble mechanism* [40, 41] (sometimes known also as *Kibble-Zureck mechanism*). In principle, it is possible to describe a continuous closed path on a plane in the physical coordinate space, connecting points where the $\Upsilon$ field settled in different VEVs, whose phases $a_\phi/v_f$ run from 0 to $2\pi$: that is legitimate just because we are dealing with a continuous symmetry. This path can be smoothly deformed into a point, provided that a singular region where $\langle\Upsilon\rangle = 0$ still exists. We can extend this discussion from plane to space, so that the simple requirement of causality provides us with a chain of points where the PQ transition did not occur and the scalar field got trapped in the false vacuum: these chains are the *axionic strings*.

Let us consider a simple axionic string, located along the z axis of a coordinate system. The $\Upsilon$ field configuration describing an axionic string can be encompassed by the ansatz

$$S(t,\vec{x}) = v_\phi f(r) e^{in\theta} \tag{3.9}$$

where $\theta = a_\phi/(2v_\phi)$. We want that the field approaches the real vacuum far away from the singular region $\vec{x} = (0,0,z)$ (when the cylindrical $r$ coordinate goes to infinity) and that it assumes the zero value at $r = 0$. This constrains $f(r)$ in such a way that

$$\begin{cases} f(r) = 1 & \text{when} \quad r \to +\infty \\ f(r) = 0 & \text{when} \quad r \to 0 \end{cases} \tag{3.10}$$

In (3.9), $n$ is the already introduced winding number (1.34), which tells us how many times we are moving around the false vacuum region. We are not going to consider any time dependence, which means $S(t,\vec{x}) = S(\vec{x})$ is a topologically stable field configuration. After all, $S(t,\vec{x})$ is a topological solution of field equations, which is stable against small perturbations by very definition. If we now consider the total energy of the system

$$E = \int d^3x \Big[ |\partial_t S|^2 + |\nabla_x S|^2 + V(S) \Big] \tag{3.11}$$

the gradient of the string will be

$$\nabla_x S = v_\phi \left[ \partial_r f(r)\hat{r} + in\frac{f(r)}{r}\hat{\theta} \right] e^{in\theta} \quad \Rightarrow \quad |\nabla_x S|^2 = v_\phi^2 \left[ (\partial_r f(r))^2 + n^2 \frac{f(r)^2}{r^2} \right] \tag{3.12}$$

The time derivative in (3.11) is null, according to our assumptions, so that

$$E = 2\pi \int_{-\infty}^{+\infty} dz \int_0^{+\infty} drr \left( v_\phi^2 \left[ (\partial_r f(r))^2 + n^2 \frac{f(r)^2}{r^2} \right] + \lambda_\phi v_\phi^2 (f(r)^2 - 1)^2 \right) \tag{3.13}$$

We notice that our theory has an intrinsic length scale $r_c = (\sqrt{\lambda_\phi} v_\phi)^{-1}$, which can be used to define the location of the boundary separating regions where $f(r) = 0$ (the string core) and $f(r) = 1$ [42]. In our calculations, we will assume for simplicity a step-function form of $f(r) = \theta(r - r_c)$. Moreover, it is quite natural to introduce an ultraviolet cut-off $L$ in the previous integral, which otherwise would diverge: this $L$ can be considered as the distance between neighbouring strings, which is expected to be, from the dynamic of Kibble mechanism, of order of the horizon scale. Thus, we can write down

$$\begin{aligned} E &= 2\pi \int_{-\infty}^{+\infty} dz \left( \int_0^{r_c} drr \left[ v_\phi^2 (\partial_r f(r))^2 + \lambda_\phi v_\phi^2 \right] + \int_{r_c}^{L} drr v_\phi^2 \left[ (\partial_r f(r))^2 + n^2 \frac{1}{r^2} \right] \right) = \\ &\approx \int_{-\infty}^{+\infty} dz \left( \pi v_\phi^2 + 2\pi n^2 v_\phi^2 \log\left( \frac{L}{r_c} \right) \right) \end{aligned} \tag{3.14}$$

and, hence, we can define the linear energy density of the string $\mu_s$ as

$$\mu_s \approx 2\pi n^2 v_\phi^2 \log\left( \frac{L}{r_c} \right) \tag{3.15}$$

where in general the contribution of the core is neglected with respect to the second one. These string configurations are extremely related to axion field: as described in [38], string loops can for instance recollapse or wiggle, giving rise to processes which radiate axion particles. It was shown that this new non-thermal mechanism creates axions with such an energy that, because of the universe expansion and the subsequent switching on of QCD potential, become non-relativistic pretty quickly.

### 3.2.2 The domain walls

When the temperature drops to the energy $\Lambda_{QCD}$, the string decay mechanism is inhibited, because a new process plays the game. Indeed, at this epoch, pions arise, generating an effective potential for axions of the form (2.72). This new term in the Lagrangian will break the $U(1)$ symmetry connecting degenerate vacua, but we still have a residual symmetry. As a matter of fact, our axion potential is periodic in the variable $a/f_a$: its minimum is reached for

$$\langle \frac{a}{2f_a} \rangle = 0 + m\pi \quad \text{with} \quad m \in \mathbb{Z} \tag{3.16}$$

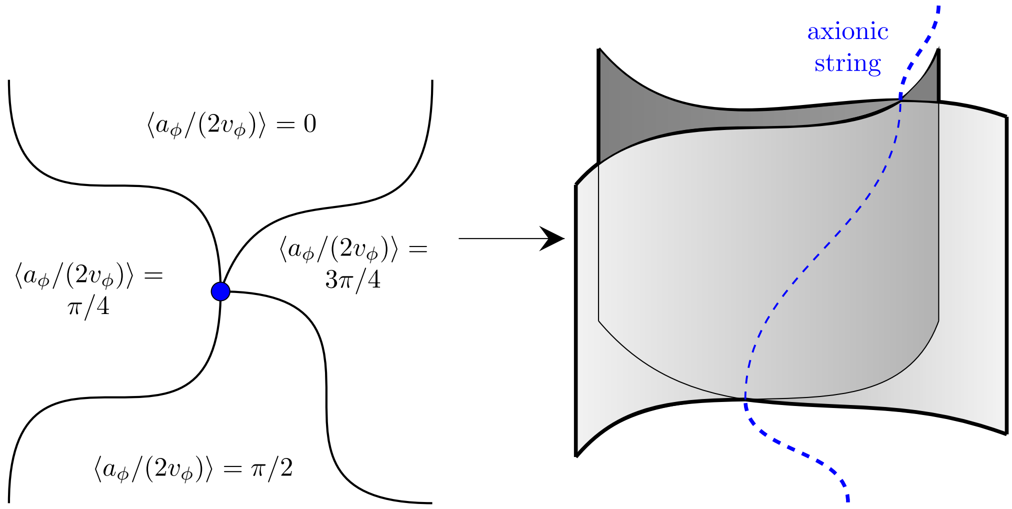



Figure 3.2: A pictorial representation of a string-wall network in case of $\mathcal{N}_{DW} = 4$: it can be appreciated how an axionic string is the boundary of four walls, which separate regions of different vacua.

so that it seems there is only one possible VEV for the axion ($a/f_a$ is an angle and, thereby, $(a/f_a) \in [0, 2\pi[$). But, here, we need to introduce the $v_\phi$ scale through the relation (3.3), which gives

$$V\left(\frac{a_\phi}{f_a}\right) \longmapsto V\left(\frac{a_\phi}{v_\phi}\right) = -f_\pi^2 m_\pi^2 \sqrt{1 - \frac{4m_u m_d}{(m_u + m_d)^2} \sin^2\left(\mathcal{N}_{DW} \frac{a_\phi}{2v_\phi}\right)} \tag{3.17}$$

The new angular variable $a/(2v_\phi)$ will minimize $V(a_\phi/v_\phi)$ for a collection of vacua

$$\langle \frac{a_\phi}{2v_\phi} \rangle = 0, \frac{\pi}{\mathcal{N}_{DW}}, \ldots, \frac{\pi(\mathcal{N}_{DW} - 1)}{\mathcal{N}_{DW}} \qquad \in [0, \pi[ \tag{3.18}$$

given exactly by $\mathcal{N}_{DW}$ values. What emerges is that the original set of $U(1)$ symmetric vacua is broken by quantum effects to a more restrict set of $\mathbb{Z}_{\mathcal{N}_{DW}}$ symmetric ones. The original vacuum field configuration, characterized by a particular value of $a_\phi/(2v_\phi)$, will now roll down towards one of the new $\mathcal{N}_{DW}$ vacua. The field configuration which interpolates two casually unrelated regions is dubbed a *domain wall*. These walls will attach to preexisting strings, which, in turn, will be surrounded by $\mathcal{N}_{DW}$ walls (figure 3.2).

The new Lagrangian describing the axion will now appear as

$$\mathcal{L}_a = \frac{1}{2}\partial_\mu a_\phi \partial^\mu a_\phi - \left(V\left(\frac{a_\phi}{f_a}\right) - V(0)\right) \tag{3.19}$$

where we have subtracted to the potential the energy of the minimum, because, as motivated in [37], it is customary to assume, as a solution to the cosmological constant problem, a null axion vacuum energy. An explicit analytical form of the domain wall field configuration can be very difficult to obtain in this specific case, but, nonetheless, we can derive some important properties of it in a simplified situation. If we consider a wall which is located along the plane $z = 0$ and which mediates between two different vacuum values, in such a way that, for instance

$$\begin{cases} a_\phi/v_\phi = 0 & \text{when} \quad z \to -\infty \\ a_\phi/v_\phi = 2\pi/\mathcal{N}_{DW} & \text{when} \quad z \to +\infty \end{cases} \tag{3.20}$$

we can simply extract the form of the energy from the previous Lagrangian. We are looking again for finite energy solutions, which have no dependence on time. Moreover, using the symmetry of this simplified problem, we consider spatial variations of the field solely along the direction perpendicular to the wall itself. With some manipulations we can write down

$$\begin{aligned} E &= \int d^3x \left[\frac{1}{2}\partial_z a_\phi \partial^z a_\phi + \left(V\left(\frac{a_\phi}{f_a}\right) - V(0)\right)\right] = \\ &= \int d^3x \frac{1}{2}\left(\partial_z a_\phi - \sqrt{2\left(V\left(\frac{a_\phi}{f_a}\right) - V(0)\right)}\right)^2 + \int d^3x \sqrt{2\left(V\left(\frac{a_\phi}{f_a}\right) - V(0)\right)}\partial_z a_\phi \end{aligned} \tag{3.21}$$

where, from the first integral, we can get the equation of motion for the domain wall field

$$\partial_z a_\phi - \sqrt{2\left(V\left(\frac{a_\phi}{f_a}\right) - V(0)\right)} = 0 \tag{3.22}$$

in such a way that we are left with an explicit expression for the energy

$$\begin{aligned} E &= \int dxdy \int_{-\infty}^{+\infty} dz \partial_z a_\phi \sqrt{2\left(V\left(\frac{a_\phi}{f_a}\right) - V(0)\right)} = \\ &= \int dxdy \int_0^{2\pi/\mathcal{N}_{DW}} da_\phi \sqrt{2f_\pi^2 m_\pi^2 \left[1 - \sqrt{1 - \frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\left(\mathcal{N}_{DW}\frac{a_\phi}{2v_\phi}\right)}\right]} = \\ &= \int dxdy \left(2\sqrt{2} f_a f_\pi m_\pi \int_0^\pi d\theta \sqrt{\left[1 - \sqrt{1 - \frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\theta}\right]}\right) \end{aligned} \tag{3.23}$$

From the previous formula, we can readily read off the domain wall surface energy

$$\begin{aligned} \sigma_W &= 2\sqrt{2} f_a f_\pi m_\pi \int_0^\pi d\theta \sqrt{\left[1 - \sqrt{1 - \frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\theta}\right]} = \\ &\approx 2\sqrt{2}\frac{1+\tilde{z}}{\sqrt{\tilde{z}}} m_a f_a^2 \int_0^\pi d\theta \sqrt{\left[1 - \sqrt{1 - \frac{4\tilde{z}}{(1+\tilde{z})^2}\sin^2\theta}\right]} \approx 8.96 m_a f_a^2 \end{aligned} \tag{3.24}$$

where we set $\tilde{z} \equiv m_u/m_d \approx 0.48$. In the next-to-last passage, we made use of (2.74), that gives the mass of the axion in terms of the pion's one. But, in a cosmological framework, we should be careful in noticing how that relation has been obtained using chiral perturbation theory: at low energy, the effective degrees of freedom which interact with axions are described by mesons. Thereby, this formula is valid only for $T < T_{QCD}$. For higher temperature, QCD couplings will be suppressed by the phenomenon of asymptotic freedom, where $g \to 0$ and perturbation theory holds. In this regime, there will be no effective potential $V(a_\phi/f_a)$ and, consequently, $m_a \to 0$, as expected for a Nambu-Goldstone boson. Nonetheless, while approaching $T_{QCD}$ from higher temperatures, before pions appear at the confinement energy, the degeneracy between the $U(1)_{PQ}$ vacua will start faltering because of instantons interactions: after all, we know that these ones are non-perturbative phenomena, whose effects are relevant just when $g \gg 1$. Thus, the anomalous constribution $\sim a_\phi tr[\mathcal{G}_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}]$ will generate an effective periodic potential through the axion-gluon triangle vertex. This discussion as a whole lets us understand how the axion mass depends on temperature and, consequently, on time, due to the modification of the effective potential, which, in its turn, is sensitive to the active degrees of freedom. This idea is summarized in the relation

$$m_a^2(T) = \left.\frac{\delta^2 \mathcal{V}(a_\phi/f_a, T)}{\delta^2 a_\phi}\right|_{a_\phi=0} = \frac{1}{f_a^2}\left.\frac{d^2\mathcal{V}(\theta, T)}{d^2\theta}\right|_{\theta=0} \equiv \frac{1}{f_a^2}\chi_{top}(T) \tag{3.25}$$

where $\mathcal{V}(a_\phi/f_a, T)$ is the effective potential, which weights up quantum corrections, too. $\chi_{top}(T)$ is instead the so-called *topological susceptibility*, which encloses temperature dependence.

Thereby, up to now, we have a universe filled with a string-wall network: differently from string configuration, where strings tend to decay by collapsing or intersecting each other, this new network where strings are the edges of walls turns out to be pretty stable. If that were the case and a QCD transition had really populated the cosmos of these topological structures, it is easy to realize that domain walls would have led to a cosmological disaster. Indeed, because of Kibble mechanism, we can estimate that there will be at least one domain wall per casual horizon, whose dimensions can be parametrized by the age of the universe $t$. The contribution to the energy density coming from domain walls will be given by $\rho_W(t) = \sigma_W/t$, so that today we would have had

$$\rho_W(t_0) = \frac{\sigma_W}{t_0} \gtrsim 2 \times 10^{-14}\frac{g}{cm^3}\left(\frac{f_a}{10^{12}GeV}\right) \tag{3.26}$$

where the age of the universe is $t_0 \approx 14Gyr$ (and $1Gyr = 3.16 \times 10^{16}s$ before present). It worth noticing that $\rho_W(t_0) \gg \rho_{\text{crit}} = 1.9h^2 10^{-29} g/cm^3$, where $\rho_{\text{crit}}$ is the well-known value of the energy density which separates hyperbolical from spherical universe geometry. $h \approx 0.7$ is related to the Hubble parameter. This means that the existence of a string-wall network would have quickly dominated the universe dynamic and would have led, nowadays, to a close universe: this is definetely inconsistent with cosmological measurements, which suggest the hypothesis of a flat cosmos.

## 3.3 Possible ways out

As well summarized in [38], there are three possible solutions to the domain wall conundrum:

- the inflationary solution;
- the introducion of a small explicit PQ breaking contribution;
- the $\mathcal{N}_{DW} = 1$ scenario.

The first one clearly relies on inflation. In a similar fashion as for the solution to the high monopole energy density, emerging from the alleged GUT phase transition, even here inflation can save us from this anomalous concentration of topological defects. Indeed, if the PQ phase transition occurred before the inflationary period, the axionic strings could be watered down by a violent phase of exponential expansion of the universe. The axion field itself would be homogenized on large scales, so that the only contribution to its mode expasion would be given by zero modes. Around the QCD temperature, the axion quanta, which are associated to oscillations of the field, will acquire a mass: nevertheless, this zero mode axion field will produce particles which are practically at rest (i.e as cold dark matter). Moreover, just because the phase of the $\Upsilon$ field will be initially homogeneous on large regions, the formation of domain walls will be suppressed.

A second less appealing solution is based on the idea that a small explicit breaking of the residual $\mathbb{Z}_{\mathcal{N}_{DW}}$ symmetry could lift the degeneracy of the $\mathcal{N}_{DW}$ vacua and lead to the decay of wall structures. This $\mathbb{Z}_{\mathcal{N}_{DW}}$ breaking term will clearly affect the original PQ symmetry: if we still

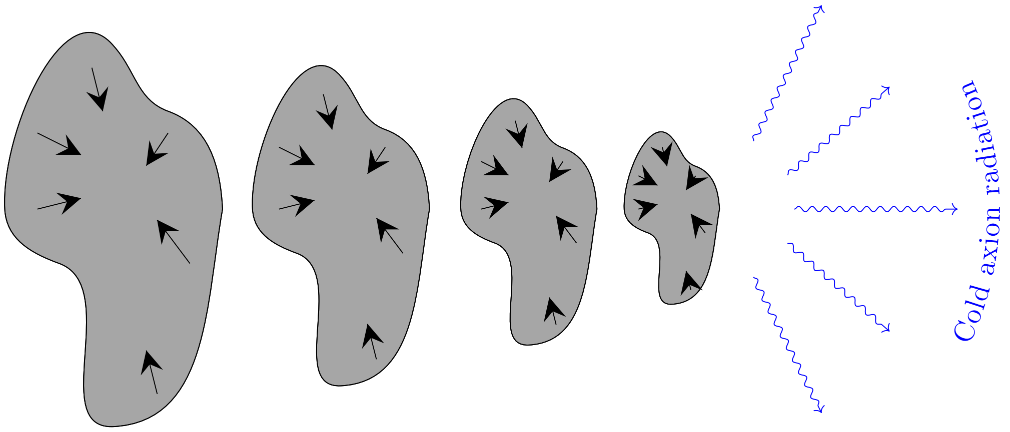



Figure 3.3: With $\mathcal{N}_{DW} = 1$ the transition below the $T_{QCD}$ temperature will generate a string-wall network made of membranes, which quickly collapse under their own surface tension. The process is expected to end up with a cold axion radiation emission.

want the PQ mechanism to work, we must require this correction to be extremely small. On the other hand, the symmetry break can not be too tiny, because the unique true vacuum should be able to take over before domain walls dominate the energy density of the universe. This set of constraints grossly limits the room in parameter space for this mechanism to appear and, even if it is completely legitimate, it does not seem one of the most attractive ways out.

A third possibility is instead to consider PQ models whose domain wall number is equal to one. In this case, it will not be necessary to require the existence of an inflationary phase after the PQ transition: this cosmological period can take place before it or can even be absent. The reasons why a $\mathcal{N}_{DW} = 1$ can save us from the domain wall catastrophe can be easily understood. First of all, it is noteworthy that, even if only a unique vacuum existed, domain walls could arise. In fact, before breaking the degeneracy of the circle of vacua, we will unavoidably have the formation of strings: around these chains of false vacua, the $S$ phase can wrap with a non-trivial winding number, exactly as before. When QCD potential switches on, field configurations characterized by $a/f_a = \pi$ could move towards the minimum using two directions: decreasing their phase towards $a/f_a = 0$ or trying to reach $a/f_a = 2\pi$, because these two values are physically identical. Therefore, a wall will interpolate field patches whose phase difference is greater than $2\pi$ nearby it and that quickly connect to the surrounding true vacuum region. Our rule of thumb will say that any string will be the edge of only one wall. Thereby, our network system made of strings and walls will degenerate into a collection of membranes and strips. We can evalutate the stability of this system by calculating the ratio between the energy in the wall surface and that stored in the boundary. The string energy will be given by $\mu_s t$, where, again, $t$ parametrizes the typical size of the string by means of the horizon scale; the wall energy will instead goes as $\sigma_W t^2$. Given that, one gets

$$R = \frac{E_{\text{wall}}}{E_{\text{string}}} = \frac{\sigma_W t}{\mu_s} \tag{3.27}$$

This relation tells us that, after a short period of time, where the string energy is greater than the wall one, the wall surface energy will dominate the dynamic of the membrane, which will collapse under its own tension. Exactly as for string decays, it is generally assumed that these *short-lived networks* are sources of non-relativistic axions. Of course, this reasoning can not be applied for $\mathcal{N}_{DW} \geq 2$.

This brief description of the role of axions in cosmology made it possible to understand how important it can be to select models respecting the condition $\mathcal{N}_{DW} = 1$. It will be explicitly computed in the following chapter that our DFSZ models do not respect this constraint. By using the formula $\mathcal{N}_{DW} = -N/X_\phi$, where $N$ will turn out to be 3 and $X_\phi$ equals $-1$ or $-1/2$ for the cubic or quartic potential respectively, one gets $\mathcal{N}_{DW}^{cub} = 3$ or $\mathcal{N}_{DW}^{quart} = 6$. The cubic model seems to move towards the direction of a possible improvement of the domain wall problem, that, nonetheless, is still present. One can try to look for different invisible axion models, as the KSVZ, which can be proved to have exactly the desired domain wall number. But as stressed in [43], this theory, where an extra heavy singlet quark is added, has the cosmological drawback of leaving a stable relic population of this heavy species, which can not decay, because not coupled to the SM sector. Of course, plenty of solutions can be explored and other viable models can be found. But if we insist on retaining a DFSZ scenario, we will show in the next section how relaxing some implicit conditions in these theories can also have, as a side effect, the resolution of the domain wall issue.

# Chapter 4

# The flavour non-universal DFSZ axion

The Yukawa interaction terms have been put aside, up to now, because they do not enter the discussion of the scalar mass spectrum and of the domain wall problem. Nevertheless, they are of capital importance in order to restrict possible axion properties, while comparing them with phenomenology. One of the most robust result of old DFSZ models was believed to be the prediction of non-null couplings of axion to nucleons, as a consequence of interaction terms with quarks. What will be proved in this chapter is that these non-vanishing axion-nucleon interactions are the outcome of assuming a flavour universal PQ charge scheme. Dropping out this hypothesis, a great amount of possible models arises, some of which enjoy the *nucleophobia* property, that makes it possible to relax some astrophysical constraints on axion mass.

## 4.1 Universal PQ charge models

To better appreciate the novelty coming from replacing the old models (where axions couple with the same strength to all fermion sectors), with the new ones, it will prove to be useful to briefly introduce the less general previous cases. Pointing out that the only difference between the two classes of models resides in the Yukawa terms, we can write down them simply as

$$\mathcal{L}_{Yuk} = y_f^u \bar{q}_{Lf} \tilde{\phi}_1 u_{Rf} + y_f^d \bar{q}_{Lf} \phi_2 d_{Rf} + \begin{cases} y_f^e \bar{l}_{Lf} \phi_1 e_{Rf} & \text{DFSZ II} \\ y_f^e \bar{l}_{Lf} \phi_2 e_{Rf} & \text{DFSZ I} \end{cases} \tag{4.1}$$

where $f$ is an index running over the three fermion sectors and with $y^u$, $y^d$ and $y^e$ referring to the nine different Yukawa couplings of the SM. Clearly, $q_L$ and $l_L$ are isospin doublets and $u_R$, $d_R$ and $e_R$ are isospin singlets, while $\tilde{\phi}_1 = i\sigma_2\phi_1$ is used to give mass to the correct component of the doublet. We finally notice that in (4.1) the possibility of having two different DFSZ versions has been highlighted, according to whether lepton masses arise from the first or the second Higgs doublet. If we want to ensure PQ invariance, our fermions should change accordingly under a PQ transformation. We remind that:

$$\phi_1' = e^{iX_1\theta}\phi_1 \qquad \phi_2' = e^{iX_2\theta}\phi_2 \qquad \phi' = e^{iX_\phi\theta}\phi \tag{4.2}$$

and we can further assume that only right fermions are PQ charged (we will extensively discuss later why this does not mean losing generality):

$$q_L' = q_L \qquad l_L' = l_L \qquad u_R' = e^{iX_u\theta}u_R \qquad d_R' = e^{iX_d\theta}d_R \qquad e_R' = e^{iX_e\theta}e_R \tag{4.3}$$

so that we get the following conditions:

$$X_u = X_1 \qquad X_d = -X_2 \qquad \begin{cases} X_e = -X_1 & \text{DFSZ II} \\ X_e = -X_2 & \text{DFSZ I} \end{cases} \tag{4.4}$$

Now it is important to go back to the final parametrization we derived in the second chapter (2.53), that, without explicitly reporting the goldstone boson fields, can be read as

$$\begin{aligned} \phi_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} c_\beta H^+ \\ (v+H)c_\beta - s_\beta \Big( S + i\frac{2v_\phi A_0}{v_f} \Big) \end{pmatrix} exp\Big\{ \frac{-2is_\beta^2 a_\phi}{v_f} \Big\} \\ \phi_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} s_\beta H^+ \\ (v+H)s_\beta + c_\beta \Big( S + i\frac{2v_\phi A_0}{v_f} \Big) \end{pmatrix} exp\Big\{ \frac{2ic_\beta^2 a_\phi}{v_f} \Big\} \end{aligned} \tag{4.5}$$

for the Higgs fields that play a role in Yukawa interactions. We remind that $v_f = \sqrt{4X_\phi^2 v_\phi^2 + v^2 s_{2\beta}}$, as usual. The phase containing the axion will appear in (4.1): this is precisely the forerunner of a possible complex phase for the mass term, that we already had to cope with. It is just this imaginary exponential which gives rise to axion-fermions interactions. Indeed, we can act with a local PQ transformation over our fermions, as in (4.3), to remove Higgs phases. The only thing to keep in mind is that now $\theta$ will be replaced by the local axion field, that is, explicitly

$$u_R' = e^{-2is_\beta^2 a(x)/v_f} u_R \qquad d_R' = e^{-2ic_\beta^2 a(x)/v_f} d_R \qquad \begin{cases} e_R' = e^{-2ic_\beta^2 a(x)/v_f} e_R & \text{DFSZ I} \\ e_R' = e^{+2is_\beta^2 a(x)/v_f} e_R & \text{DFSZ II} \end{cases} \tag{4.6}$$

This is a chiral transformation, which will generate some anomalous contributions, in a way that we have already seen and that we will better explore for more general models. Here we just want to derive the peculiar form of the fermion couplings. To see that, all we have to do is to consider how a local PQ transformation will not leave unchanged the fermion kinetic part of the Lagrangian, because the customary partial derivative will act on the exponential. Therefore, by removing the axion phase from Yukawa terms, we introduce a further contribution, which represents the aforementioned fermion interaction:

$$\mathcal{L}_{fermion-axion} = 2s_\beta^2 \frac{\partial_\mu a}{v_f} \bar{u}_{Rf} \gamma^\mu u_{Rf} + 2c_\beta^2 \frac{\partial_\mu a}{v_f} \bar{d}_{Rf} \gamma^\mu d_{Rf} + \begin{cases} 2c_\beta^2 \frac{\partial_\mu a}{v_f} \bar{e}_{Rf} \gamma^\mu e_{Rf} & \text{DFSZ I} \\ -2s_\beta^2 \frac{\partial_\mu a}{v_f} \bar{e}_{Rf} \gamma^\mu e_{Rf} & \text{DFSZ II} \end{cases} \tag{4.7}$$

where, for clear reasons, only right fermions are involved. Using the up sector as an example, we can develop our expression even more as:

$$2s_\beta^2 \frac{\partial_\mu a}{v_f} \bar{u}_{Rf} \gamma^\mu u_{Rf} = 2s_\beta^2 \frac{\partial_\mu a}{v_f} \bar{u}_f \gamma^\mu \frac{1+\gamma^5}{2} u_f = 2s_\beta^2 \frac{\partial_\mu a}{2v_f} \bar{u}_f \gamma^\mu \gamma^5 u_f \tag{4.8}$$

where the vectorial part of the fermion current never contributes, because it can be eliminated using the Dirac equation

$$2s_\beta^2 \frac{\partial_\mu a}{2v_f} \bar{u}_f \gamma^\mu u_f = -2s_\beta^2 \frac{a}{2v_f} (\partial_\mu \bar{u}_f \gamma^\mu u_f + \bar{u}_f \gamma^\mu \partial_\mu u_f) = -2is_\beta^2 \frac{a}{2v_f} (m_f \bar{u}_f u_f - m_f \bar{u}_f u_f) = 0 \tag{4.9}$$

in which the first equality holds up to boundary terms. To extract the correct coupling constants, we refer to the usual normalization, which employs the axion decay constant $f_a$

$$\mathcal{L}_{fermion-axion} = C_\psi \frac{\partial_\mu a}{2f_a} \bar{\psi} \gamma^\mu \gamma^5 \psi \tag{4.10}$$

and using the relation $f_a = v_f/2N$ of (3.3), we finally end up with

$$C_u = s_\beta^2/3 \qquad C_d = c_\beta^2/3 \qquad \begin{cases} C_e = c_\beta^2/3 & \text{DFSZ I} \\ C_e = -s_\beta^2/3 & \text{DFSZ II} \end{cases} \tag{4.11}$$

where the color anomaly value $N = 3$ (a result which will be proved in the next paragraph) has been made explicit.

## 4.2 The photon coupling

As already shown, one of the corner stone of axion models is the effective axion-gluon coupling: indeed, while acting with (4.6), we will generate an anomalous color term. Considering again as a starting point the original DFSZ theory, the latter can be derived by using (1.69), (1.70) and (1.71) so that

$$\mathcal{L}_{axion-gluon} = \frac{g_s^2}{32\pi^2}\frac{2a_\phi}{v_f} d^{ab}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}_b^{\mu\nu} \tag{4.12}$$

In particular, being the trace over different flavours, if we detone by $\mathcal{X}^q_{L/R}$ the diagonal matrix containing quark PQ charges, we can write

$$d^{ab} = tr[\mathcal{X}^q_L\{\lambda^a,\lambda^b\}] - tr[\mathcal{X}^q_R\{\lambda^a,\lambda^b\}] \tag{4.13}$$

where we used the fact that right and left fermions contribute with opposite signs. Just the relative sign is meaningful, here: a global change of sign will not have spill-over effects on physics, because the effective QCD potential is an even function of the axion field. Moreover, leptons do not couple to gluons and, so, they do not appear in the calculation. By using the already quoted $\{\lambda_a,\lambda_b\} = (1/3)\delta_{ab} + d_{abc}\lambda_c$ and summing up over the three color hues, we get

$$d^{ab} = \frac{\delta^{ab}}{3}\sum_{colors}\left(\sum_{f_L} X_{Lf} - \sum_{f_R} X_{Rf}\right) = \frac{\delta^{ab}}{3}\sum_{colors}\left(0 - 3(-s_\beta^2 - c_\beta^2)\right) = 3\delta^{ab} \tag{4.14}$$

where the factor three is the so-called *color anomaly coefficient* $N$. In the previous expression, the $f_L$ index runs over the three quark generations, while $f_R$ over the six quark singlets.

In a similar fashion, even if without any relation to the solution of the strong CP problem, other kinds of anomalies can be produced, according to whether $d^{abc}$ of (1.71) will vanish or not. Thereby, it is legitimate to believe that axions can present an effective interaction vertex with photons and $SU(2)_L$ field strength tensors. To start with, let us consider the case of electromagnetic interactions. By making use of our previous formula, we can derive for the DFSZ model

$$\begin{aligned} d &= 2\sum_{colors}\left(\sum_{f_L} X_{Lf}Q^2_{Lf} - \sum_{f_R} X_{Rf}Q^2_{Rf}\right) + 2\left(\sum_{l_L} X_{Ll}Q^2_{Ll} - \sum_{l_R} X_{Rl}Q^2_{Rl}\right) = \\ &= 6\left(3s_\beta^2\left(\frac{2}{3}\right)^2 + 3c_\beta^2\left(-\frac{1}{3}\right)^2 + \begin{cases} (-1)^2c_\beta^2 & \text{DFSZ II} \\ -(-1)^2s_\beta^2 & \text{DFSZ I} \end{cases}\right) = \begin{cases} 8 & \text{DFSZ I} \\ 2 & \text{DFSZ II} \end{cases} \end{aligned} \tag{4.15}$$

in which the second part accounts for leptonic contributions (again $l_L$ runs over the three lepton generations, but $l_R$ only over the three charged lepton singlets, because right-handed neutrinos, if existing, are not electrically charged). The results obtained are dubbed *electromagnetic anomaly coefficients* $E$. Therefore, until now, the Lagrangian containing the anomalies will happen to be

$$\mathcal{L}_{anomaly} = \frac{\alpha_s}{8\pi}\frac{a_\phi}{f_a}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}_b^{\mu\nu} + \frac{\alpha}{8\pi}\frac{E}{N}\frac{a_\phi}{f_a}\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} \tag{4.16}$$

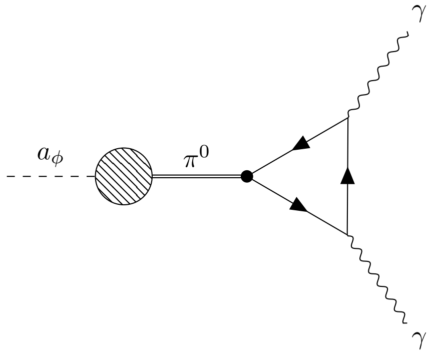



Figure 4.1: A diagrammatic representation of the axion-pion mixing. Because of that, the decay $\pi^0 \to 2\gamma$ (which is the major neutral pion decay channel, occuring with a fraction of $(98.823 \pm 0.034)\%$ [44]) generates an additional model-independent contribution to the axion coupling to photons.

where we have explicitly made use of $N = 3$ and, again, of $f_a = v_f/2N$.

All of the axion strong and electromagnetic effective interactions should be included in the just written terms. But here we have to proceed cautiously, because it is necessary to remember how the axion fate is severely connected to that of pions. If we consider again the effective Lagrangian (2.66), by expanding the $M_a$ matrix for large $f_a$, we get

$$\mathcal{L}_{\text{mass}} = v_\psi^3 tr[\Sigma M_a^\dagger + h.c] = v_\psi^3 tr\left[\Sigma\left(M_q - \frac{ia_\phi}{f_a} + \dots\right)\right] \tag{4.17}$$

so that a mass mixing between axion and $\pi^0$ emerges, when the linear $a_\phi$ term multiplies the linear pion one stemming from the $\Sigma$ expansion. This remark did not affect the calculation of axion mass, for which pion degrees of freedom had been integrated out, but now it has important consequences. Indeed, if a mixing exists, it means we are not dealing with the correct physical fields and, hence, in principle, our axion field could turn into a pion while propagating. Because we know that a $\pi^0$ can decay into a couple of photons, we have an extra contribution even to the axion-photon coupling. To assess it, we can try to eliminate at leading order the axion-pion mixing through a different quark field definition. We can generalize the transformation (2.64) into

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \longmapsto e^{i\gamma_5 \frac{a_\phi}{2f_a} Q_a} \begin{pmatrix} u \\ d \end{pmatrix} \tag{4.18}$$

with $Q_a$ a two-by-two matrix, which replaces the identity. If we choose $trQ_a = 1$, we have the correct multiplicative factor which exactly cancels the gluon anomaly. From now on, we will never consider it again in a low energy regime. This second chiral transformation acting only on up and down quarks will modify the axion electromagnetic interaction as

$$\mathcal{L}_{anomaly} = \frac{\alpha}{8\pi}\left(\frac{E}{N} - 6tr[Q_a Q_q^2]\right)\frac{a_\phi}{f_a}\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} \tag{4.19}$$

where

$$Q_q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} \tag{4.20}$$

contains the quark charges. If we now move back to the expansion of (2.66) in this new general case, where $M_a = e^{i\frac{a}{2f_a}Q_a} M_q e^{i\frac{a}{2f_a}Q_a}$, one gets

$$\mathcal{L}_{\text{mass}} = v_\psi^3 tr\left[\Sigma\left(M_q - \frac{ia_\phi}{2f_a}\{Q_a, M_q\} + \dots\right)\right] \tag{4.21}$$

so that to remove the mixing, we just need to impose $Q_a = M_q^{-1}/tr[M_q^{-1}]$. This explicit redefinition of quark fields lets us compute a complete form of the electromagnetic anomaly

$$\mathcal{L}_{anomaly} = \frac{g^2}{32\pi^2}\left(\frac{E}{N} - \frac{2}{3}\frac{4m_d + m_u}{m_d + m_u}\right)\frac{a_\phi}{f_a}\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} \tag{4.22}$$

It is noteworthy that the first term $E/N$ is model dependent, because directly calculated through the PQ charges of the theory; the second one is universal and, substituing the values of quark masses at $2GeV$, gives $C_\gamma = E/N - 1.92(4)$ (where the last digit is affected by an uncertainty arising from the determination of the quak masses ratio).

If we now try to repeat the same passages for the $SU(2)_L$ gauge symmetry, we immediately realize that $d^{abc} = 0$: indeed, only left fermions can couple to the weak force and, just because in our model they are not endowed with a PQ charge, no anomaly coefficient will arise. But let us consider a bit better the role of the weak interaction in the anomaly scenario.

### 4.2.1 The electroweak parameter $\theta_{EW}$

We saw that a supposed $\theta$ electromagnetic angle can not exist, because the term $\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu}$ is a total derivative and can be removed from the Lagrangian without any topological complication: $U(1)_{em}$, exactly as $U(1)_Y$, is an abelian gauge symmetry. The same reasoning does not apply to the rest of SM local symmetries, so that for the strong force we introduced a $\theta_{QCD}$ parameter, that is at the origin of all this work. It can not be removed by means of any transformation, because, as noticed in [45], it does not exist any current whose divergence contains only the strong anomaly with no further contribution which moves the effect of the $\theta_{QCD}$ somewhere else in the Lagrangian.

For the $SU(2)_L$ anomaly this is not true. If we consider the global baryon and lepton transformations

$$Q \longmapsto e^{iB\alpha_b}Q \qquad l \longmapsto e^{iL\alpha_L}l \tag{4.23}$$

we notice that they are classical symmetries of the theory. Nevertheless, they turn out to be anomalous. Defining $B$ as the baryon number (1/3 for baryons and zero for leptons) and $L$ as the lepton number (one for leptons and zero for baryons), we have

$$\begin{aligned}
&tr[B\{\lambda^a, \lambda^b\}] = tr[L\{\lambda^a, \lambda^b\}] = 0\\
&tr[B\{\tau^a, \tau^b\}] = \frac{\delta^{ab}}{2}B\sum_{colors}\left(\sum_{\text{left quarks}} 1\right) = 3\delta^{ab}\\
&tr[L\{\tau^a, \tau^b\}] = \frac{\delta^{ab}}{2}L\sum_{\text{left leptons}} 1 = 3\delta^{ab}\\
&tr\left[B\left(\frac{2Y^2}{4}\right)\right] = \frac{B}{2}\left[18\left(\frac{1}{3}\right)^2 - 9\left(\frac{4}{3}\right)^2 - 9\left(-\frac{2}{3}\right)^2\right] = -3\\
&tr\left[L\left(\frac{2Y^2}{4}\right)\right] = \frac{L}{2}[6(-1)^2 - 3(-2)^2] = -3
\end{aligned} \tag{4.24}$$

and, hence, the corresponding currents are not conserved at quantum level. We underscore that the extra factor 1/4 in the calculation of the anomaly for the abelian hypercharge symmetry comes from the usual parametrization of these transformations, where the hypercharge number always appears multiplied by a factor one-half as $Y/2$. Upon a baryon and lepton transformation, the weak anomalous Lagrangian will be

$$\mathcal{L}_{\theta_{EW}} = \frac{\alpha_W}{8\pi}(\theta_{EW} - 3\alpha_B - 3\alpha_L)\mathcal{W}^a_{\mu\nu}\tilde{\mathcal{W}}^{\mu\nu}_a \tag{4.25}$$

If we set $\alpha_B+\alpha_L = \theta_{EW}/3$, the electroweak vacuum angle disappears from the theory, which means it is not physical. Despite that, it is noteworthy that the anomalies (4.24) obtained for baryonic and leptonic symmetries exactly cancel each other: thus, a real symmetry of the Lagrangian, even at quantum level, will be the so-called *B-L symmetry*.

## 4.3 The coupling to matter

We have already explored on different occasions the importance of dealing with effective theories, where only the active degrees of freedom are considered and all the heavier ones are integrated out. To describe some situations of practical value, we will definetely need to use them.

But first of all, we have to rewrite the quark part of the Lagrangian in the new quark basis, given by (4.18), where the gluon anomaly disappears and the axion-pion mixing is suppressed. This can be easily achieved by considering that this phase transformation will provide a new contribution to Yukawa terms by means of the derivative in the kinetic part. Starting from (4.10), where we refer to $c_q^{UV}$ as the ultraviolet axion-quark couplings at high energy scale (e.g $f_a \sim 10^{12}$), we can write

$$\begin{aligned}\mathcal{L}_{axion-quarks} &= \frac{\partial_\mu a_\phi}{2f_a}\sum_{u,d,s,c,b,t} c_q^{UV}\bar{q}\gamma^\mu\gamma^5 q - \sum_{u,d,s,c,b,t}(\bar{q}_R M_q q_L + h.c.)\\ \longmapsto \quad & \frac{\partial_\mu a_\phi}{2f_a}\sum_{u,d}[c_q^{UV}\bar{q}\gamma^\mu\gamma^5 q - \bar{q}\gamma^\mu\gamma^5(Q_a)_q q] + \frac{\partial_\mu a_\phi}{2f_a}\sum_{s,c,b,t} c_q^{UV}\bar{q}\gamma^\mu\gamma^5 q - \sum_{u,d,s,c,b,t}(\bar{q}_R M_a q_L + h.c.)\end{aligned} \tag{4.26}$$

where the axion non-derivative coupling through the mass term $M_a$ has been reported, too, for sake of completeness. The $Q_a$ matrix contains the infrared correction to up and down quark couplings: $(Q_a)_u = c_u^{IR} = 1/(1+z)$ and $(Q_a)_d = c_d^{IR} = z/(1+z)$, with $z = m_u/m_d$. Therefore, we can redefine $\tilde{c}^{UV}_{u/d} = c^{UV}_{u/d} - c^{IR}_{u/d}$.

### 4.3.1 Pionphobia

As described in the previous chapter, the leading thermalization processes of axions for $T > T_\pi$ are $\pi + \pi \leftrightarrow \pi + a_\phi$ and $\pi + n \leftrightarrow n + a_\phi$ (where, here, $n$ stands for a general nucleon). These are related to the production of possible hot dark matter axions. To have an effective theory involving both nucleons and pions, we can not avoid to include all meson states within a mass range running from the pion mass to that of neutrons. In so doing, the approximate symmetry group that should guide us in building the effective Lagrangian will be the $SU(3)$ flavour group. This symmetry is also mirrored in the baryon sector, where the lightest states are described by an octet whose members are collected in the matrix

$$\tilde{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{\Sigma}^0 + \frac{\tilde{\Lambda}^0}{\sqrt{3}} & \sqrt{2}\tilde{\Sigma}^+ & \sqrt{2}\tilde{p} \\ \sqrt{2}\tilde{\Sigma}^- & -\tilde{\Sigma}^0 + \frac{\tilde{\Lambda}^0}{\sqrt{3}} & \sqrt{2}\tilde{n} \\ \sqrt{2}\tilde{\Xi}^- & \sqrt{2}\tilde{\Xi}^0 & -2\frac{\tilde{\Lambda}^0}{\sqrt{3}} \end{pmatrix} \tag{4.27}$$

Nevertheless, before starting our analysis, we wish to stress how these theories will be solely useful to extract some considerations up to tree level calculation. Indeed, it has been realized how a description including all $0^-$ mesons and $\frac{1}{2}^+$ baryons is unstable at loop level: that is ultimately due to superpositions and small energy gaps separating these two $SU(3)$ octets from the rest of QCD spectrum. For instance, between the proton/neutron threshold ($\sim 938MeV$) and the $\Lambda^0$ one ($\sim 1116MeV$) all members of the $1^-$ meson octet thrive, while within the energy gap running from $\Sigma$ ($\sim 1193MeV$) and $\Xi$ ($\sim 1318MeV$) has place the first element of the $\frac{3}{2}^+$ baryon decuplet $\Delta$ ($\sim 1232MeV$). Therefore, we can rely on the following picture just as a model.

When constructing a Lagrangian including baryons, it is much more convenient not to use the matrix (1.26), which transfroms as $\Sigma \longmapsto U_L \Sigma U_R^\dagger$, but its squared root $u$:

$$u^2 = \Sigma \quad \text{so that} \quad u' = \sqrt{U_L \Sigma U_R^\dagger} \equiv U_L u h^{-1} = h u U_R^\dagger \tag{4.28}$$

where $h = h(U_L, U_R, \Sigma) = (U_L \Sigma U_R^\dagger)^{-1/2} U_L \sqrt{\Sigma}$ is named *compensator field*. From the previous formula it can be noticed how $h$ belongs to the unbroken part of $SU(3)_L \times SU(3)_R$ symmetry group: it acts on the right as $h^{-1} = h^\dagger$ and on the left as $h$, in such a way that $h \in SU(3)_V$.

Moreover, a suitable representation of the baryon field $\tilde{B}$ can always be choosen in terms of a $B$ such that $B \longmapsto hBh^{-1}$. As a matter of fact, under a $SU(3)_L \times SU(3)_R$ action, the $\tilde{B}$ matrix will change as $\tilde{B}' = U_R \tilde{B}_R U_R^\dagger + U_L \tilde{B}_L U_L^\dagger$. But we can cleverly define a baryon matrix $B = (uP_R + u^\dagger P_L)\tilde{B}(P_R u^\dagger + P_L u)$, having the desired behavior.

With $u$ and $B$ we have identified a particular non-linear realization of the chiral group symmetry, because the transformation law of $u$ and $B$ involves a matrix $h$ which depends on the meson fields themselves. It can be shown that these two field representations (i.e $(\Sigma, \tilde{B})$ and $(u, B)$) are equivalent, because the physics of the $S$-matrix elements is not affected by their interchange. As a consequence, we can assume the $B$ matrix as our initial object, defined as in (4.27) in terms of entries deprived of their tilde symbols.

We can now introduce some useful objects. The first one is the *chiral connection* $\Gamma^\mu$

$$\Gamma_\mu = \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger) \tag{4.29}$$

which transforms as $\Gamma_\mu \longmapsto h\Gamma_\mu h^{-1} + h\partial h^{-1}$. The chiral connection can be employed to build a covariant derivative for the baryon field with homogeneous transformation properties

$$\mathcal{D}_\mu B \equiv \partial_\mu B + [\Gamma_\mu, B] \qquad \longmapsto \qquad (\mathcal{D}_\mu B)' = h \mathcal{D}_\mu B h^{-1} \tag{4.30}$$

The second important quantity is the *chiral vielbein* $u^\mu$

$$u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) \tag{4.31}$$

which transforms homogeneously, too. Using $u^\mu$ we can rephrase the meson kinetic part $tr[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma]$ as $tr[u_\mu u^\mu]$. Of course, all of these expressions involving derivatives of fields should be suitably tweaked, if one needed to reproduce the gauge field couplings: nevertheless, they will not enter our considerations.

Finally, in this non-linear representation, the standard quark mass term $v_\psi^3 tr[M_q \Sigma^\dagger + \Sigma M_q^\dagger]$ (where now $M_q$ is a three-by-three matrix) will be replaced by $tr[\chi_+] = v_\psi^3 tr[u^\dagger M_q u^\dagger + u M_q^\dagger u]$, which again transforms homogeneously.

We are now ready to write a pretty general effective Lagrangian, whose terms are obtained by requiring the presence of $SU(3)$ symmetry (with the exception of the explicit breaking contribution of the mass term) and CP invariance, typical of strong interactions. The last requirement can be taken into account by remembering that mesons are pseudo-scalar states. So, under a parity transformation, they will entail $P\Sigma P^\dagger = \Sigma^\dagger$. That implies $Pu_\mu P^\dagger = -g_{i\mu}u_\mu$ and $P\Gamma_\mu P^\dagger = g_{i\mu}\Gamma_\mu$, where $i$ is a spatial index (the extra minus sign of $g_{ii}$ comes out from the partial derivative change under parity). Therefore, the only possible contributions will be

$$\begin{aligned}\mathcal{L}^{effective}_{\pi-n} =& \frac{f_\pi^2}{4}tr[u_\mu u^\mu] + tr[\chi_+] + tr[\bar{B}(i\gamma^\mu\mathcal{D}_\mu - m_b)B] + \frac{D}{2}tr[\bar{B}\gamma^\mu\gamma^5\{u_\mu, B\}] + \\ &+ \frac{F}{2}tr[\bar{B}\gamma^\mu\gamma^5[u_\mu, B]] + \mathcal{L}^{int}_{\pi-a} + \mathcal{L}^{int}_{\pi-n-a} + \mathcal{L}^{int}_{n-a}\end{aligned} \tag{4.32}$$

where $m_b$ is the bare baryon mass. The $SU(3)$ axial couplings $D \approx 0.80$ and $F \approx 0.46$ can be experimentally determined through semileptonic baryon decays [46]: they fullfil the $SU(2)$ isospin symmetry constraint $F + B = g_A = 1.26$, where $g_A$ is an axial constant that controls the neutron $\beta$-decay (see next subparagraph).

Pretty interestingly, in this $SU(3)$ model the mass term $tr[\chi_+]$ gives rise not only to a pion-axion mass mixing (that we removed at first order), but the axion field can actually blend in with the other neutral pseudoscalar mesons, i.e the $\eta_8$ and $\eta_0$. However, these mixing angles are highly suppressed and do not significantly modify the physics: being the axion mass very tiny, the mixing with the lightest particles will be favoured.

We can now focus on the last three terms, which contain the axion physics: they encode the axion interactions with mesons and nucleons. We will concentrate on $\mathcal{L}^{int}_{n-a}$ in the next subparagraph, where we will develop a more precise effective theory in which only protons and neutrons are produced among QCD bound states. Thus, let us consider only interaction terms that are peculiar of high energy scales and cosmological problems, where meson degrees of freedom can still be excited. We can construct these terms following [47].

### The axion-pion interaction

Regarding $\mathcal{L}^{int}_{\pi-a}$, we should try to couple an external axion source to meson currents. These ones can be obtained differentiating the meson kinetic term of (4.32) with respect to the $SU(3)_A$ axial $\delta_{A_3}\Sigma = i\{\lambda^a, \Sigma\}\delta\theta^a_A$ and $U(1)_A$ axial $\delta_{A_1}\Sigma = 2i\Sigma\delta\theta^a_A$ variations, together with the $SU(3)_V$ vectorial $\delta_{V_3}\Sigma = i[\lambda^a, \Sigma]\delta\theta^a_V$ and $U(1)_V$ vectorial $\delta_{V_1}\Sigma = 2i\Sigma\delta\theta^a_V$ ones. Nevertheless, the last two transformations do not contribute. After all, a parity even meson current can only be associated, for parity reasons, to vectorial external sources, which can not be built with a linear term in the axion field. After these preparatory considerations, we can write down

$$\mathcal{L}^{int}_{\pi-a} = \frac{\partial_\mu a_\phi}{2f_a}c^a \underbrace{\left(\frac{if_\pi^2}{2}tr[\lambda^a(\Sigma\partial^\mu\Sigma^\dagger - \Sigma^\dagger\partial^\mu\Sigma)]\right)}_{\text{axial-multiplet meson current}} + \frac{\partial_\mu a_\phi}{2f_a}c_1 \underbrace{\left(\frac{if_\pi^2}{2}tr[\Sigma\partial^\mu\Sigma^\dagger - \Sigma^\dagger\partial^\mu\Sigma]\right)}_{\text{axial-singlet meson current}} \tag{4.33}$$

where $c^a$ and $c_1$ refer to the dependence on the ultraviolet coupling constants. If we denote by $C$ the diagonal matrix $C = diag(\tilde{c}^{UV}_u, \tilde{c}^{UV}_d, c^{UV}_s)$, we can rewrite it as

$$C = \frac{1}{3}tr[C]\mathbb{1}_{3\times3} + \frac{1}{2}tr[C\lambda_3]\lambda_3 + \frac{1}{2}tr[C\lambda_8]\lambda_8 \tag{4.34}$$

So, what enters expression (4.33) is just the projection $c^a = tr[C\lambda^a]/2$ of $C$ (with $a \in \{3, 8\}$) on the symmetry generators and $c_1 = tr[C]/3$.

To get some useful results, we just need to expand the meson matrix. The leading axion thermalization processes will comprise pion interactions: the shorter life-time of heavier mesons strongly suppresses collisions with axions, which become highly improbable. The current term can be expanded as

$$\Sigma\partial_\mu\Sigma^\dagger - \Sigma^\dagger\partial^\mu\Sigma = -2i\frac{\partial_\mu(\pi^e)\lambda^e}{f_\pi} + \frac{i}{3}\frac{\partial_\mu(\pi^e\lambda^e)^3}{f_\pi^3} - \frac{i}{f_\pi^3}(\pi^e\lambda^e)\partial_\mu(\pi^b\lambda^b)^2 + \frac{i}{f_\pi^3}(\pi^e\lambda^e)^2\partial_\mu\pi^b\lambda^b + \dots \tag{4.35}$$

The previous linear factor in $\partial_\mu(\pi^e)$ can be neglected in our reasoning: it is just responsible for a pion-axion kinetic mixing, that can be reabsorbed in the normalization of the field wave functions. In the second term, we have to carefully consider the contribution $(\pi^e\lambda^e)^3$: because pion fields commute, we have to project the product of the three $\lambda$ matrices into its completely symmetric part

$$\pi^a\pi^b\pi^c\lambda^a\lambda^b\lambda^c = \frac{1}{3!}\pi^a\pi^b\pi^c\lambda^{(a}\lambda^b\lambda^{c)} = \frac{1}{12}\pi^a\pi^b\pi^c(\{\lambda^a,\{\lambda^b,\lambda^c\}\} + \{\{\lambda^a,\lambda^b\},\lambda^c\} + \{\lambda^b,\{\lambda^a,\lambda^c\}\}) \tag{4.36}$$

Using the Gell-Mann matrices algebra of anticommutators $\{\lambda^a,\lambda^b\} = 4\delta^{ab}/3 + 2d^{abc}\lambda^c$, where we remind that $d_{ebc} = tr[\lambda_e\{\lambda_b,\lambda_c\}]/4$, we can claim for instance:

$$\{\lambda^a,\{\lambda^b,\lambda^c\}\} = \frac{8}{3}\delta^{bc}\lambda^a + \frac{8}{3}d^{abc} + 4d^{bce}d^{aef}\lambda^f \tag{4.37}$$

and from the complete symmetry among the $d^{abc}$ indeces, we can end up with

$$\begin{aligned} &\{\lambda^a,\{\lambda^b,\lambda^c\}\} + \{\{\lambda^a,\lambda^b\},\lambda^c\} + \{\lambda^b,\{\lambda^a,\lambda^c\}\} = \\ &= 8d^{abc} + \frac{8}{3}(\delta^{bc}\lambda^a + \delta^{ab}\lambda^c + \delta^{ac}\lambda^b) + 4(d^{bce}d^{aef} + d^{abe}d^{cef} + d^{ace}d^{bef})\lambda^f \end{aligned} \tag{4.38}$$

Now, employing the relation $d^{bce}d^{aef} + d^{abe}d^{cef} + d^{ace}d^{bef} = \delta^{ba}\delta^{cf} + \delta^{ac}\delta^{bf} + \delta^{cb}\delta^{af}$, the $(\pi^e\lambda^e)^3$ term will become

$$\begin{aligned} \frac{i}{3}\frac{\partial_\mu(\pi\cdot\lambda)^3}{f_\pi^3} &= \frac{i}{36f_\pi^3}\partial_\mu[8d^{abc}\pi^a\pi^b\pi^c + 12(\pi\cdot\pi)(\pi\cdot\lambda)] = \\ &= \frac{i}{36f_\pi^3}\partial_\mu(8d^{abc}\pi^a\pi^b\pi^c) + \frac{i}{3f_\pi^3}[(\pi\cdot\pi)(\partial_\mu\pi\cdot\lambda) + 2(\pi\cdot\partial_\mu\pi)(\pi\cdot\lambda)] \end{aligned} \tag{4.39}$$

To extract the pion contributions we have first of all to restrict the indeces $a$, $b$ and $c$ to the subgroup of Gell-Mann matrices related to pion fields, i.e $a, b, c \in \{1, 2, 3\}$. If we consider that the unique non-null coefficients of the symmetric tensor $d_{abc}$, with at least two indeces in the subset $\{1, 2, 3\}$ [48], are $d_{118} = d_{228} = d_{338} = 1/\sqrt{3}$, the first term in the previous formula will not contribute.

For the third addend of (4.35), we could write in a similar fashion:

$$-\frac{i}{f_\pi^3}(\pi\cdot\lambda)\partial_\mu(\pi\cdot\lambda)^2 = -\frac{i}{f_\pi^3}\left[\frac{8}{3}\pi\cdot\partial_\mu\pi(\pi^e\lambda^e) + 2d^{abc}(\partial_\mu\pi^a\pi^b + \pi^a\partial_\mu\pi^b)\pi^e\lambda^e\lambda^c\right] \tag{4.40}$$

Again, $a, b, e = i$ with $i \in \{1, 2, 3\}$, while $c$ can just be equal to 8. Making use of the known expression for the product of Gell-Mann matrices specialized to our case

$$\lambda^e\lambda^c = \frac{2}{3}\delta^{ec}\mathbb{1}_{3\times3} + (if^{ecb} + d^{ecb})\lambda^b = \frac{2}{3}\delta^{i8}\mathbb{1}_{3\times3} + (if^{i8b} + d^{i8b})\lambda^b = d^{ii8}\lambda^i \tag{4.41}$$

we will be able to get

$$-\frac{i}{f_\pi^3}(\pi\cdot\lambda)\partial_\mu(\pi\cdot\lambda)^2 = -\frac{2i}{f_\pi^3}(\pi\cdot\pi)(\pi\cdot\partial_\mu\pi) \tag{4.42}$$

Calculations for the last term of (4.35) can be carried out hand in hand, so that we get

$$\frac{i}{f_\pi^3}(\pi^e\lambda^e)^2\partial_\mu\pi^b\lambda^b = \frac{i}{f_\pi^3}(\pi\cdot\pi)\partial_\mu\pi\cdot\lambda \tag{4.43}$$

Collecting all of these results, the building block of our axion-meson currents will happen to be

$$\Sigma\partial_\mu\Sigma^\dagger - h.c = \frac{4}{3}\frac{i}{f_\pi^3}[\partial_\mu\pi^e(\pi\cdot\pi) - \pi^e(\pi\cdot\partial_\mu\pi)]\lambda^e\ldots \tag{4.44}$$

The axial-singlet meson current can not contribute, owing to $tr[\lambda^e]=0$. Furthermore, we have to take into account that, with $e\in\{1,2,3\}$, also the index $a$ of (4.33) is restricted to the same subset, because of the property $tr[\lambda_a\lambda_e]=2\delta_{ae}$. As a consequence, just because $c^a\neq 0$ if and only if $a=3$, we will have

$$\mathcal{L}^{int}_{\pi-a} = \frac{\partial_\mu a_\phi}{f_a}\frac{\tilde{c}_u^{UV}-\tilde{c}_d^{UV}}{3f_\pi}[\pi^0(\pi^+\partial_\mu\pi^- + \pi^-\partial_\mu\pi^+) - 2\pi^+\pi^-\partial_\mu\pi^0] \tag{4.45}$$

as stated in [49].

### The axion-pion-nucleon interaction

Moving onto the interaction term $\mathcal{L}^{int}_{\pi-n-a}$, we can proceed similarly. We can extract the Noether currents from the mixed pion-baryon terms in the Lagrangian (4.32) and couple them to an axion external source. Terms proportional to $\gamma^5$, where the chiral vielbein appears, will contribute with a Noether current derived from a vectorial transformation for parity invariance reasons. In this special case $u' = UuU^\dagger$, just as $\Sigma$, so that $\delta_{V_3}u = i[\lambda^a,u]\delta\theta^a_V$. The remaining mixed term, represented by the contribution with the chiral connection factor, will instead enter through an axial variation. From relations

$$\begin{cases}\delta_{A_3}\Sigma = i\{\lambda^a,\Sigma\}\delta\theta^a_A = i\{\lambda^a,u^2\}\delta\theta^a_A = i\{\lambda^a,u\}u\delta\theta^a_A + iu\{\lambda^a,u\}\delta\theta^a_A\\ \delta_{V/A}\Sigma = \delta_{V/A}u^2 = \delta_{V/A}uu + u\delta_{V/A}u\end{cases} \tag{4.46}$$

we can guess the transformation law $\delta_{V_3}u = i\{\lambda^a,u\}\delta\theta^a_A$ for the axial part. Therefore, we will eventually obtain:

$$\begin{aligned}\mathcal{L}^{int}_{\pi-n-a} = {} & \frac{\partial_\mu a_\phi}{2f_a}c^a\underbrace{\left(-\frac{1}{2}tr[\bar{B}\gamma^\mu[(u^\dagger\lambda^a u - u\lambda^a u^\dagger),B]]\right)}_{\text{vectorial meson-baryon current}} + \\ & + \frac{\partial_\mu a_\phi}{2f_a}c^a\underbrace{\left(-\frac{D}{2}tr[\bar{B}\gamma^\mu\gamma^5\{(u^\dagger\lambda^a u + u\lambda^a u^\dagger),B\}] - \frac{F}{2}tr[\bar{B}\gamma^\mu\gamma^5[(u^\dagger\lambda^a u + u\lambda^a u^\dagger),B]]\right)}_{\text{axial meson-baryon current}}\end{aligned} \tag{4.47}$$

Of course, $c^a = tr[C\lambda^a]/2$ will still be valid. From this expression, it immediately follows that $U(1)_{V/A}$ variations can not be useful in this discussion: if $\lambda^a$ is replaced by the identity, $\mathcal{L}^{int}_{\pi-n-a}$ will vanish. In addition, the axial meson-baryon current can not generate a linear pion term, so that we will exclude it from our considerations. What is missing is just the expansion of the first contribution, in order to reproduce interaction verteces involving only pions and nucleons. As a matter of fact, we will get

$$u^\dagger \lambda^a u - u\lambda^a u^\dagger = \frac{i}{f_\pi}[\lambda^a, \lambda^b]\pi^b + \ldots = -\frac{2}{f_\pi} f_{abc}\pi^b \lambda^c \tag{4.48}$$

where $f_{abc}$ are the structure constants entering $[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c$. In the previous formula, the index $b$ can be immediately restricted to the set $\{1, 2, 3\}$. To select the proton and neutron component, while multiplying for the $B$ matrix from the right and $\bar{B}$ from the left, $\lambda_c$ can only have a $c \in \{1, 2, 3, 8\}$. Looking at the values of the structure constants [48], the only possible ones for $c$ are $\{1, 2, 3\}$: that fixes $a$ to the same set. Consequently,

$$u^\dagger \lambda^a u - u\lambda^a u^\dagger = 2i(f_{a23}\pi^2 + f_{a13}\pi^1)\lambda^3/f_\pi + 2i(f_{a32}\pi^3 + f_{a12}\pi^1)\lambda^2/f_\pi + 2i(f_{a31}\pi^3 + f_{a21}\pi^2)\lambda^1/f_\pi$$

$$\begin{aligned}
a = 1 \qquad & u^\dagger \lambda^1 u - u\lambda^1 u^\dagger = -(2f_{123}\pi^2)\lambda^3/f_\pi - (2f_{132}\pi^3)\lambda^2/f_\pi \\
a = 2 \qquad & u^\dagger \lambda^2 u - u\lambda^2 u^\dagger = -(2f_{213}\pi^1)\lambda^3/f_\pi - (2f_{231}\pi^3)\lambda^1/f_\pi \\
a = 3 \qquad & u^\dagger \lambda^3 u - u\lambda^3 u^\dagger = -(2f_{312}\pi^1)\lambda^2/f_\pi - (2f_{321}\pi^2)\lambda^1/f_\pi
\end{aligned} \tag{4.49}$$

where $f_{123} = 1$. Because only $c^3 \neq 0$, just the last line will contribute in the sum over the $a$ index. It can be similarly shown that the second term of the commutator, i.e $\bar{B}\gamma^\mu B(u^\dagger \lambda^a u - u\lambda^a u^\dagger)$, does not give relevant contributions to our problem. Selecting solely the interesting products while multiplying by the baryon matrices, one ultimately gets

$$\mathcal{L}^{int}_{\pi-n-a} = \frac{i\partial_\mu a_\phi}{f_a f_\pi} \frac{\tilde{c}_u^{UV} - \tilde{c}_d^{UV}}{2\sqrt{2}} (\pi^- \bar{n}\gamma^\mu p - \pi^+ \bar{p}\gamma^\mu n) \tag{4.50}$$

again in full agreement with [49]. For sake of clarity, we spell out how we used the definitions $\pi^1 = (\pi^+ + \pi^-)/\sqrt{2}$ and $\pi^2 = -i(\pi^+ - \pi^-)/\sqrt{2}$, consistent with the initial form of $\Sigma$ in (1.26).

What clearly emerges from the previous calculations is that reactions involving both axions and pions are proportional to the coefficient $\tilde{c}_u^{UV} - \tilde{c}_d^{UV}$. This feature is fundamental, from a model building point of view, because it suggests us a possible way of evading cosmological constraints where dark matter interactions with pions are relevant. To suppress pions couplings, opening the possibility for a so-called *pionphobic* behaviour, one should just require

$$\tilde{c}_u^{UV} - \tilde{c}_d^{UV} = 0 \quad \Rightarrow \quad c_u^{UV} - c_d^{UV} = \frac{1-z}{1+z} \tag{4.51}$$

in accordance with [50]. Every pionphobic model must satisfy the previous constraint.

### 4.3.2 Nucleophobia

When facing astrophysical problems involving the search for axions, it is often enough to consider a low energy theory, where the only particles with a role are the stable ones: neutrons (with a pretty long decay time), protons, electrons and, clearly, the axions. In this context, energies are so low that pions are not produced, too. Being electrons elementary particles, their low energy couplings can be readily read off from the ultraviolet Lagrangian (up to some running effects); the previous reasoning is instead not applicable to protons and neutrons: just because they are hadrons, that is quark bound states, their coupling constants can solely be obtained by matching the effective theory with the ultraviolet one (look at [30]).

We would like to point out that, in this low energy scenario, we know we can rely on the approximate global symmetry $SU(2)_V$ of isospin, which mixes neutrons and protons. We do not need to consider a model based on the $SU(3)$ group, because all the unstable degrees of freedom (that we had to account for in the previous paragraph) can not be excited. In so doing, we obtain a twofold advantage: we can use a more precise symmetry group of nature than the $SU(3)$ one and, furthermore, we are able to avoid the problem of loop instabilities. The second point, in particular, makes the isospin-based model a really reliable theory.

To start with, we can try to highlight the isospin symmetry at high energy by rewriting the first term of the Lagrangian (4.26) as

$$\begin{aligned}\mathcal{L}_{axion-up/down} &= \frac{\partial_\mu a_\phi}{2f_a}\begin{pmatrix}\bar{u} & \bar{d}\end{pmatrix}\begin{pmatrix}\tilde{c}_u^{UV}\gamma^\mu\gamma^5 & 0\\ 0 & \tilde{c}_d^{UV}\gamma^\mu\gamma^5\end{pmatrix}\begin{pmatrix}u\\ d\end{pmatrix} = \\ &= \frac{\partial_\mu a_\phi}{2f_a}\begin{pmatrix}\bar{u} & \bar{d}\end{pmatrix}\left[\frac{\tilde{c}_u^{UV}+\tilde{c}_d^{UV}}{2}\mathbb{1}_{2\times 2}\gamma^\mu\gamma^5 + \frac{\tilde{c}_u^{UV}-\tilde{c}_d^{UV}}{2}\sigma^3\gamma^\mu\gamma^5\right]\begin{pmatrix}u\\ d\end{pmatrix}\end{aligned} \tag{4.52}$$

which clearly stresses the isospin structure. If we now set $c_V^{UV} = (\tilde{c}_u^{UV} + \tilde{c}_d^{UV})/2$ and $c_A^{UV} = (\tilde{c}_u^{UV} - \tilde{c}_d^{UV})/2$, we can finally obtain

$$\begin{aligned}\mathcal{L}_{axion-quarks} = &\frac{\partial_\mu a_\phi}{2f_a}\left[c_V^{UV}(\bar{u}\gamma^\mu\gamma^5 u + \bar{d}\gamma^\mu\gamma^5 d) + c_A^{UV}(\bar{u}\gamma^\mu\gamma^5 u - \bar{d}\gamma^\mu\gamma^5 d)\right] + \\ &+ \frac{\partial_\mu a_\phi}{2f_a}\sum_{s,c,b,t} c_q^{UV}\bar{q}\gamma^\mu\gamma^5 q - \sum_{u,d,s,c,b,t}(\bar{q}_R M_a q_L + h.c.)\end{aligned} \tag{4.53}$$

Moving to a non-relativistic regime, the effective theory can be built up using the only existing symmetry group: the approximate isospin symmetry $SU(2)_V$. All terms compatible with this symmetry enter the low energy Lagrangian:

$$\mathcal{L}_{effective} = \bar{N}v^\mu\partial_\mu N + g_V\bar{N}\gamma^\mu J_\mu^V N + \bar{N}\gamma^\mu\gamma^5 J_\mu^A N + a tr[M_a]\bar{N}N + b\bar{N}M_a N + \dots \tag{4.54}$$

where $a$ and $b$ are some constants and dots stand for extra terms with respect to leading order contributions, which are not essential for our task. $N = \begin{pmatrix}p\\ n\end{pmatrix}$ is clearly the nucleon isospin doublet, while $v^\mu$ the nucleon quadrivelocity (a fixed four-vector, which gives the non-relativistic limit of the $\gamma^\mu$ matrices). The presence of it can be traced back to the *heavy baryon formalism*, where the momentum $p^\mu$ is recast as $p^\mu = m_n v^\mu + k^\mu$ (with $k^\mu \ll m_n$ a residual momentum). In this limit, the usual Dirac propagator will be

$$\frac{i(\not{p} + m_n)}{p^2 - m^2 + i\epsilon} \approx i\left(\frac{1+\not{v}}{2}\right)\frac{1}{v\cdot k} + \mathcal{O}\left(\frac{k^2}{m_n}\right) \tag{4.55}$$

in which we made use of $v^\mu v_\mu = 1$ and where $P_\pm^v = (1\pm\not{v})/2$ are velocity projectors. These latters define the velocity eigenstates

$$N_v = e^{imx\cdot v}P_+^v N \qquad n_v = e^{imx\cdot v}P_-^v N \tag{4.56}$$

In the nucleon rest frame, where $v^\mu = (1,\vec{0})$, $N_v$ and $n_v$ coincide respectively with the large and the small Dirac components. The $n_v$ field can be integrated out using its equations of motion, so that, eventually, the kinetic term correctly reproducing the non-relativistic propagator (4.55) will be $\bar{N}_v(v\cdot\partial)N_v$. In the following we will remove the subscript of velocity eigenstate from $N_v$ and we will simply refer to it as $N$, remembering that only the large component plays the game in the non-relativitic limit.

Together with the kinetic part, the Lagrangian presents two addends involving $M_a$, whose expansion does not generate linear axion contributions, and two terms where the nucleon current is coupled to external sources. The first one includes a vector current $J^V_\mu$. Nevertheless, we will set $J^V_\mu = 0$, because there is no way of getting a vectorial current linear in the axion field, which is a pseudo-scalar particle by very definition. Then, the nucleon axial current $\bar{N}\gamma^\mu\gamma^5 N$ is associated to the most general current respecting the isospin symmetry, $J^A_\mu$, which can be decomposed in a singlet and triplet contribution

$$J^A_\mu = J^{sing}_\mu + J^{tripl}_\mu = g^q_s J^q_{s\mu} \mathbb{1}_{2\times 2} + g_A J^i_{A\mu}\sigma^i \tag{4.57}$$

$J^{sing}_\mu$ couples the axion source to an axial-singlet nucleon current. Moreover, since we do not have charged external currents, we need to require $J^{1/2}_{A\mu} = 0$. In so doing, we are left with an axial isospin-triplet current, with which only the antisymmetric up and down combination will finally interact, and with an axial isospin-singlet current, where the couplings of the symmetric up and down part and of all remaining quarks will enter. Therefore, we can envisage that

$$J^q_{s\mu} = c^Q_q \frac{\partial_\mu a_\phi}{2f_a} \quad \text{with } q = (V, s, c, b, t) \quad J^3_{A\mu} = c^Q_A \frac{\partial_\mu a_\phi}{2f_a} \tag{4.58}$$

where $V$ and $A$ refer to the particular up and down combinations discussed above. It worth noticing that the quark couplings, extracted from the unknown constants $g^q_s$ and $g_A$ for reasons of convenience, are here evalutated at the QCD scale $Q = 2GeV$. To match the two theories, we should ultimately be able to account for the running of Yukawa couplings from the ultraviolet scale $f_a$ to the infrared $Q$ threshold.

To get an expression for $g^q_s$ and $g_A$ we can simply compare the nucleon matrix elements of the two Lagrangians. Remembering the basic ingredients of the isospin algebra of nucleons

$$\begin{gathered}[I_i, I_j] = -\epsilon_{ijk} I_k \text{ and } I^\dagger_i = I_i \qquad I^\pm = (I_1 \pm I_2)/2 \\ I_3|p\rangle = +1/2|p\rangle \qquad I_3|n\rangle = -1/2|n\rangle \qquad |p\rangle = I^+|n\rangle \qquad |n\rangle = I^-|p\rangle \qquad \langle n|p\rangle = 0\end{gathered} \tag{4.59}$$

we can write the proton matrix element for the effective Lagrangian (assessing only the interesting terms involving the axial current) as

$$\begin{aligned}\langle p|\mathcal{L}_{effective}|p\rangle &= g^q_s \langle p|\bar{N}\gamma^\mu\gamma^5 (J^q_{s\mu}\mathbb{1}_{2\times 2})N|p\rangle + g_A\langle p|\bar{N}\gamma^\mu\gamma^5(J^i_{A\mu}\sigma^i)N|p\rangle = \\ &= g^q_s\langle p|\bar{p}\gamma^\mu\gamma^5(J^q_{s\mu})p|p\rangle + g^q_s\langle n|I^-\bar{n}\gamma^\mu\gamma^5(J^q_{s\mu})nI^+|n\rangle + g_A\langle p|\bar{p}\gamma^\mu\gamma^5(J^3_{A\mu})p|p\rangle + \\ &\quad - g_A\langle n|I^-\bar{n}\gamma^\mu\gamma^5(J^3_{A\mu})nI^+|n\rangle = \frac{\partial_\mu a_\phi}{2f_a} S^\mu_p (g^q_s c^Q_q + g_A c^Q_A)\end{aligned} \tag{4.60}$$

In the preceding expression we have defined the proton spin $S^\mu_p = \langle p|\bar{p}\gamma^\mu\gamma^5 p|p\rangle$: Lorentz and parity invariant arguments lead us to this identification up to a constant, which can always be reabsorbed in our unknown multiplicative factors. The same procedure is applicable to neutrons

$$\langle n|\mathcal{L}_{effective}|n\rangle = \frac{\partial_\mu a_\phi}{2f_a} S^\mu_n (g^q_s c^Q_q - g_A c^Q_A) \tag{4.61}$$

where clearly $S^\mu_n = \langle n|\bar{n}\gamma^\mu\gamma^5 n|n\rangle$.

The nucleon matrix elements can now be computed for the ultraviolet theory in a similar fashion

$$\begin{aligned}&\langle p|\mathcal{L}_{axion-quarks}|p\rangle = \\ &= \frac{\partial_\mu a_\phi}{2f_a}\left[c^{UV}_V\langle p|(\bar{u}\gamma^\mu\gamma^5\bar{u} + \bar{d}\gamma^\mu\gamma^5\bar{d})|p\rangle + c^{UV}_A\langle p|(\bar{u}\gamma^\mu\gamma^5\bar{u} - \bar{d}\gamma^\mu\gamma^5\bar{d})|p\rangle + \sum_{s,c,b,t} c^{UV}_q\langle p|\bar{q}\gamma^\mu\gamma^5 q|p\rangle\right] = \\ &= \frac{\partial_\mu a_\phi}{2f_a}\left[c^{UV}_V S^\mu_p(\Delta u + \Delta d) + c^{UV}_A S^\mu_p(\Delta u - \Delta d) + \sum_{s,c,b,t} c^{UV}_q S^\mu_p \Delta q\right]\end{aligned} \tag{4.62}$$

in which we have introduced the quark contribution $\Delta q$ to proton spin as $\langle p|\bar{q}\gamma^\mu\gamma^5 q|p\rangle \equiv S_p^\mu \Delta q$. By noticing the relation between neutron and proton matrix elements

$$\begin{aligned} &S_n^\mu \Delta u \equiv \langle n|\bar{u}\gamma^\mu\gamma^5 u|n\rangle = \langle p|\bar{d}\gamma^\mu\gamma^5 d|p\rangle = S_p^\mu \Delta d \\ &S_n^\mu \Delta d = S_p^\mu \Delta u \qquad S_n^\mu \Delta q = S_p^\mu \Delta q \quad \text{with } q = s, c, b, t \end{aligned} \tag{4.63}$$

dictated by isospin symmetry, we can similarly get

$$\langle n|\mathcal{L}_{axion-quarks}|n\rangle = \frac{\partial_\mu a_\phi}{2f_a}\left[c_V^{UV} S_n^\mu(\Delta d + \Delta u) + c_A^{UV} S_n^\mu(\Delta d - \Delta u) + \sum_{s,c,b,t} c_q^{UV} S_n^\mu \Delta q\right] \tag{4.64}$$

Once the couplings of the ultraviolet Lagrangian are decreased up to the $Q$ energy, we can compare equations (4.60) with (4.62) or (4.61) with (4.64) in order to get

$$g_A = \Delta u - \Delta d \qquad g_s^V = \Delta u + \Delta d \qquad g_s^q = \Delta q \quad \text{with} \quad q = s, c, b, t \tag{4.65}$$

All of these calculations let us rewrite the third addend of (4.54) in the form

$$\mathcal{L}_{effective} = \frac{\partial_\mu a_\phi}{2f_a}\bar{N}\gamma^\mu\gamma^5\left[\frac{\tilde{c}_u^Q - \tilde{c}_d^Q}{2}(\Delta u - \Delta d)\sigma^3 + \frac{\tilde{c}_u^Q + \tilde{c}_d^Q}{2}(\Delta u + \Delta d) + \sum_{s,c,b,t} c_q^Q \Delta q\right]N + \dots \tag{4.66}$$

from which we can extract a first formula for nucleon couplings:

$$C_p = \tilde{c}_u^Q \Delta u + \tilde{c}_d^Q \Delta d + \sum_{s,c,b,t} c_q^Q \Delta q \qquad C_n = \tilde{c}_d^Q \Delta u + \tilde{c}_u^Q \Delta d + \sum_{s,c,b,t} c_q^Q \Delta q \tag{4.67}$$

The first problem to deal with is obtaining an explicit value for $\Delta q$. The quantity $g_A = \Delta u - \Delta d = 1.2723(23)$ can be measured from the $\beta$-decay with great precision, as we already brought up in the previous subparagraph. On the other hand, some of the remaining ones can be derived by lattice simulation: $\Delta u + \Delta d = 0.521(53)$ and $\Delta s = -0.026(4)$. For the charm contribution only an upper and lower bound is known (i.e $\Delta c = \pm 0.004$). Nevertheless, this latter, together with the bottom and top terms, can be neglected, because we expect them to enter very little in the nucleon spin magnitude. All that gives us

$$\begin{aligned} C_p &= 0.897(27)\tilde{c}_u^Q - 0.376(27)\tilde{c}_d^Q - 0.026(4)c_s^Q \\ C_n &= 0.897(27)\tilde{c}_d^Q - 0.376(27)\tilde{c}_u^Q - 0.026(4)c_s^Q \end{aligned} \tag{4.68}$$

The second issue is that our formula should account for the running of Yukawa couplings between the two energy scales typical of the ultraviolet and effective theory respectively. As a matter of fact, all of the couplings, in the previous expression, are evalutated at $Q$ and not in terms of the original quark Lagrangian. Going on following the discussion sketched in [30], one should keep in mind that the QFT running phenomenon is intimately related to ultraviolet fluctuations of virtual particles: at high energy scale all quark masses can be disregarded and an $SU(n_f)$ group emerges (where $n_f$ is the number of active quarks). Considering the diagonal matrix of axion-quark couplings for this situation, we will have

$$C = diag(\underbrace{c_u, c_d, ..., c_{n_f}}_{n_f}) = \frac{1}{n_f}tr[C]\mathbb{1}_{n_f\times n_f} + \frac{1}{r}\sum_{j=1}^{r} tr[CT_j]\lambda_j \tag{4.69}$$

where $T_j$ are the generator of the Cartan subalgebra of $SU(n_f)$ and $r$ is the dimension of the algebra itself (the *rank*). From this expression we can extract the coupling of the singlet current in this context. If $j^\mu_q$ are the individual quark currents, one can recognize the generalized axial singlet current $j^\mu_\Sigma = \sum_q j^\mu_q$. As we discussed in the first chapter, this current is anomalous: it is not protected by conservation laws and, thereby, it acquires an anomalous dimension in the renormalization procedure. Therefore,

$$\begin{gathered}\frac{\partial_\mu a_\phi}{2f_a}\sum_q j^\mu_q = \frac{\partial_\mu a_\phi}{2f_a}\Bigg[\underbrace{\sum_q\left(c_q - \frac{\sum_{q'} c_{q'}}{n_f}\right)j^\mu_q}_{\text{axial multiplet current}} + \underbrace{\frac{\sum_{q'} c_{q'}}{n_f}j^\mu_\Sigma}_{\text{axial singlet current}}\Bigg] \underset{\text{running}}{\longmapsto} \\ \frac{\partial_\mu a_\phi}{2f_a}\left[\sum_q\left(c_q - \frac{\sum_{q'} c_{q'}}{n_f}\right)j^\mu_q + Z(Q)\frac{\sum_{q'} c_{q'}}{n_f}j^\mu_\Sigma\right] = \frac{\partial_\mu a_\phi}{2f_a}\sum_q \underbrace{\left[c_q + (Z(Q)-1)\frac{\sum_{q'} c_{q'}}{n_f}\right]}_{c^Q_q} j^\mu_q\end{gathered} \tag{4.70}$$

where $Z(Q)$ is the renormalization constant of the axial singlet current (see [51]). We would like to relate couplings at two different energy scales, removing the bare parameters of the Lagrangian from formulae. For two couplings at scales $Q$ and $Q_0$ respectively, we can write

$$\begin{cases} c^Q_q = c_q + (Z(Q)-1)\frac{\sum_{q'} c_{q'}}{n_f} \\ c^{Q_0}_q = c_q + (Z(Q_0)-1)\frac{\sum_{q'} c_{q'}}{n_f} \end{cases} \tag{4.71}$$

By summing up over $q$ on the two sides of the second equation of the system, we will get $\sum_q c^{Q_0}_q = Z(Q_0)\sum_{q'} c_{q'}$. Subtracting the two equations and replacing the summmation over bare couplings with the relation derived above, one can finally obtain the running formula

$$c^Q_q = c^{Q_0}_q + \left(\frac{Z(Q)}{Z(Q_0)} - 1\right)\frac{\sum_{q=1}^{n_f} c^{Q_0}_q}{n_f} \tag{4.72}$$

It is important to point out that $j^\mu_\Sigma$ only renormalizes multiplicatively, so that the running process can be pretty simplified. If we had not removed through a quark redefinition the term proportional to $\sim \mathcal{G}\tilde{\mathcal{G}}$, we should have had to consider its mixing with $\partial_\mu j^\mu_\Sigma$ (an operator with equal dimensions and the same quantum numbers). Moving from $f_a$ towards $Q$, crossing the top and bottom thresholds (of $m_b \approx 4.18GeV$ and $m_t \approx 172.44GeV$ respectively), where the number $n_f$ significantly changes, one can correct our preceding coupling formulae as:

$$\begin{aligned} C_p &= -0.47(3) + 0.88(3)C_u - 0.39(2)C_d - 0.038(5)C_s - 0.012(5)C_c - 0.009(2)C_b - 0.0035(4)C_t \\ C_n &= -0.02(3) + 0.88(3)C_d - 0.39(2)C_u - 0.038(5)C_s - 0.012(5)C_c - 0.009(2)C_b - 0.0035(4)C_t \end{aligned} \tag{4.73}$$

where we have finally set $c^{UV}_q = C_q$ and where the constant term derives from the infrared corrections (with $z = m_u/m_d = 0.48(3)$). The values of $Z(Q)$ at different scales have been computed in [30] from the anomalous dimension equation, i.e the differential equation satisfied by $Z(Q)$. It worth noticing how the main contribution to neutron and proton couplings comes from up and down quarks, dubbed *valence quarks*, which determine the quantum numbers of the two hadrons. All heavier quarks enter the expression as virtual particles produced by quantum vacuum fluctuations (*quarks of the sea*).

But now we get all the tools together to explore an extra possibility: the *nucleophobic behavior*, which means that axions interact feebly with protons and neutrons. The achievement of this result is not just a theoretical pastime, but it has profound consequences on some axion properties which can be extracted from astrophysical measurements.

If we add and subtract the two previous formulae, we clearly obtain

$$\begin{cases} C_p + C_n = 0.050(5)(C_u + C_d - 1) - 2\delta \\ C_p - C_n = 1.273(2)\left(C_u - C_d - \frac{1-z}{1+z}\right) \end{cases} \tag{4.74}$$

where we have set $\delta = 0.038(5)C_s + 0.012(5)C_c + 0.009(2)C_b + 0.0035(4)C_t$. From that, it is quite evident how the requirement for $C_p$ and $C_n$ to simultaneously vanish is equivalent to $C_p + C_n = 0$ and $C_p - C_n = 0$, which implies

$$C_u = \frac{1}{1+z} + 2\delta \qquad C_d = \frac{z}{1+z} + 2\delta \tag{4.75}$$

If we decide to neglect the tiny $\delta$ value in the previous expressions, these conditions of nucleophobia can be recast as

$$C_u + C_d = 1 \qquad C_u = \frac{1}{1+z} \approx \frac{2}{3} \tag{4.76}$$

Incidentally, it worth noticing that nucleophobic conditions automatically imply pionphobia. The requirements (4.75) directly satisfy (4.51). Moreover, this statement is untouched by our running considerations: indeed, the difference $\tilde{c}_u^{UV} - \tilde{c}_d^{UV}$ is the multiplicative coupling of the non-singlet current in equation (4.66), which is free from anomalies. The conservation law of this current protects our pionphobia constraint from renormalization. If that were not convincing enough, one can simply observe how $\delta$ corrections coming from sea quarks simplify by subtracting $C_u$ and $C_d$ in (4.75).

From here, it lucidly emerges the dire need to look for different DFSZ models, if one wishes to endow the axion with a nucleophobic behaviour. In fact, it is straightforward to verify that the foregoing relations do not hold for (4.11), where $C_u \leq 1/3$ and $C_u + C_d = 1/3$. Nonetheless, it emerges that this ubiquitous factor one third, in both relations, is ultimately related to the number of quark generations which couple to PQ symmetry in the same way. Thereby, it can be envisaged that the only possible way out of this situation is relaxing the hypothesis of universal PQ coupling among the three generations (as first noticed in [52]). This will give life to a plethora of new possible models, some of which nucleophobic.

As it will be shown, these generalized DFSZ models also offer the possibility of removing the coupling to electrons in a very simple way, even though this is not a previlege reserved to these theories. An axion that does not interact with electrons is dubbed *electrophobic*. If both nucleophobia and electrophobia are present, we speak of an *astrophobic axion*.

## 4.4 The $2+1$ DFSZ model

### 4.4.1 The PQ charge pattern

For sake of clarity, we report here our UV reference Lagrangian involving the axion:

$$\begin{aligned} \mathcal{L}_a = \frac{1}{2}\partial_\mu a_\phi \partial^\mu a_\phi + \mathcal{L}_{anomaly} + \frac{\partial_\mu a_\phi}{2f_a}\bar{u}_i\gamma^\mu(C^V_{u_iu_j} + C^A_{u_iu_j}\gamma^5)u_j + \\ + \frac{\partial_\mu a_\phi}{2f_a}\bar{d}_i\gamma^\mu(C^V_{d_id_j} + C^A_{d_id_j}\gamma^5)d_j + \frac{\partial_\mu a_\phi}{2f_a}\bar{e}_i\gamma^\mu(C^V_{e_ie_j} + C^A_{e_ie_j}\gamma^5)e_j \end{aligned} \tag{4.77}$$

where $\mathcal{L}_{anomaly}$ contains the anomalous contributions. The last three addends are the derivative axion couplings to fermions which we want to determine: we have written down them in the most general form by including the vectorial fermion current, too. Studying the basic DFSZ model, we have seen how these couplings intimately depend on the PQ charge assignment, that is the only essential aspect of the previous DFSZ theory we are going to tweak. In particular, we will explore a next-to-minimal model with a $2+1$ flavour structure, where only two generations have universal PQ charges: this hypothesis will be enough in order to derive some interesting results.

To redefine the charge assignment of the theory, let us consider a Yukawa Lagrangian with twelve Higgs fields given by $\mathcal{L}_{Yuk} = \mathcal{L}_u + \mathcal{L}_d + \mathcal{L}_l$, with:

$$\mathcal{L}_u = y^u_{33}\overline{q}_{L3}u_{R3}\phi_{A_1} + y^u_{3a}\overline{q}_{L3}u_{Ra}\phi_{A_2} + y^u_{a3}\overline{q}_{La}u_{R3}\phi_{A_3} + y^u_{ab}\overline{q}_{La}u_{Rb}\phi_{A_4} + h.c. \tag{4.78}$$

$$\mathcal{L}_d = y^d_{33}\overline{q}_{L3}d_{R3}\tilde{\phi}_{A_5} + y^d_{3a}\overline{q}_{L3}d_{Ra}\tilde{\phi}_{A_6} + y^d_{a3}\overline{q}_{La}d_{R3}\tilde{\phi}_{A_7} + y^d_{ab}\overline{q}_{La}d_{Rb}\tilde{\phi}_{A_8} + h.c. \tag{4.79}$$

$$\mathcal{L}_l = y^e_{33}\overline{l}_{L3}e_{R3}\tilde{\phi}_{A_9} + y^e_{3a}\overline{l}_{L3}e_{Ra}\tilde{\phi}_{A_{10}} + y^e_{a3}\overline{l}_{La}e_{R3}\tilde{\phi}_{A_{11}} + y^e_{ab}\overline{l}_{La}e_{Rb}\tilde{\phi}_{A_{12}} + h.c. \tag{4.80}$$

Here, we have $a, b = 1, 2$ running over the two families with the same PQ charges. On the other hand, the subscript $A_i$ can take on its value in the set $\{1, 2\}$, which means that, according to the model, all of the twelve Higgs fields will be eventually identified with one of the two original Higgs doublets. From the previous Lagrangian, in order to ensure PQ invariance, we can readily read the PQ charges associated to each fermion. However, we have to consider that there is a certain arbitrariness in the way these charges can be assigned, so that we should try to benefit from it in the most convenient form.

First of all, due to the anomaly free $B-L$ symmetry, we can always relate two phases between quark and lepton fields. If $q'_{L3} = q_{L3}e^{iX_{q_{L3}}\alpha}$ and $l'_{L3} = l_{L3}e^{iX_{l_{L3}}\alpha}$ (where phases are independent because of different PQ charges), we can act with a global $B-L$ symmetry, which transforms quark and lepton phases simultaneously:

$$\begin{cases} \alpha_{q_{L3}} = X_{q_{L3}}\alpha \longmapsto \alpha'_{q_{L3}} = X_{q_{L3}}\alpha + \beta/3 \\ \alpha_{l_{L3}} = X_{l_{L3}}\alpha \longmapsto \alpha'_{l_{L3}} = X_{l_{L3}}\alpha + \beta \end{cases} \tag{4.81}$$

In order to have $\alpha'_{q_{L3}} = r\alpha'_{l_{L3}}$, where $r$ is the ratio between the hypercharges of the two fields (i.e $r = Y_{l_{L3}}/Y_{q_{L3}}$), we just need to choose $\beta = 3(X_{q_{L3}} - rX_{l_{L3}})\alpha/(3r-1)$. Then, we can use the hypercharge symmetry $U(1)_Y$, which is again anomaly free, to remove both of these phases at once. This discussion, as a whole, enables us to understand how we are always free to set $X_{q_{L3}} = X_{l_{L3}} = 0$, without loss of generality. Once fixed that, the rest of charges will be given by:

$$\begin{aligned} X_{u_{Ra}} &= -X_{A_2}\,, \qquad & X_{d_{Ra}} &= X_{A_6}\,, \qquad & X_{e_{Ra}} &= X_{A_{10}}\,, \\ X_{u_{R3}} &= -X_{A_1}\,, \qquad & X_{d_{R3}} &= X_{A_5}\,, \qquad & X_{e_{R3}} &= X_{A_9}\,, \\ X_{q_{La}} &= -X_{A_1} + X_{A_3}\,, \qquad & X_{l_{La}} &= X_{A_9} - X_{A_{11}} \end{aligned} \tag{4.82}$$

All of combinations that can be generated modifying the $A_i$ values are collected in table 4.1. We observe that the models presented in the table are organized in couples, whose members just differ for the arrangement of the Higgs doublets in the leptonic sector. Moreover, some theories are obtained from others just exchanging partially or entirely $\phi_1$ with $\phi_2$, as for $M2$ and $M1'$, where the interchange is complete. The class of models dubbed as $M1$ and $M2$ (with all variants indicated with a prime) is the only one where charge univerality is broken among left quarks. The variants of $M3$ and $M4$ models differ from the rest of the theories because the universal PQ pattern is preserved only in one sector between the right up and right down quarks ones. In the $M5$ class, the universality is broken in both of them.

| | $\mathbf{A_{1-4}}$ | $\mathbf{A_{5-8}}$ | $\mathbf{A_{9-12}}$ | | $\mathbf{A_{1-4}}$ | $\mathbf{A_{5-8}}$ | $\mathbf{A_{9-12}}$ |
|---|---|---|---|---|---|---|---|
| **M1** | 1122 | 2211 | 1122 | $\mathbf{M4''}$ | 1111 | 1212 | 1122 |
| **M2** | 1122 | 2211 | 2211 | $\mathbf{M4'''}$ | 1111 | 1212 | 2211 |
| $\mathbf{M1'}$ | 2211 | 1122 | 1122 | $\mathbf{M4_{bis}}$ | 2222 | 2121 | 1122 |
| $\mathbf{M2'}$ | 2211 | 1122 | 2211 | $\mathbf{M4'_{bis}}$ | 2222 | 2121 | 2211 |
| **M3** | 2121 | 1111 | 1122 | $\mathbf{M4''_{bis}}$ | 1111 | 2121 | 1122 |
| $\mathbf{M3'}$ | 2121 | 1111 | 2211 | $\mathbf{M4'''_{bis}}$ | 1111 | 2121 | 2211 |
| $\mathbf{M3''}$ | 1212 | 2222 | 1122 | **M5** | 1212 | 2121 | 1122 |
| $\mathbf{M3'''}$ | 1212 | 2222 | 2211 | $\mathbf{M5'}$ | 1212 | 2121 | 2211 |
| $\mathbf{M3_{bis}}$ | 1212 | 1111 | 1122 | $\mathbf{M5''}$ | 2121 | 1212 | 1122 |
| $\mathbf{M3'_{bis}}$ | 1212 | 1111 | 2211 | $\mathbf{M5'''}$ | 2121 | 1212 | 2211 |
| $\mathbf{M3''_{bis}}$ | 2121 | 2222 | 1122 | $\mathbf{M5_{bis}}$ | 1212 | 1212 | 1122 |
| $\mathbf{M3'''_{bis}}$ | 2121 | 2222 | 2211 | $\mathbf{M5'_{bis}}$ | 1212 | 1212 | 2211 |
| **M4** | 2222 | 1212 | 1122 | $\mathbf{M5''_{bis}}$ | 2121 | 2121 | 1122 |
| $\mathbf{M4'}$ | 2222 | 1212 | 2211 | $\mathbf{M5'''_{bis}}$ | 2121 | 2121 | 2211 |
| **DFSZ − I** | 2222 | 1111 | 1111 | **DFSZ − II** | 2222 | 1111 | 2222 |

Table 4.1: 2+1 DFSZ models in comparison with the simpler DFSZ-I and DFSZ-II structure

Using rules (4.82), one can easily derive the charge configurations for each of the twenty-eight cases, as shown in table 4.2. From the calculation of the color anomaly, the table illustrates how the four theories in class $M5_{\text{bis}}$ must be excluded due to $N = 0$: they are not real QCD axion models, because, if no color anomaly exists, we can not reabsorb the $\theta_{QCD}$ parameter, making it unphysical. The remaining models present two types of $N$, up to the sign. Thus, according to (3.3), the relation between $f_a$ and $v_f$ will be $f_a = v_f/2$ for $N = 1$ or $f_a = v_f/4$ for $N = 2$. We know that the domain wall number is a positive quantity by very definition, because it relates two energy scales. Therefore, if the color anomaly turns out to be negative, the extra minus sign will not enter the $\mathcal{N}_{DW}$ definition. Actually, it will not be able to affect physics at all: as we said, anomalous terms generate the effective axion potential, that is even in the axion field.

After this small clarification, we are ready to explicitly compute an expression for $\mathcal{L}_{anomaly}$. Differently from the basic DFSZ models, now the assignment of PQ charges also introduces an effective coupling between axions and electroweak gauge bosons, because here left fermions are PQ charged, too. Just as for $E$ and $N$, we can define an *electroweak anomaly coefficient* $W$, which can be assessed as

$$W\delta^{ab} = tr[\mathcal{X}_L^q\{\tau^a,\tau^b\}] + tr[\mathcal{X}_L^l\{\tau^a,\tau^b\}] = \frac{\delta^{ab}}{2}\left(\sum_{\text{colors}}\sum_{f_L} X_{Lf} + \sum_{l_L} X_{Ll}\right) \tag{4.83}$$

where again $\mathcal{X}_L^q$ and $\mathcal{X}_L^l$ are the diagonal matrices containing PQ charges for quarks and leptons respectively. Thereby, before the breaking of the electroweak scale, we will have

$$\mathcal{L}_{anomaly} = \frac{a_\phi}{f_a}\frac{\alpha_s}{8\pi}\mathcal{G}_a^{\mu\nu}\tilde{\mathcal{G}}_{\mu\nu}^a + \frac{a_\phi}{f_a}\frac{\alpha_Y}{8\pi}\frac{Y}{N}\mathcal{B}^{\mu\nu}\tilde{\mathcal{B}}_{\mu\nu} + \frac{a_\phi}{f_a}\frac{\alpha_{ew}}{8\pi}\frac{W}{N}\mathcal{W}_a^{\mu\nu}\tilde{\mathcal{W}}_{\mu\nu}^a \tag{4.84}$$

in which the second term is associated to the hypercharge anomaly $Y$. When we cross the energy at which electroweak gauge bosons become massive, the only surviving symmetries are $SU(3)_c$ and $U(1)_{em}$. The field strength tensors $\mathcal{B}_{\mu\nu}$ and $\mathcal{W}_{\mu\nu}^a$ are not good degrees of freedom to describe the theory any more. We can use instead the well-known photon $\mathcal{F}_{\mu\nu}$ and Z-boson $\mathcal{Z}_{\mu\nu}$ field strength

| | | $\mathbf{X_{u_{R3}}}$ | $\mathbf{X_{u_{Ra}}}$ | $\mathbf{X_{d_{R3}}}$ | $\mathbf{X_{d_{Ra}}}$ | $\mathbf{X_{q_{L3}}}$ | $\mathbf{X_{q_{La}}}$ | $\mathbf{X_{e_{R3}}}$ | $\mathbf{X_{e_{Ra}}}$ | $\mathbf{X_{l_{L3}}}$ | $\mathbf{X_{l_{La}}}$ | **N** | **W** | **Y** | **E** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **left quarks** | $M1$ | $-X_1$ | $-X_1$ | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | +1 | 4 | −10/3 | 2/3 |
| | $M2$ | $-X_1$ | $-X_1$ | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | +1 | 8 | −16/3 | 8/3 |
| | $M1'$ | $-X_2$ | $-X_2$ | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | −1 | −8 | 16/3 | −8/3 |
| | $M2'$ | $-X_2$ | $-X_2$ | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | −1 | −4 | 10/3 | −2/3 |
| **right up quarks** | $M3$ | $-X_2$ | $-X_1$ | $X_1$ | $X_1$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | +1 | −2 | 2/3 | −4/3 |
| | $M3'$ | $-X_2$ | $-X_1$ | $X_1$ | $X_1$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | +1 | 2 | −4/3 | 2/3 |
| | $M3''$ | $-X_1$ | $-X_2$ | $X_2$ | $X_2$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | −1 | −2 | 4/3 | −2/3 |
| | $M3'''$ | $-X_1$ | $-X_2$ | $X_2$ | $X_2$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | −1 | 2 | −2/3 | 4/3 |
| | $M3_{bis}$ | $-X_1$ | $-X_2$ | $X_1$ | $X_1$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | +2 | −2 | 10/3 | 4/3 |
| | $M3'_{bis}$ | $-X_1$ | $-X_2$ | $X_1$ | $X_1$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | +2 | 2 | 4/3 | 10/3 |
| | $M3''_{bis}$ | $-X_2$ | $-X_1$ | $X_2$ | $X_2$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | −2 | −2 | −4/3 | −10/3 |
| | $M3'''_{bis}$ | $-X_2$ | $-X_1$ | $X_2$ | $X_2$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | −2 | 2 | −10/3 | −4/3 |
| **right down quarks** | $M4$ | $-X_2$ | $-X_2$ | $X_1$ | $X_2$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | +1 | 2 | 8/3 | 14/3 |
| | $M4'$ | $-X_2$ | $-X_2$ | $X_1$ | $X_2$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | +1 | −2 | 14/3 | 8/3 |
| | $M4''$ | $-X_1$ | $-X_1$ | $X_2$ | $X_1$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | −1 | 2 | −14/3 | −8/3 |
| | $M4'''$ | $-X_1$ | $-X_1$ | $X_2$ | $X_1$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | −1 | −2 | −8/3 | −14/3 |
| | $M4_{bis}$ | $-X_2$ | $-X_2$ | $X_2$ | $X_1$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | +2 | 2 | 10/3 | 16/3 |
| | $M4'_{bis}$ | $-X_2$ | $-X_2$ | $X_2$ | $X_1$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | +2 | −2 | 16/3 | 10/3 |
| | $M4''_{bis}$ | $-X_1$ | $-X_1$ | $X_1$ | $X_2$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | −2 | 2 | −16/3 | −10/3 |
| | $M4'''_{bis}$ | $-X_1$ | $-X_1$ | $X_1$ | $X_2$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | −2 | −2 | −10/3 | −16/3 |
| **$u_R$ and $d_R$** | $M5$ | $-X_1$ | $-X_2$ | $X_2$ | $X_1$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | +1 | −2 | 8/3 | 2/3 |
| | $M5'$ | $-X_1$ | $-X_2$ | $X_2$ | $X_1$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | +1 | 2 | 2/3 | 8/3 |
| | $M5''$ | $-X_2$ | $-X_1$ | $X_1$ | $X_2$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | −1 | −2 | −2/3 | −8/3 |
| | $M5'''$ | $-X_2$ | $-X_1$ | $X_1$ | $X_2$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | −1 | 2 | −8/3 | −2/3 |
| | $M5_{bis}$ | $-X_1$ | $-X_2$ | $X_1$ | $X_2$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | 0 | −2 | 2 | 0 |
| | $M5'_{bis}$ | $-X_1$ | $-X_2$ | $X_1$ | $X_2$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | 0 | 2 | 0 | 2 |
| | $M5''_{bis}$ | $-X_2$ | $-X_1$ | $X_2$ | $X_1$ | 0 | 0 | $X_1$ | $X_1$ | 0 | $X_1 - X_2$ | 0 | −2 | 0 | −2 |
| | $M5'''_{bis}$ | $-X_2$ | $-X_1$ | $X_2$ | $X_1$ | 0 | 0 | $X_2$ | $X_2$ | 0 | $X_2 - X_1$ | 0 | 2 | −2 | 0 |
| | *DFSZI* | $-X_2$ | $-X_2$ | $X_1$ | $X_1$ | 0 | 0 | $X_1$ | $X_1$ | 0 | 0 | -3 | 0 | -8 | -8 |
| | *DFSZII* | $-X_2$ | $-X_2$ | $X_1$ | $X_1$ | 0 | 0 | $X_2$ | $X_2$ | 0 | 0 | -3 | 0 | -2 | -2 |

Table 4.2: In this table the charge assignment of different 2+1 DFSZ models are shown. The theories have been grouped according to the sector where the PQ universality is broken. In the last two rows we report the old DFSZ models, where calculations have been done with a Yukawa Lagrangian compatible with that of this paragraph (and slightly different from that one at the beginning of the chapter, how charges display). In the four columns on the right, the anomaly coefficients associated to the $SU(3)_c$ color symmetry (N), to the $SU(2)_L$ weak isospin group (W), to the $U(1)_Y$ hypercharge phase transformation (Y) and to the residual $U(1)_{em}$ symmetry (E) are presented. We stress how the four hightlighted rows (running from $M5_{bis}$ to $M5'''_{bis}$) must be discarded because of $N = 0$: they do not represent viable QCD axion models. It also worth noticing how the simple rule $E = W + Y$ relating different anomalies holds.

tensors, which are related to the previous ones through

$$\begin{cases} \mathcal{B}_{\mu\nu} = -s_{\theta_W}\mathcal{Z}_{\mu\nu} + c_{\theta_W}\mathcal{F}_{\mu\nu} \\ \mathcal{W}^{(3)}_{\mu\nu} = c_{\theta_W}\mathcal{Z}_{\mu\nu} + s_{\theta_W}\mathcal{F}_{\mu\nu} \end{cases} \qquad \begin{cases} \mathcal{W}^{(2)}_{\mu\nu} = \frac{\mathcal{W}^+_{\mu\nu}+\mathcal{W}^-_{\mu\nu}}{\sqrt{2}} \\ \mathcal{W}^{(1)}_{\mu\nu} = \frac{\mathcal{W}^+_{\mu\nu}-\mathcal{W}^-_{\mu\nu}}{\sqrt{2}i} \end{cases} \tag{4.85}$$

where we have also reminded the form of the charge eigenstates $\mathcal{W}^{\pm}$. Of course, $\theta_W$ is the Weinberg angle.

By plugging these expressions into equation (4.84), one gets

$$\begin{aligned} \mathcal{L}_{anomaly} = &\frac{a_\phi}{f_a}\frac{\alpha_s}{8\pi}\mathcal{G}^{\mu\nu}_a\tilde{\mathcal{G}}^a_{\mu\nu} + \frac{a_\phi}{f_a}\frac{\alpha_Y}{8\pi}\frac{Y}{N}(-s_{\theta_W}\mathcal{Z}_{\mu\nu} + c_{\theta_W}\mathcal{F}_{\mu\nu})(-s_{\theta_W}\tilde{\mathcal{Z}}^{\mu\nu} + c_{\theta_W}\tilde{\mathcal{F}}^{\mu\nu}) + \\ &+ \frac{a_\phi}{f_a}\frac{\alpha_{ew}}{8\pi}(c_{\theta_W}\mathcal{Z}_{\mu\nu} + s_{\theta_W}\mathcal{F}_{\mu\nu})(c_{\theta_W}\tilde{\mathcal{Z}}^{\mu\nu} + s_{\theta_W}\tilde{\mathcal{F}}^{\mu\nu}) + \frac{2a_\phi}{f_a}\frac{\alpha_{ew}}{8\pi}\frac{W}{N}\mathcal{W}^{\mu\nu}_+\tilde{\mathcal{W}}^-_{\mu\nu} \end{aligned} \tag{4.86}$$

in which we made use of the identity $\mathcal{W}^{\mu\nu}_+\tilde{\mathcal{W}}^-_{\mu\nu} = \mathcal{W}^{\mu\nu}_-\tilde{\mathcal{W}}^+_{\mu\nu}$, thanks to the Levi-Civita symbol property $\epsilon^{\mu\nu\rho\sigma} = \epsilon^{\rho\sigma\mu\nu}$. Now, remembering that the electric charge $e$ is related to the hypercharge $g'$ (inside $\alpha_Y$) and the electroweak coupling $g$ (in $\alpha_{ew}$) through the relation $e^2 = g'^2 c^2_{\theta_W} = g^2 s^2_{\theta_W}$, we can write

$$\begin{aligned} \mathcal{L}_{anomaly} = &\frac{a_\phi}{f_a}\frac{\alpha_s}{8\pi}\mathcal{G}^{\mu\nu}_a\tilde{\mathcal{G}}^a_{\mu\nu} + \frac{a_\phi}{f_a}\frac{\alpha_{em}}{8\pi}\frac{Y+W}{N}\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} + \frac{a_\phi}{f_a}\frac{\alpha_Y s^2_{\theta_W} Y + \alpha_{ew} c^2_{\theta_W} W}{8\pi N}\mathcal{Z}_{\mu\nu}\tilde{\mathcal{Z}}^{\mu\nu} + \\ &- s_{2\theta_W}\frac{a_\phi}{f_a}\frac{\alpha_Y Y - \alpha_{ew} W}{8\pi N}\mathcal{F}_{\mu\nu}\tilde{\mathcal{Z}}^{\mu\nu} + \frac{2a_\phi}{f_a}\frac{\alpha_{ew}}{8\pi}\frac{W}{N}\mathcal{W}^{\mu\nu}_+\tilde{\mathcal{W}}^-_{\mu\nu} \end{aligned} \tag{4.87}$$

so that, comparing with the usual expression (4.16), we deduce the anomaly relation $E = W + Y$. For sake of completeness, we remind that this is not the more useful anomalous Lagrangian to employ in calculation: we have already seen how the gluon coupling can be reabsorbed in a redefinition of the up and down quark fields, introducing an extra model-independent contribution in the electromagnetic coupling.

We would like to emphasize that the effective interaction with the Z-boson and the mixed axion-Z-photon vertex are not peculiar of a generalized DFSZ model: these would survive even if $W = 0$. What is a really distinctive mark of these theories is the new decay mode of axion in two charged $W^{\pm}$ bosons. Even if interesting from a theoretical point of view, this new effective coupling with charged electroweak bosons is very unlikely to produce any testable results, being the vertex suppressed by the tiny value of the $\alpha_{ew}$ constant, typical of weak interactions.

### 4.4.2 The fermion interactions

The most fundamental aspect of these new models resides in the derivation of fermion interactions, that now we are going to develop. We have already discussed how the axion coupling to fermions can be brought to light by means of a chiral redefinition of fermion fields, which moves the Higgs doublet phases from the Yukawa term to the anomaly. Nevertheless, being the transformation local, a derivative axion term emerges from the kinematic part of the Lagrangian. Now, we just have to consider that PQ charges are not universal. We will explicitly carry out the calculation just for the M1 model of table 4.2, that will be used as an example. From the kinetic term we will have:

$$
\begin{aligned}
\mathcal{L}_{fermion-axion} &= \bar{u}_{Rm} i\gamma^\mu \partial_\mu \left(\frac{2iX_{u_{Rmn}} a_\phi}{v_f}\right) u_{Rn} + \bar{u}_{Lm} i\gamma^\mu \partial_\mu \left(\frac{2iX_{u_{Lmn}} a_\phi}{v_f}\right) u_{Ln} + \\
&+ \bar{d}_{Rm} i\gamma^\mu \partial_\mu \left(\frac{2iX_{d_{Rmn}} a_\phi}{v_f}\right) d_{Rn} + \bar{d}_{Lm} i\gamma^\mu \partial_\mu \left(\frac{2iX_{d_{Lmn}} a_\phi}{v_f}\right) d_{Ln} + \\
&+ \bar{e}_{Rm} i\gamma^\mu \partial_\mu \left(\frac{2iX_{e_{Rmn}} a_\phi}{v_f}\right) e_{Rn} + \bar{e}_{Lm} i\gamma^\mu \partial_\mu \left(\frac{2iX_{e_{Lmn}} a_\phi}{v_f}\right) e_{Ln} = \\
&= \bar{u}_m \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} (-X_{u_{Rmn}}(1+\gamma^5) - X_{u_{Lmn}}(1-\gamma^5)) u_n + \\
&+ \bar{d}_m \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} (-X_{d_{Rmn}}(1+\gamma^5) - X_{d_{Lmn}}(1-\gamma^5)) d_n + \\
&+ \bar{e}_m \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} (-X_{e_{Rmn}}(1+\gamma^5) - X_{e_{Lmn}}(1-\gamma^5)) e_n
\end{aligned}
\tag{4.88}
$$

where, using the definition of $f_a$, there are no extra multiplicative factors at the coupling level, because $N = 1$. But we have to remember that quarks appearing in the foregoing expression are not mass eigenstates: one has to diagonalize the Yukawa matrix for fermions through a bi-unitary transformation. This latter will redefine our states as $u'_{Rm} = U^{(u)}_{mn} u_{Rn}$, $u'_{Lm} = V^{(u)}_{mn} u_{Ln}$, $d'_{Rm} = U^{(d)}_{mn} d_{Rn}$, $d'_{Lm} = V^{(d)}_{mn} d_{Ln}$, $e'_{Rm} = U^{(e)}_{mn} e_{Rn}$ and $e'_{Lm} = V^{(e)}_{mn} e_{Ln}$. Taking this into account and considering that

$$
X_{u_L} = X_{d_L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{1} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\tag{4.89}
$$

$$
X_{e_L} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathbb{1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\tag{4.90}
$$

we can readily write

$$
\begin{aligned}
\mathcal{L}_{fermion-axion} &= \bar{u}'_i \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} (-U^{(u)}_{im} X_{u_{Rmn}} U^{(u)\dagger}_{nj} (1+\gamma^5) - V^{(u)}_{im} X_{u_{Lmn}} V^{(u)\dagger}_{nj} (1-\gamma^5)) u'_j + \\
&+ \bar{d}'_i \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} (-U^{(d)}_{im} X_{d_{Rmn}} U^{(d)\dagger}_{nj} (1+\gamma^5) - V^{(d)}_{im} X_{d_{Lmn}} V^{(d)\dagger}_{nj} (1-\gamma^5)) d'_j + \\
&+ \bar{e}'_i \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} (-U^{(e)}_{im} X_{e_{Rmn}} U^{(e)\dagger}_{nj} (1+\gamma^5) - V^{(e)}_{im} X_{e_{Lmn}} V^{(e)\dagger}_{nj} (1-\gamma^5)) e'_j = \\
&= \bar{u}'_i \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} \{(X_1 - 1)\delta_{ij} + \epsilon^{u_L}_{ij} + \gamma^5[(X_1 + 1)\delta_{ij} - \epsilon^{u_L}_{ij}]\} u'_j + \\
&+ \bar{d}'_i \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} \{(-X_2 - 1)\delta_{ij} + \epsilon^{d_L}_{ij} + \gamma^5[(-X_2 + 1)\delta_{ij} - \epsilon^{d_L}_{ij}]\} d'_j + \\
&+ \bar{e}'_i \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} \{(-X_1 + 1)\delta_{ij} - \epsilon^{e_L}_{ij} + \gamma^5[(-X_1 - 1)\delta_{ij} + \epsilon^{e_L}_{ij}]\} e'_j
\end{aligned}
\tag{4.91}
$$

where we have set $\epsilon^{u_L}_{ij} = V^{(u)}_{i3} V^{(u)\dagger}_{3j}$, $\epsilon^{d_L}_{ij} = V^{(d)}_{i3} V^{(d)\dagger}_{3j}$ and $\epsilon^{e_L}_{ij} = V^{(e)}_{i3} V^{(e)\dagger}_{3j}$. Remembering that the diagonal vectorial part of the interaction never contributes (as we already justified previously), we can easily extract the relevant fermion-axion couplings for the M1 model. These are reported in 4.3, together with all the other ones.

What immediately appears to be peculiar of these generalized models is the presence of flavour-violationg couplings, which can in principle induce decays such as $K^{\pm} \to \pi^{\pm} + a_{\phi}$, $\mu^{\pm} \to e^{\pm} + a_{\phi}$ or $\tau^{\pm} \to e^{\pm} + a_{\phi}$, just to mention some possibilities. When these verteces appear, they are always proportional to the misalignment factor $\epsilon_{ij}^{f_L} = V_{i3}^{(f)} V_{3j}^{(f)\dagger}$ or $\epsilon_{ij}^{f_R} = U_{i3}^{(f)} U_{3j}^{(f)\dagger}$. Thus, in order to be able to say something about these new processes, one should make some hypothesis on the form of these objects. As described in [52], from the unitarity properties of $U$ and $V$, it immediately follows that the diagonal entries of $\epsilon^f$ satisfy $0 \le \epsilon_{ii}^f \le 1$ and $\sum_i \epsilon_{ii}^f = 1$, while for the off-diagonal ones the relation $|\epsilon_{i\neq j}^f| = \sqrt{\epsilon_{ii}^f \epsilon_{jj}^f}$ holds true just by the very definition. Once these constraints are fulfilled, one can either treat these $\epsilon_{ij}^f$ as free parameters or speculate about their order of magnitude. Given the lack of an explicit structure for $U$ and $V$, for the left quark sector the only reasonable thing we can imagine is that $V^{(u)}$ and $V^{(d)}$ are almost CKM-like. We remind that $V_{CKM} = V^{(u)} V^{(d)\dagger}$ and that, in the Wolfenstein parametrization, $V_{CKM}$ takes the form

$$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{4.92}$$

where $\lambda \approx 0.2$ and $A, \rho^2 + \eta^2 \approx \mathcal{O}(1)$. So, we can envisage that

$$\epsilon_{11}^{u_L/d_L} = V_{13}^{(u)/(d)} V_{31}^{(u)/(d)\dagger} \approx \lambda^6 \,, \qquad \epsilon_{22}^{u_L/d_L} = V_{23}^{(u)/(d)} V_{32}^{(u)/(d)\dagger} \approx \lambda^4 \,, \tag{4.93}$$

$$\epsilon_{33}^{u_L/d_L} = V_{33}^{(u)/(d)} V_{33}^{(u)/(d)\dagger} \approx 1, \tag{4.94}$$

$$\epsilon_{12}^{u_L/d_L} = \epsilon_{21}^{u_L/d_L} = V_{13}^{(u)/(d)} V_{32}^{(u)/(d)\dagger} \approx \lambda^5 \,, \qquad \epsilon_{13}^{u_L/d_L} = \epsilon_{31}^{u_L/d_L} = V_{13}^{(u)/(d)} V_{31}^{(u)/(d)\dagger} \approx \lambda^3 \,,$$
$$\epsilon_{23}^{u_L/d_L} = \epsilon_{32}^{u_L/d_L} = V_{23}^{(u)/(d)} V_{33}^{(u)/(d)\dagger} \approx \lambda^2 \tag{4.95}$$

in which the last set of equalities (4.95) can equally well be estimated using the unitary relations mentioned above for the off-diagonal misalignments.

These left misalignment parameters have a central role in the first four models of table 4.3. Nevertheless, because of the sign of color anomalies, the photon coupling can assume only two values and, even if fermion couplings slightly vary, there are only two classes of theories with a different phenomenology: those with $C_\gamma = 1.25$ or with $C_\gamma = 0.75$. According to our preceding estimation, we are being said that quark flavour violating verteces are highly suppressed by (4.95), while diagonal couplings for the first two generations essentially depend on PQ charges, because contributions in (4.93) are tiny corrections. If we neglect these latter, we see that the first nucleophobic constraint $C_u + C_d = 1$ holds, while $C_u \approx 2/3$ can be achieved by tuning the value of $c_\beta^2$ or $s_\beta^2$, depending on the case.

If we require electrophobia, too, all we have to impose is $C_{e_1} = 0$, which means $\epsilon_{11}^{e_L} = c_\beta^2 \approx 2/3$ (for M1-like models) or $\epsilon_{11}^{e_L} = s_\beta^2 \approx 1/3$ (for M2-like cases), because of the aforementioned nucleophobic conditions. A so large value of $\epsilon_{11}^{e_L}$ could be explained by physics beyond the SM and, in particular, by a consistently high mixing angle in the PMNS matrix, which should be related to $V^{(e)}$. The remaining diagonal misalignments of the leptonic sector can be fixed by the experimental constraint on the *muon-to-electron conversion* (a suppressed process in the SM scenario) and by the condition $\sum_i \epsilon_{ii}^f = 1$.

If we move onto the following models, we clearly observe that only three cases are phenomenologically new among the huge number of displayed possibilities: these ones have $C_\gamma = 3.25, 6.59$

| | | Parameters | | | Fermion couplings | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\mathcal{N}_{\mathbf{DW}}$ | $\mathbf{E/N}$ | $\lvert\mathbf{C}_\gamma\rvert$ | $\mathbf{C^A_{u_i}}$ | $\mathbf{C^A_{d_i}}$ | $\mathbf{C^A_{e_i}}$ | $\mathbf{C^{V/A}_{u_i\neq u_j}}$ | $\mathbf{C^{V/A}_{d_i\neq d_j}}$ | $\mathbf{C^{V/A}_{e_i\neq e_j}}$ |
| **left quarks** | $M1$ | $-1/X_\phi$ | 2/3 | 1.25 | $c_\beta^2-\epsilon_{ii}^{u_L}$ | $s_\beta^2-\epsilon_{ii}^{d_L}$ | $-c_\beta^2+\epsilon_{ii}^{e_L}$ | $\pm\epsilon_{ij}^{u_L}$ | $\pm\epsilon_{ij}^{d_L}$ | $\mp\epsilon_{ij}^{e_L}$ |
| | $M2$ | $-1/X_\phi$ | 8/3 | 0.75 | $c_\beta^2-\epsilon_{ii}^{u_L}$ | $s_\beta^2-\epsilon_{ii}^{d_L}$ | $s_\beta^2-\epsilon_{ii}^{e_L}$ | $\pm\epsilon_{ij}^{u_L}$ | $\pm\epsilon_{ij}^{d_L}$ | $\pm\epsilon_{ij}^{e_L}$ |
| | $M1'$ | $1/X_\phi$ | 8/3 | 0.75 | $s_\beta^2-\epsilon_{ii}^{u_L}$ | $c_\beta^2-\epsilon_{ii}^{d_L}$ | $c_\beta^2-\epsilon_{ii}^{e_L}$ | $\pm\epsilon_{ij}^{u_L}$ | $\pm\epsilon_{ij}^{d_L}$ | $\pm\epsilon_{ij}^{e_L}$ |
| | $M2'$ | $1/X_\phi$ | 2/3 | 1.25 | $s_\beta^2-\epsilon_{ii}^{u_L}$ | $c_\beta^2-\epsilon_{ii}^{d_L}$ | $-s_\beta^2+\epsilon_{ii}^{e_L}$ | $\pm\epsilon_{ij}^{u_L}$ | $\pm\epsilon_{ij}^{d_L}$ | $\mp\epsilon_{ij}^{e_L}$ |
| **right up quarks** | $M3$ | $-1/X_\phi$ | -4/3 | 3.25 | $-s_\beta^2+\epsilon_{ii}^{u_R}$ | $s_\beta^2$ | $-c_\beta^2+\epsilon_{ii}^{e_L}$ | $\epsilon_{ij}^{u_R}$ | 0 | $\mp\epsilon_{ij}^{e_L}$ |
| | $M3'$ | $-1/X_\phi$ | 2/3 | 1.25 | $-s_\beta^2+\epsilon_{ii}^{u_R}$ | $s_\beta^2$ | $s_\beta^2-\epsilon_{ii}^{e_L}$ | $\epsilon_{ij}^{u_R}$ | 0 | $\pm\epsilon_{ij}^{e_L}$ |
| | $M3''$ | $1/X_\phi$ | 2/3 | 1.25 | $-c_\beta^2+\epsilon_{ii}^{u_R}$ | $c_\beta^2$ | $c_\beta^2-\epsilon_{ii}^{e_L}$ | $\epsilon_{ij}^{u_R}$ | 0 | $\pm\epsilon_{ij}^{e_L}$ |
| | $M3'''$ | $1/X_\phi$ | -4/3 | 3.25 | $-c_\beta^2+\epsilon_{ii}^{u_R}$ | $c_\beta^2$ | $-s_\beta^2+\epsilon_{ii}^{e_L}$ | $\epsilon_{ij}^{u_R}$ | 0 | $\mp\epsilon_{ij}^{e_L}$ |
| | $M3_{bis}$ | $-2/X_\phi$ | 2/3 | 1.25 | $(c_\beta^2-\epsilon_{ii}^{u_R})/2$ | $s_\beta^2/2$ | $(-c_\beta^2+\epsilon_{ii}^{e_L})/2$ | $\epsilon_{ij}^{u_R}/2$ | 0 | $\mp\epsilon_{ij}^{e_L}/2$ |
| | $M3'_{bis}$ | $-2/X_\phi$ | 5/3 | 0.26 | $(c_\beta^2+\epsilon_{ii}^{u_R})/2$ | $s_\beta^2/2$ | $(s_\beta^2-\epsilon_{ii}^{e_L})/2$ | $\epsilon_{ij}^{u_R}/2$ | 0 | $\pm\epsilon_{ij}^{e_L}/2$ |
| | $M3''_{bis}$ | $2/X_\phi$ | 5/3 | 0.26 | $(s_\beta^2-\epsilon_{ii}^{u_R})/2$ | $c_\beta^2/2$ | $(c_\beta^2-\epsilon_{ii}^{e_L})/2$ | $\epsilon_{ij}^{u_R}/2$ | 0 | $\pm\epsilon_{ij}^{e_L}/2$ |
| | $M3'''_{bis}$ | $2/X_\phi$ | 2/3 | 1.25 | $(s_\beta^2+\epsilon_{ii}^{u_R})/2$ | $c_\beta^2/2$ | $(-s_\beta^2+\epsilon_{ii}^{e_L})/2$ | $\epsilon_{ij}^{u_R}/2$ | 0 | $\mp\epsilon_{ij}^{e_L}/2$ |
| **right down quarks** | $M4$ | $-1/X_\phi$ | 14/3 | 6.59 | $c_\beta^2$ | $-c_\beta^2+\epsilon_{ii}^{d_R}$ | $s_\beta^2-\epsilon_{ii}^{e_L}$ | 0 | $\epsilon_{ij}^{d_R}$ | $\pm\epsilon_{ij}^{e_L}$ |
| | $M4'$ | $-1/X_\phi$ | 8/3 | 0.75 | $c_\beta^2$ | $-c_\beta^2+\epsilon_{ii}^{d_R}$ | $-c_\beta^2+\epsilon_{ii}^{e_L}$ | 0 | $\epsilon_{ij}^{d_R}$ | $\mp\epsilon_{ij}^{e_L}$ |
| | $M4''$ | $1/X_\phi$ | 8/3 | 0.75 | $s_\beta^2$ | $-s_\beta^2+\epsilon_{ii}^{d_R}$ | $-s_\beta^2+\epsilon_{ii}^{e_L}$ | 0 | $\epsilon_{ij}^{d_R}$ | $\mp\epsilon_{ij}^{e_L}$ |
| | $M4'''$ | $1/X_\phi$ | 14/3 | 6.59 | $s_\beta^2$ | $-s_\beta^2+\epsilon_{ii}^{d_R}$ | $c_\beta^2-\epsilon_{ii}^{e_L}$ | 0 | $\epsilon_{ij}^{d_R}$ | $\pm\epsilon_{ij}^{e_L}$ |
| | $M4_{bis}$ | $-2/X_\phi$ | 8/3 | 0.75 | $c_\beta^2/2$ | $(s_\beta^2-\epsilon_{ii}^{d_R})/2$ | $(s_\beta^2-\epsilon_{ii}^{e_L})/2$ | 0 | $-\epsilon_{ij}^{d_R}/2$ | $\pm\epsilon_{ij}^{e_L}/2$ |
| | $M4'_{bis}$ | $-2/X_\phi$ | 5/3 | 0.26 | $c_\beta^2/2$ | $(s_\beta^2-\epsilon_{ii}^{d_R})/2$ | $(-c_\beta^2+\epsilon_{ii}^{e_L})/2$ | 0 | $-\epsilon_{ij}^{d_R}/2$ | $\mp\epsilon_{ij}^{e_L}/2$ |
| | $M4''_{bis}$ | $2/X_\phi$ | 5/3 | 0.26 | $s_\beta^2/2$ | $(c_\beta^2-\epsilon_{ii}^{u_R})/2$ | $(-s_\beta^2+\epsilon_{ii}^{e_L})/2$ | 0 | $-\epsilon_{ij}^{d_R}/2$ | $\mp\epsilon_{ij}^{e_L}/2$ |
| | $M4'''_{bis}$ | $2/X_\phi$ | 8/3 | 0.75 | $s_\beta^2/2$ | $(c_\beta^2-\epsilon_{ii}^{u_R})/2$ | $(c_\beta^2-\epsilon_{ii}^{e_L})/2$ | 0 | $-\epsilon_{ij}^{d_R}/2$ | $\pm\epsilon_{ij}^{e_L}/2$ |
| $\mathbf{u_R}$ **and** $\mathbf{d_R}$ | $M5$ | $-1/X_\phi$ | 2/3 | 1.25 | $-c_\beta^2+\epsilon_{ii}^{u_R}$ | $-s_\beta^2+\epsilon_{ii}^{d_R}$ | $-c_\beta^2+\epsilon_{ii}^{e_L}$ | $\epsilon_{ij}^{u_R}$ | $\epsilon_{ij}^{d_R}$ | $\mp\epsilon_{ij}^{e_L}$ |
| | $M5'$ | $-1/X_\phi$ | 8/3 | 0.75 | $-c_\beta^2+\epsilon_{ii}^{u_R}$ | $-s_\beta^2+\epsilon_{ii}^{d_R}$ | $s_\beta^2-\epsilon_{ii}^{e_L}$ | $\epsilon_{ij}^{u_R}$ | $\epsilon_{ij}^{d_R}$ | $\pm\epsilon_{ij}^{e_L}$ |
| | $M5''$ | $1/X_\phi$ | 8/3 | 0.75 | $-c_\beta^2-\epsilon_{ii}^{u_R}$ | $-s_\beta^2-\epsilon_{ii}^{d_R}$ | $c_\beta^2-\epsilon_{ii}^{e_L}$ | $\epsilon_{ij}^{u_R}$ | $\epsilon_{ij}^{d_R}$ | $\pm\epsilon_{ij}^{e_L}$ |
| | $M5'''$ | $1/X_\phi$ | 2/3 | 1.25 | $-c_\beta^2-\epsilon_{ii}^{u_R}$ | $-s_\beta^2-\epsilon_{ii}^{d_R}$ | $-s_\beta^2+\epsilon_{ii}^{e_L}$ | $\epsilon_{ij}^{u_R}$ | $\epsilon_{ij}^{d_R}$ | $\mp\epsilon_{ij}^{e_L}$ |
| | $DFSZI$ | $3/X_\phi$ | 8/3 | 0.75 | $c_\beta^2/3$ | $s_\beta^2/3$ | $s_\beta^2/3$ | 0 | 0 | 0 |
| | $DFSZII$ | $3/X_\phi$ | 2/3 | 1.25 | $c_\beta^2/3$ | $s_\beta^2/3$ | $-c_\beta^2/3$ | 0 | 0 | 0 |

Table 4.3: The table displays the axion-fermion couplings for the different theories. The first three columns show respectively the domain wall number $N_{DW}$, the color to electromagnetic anomaly ratio $E/N$ and the consequent photon coupling $C_\gamma$, computed with the formula $C_\gamma = E/N - 1.92$. The highlighted rows are those associated to models where nucleophobia is feasible. We would like to underscore once more, that for $N_{DW}$ we simply applied the definition $N_{DW} = -N/X_\phi$. Therefore, the domain wall number could show up under the guise of a negative quantity, on occasions, depending on $N$. This unwelcome minus sign must be understood as a harmless factor, to eventually collect away from the $N_{DW}$ definition: it will just affect the anomalous axion-gluon coupling, which is physically insensitive to an overall sign redefinition.

and 0.26. The last value is associated to $M'_{3\text{bis}}$, $M''_{3\text{bis}}$, $M'_{4\text{bis}}$ and $M''_{3\text{bis}}$, but, for all of them, nucleophobia is unfeasible. For the remaining cases, the situation is a bit different from the previous ones. Indeed, to get a nucleophobic theory with $C_u + C_d = 1$, we just have to impose $\epsilon_{11}^{u_R} = 1$ or $\epsilon_{11}^{d_R} = 1$, depending on whether we are dealing with the $M3$ or $M4$ class. That is perfectly legitimate, because, in these cases, there is no working hypothesis on the structure of $\epsilon_{ij}^{f_R}$ coming from electroweak considerations. The second nucleophobic condition is instead fulfilled if $s_\beta^2$ or $c_\beta^2$ are suitable chosen. By fixing the $\beta$ angle, we will automatically determine the $\epsilon_{11}^{e_L}$ value which ensures electrophobia exactly as we have already shown.

Before exploring the experimental falls-out of nucleophobic models, we would like to point out that, by relaxing the PQ univerality condition, we were also able to pave the way for the solution of another open problem: the domain wall issue. In table 4.3 the different domain wall numbers associated to each model are presented.

A particularly appealing class is that which shows a $\mathcal{N}_{DW} = -1/X_\phi$. If we consider a DFSZ theory with a cubic potential, we readily obtain a possible solution to the flourishing of topological defects in a post-inflationary PQ breaking mechanism. We already stressed how this one is not the only viable solution to the problem: despite that, it is noteworthy how a cubic nucleophobic model can be naturally endowed with interesting cosmological properties, too.

## 4.5 Astrophobia and phenomenology

It is pretty clear that the solution of the domain wall conundrum is just a side-effect of astrophobic models. The major outcome of considering theories where couplings to electrons and nucleons are suppressed is the possibility of raising the axion mass up to the so-called *heavy axion window*: $20meV \lesssim m_a \lesssim 200meV$. We will show how axion properties can be extremely constrained by astrophysical observations and, together with particle physics considerations, can be used to test the astrophobic scenario as a possible candidate for physics beyond the standard model.

### 4.5.1 Astrophysical bounds

The astrophysical observations which can be employed to fix some limits on axion interaction properties are a huge number and we are not going to cover all of them here. Nevertheless, for a fairly complete discussion, we are going to take into account four of them:

- the branching ratio in globular clusters;
- the hot dark matter boundary;
- the white dwarf cooling anomalies;
- the burst duration of the supernovae neutrino signal.

The key idea of all astrophysical measurements is that, if axions existed, they would provide a significant channel through which galactic bodies could release their energy, together with the standard known cooling processes. The first test we have recalled is based on the observation that the effective axion-photon coupling never disappears in our models. As described in [53], globular clusters populating the Milky Way Galactic Halo (a nearly spherical volume of stars and dust surrounding galaxies) can be used as a constraint for the photon coupling parameter, that is customarily defined as $g_{a\gamma} = \alpha_{em}C_\gamma/(2\pi f_a)$. A Globular Cluster is a tightly gravitationally bound system of stars (they are among the oldest structures of our galaxy). According to their formation mechanism, they are expected to have the same origin and practically identical features.

The great amount of stars they can host (a few milion of them) makes very easy to spot different stellar evolution phases: each of them will be very well populated and simply distinguished from the others. That provides a relevant statistical sample, which renders globular clusters good candidates for a precise analysis. In particular, one can make out: the main sequence, related to the core hydrogen burning phase; the red giant branch (RGB), during which the gravitational force is balanced by the radiatio pressure, obtained by burning the hydrogen external shell; the horizontal branch (HB), where the helium presents in the core is consumed. For a statistically wide family of stars, the number of members of each region is related to lifetime, which, in turn, is affected by the efficiency of all energy loss channels. Employing this observation, a boundary of $|g_{a\gamma}| < 6.6 \times 10^{-11} GeV^{-1}$ was obtained by measuring for each globular cluster the *R parameter*, defined as $R = N_{HB}/N_{RGB}$, i.e the ratio between the HB and the RGB stars. Indeed, the latters are expected to be less sensitive to the axion-photon channel with respect to the formers because of their temperature, which makes the bremsstrahlung processes a definitely more efficient cooling mechanism.

A second boundary comes from hot dark matter axions. As we know, the majority of alleged dark matter should be cold, according to galaxy scale considerations. Nevertheless, variations in the fraction of hot dark matter can produce testable consequences on structure formation, anisotropy in the cosmic microwave background (CMB) or even in the abundance of species generated during the primordial nucleosynthesis. Both [39] and [54], for example, point out how the presence of hot axion relic can affect the customary effective number of relativistic degrees of freedom $N_{eff} = 3.046$ (given by the three neutrino species, together with a relativistic correction of 0.046): as a consequence, a lot of astrophysical observables should keep track of hot axions through a $\Delta N_{eff}$. In the early universe, with a temperature in the interval $T_{QCD} < T < T_\pi$, the main thermalization mechanism for axions is given by pion interactions. Computing the efficiency of these reactions, one can get the axion decoupling temperature and the corresponding effective degrees of freedom at that energy: from here, the desired information on the CMB or other astrophysical quantities can be extracted. In particular, the presented reasoning has been used in [55] to constrain the axion mass with an upper bound of $m_a \lesssim 0.8eV$. In spite of that, one should consider that, in a nucleophobic model, the axion-pion couplings will be highly suppressed, too: as stated by [56] and as we have explicitly shown, all of pion interactions are proportional to $C_u - C_d \approx 0$. This suggests that bounds from hot dark matter considerations could be grossly relaxed (even if this will not affect our analysis).

The remaining more stringent constraints rely on interactions of axions with fermions. To compare our models with physical data, it will prove to be much more useful to rewrite our Yukawa Lagrangian in a slighlty different way. We can start from a general fermion contribution and manipulate it as

$$\begin{aligned} \mathcal{L}_{fermion-axion} &= \bar{f}'_i \gamma^\mu \frac{\partial_\mu a_\phi}{2f_a} (C^V_{f_i f_j} + \gamma^5 C^A_{f_i f_j}) f'_j = \\ &= -\partial_\mu \bar{f}'_i \gamma^\mu \frac{a_\phi}{2f_a} (C^V_{f_i f_j} + \gamma^5 C^A_{f_i f_j}) f'_j - \bar{f}'_i \frac{a_\phi}{2f_a} (C^V_{f_i f_j} - \gamma^5 C^A_{f_i f_j}) \gamma^\mu \partial_\mu f'_j = \\ &= -i \frac{a_\phi}{2f_a} \bar{f}'_i ((m_{f_i} - m_{f_j}) C^V_{f_i \neq f_j} + (m_{f_i} + m_{f_j}) \gamma^5 C^A_{f_i f_j}) f'_j \end{aligned} \tag{4.96}$$

which explicitly shows how fermion verteces are proportional to the respective fermion masses. In particular, the complete effective Lagrangian turns out to be

$$\mathcal{L} = \frac{\partial_\mu a_\phi}{2f_a} (C_p \bar{p} \gamma^\mu \gamma^5 p + C_n \bar{n} \gamma^\mu \gamma^5 n + C_e \bar{e} \gamma^\mu \gamma^5 e) + \frac{a_\phi}{f_a} \frac{\alpha_{em}}{8\pi} C_\gamma \mathcal{F}_{\mu\nu} \tilde{\mathcal{F}}^{\mu\nu} \tag{4.97}$$

which presents only diagonal terms: an useful quantity parametrizing fermion verteces will be $g_{af} = C^A_f m_f / f_a$, as it can be readily read off from (4.96). The effective Lagrangian shown above will be extensively satisfactory to study astrophysical processes.

Information about the axion-electron coupling can be derived from the white dwarf luminosity function, which describes the number of stars per unit volume and luminosity interval. The basic idea of this experimental method, presented in [57], relies on the fact that white dwarfs (WD) are the end products of star lifes: their evolution can not be described in terms of thermonuclear reactions, but just as a gravothermal process of cooling. In this scenario, an additional cooling channel involving axions could significantly affect the shape of the luminosity function itself. From a historical point of view, the introduction of new cooling sources had been a great step forward in an eventual insight of the so-called *cooling anomaly problem*, according to which stars seem to cool faster than predicted. These studies make it possible to fix a bound for $g_{ae}$: as a matter of fact, the interaction with electrons is, reasonably, the prevailing axion process here. This one manifests itself through the bremsstrahlung reaction $e + N \to e + N + a$, where electrons are slowed down by the electromagnetic field of nuclei: the outgoing electron can release part of its energy through axions production. In [58] the bound on the electron coupling has been fixed to

$$|g_{ae}| < 2.7 \times 10^{-13} \tag{4.98}$$

Finally, we are going to consider an astrophysical phenomenon which will let us derive a limit on the axion-nucleon coupling. As claimed in [59], an efficient measurement of it can come from supernovae (SN). This violent event, associated to the core collape of a massive star, will be followed by a cooling phase of the nascent neutron star, where a lot of energy is released through a neutrino signal. Nevertheless, reactions of nucleon-nucleon axion bremsstrahlung (i.e $N + N \to N + N + a$) can provide an extra cooling mechanism, which will hasten the neutrino emission, leading to fewer events over a shorter time. By computing the matrix element for the process, one notices that the leading contribution is proportional to the non-obvious combination $g_{ap}^2 + g_{an}^2$ of the two nucleons involved in the reactions. From that, the amount of energy loss due to axions flux can be estimated. In [58], this procedure applied to the neutrino signal of SN 1987A (the closest observed supernovae from the telescope invention) resulted in the constraint

$$g_{ap}^2 + g_{an}^2 < 3.6 \times 10^{-19} \tag{4.99}$$

To extract some phenomenology, we are going to consider the four nucleophobic models $M1$, $M2$, $M3$ and $M4$, the two non-universal and non-nucleophobic $M_{3bis}$ and $M''_{3bis}$ theories and, finally, the old DFSZI and DFSZII ones, to better appreciate the differences. We would like to stress how the $g_{a\gamma}$ and hot dark matter boundaries affect all of the models in a simple way; what should be treated case by case are constraints on matter couplings.

To start with, we are going to consider the nucleophobic class. We have already observed how the electron coupling can be easily and completely removed by means of a right choise of $\beta$: if this was the only constraint, we would always have enough freedom to raise the axion mass up to the hot dark matter bound. However, a stronger limit comes from the coupling to nucleons: it can be suppressed by nucleophobic conditions, but not removed once for all, because of the small surving $\delta$ correction. In order to impose nucleophobia, we can set $C_u = 2/3$ and $C_d = 1/3$. Then, we can evalutate the $\delta$ contribution using $\delta = 0.038(5)C_s + 0.012(5)C_c + 0.009(2)C_b + 0.0035(4)C_t$. For the $M1$ and $M2$ models, the four remaining couplings which appear in the formula can be easily assessed using (4.93) and (4.94) for the CKM-like matrix elements; for $M3$ and $M4$, instead, we have no hypothesis on the $\epsilon_{ii}^{f_R}$ terms, but we know that $\epsilon_{11}^{u_R/d_R} = 1$ should hold true to achieve nucleophobia. This latter, together with $\sum_i \epsilon_{ii}^{f_L} = 1$, requires $\epsilon_{22}^{u_R/d_R} = \epsilon_{33}^{u_R/d_R} \approx 0$. From (4.74), we can obtain some expressions for $C_p$ and $C_n$, which directly enter our condition (4.99) as

$$(C_p^2 m_p^2 + C_n^2 m_n^2) - 3.6 \times 10^{-19} f_a^2 = 0 \tag{4.100}$$

where we have saturated the bound and in which, clearly, $m_p$ and $m_n$ are the proton and neutron masses respectively. By using (2.75), one gets a relation for the maximally allowed axion mass, which turns out to fall into the heavy axion mass window. The results are $m_a = 0.20eV$ for $M1/M2$, $m_a = 0.25eV$ for $M3$ and $m_a = 0.12eV$ for $M4$.

For the models $M_{3bis}$ and $M''_{3bis}$, the electron coupling constraint can again be evaded by means of a prudent choise of the mixing angle in the PMNS sector. The sole bound is again the nucleon one. Nevertheless, here the analysis is pretty different, because nucleophobia cannot be achieved: this will result in a freedom in the $\beta$ value, that, in principle, is not fixed by any other criterion. Moreover, here, we will set $\epsilon_{11}^{u_R} = 0$, $\epsilon_{22}^{u_R} = 0$ and $\epsilon_{33}^{u_R} = 1$, where the relation $\sum_i \epsilon_{ii}^{u_R} = 1$ must hold independently on the CKM-like form of the $\epsilon$ parameters. In these conditions, we have $C_n + C_p = 1/2$. It can be directly verified that, by choosing the $\epsilon_{ii}^{u_R}$ terms this way, we are able to lift the axion mass the most. Proceeding like before, we will finally get trickier expressions for the nucleon bound (4.100), which is now a function of $\beta$:

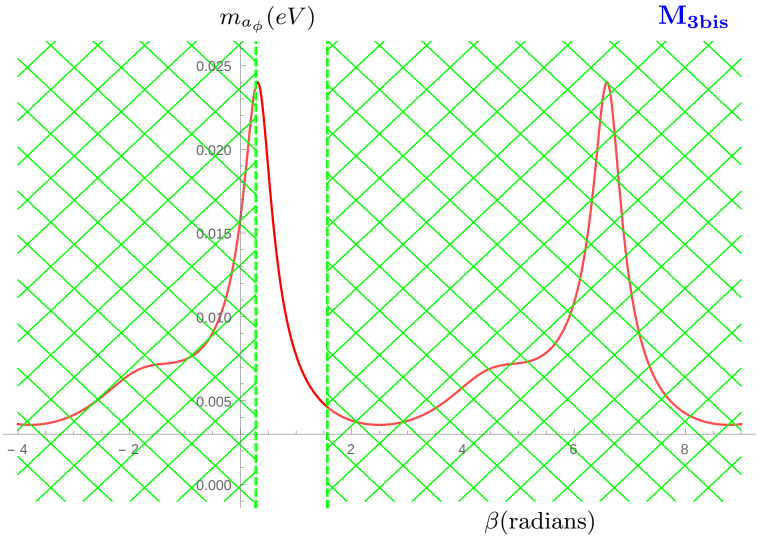
$$\begin{aligned}
&0.196 + 0.202c_\beta^2 + c_\beta(-0.353 - 0.315s_\beta) + 0.162s_\beta + 0.197s_\beta^2 - 1.17\times10^{-5}/m_a^2 = 0 \quad M_{3bis}\\
&0.196 + 0.197c_\beta^2 + c_\beta(-0.353 - 0.315s_\beta) + 0.353s_\beta + 0.202s_\beta^2 - 1.17\times10^{-5}/m_a^2 = 0 \quad M''_{3bis}
\end{aligned} \tag{4.101}$$

The plots for these two equations are shown in figure 4.2. Even if the range of $\beta$ can seem to be arbitrary, we have to remember that this value directly appears in the Lagrangian through the Yukawa couplings, which must guarantee a perturbative expansion both at high and low energy, as explained in [60]. We will not go through this delicate topic here, but we will assume the benchmark interval $0.28 < \tan\beta < 140$ [61]. This range has been obtained by computing the $\beta$-function for the Yukawa coupling of the light Higgs of a 2HDM model and by requiring the condition $y^2/4\pi < 1$ at different energy scales. The lower bound is common to all possible 2HDM theories, i.e type I, II, Leptonic-specific and Flipped (which vary for the position of the two Higgs doublet in the Yukawa sector: see [62] for details). The upper bound is of $\tan\beta < 140$ just for the Flipped model, whereas it is of $\tan\beta < 350$ for the Leptonic-specific one. Just because, as claimed in [61], further constraints can arise requiring the perturbative unitarity of other coupling constants, we will restrict ourselves at least to the tightest range of values between the two ones. It can be questioned that our models are not 2HDM-like, because DFSZ theories add a complex singlet field to the spectrum and, in addition, they impose a PQ symmetry, which prohibits the emergence of some terms in the Lagrangian. Nevertheless, for our analysis, it will be enough to consider this interval as a point of reference. After all, if we remember that $v_\phi \gg v$, the additional heavy scalar $\rho$ almost decouples from the calculation of the $\beta$-function corrections at energy lower than $v_\phi$ and the axion couplings are highly suppressed. In this way, DFSZ models can be considered like a subclass of 2HDM models, without grossly affecting our benchmark range for $\tan\beta$.

Taking this into account, we can graphically derived the highest axion mass value achievable in these theories. That turns out to show up in the limit $\tan\beta \to 0.28$ for the $M_{3bis}$ case and for $\tan\beta \approx 3.16$ in $M''_{3bis}$, with two pretty similar results of $m_a \approx 24meV$.

We finally move onto the ordinary DFSZ models, in order to underscore their departure from the new cases we have presented. The considerations done for nucleon couplings still hold in these situations. But now we have no freedom to remove the electron coupling (unless we choose a particular value of $\beta$), so that the bound

$$C_e m_e = 2.7\times10^{-13} f_a \tag{4.102}$$



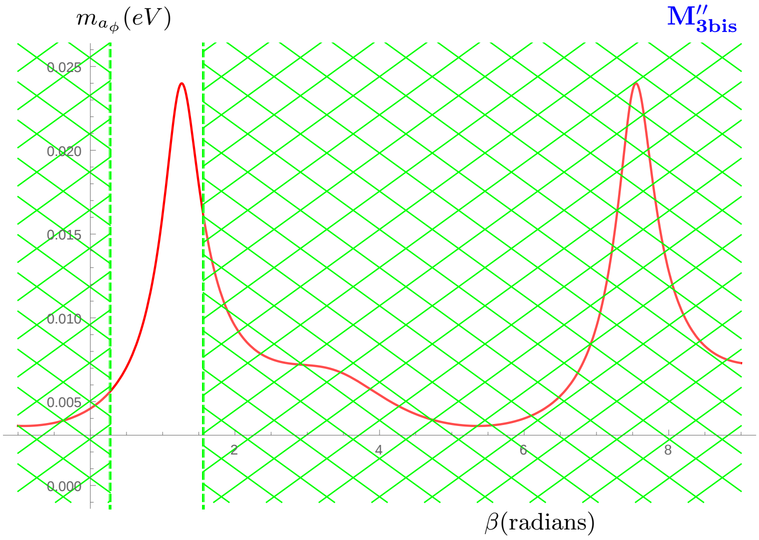




Figure 4.2: *The two plots show the axion mass dependence on the $\beta$ parameter (radians) for the $M_{3bis}$ model (the upper graph) and the $M''_{3bis}$ one (the lower graph). The green grid highlights regions of $\beta$ excluded by the perturbative interval in a reference period of the function.*

must be satisfied, too. By doing that, we are led to a system of equations

$$\begin{aligned} &\begin{cases} 2.99 \times 10^{-11} m_a s_\beta^2 - 2.7 \times 10^{-13} = 0 & \text{DFSZII} \\ 0.183 - 0.170 c_{2\beta} + 0.0397 c_{4\beta} - 1.17 \times 10^{-5}/m_a^2 = 0 \end{cases} \\ &\begin{cases} 2.99 \times 10^{-11} m_a c_\beta^2 - 2.7 \times 10^{-13} = 0 & \text{DFSZI} \\ 0.183 - 0.170 c_{2\beta} + 0.0397 c_{4\beta} - 1.17 \times 10^{-5}/m_a^2 = 0 \end{cases} \end{aligned} \tag{4.103}$$

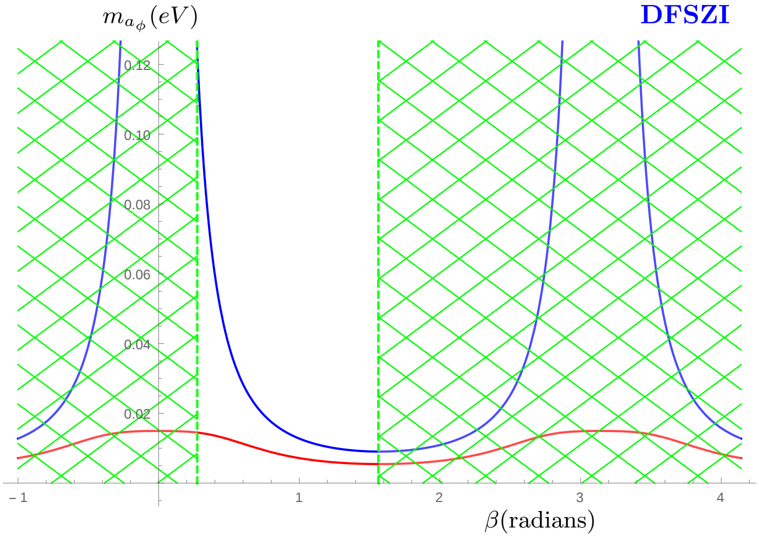
whose plots are displayed in figure 4.3.

In the DFSZI case, the plot reveals how the electron coupling constraint (blue line) does not really provide a further bound: one can just consider the best value for $m_a$ inside the unitary interval given by the nucleon bound (red curve). The maximal axion mass compatible with this model is $m_a \approx 14meV$ in the limit $\tan\beta \to 0.28$. For the DFSZ II version, we can not rise the axion mass up to this value of $\beta$ without violating the restriction on $g_{ae}$: looking at the curves intersection in the graph, one can get $m_a \approx 12meV$. It worth noticing that the blue curve presents some points where it diverges: these ones correspond to the $\beta$ values which annul the axion-electron coupling. On the contrary, the red curve always has a different behaviour: there are no divergences, because the nucleon coupling can be suppressed, but never removed, as a result of the non-vanishing $\delta$ correction.

In figure 4.4, all of the outcomes we have quoted are summarized. Here, the different constraints that we have described in this paragraph are considered. While plotting the $|g_{a\gamma}|$ coupling as a function of $m_a$, one should take into account the upper limit imposed by the $R$ parameter and other restrictions on axion mass. One of these comes from the model independent bound $m_a \lesssim 0.8eV$; then, there are the model dependent constraints from WD and SN studies, which are graphically displayed by colorful bullets or stars truncating the curves. The PQ non-universal models, which do not enjoy nucleophobia, can just slighlty increase the maximal value of axion mass with respect to the old paradigm, provided that the $\beta$ parameter is properly tuned and electrophobia is required. Hence, the really essential feature to reach the heavy axion mass window resides in the ability of removing the nucleon coupling.

First of all, it worth noticing that the nucleophobic scenario can easily accomodate the recent fits for $g_{ae}$ and $g_{a\gamma}$ from a combined analysis of HB and WD [58]: $\bar{g}_{ae} = 1.5 \times 10^{-13}$ and $\bar{g}_{a\gamma} = 0.14 \times 10^{-10} GeV^{-1}$. If these values are in tension with the SN bounds in the old DFSZ theories, this is not true for the nucleophobic ones.

In addition, these models are very interesting from an experimental viewpoint, because to be revealed they ask for a sensitivity to $|g_{a\gamma}|$ which is just slightly greater than the current one. Therefore, the next generation of axion helioscopes (which employ the axion-photon interaction to detect possible axions coming from the Sun), such as IAXO and its upgrade IAXO+, will be crucial to constrain or discover an astrophobic axion species.



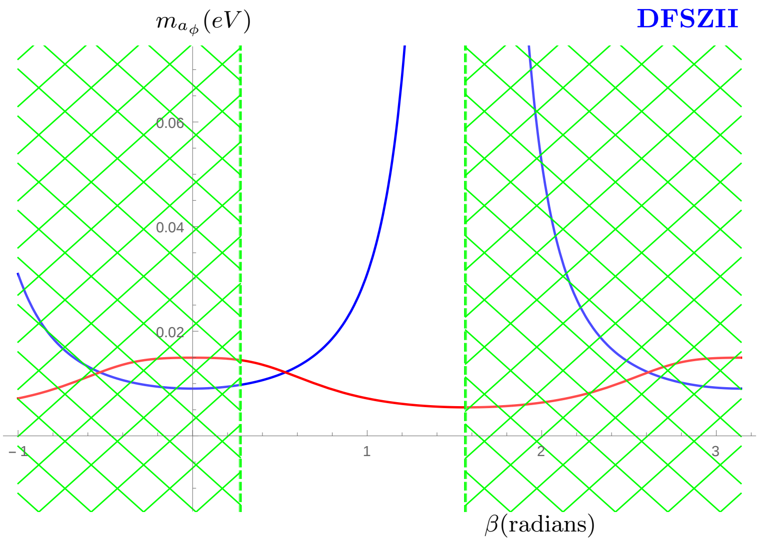




Figure 4.3: *The two plots show the axion mass dependence on the $\beta$ parameter (radians) given by the bound on $g_{ae}$ (blue curve) and on the nucleon coupling (red curve). The upper graph refers to the DFSZI model, while the lower one to the DFSZII theory. The green grid highlights regions of $\beta$ excluded by the perturbative interval in a reference period of the function.*

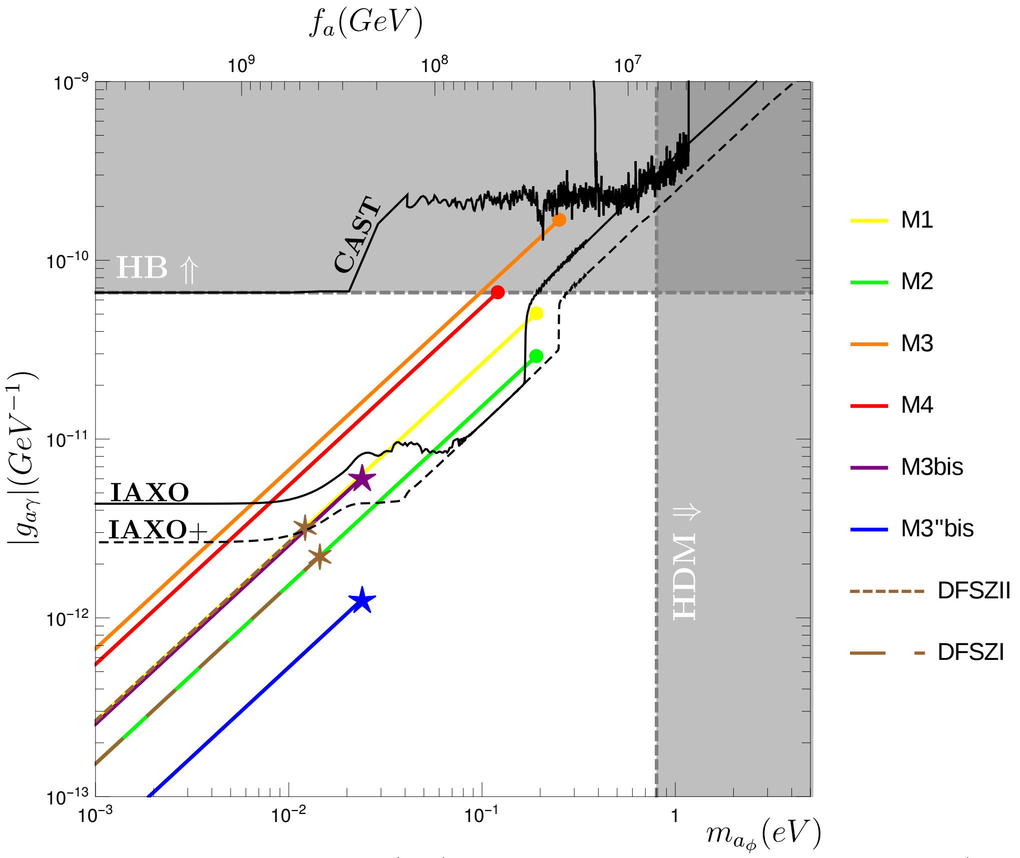



Figure 4.4: *The graph presents the $|g_{a\gamma}|$ coupling as a function of the axion mass (or its decay constant) in a double logarithmic scale. Each straight line is truncated when the upper bound on the axion mass from $g_{ae}$ and $g_{ap}^2 + g_{an}^2$ is saturated. The horizontal and vertical gray regions represent the excluded range of values obtained from the R parameter and the hot axion limit respectively. It worth noticing how nucleophobic models are able to reach the heavy axion window, making them testable by future experiments. The black lines highlight the sensitivity of some next-generation axion helioscopes (i.e IAXO and IAXO+) to $|g_{a\gamma}|$ with respect to the old CAST helioscope.*

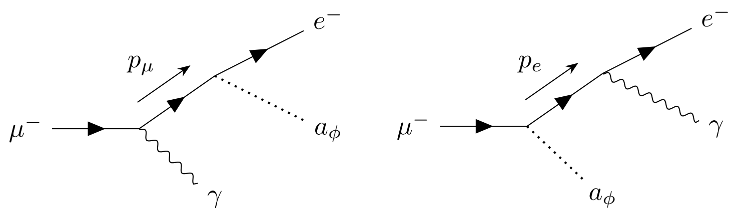



Figure 4.5: The two diagrams contributing to the muon decay $\mu^- \to e^- + \gamma + a_\phi$. They differ by the position of the two interaction verteces and, as a consequence, by the exchange of a virtual muon or electron respectively.

### 4.5.2 Precision Flavour Experiments

An unavoidable property of PQ non-universal models is the appearence of flavour changing verteces. This feature offers a further possibility to constrain and test these theories through precise particle physics experiments.

In the lepton sector, the strongest bound on flavour violating processes stems from the *muon to electron transition*. As already mentioned, this decay is not forbidden by the SM, but highly suppressed: a neutrino oscillation mechanism will render this process viable, but with a branching ratio $\mathcal{B}(\mu^+ \to e^+) \leq 10^{-54}$ [63] (extremely below current experimental bounds). On the contrary, physics beyond the standard model could facilitate this channel: that is the case with our axion models, which breaks family symmetry. To derive some comparable results, we need to get an estimation for the free parameter $\epsilon_{22}^{e_L}$, which can be extracted from family violating transition involving the second lepton generation. The muon-to-electron transition exactly fits the bill. An upper bound on the muon decay into an electron and a pseudoscalar particle $f$ is given in [64] to the value of $\mathcal{B}(\mu^+ \to e^+ f) \leq 2.6 \times 10^{-6}$. All pseudoscalar particles inducing flavour violation are dubbed *familons*: in our generalized models, the axion is a particular kind of familon. Nevertheless, how explained in [65], this tight constraint on $\mathcal{B}(\mu^+ \to e^+ f)$ has been derived under the assumption of a vectorial coupling of familons to SM particles, in order to deal with some technical complications. Just because our Yukawa Lagrangian shows both axial and vectorial couplings, this upper limit can not be used in our reasoning.

What we can do is considering the more stringent experimental constraint $\mathcal{B}(\mu^+ \to e^+ + \gamma + f) < 1.1 \times 10^{-9}$, where leptons of both chiralities take part in it: this will give rise to a weaker limit on the familon coupling for this process. This latter was obtained in [64] from the previous condition:

$$F_{\mu e} = \frac{2 f_a}{\sqrt{|C_{e\mu}^V|^2 + |C_{e\mu}^A|^2}} > 3.1 \times 10^9 GeV \tag{4.104}$$

Remembering that $|C_{e\mu}^V|^2 = |C_{e\mu}^A|^2 = \epsilon_{11}^{e_L} \epsilon_{22}^{e_L}$, we can constrain the axion mass as

$$m_a < 0.2eV \sqrt{\frac{1.7 \times 10^{-4}}{\epsilon_{11}^{e_L} \epsilon_{22}^{e_L}}} \tag{4.105}$$

Not to have a constraint stronger than SN bounds, we can require the left-hand side of the previous expression to be greater than the tighter bound $m_a < 0.25eV$ of the $M3$ model, where $\epsilon_{11}^{e_L}$ should be fixed to $1/(1+z)$ ($z \approx 0.48$) to reach electrophobia. This implies

$$\epsilon_{22}^{e_L} < 1.6 \times 10^{-4} \tag{4.106}$$

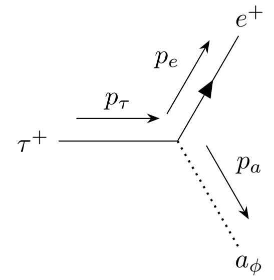



Figure 4.6: Diagram of the flavour changing $\tau^+$ decay: $\tau^+ \to e^+ + a_\phi$

Because of $\sum_i \epsilon_{ii}^{e_L} = 1$, $\epsilon_{33}^{e_L}$ will be determined, too, depending on the value of $\epsilon_{11}^{e_L}$. Therefore, we needed an experimental input to estimate all of the diagonal leptonic $\epsilon$ parameters: for this purpose, we employed one of the most precisely measured process.

Once that our variables have been fixed, we can explicitly compute the branching ratio for some interesting flavour violating decays. Let us consider the important $\tau^+ \to e^+ + a_\phi$. In order to compare experimental results with our theory, we need to calculate the decay rate given by the simple diagram in 4.6.

Starting from the Yukawa Lagrangian contributing to the decay

$$\mathcal{L}_{\tau\to e} = \bar{e}\gamma^\mu \frac{\partial_\mu a_\phi}{2f_a}(C^V_{e\tau} + \gamma^5 C^A_{e\tau})\tau = -\bar{e}\frac{ia_\phi}{2f_a}((m_e - m_\tau)C^V_{e\tau} + \gamma^5(m_e + m_\tau)C^A_{e\tau})\tau \tag{4.107}$$

we can readily obtain the corresponding matrix element from the LSZ reduction formula

$$\mathcal{M}_{if} = -\frac{i}{2f_a}\bar{v}_r(p_e)_\alpha((m_e - m_\tau)C^V_{e\tau} + \gamma^5(m_e + m_\tau)C^A_{e\tau})_{\alpha\beta} v_s(p_\tau)_\beta \tag{4.108}$$

Using the shorthand notation $\Delta m = (m_e - m_\tau)$ and $\Sigma m = (m_e + m_\tau)$, we can sum up over the final spin polaritazions and average over the initial ones. By employing the spin sum rules for the antiparticle spinor state $v_r$, one gets:

$$\begin{aligned}
&\frac{1}{2}\sum_{r,s} |\mathcal{M}_{if}|^2 = \\
&= \sum_{r,s} \frac{1}{8f_a^2}\bar{v}_s(p_\tau)_\rho(\Delta m C^V_{\tau e} - \gamma^5 \Sigma m C^A_{\tau e})_{\rho\sigma} v_r(p_e)_\sigma \bar{v}_r(p_e)_\alpha(\Delta m C^V_{e\tau} + \gamma^5\Sigma m C^A_{e\tau})_{\alpha\beta} v_s(p_\tau)_\beta = \\
&= \frac{1}{8f_a^2} tr[(\Delta m C^V_{e\tau} - \gamma^5 \Sigma m C^A_{e\tau})(\not{p}_e - m_e)(\Delta m C^V_{e\tau} + \gamma^5 \Sigma m C^A_{e\tau})(\not{p}_\tau - m_\tau)] = \\
&= \frac{1}{8f_a^2} tr[(\Delta m C^V_{e\tau})^2(\not{p}_e \not{p}_\tau) + (\Delta m C^V_{e\tau})^2 m_e m_\tau - (\Sigma m C^A_{e\tau})^2 \gamma^5 \not{p}_e \gamma^5 \not{p}_\tau - (\Sigma m C^A_{e\tau})^2 m_e m_\tau] = \\
&= \frac{1}{8f_a^2}[(\Delta m C^V_{e\tau})^2(4p_e \cdot p_\tau + 4m_e m_\tau) + (\Sigma m C^A_{e\tau})^2(4p_e \cdot p_\tau - 4m_e m_\tau)]
\end{aligned} \tag{4.109}$$

Because of the condition $(C^V_{e\tau})^2 = (C^A_{e\tau})^2$, the model dependent coupling can be collected. Then, we can proceed with some elementary kinematic manipulations. Using $p^\tau_\mu = (m_\tau, 0)$, we get $p_e \cdot p_\tau = m_\tau E_e$. To compute $E_e$, we know that

$$\begin{cases} \vec{p}_\tau = \vec{0} = \vec{p}_a + \vec{p}_e \quad \Rightarrow \quad |p_e| = |p_a| \\ E_\tau = m_\tau = E_e + E_a \end{cases} \tag{4.110}$$

Considering both the definition $E_e = \sqrt{m_e^2 + p_e^2}$ and the previous conservation laws, we can write

$$m_\tau - \sqrt{m_a^2 + p_a^2} = \sqrt{m_e^2 + p_e^2} \quad \Rightarrow$$
$$p_e = \sqrt{\left(\frac{m_\tau^2 - m_e^2 + m_a^2}{2m_\tau}\right)^2 - m_a^2} \approx \frac{m_\tau}{2}\left(1 - \frac{m_e^2}{m_\tau^2}\right) \qquad E_e \approx \frac{m_\tau}{2}\left(1 - \frac{m_e^2}{m_\tau^2}\right) \tag{4.111}$$

where the approximate formulae originate from the invisible axion limit $m_a \to 0$. In so doing, we can state

$$\frac{1}{2}\sum_{r,s} |\mathcal{M}_{if}|^2 = \frac{m_\tau^4}{2f_a^2}(\epsilon_{13}^{e_L})^2\left(1 - \frac{m_e^2}{m_\tau^2}\right)^2 \tag{4.112}$$

This expression will enter the decay rate formula (once the conservation conditions will be taken into account by solving the four-dimensional Dirac delta):

$$\Gamma(\tau^+ \to e^+ + a_\phi) = \frac{1}{2m_\tau}\int \frac{d^3p'_e}{(2\pi)^3 2E'_e}\int \frac{d^3p'_a}{(2\pi)^3 2E'_a}(2\pi)^4\delta^{(4)}(p_\tau - p'_e - p'_a)\left[\frac{1}{2}\sum_{r,s}|\mathcal{M}_{if}|^2\right] =$$
$$= \frac{4\pi}{8m_\tau(2\pi)^2}\int \frac{d|p'_e||p'_e|^2}{E'_a(p'_e)E'_e}\delta^{(0)}(m_\tau - E'_e - E'_a(p'_e))\left[\frac{1}{2}\sum_{r,s}|\mathcal{M}_{if}|^2\right] =$$
$$= \frac{1}{8m_\tau\pi}\int \frac{dE'_e|p'_e|}{E'_a(p'_e)}\frac{\delta^{(0)}(E'_e - E_e)}{|1 + E_e/E_a(p_e)|}\left[\frac{1}{2}\sum_{r,s}|\mathcal{M}_{if}|^2\right] = \frac{1}{8m_\tau\pi}\frac{|p_e|}{E_e + E_a(p_e)}\left[\frac{1}{2}\sum_{r,s}|\mathcal{M}_{if}|^2\right] =$$
$$= \frac{m_\tau^3}{32\pi f_a^2\hbar}(\epsilon_{11}\epsilon_{22})\left(1 - \frac{m_e^2}{m_\tau^2}\right)^3 \tag{4.113}$$

where in the last passage we have inserted back an $\hbar$ (set to one), to recover the correct physical dimensions. Now, making use of $m_\tau = 1.78GeV$ and $\tau_{tot}^\tau = 1/\Gamma_{tot}^\tau = 290.3 \times 10^{-15}s$, together with $m_e = 0.5MeV$, we end up with

$$\mathcal{B}(\tau^+ \to e^+ + a_\phi) = \frac{m_\tau^3}{32\pi f_a^2\hbar}\frac{(\epsilon_{11}^{e_L}\epsilon_{33}^{e_L})}{\Gamma_{tot}^\tau}\left(1 - \frac{m_e^2}{m_\tau^2}\right)^3 = 6.6 \times 10^{-6}\left(\frac{m_a}{0.2eV}\right)^2 \tag{4.114}$$

in which we notice how the product $\epsilon_{11}\epsilon_{33} \approx z/(1+z)^2$ is essentially model-independent. Comparing it with the current bound $\mathcal{B}(\tau^- \to e^- f) < 2.6 \times 10^{-3}$ [64], we see that we are three orders of magnitude below the experimental limit: thereby, such a process is still viable and not excluded by high precision measurements. In addition, family violating transitions, such as $\tau^- \to \mu^- + a_\phi$, will be grossly suppressed by the tiny $\epsilon_{22}^{e_L}$ value. One can calculate

$$\begin{cases} \mathcal{B}(\tau^- \to \mu^- + a_\phi) = 3.2 \times 10^{-9} & \text{for} \quad M1/M3 \quad \text{with} \quad \epsilon_{11}^{e_L} \approx 1/(1+z) \\ \mathcal{B}(\tau^- \to \mu^- + a_\phi) = 1.6 \times 10^{-9} & \text{for} \quad M2/M4 \quad \text{with} \quad \epsilon_{11}^{e_L} \approx z/(1+z) \end{cases} \tag{4.115}$$

where there is no doubt true that they are compatible with the bound $\mathcal{B}(\tau^- \to \mu^- + f) < 4.6 \times 10^{-3}$ [64].

Other relevant constraints can arise considering the quark sector. An important phenomenon to take into account is the decay $K^- \to \pi^- + a_\phi$ (figure 4.7), which is made possible by the following Yukawa interaction:

$$\mathcal{L}_{s\to d} = \bar{d}\gamma^\mu\frac{\partial_\mu a_\phi}{2f_a}(C_{ds}^V + \gamma^5 C_{ds}^A)s \tag{4.116}$$

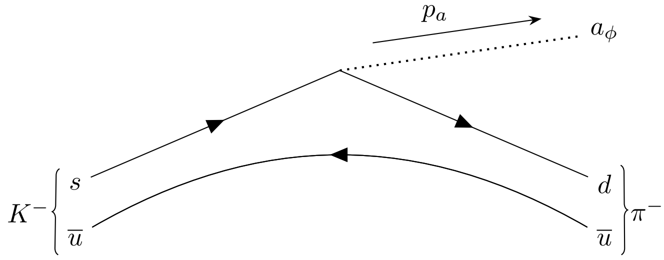



Figure 4.7: Diagram of the flavour violating $K^-$ decay: $K^- \to \pi^- + a_\phi$

because the up quark is just a passive spectator. To get the $\mathcal{M}_{if}$ relevant for the process, we have to evalutate the following matrix element:

$$\langle \pi^-, a_\phi | \left[ \bar{d}\gamma^\mu \frac{\partial_\mu a_\phi}{2f_a}(C^V_{ds} + \gamma^5 C^A_{ds})s \right] |K^-\rangle \tag{4.117}$$

In momentum space, the axion will just contribute with its momentum through the four-derivative. Morever, we notice that parity invariance reasonings lead us to drop out the quark axial current. Indeed, we know that pseudo-scalar meson states and the axion will transform as $\hat{P}|K^-\rangle = -|K^-\rangle$, $\hat{P}|\pi^-\rangle = -|\pi^-\rangle$ and $\hat{P}|a_\phi\rangle = -|a_\phi\rangle$, but the two pseudo-particles state $|\pi^-, a_\phi\rangle$ will be parity even in the simplest case of zero total orbital angular momentum. Remembering the transformation of Dirac fermions under parity $\hat{P}\psi = \gamma^0\psi$, we can easily realise:

$$\begin{aligned}\hat{P}(\partial_\mu a_\phi \langle \pi^-, a_\phi | \bar{d}\gamma^\mu\gamma^5 C^A_{ds} s|K^-\rangle) = g_{0i}\partial_\mu(-a_\phi)\langle \pi^-, a_\phi|\bar{d}\gamma^0\gamma^\mu\gamma^5\gamma^0 C^A_{ds}s(-|K^-\rangle) = \\ = g_{0i}\partial_\mu a_\phi \langle \pi^-, a_\phi|\bar{d}(-g_{0i})\gamma^\mu\gamma^5(\gamma^0)^2 C^A_{ds}s|K^-\rangle = -\partial_\mu a_\phi\langle \pi^-, a_\phi|\bar{d}\gamma^\mu\gamma^5 C^A_{ds}s|K^-\rangle\end{aligned} \tag{4.118}$$

which implies $\langle \pi^- |\bar{d}\gamma^\mu\gamma^5 C^A_{ds}s|K^-\rangle = 0$.

What we need to do is just assessing the vectorial part of the matrix element. We can envisage its most general form just by means of considerations based on Lorentz invariance:

$$\langle \pi^-|\bar{d}\gamma^\mu s|K^-\rangle = (p^\mu_s + p^\mu_d)\mathcal{F}_{K1} + (p^\mu_s - p^\mu_d)\mathcal{F}_{K2} \tag{4.119}$$

up to two form factors $\mathcal{F}_{K1}$ and $\mathcal{F}_{K2}$, which must be evalutated by numerical and non-perturbative techniques. They can just be functions of Lorentz scalars, i.e the kaon and pion mass, together with $p_a^2$. There are not additional independent scalars that can be built up with the three particle momenta if one set the axion mass to zero (as we are going to do, using once more the invisible axion limit). That said, we can move back to the evalutation of the matrix $\mathcal{M}_{if}$, which will take the form

$$\mathcal{M}_{if} = \frac{C^V_{ds}}{2f_a}\left[ p_{a\mu}(p^\mu_s + p^\mu_d)\mathcal{F}_{K1} + p_{a\mu}(p^\mu_s - p^\mu_d)\mathcal{F}_{K2} \right] \tag{4.120}$$

Using the conservation law $p^\mu_s - p^\mu_d = p^\mu_a$ and setting the axion mass approximately to zero (i.e $p_{a\mu}p^{a\mu} \approx 0$), we see that the term proportional to $\mathcal{F}_{K2}$ can be removed from the calculation. The remaining contribution can be easily written down as

$$\mathcal{M}_{if} = \frac{C^V_{ds} m^2_K}{2f_a}\left(1 - \frac{m^2_\pi}{m^2_K}\right)\mathcal{F}_{K1} \tag{4.121}$$

From this point on, the computation proceeds exactly as shown above, finally leading us to the result

$$\mathcal{B}(K^- \to \pi^- + a_\phi) = \frac{m^3_K}{64\pi f^2_a \hbar}\frac{\epsilon^{d_R/d_L}_{ds}}{\Gamma^\tau_{tot}}\left(1 - \frac{m^2_\pi}{m^2_K}\right)^3 = 1.1 \times 10^{-9}\left(\frac{m_a}{0.2eV}\right)^2\left(\frac{\epsilon^{d_R/d_L}_{dd}\epsilon^{d_R/d_L}_{ss}}{\lambda^{10}}\right)$$

where $\lambda \approx 0.2$ is the already introduced CKM parameter. Moreover, $\epsilon_{ds}^{d_R}$ refers to $M4$ and $\epsilon_{ds}^{d_L}$ to $M1/M2$. For the form factor, we have used $\mathcal{F}_{K1}(p_a^2 \approx 0) = 1$. Strictly speaking, this is correct only for exact SU(3) flavour symmetry, but corrections to one are known to go as the ratio of quark masses and, hence, they can be ignored at this level of approximation. For $M3$ we have $C_{ds}^V = 0$, so that the process is not allowed and the current bound $\mathcal{B}(K^+ \to \pi^+ + a_\phi) < 7.3 \times 10^{-11}$ from [66] is immediately satisfied. On the contrary, for the $M4$ model the previous formula fixes a constraint on $\epsilon_{ss}^{d_R}$, for which there are no theoretical considerations to derive an estimation. Taking into account that $\epsilon_{dd}^{d_R}$ must be approximately one and using $m_a \approx 0.12eV$, we get $\epsilon_{ss}^{d_R} < 1.9 \times 10^{-8}$, which leaves us with a pretty unrealistic model. On the other hand, for the $M1/M2$ theories the $\epsilon^{d_L}$ parameters are CKM-like, so that $\epsilon_{dd}^{d_L}\epsilon_{ss}^{d_L}/\lambda^{10} \approx 1$. This provides us with an axion mass bound of $m_a < 0.05eV$, comparable with the astrophysical ones. But this does not mean that the main slice of the heavy axion mass window must be cut off. We should not forget that our model building form of $\epsilon$ is just a working hypothesis: we know that the left-handed $\epsilon$ formulae must be related to the CKM sector, but, in practise, we miss any explicit expression for them. These precision flavour experiments should actually be used to measure these Yukawa couplings, instead of further constraining the axion mass.

In a similar fashion, one can derive the branching ratio for other meson decays. For example, it can be considered the $B^+ \to K^+ + a_\phi$ reaction, whose experimental bound is $\mathcal{B}(B^+ \to K^+ + f) < 4.9 \times 10^{-5}$ from [67]. Remembering that the meson $B^+$ $(u\bar{b})$ has a mass $m_B = 5279.25MeV$ and a life-time $\tau_B = 1.64 \times 10^{-12}s$, we can evalutate its branching ratio

$$\mathcal{B}(B^+ \to K^+ + a_\phi) = \frac{m_B^3}{64\pi f_a^2 \hbar} \frac{\epsilon_{bs}^{d_R/d_L}}{\Gamma_{tot}^{\tau}} \left(1 - \frac{m_K^2}{m_B^2}\right)^3 \mathcal{F}_B^2 = 3.8 \times 10^{-7} \left(\frac{m_a}{0.2eV}\right)^2 \left(\frac{\epsilon_{bb}^{d_R/d_L}\epsilon_{ss}^{d_R/d_L}}{\lambda^4}\right) \tag{4.122}$$

where $\mathcal{F}_B$ is again the form factor defined by $\langle K^-|\bar{s}\gamma^\mu b|B^-\rangle = (p_s^\mu + p_b^\mu)\mathcal{F}_B$: its value is known to be $\mathcal{F}_B = 0.33$. The result is clearly not ruled out by experiments even in this case. Despite that, the less tight constraint on this decay mode will impose bounds that are less stringent than those previously derived: for instance, for $M1/M2$ we obtain $m_a < 1.8eV$, which is higher than the hot dark matter limit.

# Conclusions

The axion idea can solve two disturbing problems of contemporary physics at one shot. From one side, it can naturally justify why strong interactions do not violate parity and time-reversal symmetry and, therefore, it can give an answer to the strong CP problem in a pretty elegant way. On the other hand, being stable and weakly interacting particles, axions turn out to be excellent dark matter candidates.

Because of their light masses, just as for neutrinos, particle accelerators are not useful to directly detect axions. Nevertheless, these new particles must be embodied in some minimal extensions of the standard model. Hence, the particle content of these BSM theories can be potentially measured and constrained, thus offering an indirect proof of axion existence. In Chapter 2, we revised the famous DFSZ model, enlarging the SM with a second Higgs doublet and a Higgs singlet. In particular, we pinned down and examined the consequences of slightly changing the quartic c-term of the potential with a cubic one. As expected, the effects of this modification resulted in tiny corrections of mass formulae with respect to the previous model, without any testable impact on phenomenology at MeV scale: the high value of the $v_\phi$ threshold, required to make the axion invisible, overshadows any possible departure of the cubic model from the quartic one. Despite that, we considered in this new setup the mass spectrum of the theory for different values of the parameters $a, b$ and $c$, which relate the Higgs doublets sector with the singlet one. As already claimed in previous works, these scenaries are particularly appealing, because they provide a light spectrum of particles (one of which identifiable with the SM Higgs), which could be in principle detected at LHC or in future particle accelerators able to reach the $100TeV$ regime. Moreover, we noticed the presence of an extra viable situation which was absent in the quartic model, given by the *quasi-free singlet limit*. Here, the Higgs singlet self-coupling $\lambda_\phi$ is simply set to zero, opening up the possibility for the cubic potential term to dominate. But, so far, the interest in this case is again purely theoretical, because the majority of particle masses reside in the $v_\phi$ energy region. By way of conclusion, the realization of custodial symmetry was taken into account, too. To further contrain the model, a brief description of the electroweak precision test was developed and the bounds imposed by the Peskin-Takeuchi parameter $T$ for the more interesting quasi-custodial case were considered.

Even though the consequences on the mass spectrum are essentially negligible, as described in Chapter 3, the replacement of the quartic c-term with the cubic interaction can affect the periodicity conditions of the effective potential for the axion phase. We observed how that adjustment was able to narrow down the number of degenerate vacuum states in which the axion field can settle in after the emergence of a QCD potential around the scale $T_{QCD} \sim$ 200-300$MeV$. Thereby, we pointed out how a cubic DFSZ model seems to soften the domain wall cosmological puzzle.

The effects of that are even more enhanced in the generalized class of theories which we took into account in Chapter 4. In particular, this section is devoted to a precise analysis of axion interactions with gauge bosons, through the quantum anomaly contributions, and with fermions, by means of the Yukawa sector. Although axion models able to erase the axion couplings to electrons and photons were already designed in a pretty simple conceptual framework, until now it was thought that a well-established result of DFSZ models was the prediction of a non-vanishing axion-nucleon coupling. What we explicitly showed is that by dropping out the hypothesis of a family universal PQ charge pattern, the interplay between axion and nucleons can be highly suppressed and axions can be endowed with the property of nucleophobia. Two fundamental conditions for that to be pursued were derived: the axion verteces $C_u$ and $C_d$ with the up and down quark respectively are required to satisfy the relations $C_u + C_d = 1$ and $C_u = 2/3$. Working in a next-to-minimal non-universal 2+1 flavour structure, we derived all possible models varying for their charge configurations. Among all them, we spotted a particular subset of theories whose fermion couplings make nucleophobia achievable. What was observed is how for these axion models some recent fits of $\bar{g}_{ae} = 1.5 \times 10^{-13}$ and $g_{a\gamma} = 0.14 \times 10^{-10} GeV^{-1}$ can be much more easily accomodate with respect to the universl DFSZ scenario. Moreover, by very definition, the bound on protons and nucleons interactions coming from neutrino burst duration signal of supernovae 1987A of $g_{ap}^2 + g_{an}^2 < 3.6 \times 10^{-19}$ can be evaded: that discloses the possibility of reaching the heavy axion mass window $20meV < m_a < 200meV$ (in contrast with the previous bound of $m_a < 20meV$). Thereby, the next generations of helioscopes, such as IAXO, will be crucial in order to test nucleophobic axions.

Once we enable the PQ charge pattern to violate family symmetry, the axion becomes a particular kind of familon. Consequently, we highlighted at the end of the third chapter how a 2+1 DFSZ model can legitimate flavour changing reactions. That paves the way to the possibility of extracting important information about axion physics not only from astrophysical observations, but also from particle physics experiments. Indeed, the limits on the branching ratio of important decay processes, such as $\mathcal{B}(\tau^- \to e^- f) < 2.6 \times 10^{-3}$ or $\mathcal{B}(K^+ \to \pi^+ + a_\phi) < 7.3 \times 10^{-11}$, can be used to constrain the strength of the flavour violating verteces, which are ultimately related to the PMNS and CKM matrix elements.

Despite the growing precision which has been obtained by experimental measurements, this work illustrated how there is still enough theoretical freedom to mediate between our conceptual constructions and phenomenological outcomes. Furthermore, we can probably say that, because of its ability of conveying together efforts coming from all conceivable research areas, like cosmology, astrophysics, high energy frontiers or physics of flavour violation, the experimental and speculative axion hunting remains one of the most fascinating challenge of contemporary physics.

# Appendix A

# Some formal developments about chiral anomalies

In this section we are going to analyse in some detail the derivation of the anomalous term, showing up in chiral theories, from different but complementary points of view. The first part aims to obtain the Adler-Bardeen theorem by explicitly assessing the triangle diagram, responsible for the anomalous contribution. This perturbative approach to anomalies will make use of the dimensional regularization procedure. Here, the problem of handling the $\gamma^5$ matrix will be faced by means of the self-consistent Breitenlohner-Maison prescription. The second part will counter the anomaly issue from a non-perturbative perspective, directly dealing with quadratic fermion path integrals. These ill-defined expressions will be regularized using the general $\zeta$-function procedure.

## A.1 The dimensional regularization viewpoint

Let us consider a quantum field theory comprising a massless Dirac spinor, which couples to a non-abelian gauge field in the customary way:

$$\mathcal{L} = -\frac{1}{2} tr[\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}] + \bar{\psi} i \not{\mathcal{D}} \psi \tag{A.1}$$

with $\mathcal{D}_\mu = \partial_\mu - igA^a_\mu \tau^a_F$ and $tr[\tau^a_F \tau^b_F] = \delta^{ab}/2$. The field strength tensor is defined as usual: $\mathcal{G}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. From this Lagrangian, we can directly extract the Feynman rules of this problem. Moreover, by means of the action principle, we can derive the equations of motion

$$\begin{cases} [\partial_\mu \delta^{ab} + g f^{abc} A^c_\mu]\mathcal{G}^{\mu\nu b} = -g j^{\nu a} = -g\bar{\psi}\gamma^\nu \tau^a_F \psi \\ i\not{\mathcal{D}}\psi = 0 \end{cases} \tag{A.2}$$

We know that $j^{\nu a}$ is the fermion vector current, while the fermion axial current is defined as $j^{\nu a}_5 = \bar{\psi}\gamma^\nu\gamma^5\tau^a_F\psi$. The purely gluonic current $\mathfrak{J}^{\nu a} = f^{abc}A^c_\mu \mathcal{G}^{\mu\nu b}$ can be read off from (A.2). If we endow our fermions with an extra non-abelian global symmetry with generators $\mathcal{X}^a$, we could build a general axial current associated to it, i.e $J^{a\nu}_5 = \bar{\psi}\gamma^\nu\gamma^5\mathcal{X}^a_F\psi$.

Actually, the perturbative approach to anomalies starts by considering the evalutation of a peculiar matrix element: $\langle k_1, r; k_2, s | J^{a\lambda}_5(x)|0\rangle$. Here, $k_1$ and $k_2$ are the external momenta of two quanta of $A_\mu$, whereas $r$ and $s$ the respective polarization states. It is crucial to remember how, in perturbation theory, external fields are to be considered as asymptotically free: therefore, the whole normal mode expansion machinery will hold true for

$$A_\mu^{in}(x) = \lim_{x_0 \to -\infty} A_\mu(x) \qquad A_\mu^{out}(x) = \lim_{x_0 \to +\infty} A_\mu(x) \tag{A.3}$$

If $k_1$ and $k_2$ refer to gluon momenta, we will have to project our physical situation into a high energy regime, where $g \ll 1$.

We can start developing our object, so as to obtain the LSZ reduction formula in coordinate space. To start with, we can notice

$$\mathcal{M}^{a\lambda}_{k_1,r;k_2,s}(z) = \langle k_1, r; k_2, s|J_5^{a\lambda}(z)|0\rangle = \langle 0|g_{r,out}(k_1)g_{s,out}(k_2)J_5^{a\lambda}(z)|0\rangle \tag{A.4}$$

For instance, we could write $g_{r,out}(k_1) = (g_{r,out}(k_1) - g_{r,in}(k_1)) + g_{r,in}(k_1)$: factors containing one $g_{r,in}(k_1)$ or $g_{s,in}(k_2)$ will vanish when the annihilation operators will act on the right vacuum state of (A.4). In particular, making use of the inversion relation between the annihilation operator $g_r^{in/out}(k_1)$ and the vector field $A_\mu^{in/out}(x)$, we could state:

$$\begin{aligned} g_{r,out}(k_1) - g_{r,in}(k_1) &= -\varepsilon_r^{a\mu}(k_1) \int d\mathbf{x}(e^{ik_1^{out}\cdot x} i\overleftrightarrow{\partial}_{x_0} A^{out}_{a\mu}(x) - e^{ik_1^{in}\cdot x} i\overleftrightarrow{\partial}_{x_0} A^{in}_{a\mu}(x)) = \\ &= -\varepsilon_r^{a\mu}(k_1) \int d\mathbf{x}\left(\lim_{x_0\to+\infty} - \lim_{x_0\to-\infty}\right)(e^{ik_1\cdot x} i\overleftrightarrow{\partial}_{x_0} A^a_\mu(x)) = -\varepsilon_r^{a\mu}(k_1)\int d^4x \partial_{x_0}(e^{ik_1\cdot x} i\overleftrightarrow{\partial}_{x_0} A^a_\mu(x)) \end{aligned} \tag{A.5}$$

Therefore, equation (A.4) will become:

$$\begin{aligned} \mathcal{M}^{a\lambda}_{k_1,r;k_2,s}(z) &= i^2 \varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2) \int d^4x d^4y \partial_{x_0}\partial_{y_0}[e^{ik_1\cdot x + ik_2 \cdot y}\langle 0|\overleftrightarrow{\partial}_{x_0} A^b_\mu(x) \overleftrightarrow{\partial}_{y_0} A^c_\nu(y) J_5^{a\lambda}(z)|0\rangle] = \\ &= -\varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2) \int d^4x d^4y e^{ik_1\cdot x + ik_2\cdot y}(\omega_1^2 + \partial^2_{x_0})(\omega_2^2 + \partial^2_{y_0})\langle 0|A^b_\mu(x)A^c_\nu(y)J_5^{a\lambda}(z)|0\rangle = \\ &= -\varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2) \int d^4x d^4y e^{ik_1\cdot x + ik_2\cdot y}(|\mathbf{k}_1|^2 + \partial^2_{x_0})(|\mathbf{k}_2|^2 + \partial^2_{y_0})\langle 0|A^b_\mu(x)A^c_\nu(y)J_5^{a\lambda}(z)|0\rangle = \\ &= -\varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2) \int d^4x d^4y e^{ik_1\cdot x + ik_2\cdot y}(-\vec{\nabla}^2_x + \partial^2_{x_0})(-\vec{\nabla}^2_y + \partial^2_{y_0})\langle 0|A^b_\mu(x)A^c_\nu(y)J_5^{a\lambda}(z)|0\rangle = \\ &= -\varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2) \int d^4x d^4y e^{ik_1\cdot x + ik_2\cdot y}\Box_x\Box_y\langle 0|A^b_\mu(x)A^c_\nu(y)J_5^{a\lambda}(z)|0\rangle \end{aligned} \tag{A.6}$$

Moving from second to third line, we have taken into account the mass shell relation $\omega^2 = |\mathbf{k}|^2$ for massless particles and, in the following passage, we automatically integrated by parts twice, moving the operator $\vec{\nabla}^2$ from the plane wave function to the vector field.

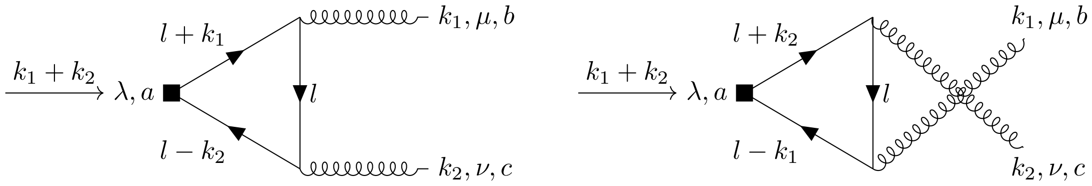
If we now apply the equation of motion (A.2) for the four vector potential, we will get

$$\langle k_1, r; k_2, s|J_5^{a\lambda}(z)|0\rangle = -g^2\varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2)\int d^4x d^4y e^{ik_1\cdot x + ik_2\cdot y}\langle 0|j^b_\mu(x)j^c_\nu(y)J_5^{a\lambda}(z)|0\rangle + \mathcal{O}(g^2) \tag{A.7}$$

The gluon current $\mathfrak{J}^{\nu a}$, in case it contributes to (A.7), will enter higher order terms, because gluons can only attach to fermion lines of an axial current by means of at least an extra $g$ vertex.

Remembering that the Fourier transform of a matrix element is given by

$$\mathfrak{F}(\langle 0|j^b_\mu(x)j^c_\nu(y)J_5^{a\lambda}(z)|0\rangle) = (2\pi)^4\delta^4(p+q-w)\widetilde{\mathcal{M}}^{abc\lambda}_{\mu\nu}(p,q,w) \tag{A.8}$$




(a) (b)

Figure A.1: Lowest order contributions to the amplitude $\widetilde{\mathcal{M}}^{abc\lambda}_{\mu\nu}(p,q,p+q)$, given by a fermionic triangle A.1a and the indistinguishable process A.1b, where $k_1, \mu, b \leftrightarrow k_2, \nu, c$. The black square corresponds to a $\gamma^5$ matrix insertion.

where $w$ is the axial current momentum, we can plug the previous identity in (A.7) and, hence, after some integrations, we will end up with:

$$\langle k_1, r; k_2, s | J_5^{a\lambda}(z)|0\rangle = -g^2 \varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2)\widetilde{\mathcal{M}}^{abc\lambda}_{\mu\nu}(k_1, k_2, w = k_1+k_2)e^{i(k_1+k_2)z} + \mathcal{O}(g^2) \quad \text{(A.9)}$$

The Feynman diagrams contributing to $\widetilde{\mathcal{M}}^{abc\lambda}_{\mu\nu}(p,q,w=p+q)$ at lower order in $g$ are shown in figure A.1. But if we want to derive the Adler-Bardeen theorem, as set out in (1.64), we have to consider the four divergence of (A.9):

$$\langle k_1, r; k_2, s | \partial_\lambda J_5^{a\lambda}(z)|0\rangle = -ig^2 \varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2)(k_1+k_2)_\lambda\widetilde{\mathcal{M}}^{abc\lambda}_{\mu\nu}(k_1, k_1, k_1+k_2)e^{i(k_1+k_2)z} + \mathcal{O}(g^2) \quad \text{(A.10)}$$

The integrals which enable us to assess the two triangle diagrams are given by:

$$\widetilde{\mathcal{M}}^{\lambda\mu\nu}_{abc}(p,q,p+q) = -\int \frac{d^4l}{(2\pi)^4} tr\left[\frac{i\mathcal{X}_a}{\not{l}-\not{k}_2}\gamma^\lambda\gamma^5\frac{i\tau^b}{\not{l}+\not{k}_1}\gamma^\mu\frac{i\tau_c}{\not{l}}\gamma^\nu + \frac{i\mathcal{X}_a}{\not{l}-\not{k}_1}\gamma^\lambda\gamma^5\frac{i\tau_c}{\not{l}+\not{k}_2}\gamma^\nu\frac{i\tau_b}{\not{l}}\gamma^\mu\right] =$$
$$= -\int \frac{d^4l}{(2\pi)^4} tr\left[\omega^{abc}\frac{i(\not{l}-\not{k}_2)}{(l-k_2)^2}\gamma^\lambda\gamma^5\frac{i(\not{l}+\not{k}_1)}{(l+k_1)^2}\gamma^\mu\frac{i\not{l}}{l^2}\gamma^\nu + \omega^{acb}\frac{i(\not{l}-\not{k}_1)}{(l-k_1)^2}\gamma^\lambda\gamma^5\frac{i(\not{l}+\not{k}_2)}{(l+k_2)^2}\gamma^\nu\frac{i\not{l}}{l^2}\gamma^\mu\right] \quad \text{(A.11)}$$

with $\omega^{abc} = tr[\mathcal{X}^a\tau^b\tau^c]$ and where indeces of the fundamental representation (entering spinor propagators) have been understood. The overall minus sign accounts for the presence of fermion loops. To be honest, we overlooked the small imaginary part $+i\epsilon$, which should have accompanied any denominators to account for mass poles: that was done just to lighten the notation.

**The Breitenlohner-Maison prescription** As already observed in the first chapter, this integral is linearly divergent. That means a regularization procedure is called for to give a mathematical significance to the foregoing expression. An extremely powerful instrument turns out to be dimensional regularization, which can be applied to any situation of interest, owning to its ability of sustaining non-abelian gauge invariance, too. This procedure was employed in chiral theories in [68], for instance. But a non-trivial issue, which one immediatly bumps into, is handling the $\gamma_5$ matrix. To pin down this problem, in this paragraph we are going to work directly in euclidean notation for matter of convenience: a Minkowski generalization, if needed, will be straightforward.

This delicate task involving $\gamma_5$ was faced in [69], where it was pointed out how this matrix is not the sole element rooted in four-dimensional space. In a dimensional regularization framework, Lorentz covariant tensors, such as $\gamma_\mu$, $l_\mu$, $\delta_{\mu\nu}\ldots$, are extended to $d$ dimensions by treating them as formal objects with some algebraic properties. But some problems arise even when dealing with the $\epsilon_{\mu\nu\rho\sigma}$-tensors, which is genuinely 4-dimensional, as shown by

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4} = \sum_{\pi\in S_4} sgn\pi \prod_{i=1}^{4} \delta_{\mu_i\nu_{\pi(i)}} \qquad d=4 \tag{A.12}$$

This property can just hold in $d = 4$. Not to abandon the possibility of reducing products of several $\epsilon$-tensors in algebraic manipulation of integrals, Breitenlohner and Maison suggested to recognize the special role of $d = 4$, while extending tensors to arbitrary $d$. That can be achieved by introducing, together with $d$-dimensional objects, also 4- and $(d-4)$-dimensional ones. Thereby, we will decompose a $d$-dimensional tensor as:

$$\underbrace{l_\mu}_{d\text{-dimensional}} = \underbrace{\hat{l}_\mu}_{(d-4)\text{-dimensional}} + \underbrace{\bar{l}_\mu}_{4\text{-dimensional}} \tag{A.13}$$

The $d$-dimensional symbols $\gamma_\mu$, $l_\mu$, $\delta_{\mu\nu}$, $\epsilon_{\mu\rho\sigma\tau}$ and $\mathbb{1}$ will satisfy the properties [69]:

$$\begin{array}{ccccc} \delta_{\mu\nu} = \delta_{\nu\mu} & \delta_{\mu\rho}\delta_{\rho\nu} = \delta_{\mu\nu} & \delta_{\mu\nu}l_\nu = l_\mu & \delta_{\mu\nu}\gamma_\nu = \gamma_\mu & \\ \delta_{\mu\nu}\epsilon_{\nu\rho\sigma\tau} = \epsilon_{\mu\rho\sigma\tau} & \{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1} & \mathbb{1}\gamma_\mu = \gamma_\mu\mathbb{1} & \delta_{\mu\mu} = d & tr\mathbb{1} = 4 \end{array} \tag{A.14}$$

while the $(d-4)$-dimensional ones $\hat{\gamma}_\mu$, $\hat{l}_\mu$ and $\hat{\delta}_{\mu\nu}$ are assumed to obey:

$$\begin{array}{c} \delta_{\mu\nu}\hat{\delta}_{\nu\sigma} = \hat{\delta}_{\mu\nu}\hat{\delta}_{\nu\sigma} = \hat{\delta}_{\mu\sigma} \qquad \hat{\delta}_{\mu\nu} = \hat{\delta}_{\nu\mu} \qquad \hat{\delta}_{\mu\nu}l_\nu = \hat{l}_\mu \qquad \hat{\delta}_{\mu\nu}\gamma_\nu = \hat{\gamma}_\mu \\ \epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4} = \sum_{\pi\in S_4} sgn\pi \prod_{i=1}^{4}(\delta_{\mu_i\nu_{\pi(i)}} - \hat{\delta}_{\mu_i\nu_{\pi(i)}}) = \sum_{\pi\in S_4} sgn\pi \prod_{i=1}^{4}\bar{\delta}_{\mu_i\nu_{\pi(i)}} \end{array} \tag{A.15}$$

This set of identities suffices to prove:

$$\begin{array}{ccc} \delta_{\mu\nu}\hat{\gamma}_\nu = \hat{\delta}_{\mu\nu}\hat{\gamma}_\nu = \hat{\gamma}_\mu & \{\gamma_\mu,\hat{\gamma}_\nu\} = \{\hat{\gamma}_\mu,\hat{\gamma}_\nu\} = 2\hat{\delta}_{\mu\nu}\mathbb{1} & \hat{\delta}_{\mu\nu}\epsilon_{\nu\rho\sigma\tau} = 0 \\ \epsilon_{\mu_1\mu_2\mu_3\mu_4} = sgn\pi\epsilon_{\mu_{\pi(1)}\mu_{\pi(2)}\mu_{\pi(3)}\mu_{\pi(4)}} & \hat{\delta}_{\mu\mu} = n-4 & tr\gamma_\mu = 0 \end{array} \tag{A.16}$$

Note how $\epsilon$ just exists in one version. The same is true for $\gamma_5$. Actually, it can be noticed that there are two properties that characterise the fifth gamma matrix in four dimensions:

$$i)\ tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4\epsilon_{\mu\nu\rho\sigma} \qquad ii)\ \{\gamma_\mu,\gamma_5\} = 0 \tag{A.17}$$

which are incompatible for $d \neq 4$: just one can be used to define a $d$-dimensional $\gamma_5$. But if we mantain the second one, together with the reasonable cyclicity property of trace operations, we are unavoidably led to a contradiction.

*Proof.* We can simply start by considering the simple object $tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma]$. If we just make use of the fundamental property $\gamma_\mu\gamma_\mu = \delta_{\mu\mu} = d$, together with the trace cyclicity and $\{\gamma_\mu,\gamma_5\} = 0$, our initial gamma trace can be manipulated as:

$$\begin{aligned} &dtr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma(\gamma_\lambda\gamma_\lambda)] = -tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\lambda\gamma_\mu] = \\ &= -2\delta_{\mu\lambda}tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda] + tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\mu\gamma_\lambda] = -2\delta_{\mu\lambda}tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda] - tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\mu\gamma_\lambda\gamma_\nu] = \\ &= -2\delta_{\mu\lambda}tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda] - 2\delta_{\nu\lambda}tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\mu] + tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\lambda] = \\ &= -2\delta_{\mu\lambda}tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda] - 2\delta_{\nu\lambda}tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\mu] - tr[\gamma_5\gamma_\sigma\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho] = \\ &= -2\delta_{\mu\lambda}tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda] - 2\delta_{\nu\lambda}tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\mu] - 2\delta_{\rho\lambda}tr[\gamma_5\gamma_\sigma\gamma_\lambda\gamma_\mu\gamma_\nu] + tr[\gamma_5\gamma_\sigma\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda] = \\ &= -2\delta_{\mu\lambda}tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda] - 2\delta_{\nu\lambda}tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\mu] - 2\delta_{\rho\lambda}tr[\gamma_5\gamma_\sigma\gamma_\lambda\gamma_\mu\gamma_\nu] - tr[\gamma_5\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda\gamma_\sigma] = \\ &= -2\delta_{\mu\lambda}tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda] - 2\delta_{\nu\lambda}tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\mu] - 2\delta_{\rho\lambda}tr[\gamma_5\gamma_\sigma\gamma_\lambda\gamma_\mu\gamma_\nu] - 2\delta_{\sigma\lambda}tr[\gamma_5\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho] + \\ &\quad + tr[\gamma_5\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda] \end{aligned}$$

If we further take into account the relations $\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}$ and $tr[\gamma_5\gamma_\alpha, \gamma_\beta] = 0$ (which is a direct consequence of $\{\gamma_\mu, \gamma_5\} = 0$), we could finally get

$$\begin{aligned}
2dtr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] &= -2tr[\gamma_5\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] - 2tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\nu\gamma_\mu] - 2tr[\gamma_5\gamma_\sigma\gamma_\rho\gamma_\mu\gamma_\nu] - 2tr[\gamma_5\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\rho] \\
2dtr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] &= 4tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] - 2tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\nu\gamma_\mu] - 2tr[\gamma_5\gamma_\sigma\gamma_\rho\gamma_\mu\gamma_\nu] \\
2dtr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] &= 4tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] + 4tr[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu] \\
(4-d)tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] &= 0
\end{aligned} \tag{A.18}$$

Therefore, if $\{\gamma_\mu, \gamma_5\} = 0$ for arbitrary $d$, the previous relation tells us $tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 0$ for $d \neq 4$. But along with being a proof of incompatibility of the two $\gamma_5$ definitions, here we are also said that no smooth limit exists so as to recover

$$\lim_{d\to 4} tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4\epsilon_{\mu\nu\rho\sigma} \neq 0 \tag{A.19}$$

As a consequence, $\{\gamma_\mu, \gamma_5\} = 0$ is not a good identity to extend to general $d$ without jeopardising the entrenched 4-dimensional $\gamma_5$ properties. □

Thereby, Breitenlohner and Maison realized $tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4\epsilon_{\mu\nu\rho\sigma}$ should have been used as the right defining relation for $\gamma_5$: $\{\gamma_\mu, \gamma_5\} = 0$ just turns out to be an accident of four dimensions. One has to recognize $\gamma_5$ is a deep-seated 4-dimensional tensor, which will not anti-commute with $d$-dimensional gamma matrices. As a matter of fact, by consistently assuming $\gamma_5 = (4!)^{-1}\epsilon_{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$, [69] succeded in proving

$$\{\gamma_\mu, \gamma_5\} = \{\hat{\gamma}_\mu, \gamma_5\} = 2\hat{\gamma}_\mu\gamma_5 \qquad \gamma_5^2 = \mathbb{1} \tag{A.20}$$

The first relation will play a crucial rule in the emergence of the anomaly. Moreover, it straightforwardly stems from it how:

$$\{\hat{\gamma}_\mu, \gamma_5\} = \hat{\gamma}_\mu\gamma_5 + \gamma_5\hat{\gamma}_\mu = 2\hat{\gamma}_\mu\gamma_5 \quad \Rightarrow \quad [\hat{\gamma}_\mu, \gamma_5] = 0 \tag{A.21}$$

**Back to the integral** After this brief interlude, which gave us the necessary tools to counter the $\gamma_5$ issue, we can return to our Minkowski integral (A.11), dimensionally regularized:

$$\begin{aligned}
I^{\mu\nu}_{abc}(k_1, k_1) = i\mu^{4-2\omega} &\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \omega_{abc} \frac{tr[(\not{l} - \not{k}_2)(\not{k}_1 + \not{k}_2)\gamma^5(\not{l} + \not{k}_1)\gamma^\mu \not{l}\gamma^\nu]}{(l-k_2)^2(l+k_1)^2 l^2} + \\
+ i\mu^{4-2\omega} &\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \omega_{acb} \frac{tr[(\not{l} - \not{k}_1)(\not{k}_1 + \not{k}_2)\gamma^5(\not{l} + \not{k}_2)\gamma^\nu \not{l}\gamma^\mu]}{(l-k_1)^2(l+k_2)^2 l^2}
\end{aligned} \tag{A.22}$$

in which $\mu$ is the usual ultraviolet regulator, that we use to preserve physical dimensions and $d = 2\omega \in \mathbb{C}$. Of course, $\gamma^5$ is handled as made clear in the previous paragraph (with a straightforward translation to Minkowski formalism). We also remind how all physical momenta $k_1$ and $k_2$ remain four dimensional vectors: just the loop momentum $l$ is analytically extended to $2\omega$.

To reduce our integral to the really contributing part, we can rewrite $(\not{k}_1 + \not{k}_2)\gamma^5$ in the first addend in a clever way:

$$(\not{k}_1 + \not{k}_2)\gamma^5 = (\not{l} + \not{k}_1)\gamma^5 - (\not{l} - \not{k}_2)\gamma^5 = -\gamma^5(\not{l} + \not{k}_1) - (\not{l} - \not{k}_2)\gamma^5 + 2\gamma^5(\hat{\not{l}} + \hat{\not{k}}_1) \tag{A.23}$$

where we used $\{\hat{\gamma}^\mu, \gamma^5\} = 2\hat{\gamma}^\mu\gamma^5$ and in which:

$$\hat{\gamma}^\mu l_\mu = \gamma_\alpha \hat{g}^{\alpha\mu} g_{\mu\beta} l^\beta = \gamma_\alpha \hat{g}^{\alpha\mu} \hat{g}_{\mu\beta} l^\beta = \hat{\gamma}^\mu \hat{l}_\mu = \hat{\not{l}} \tag{A.24}$$

An equivalent manipulation can be pursed for the second addend, with $k_1, \mu, b \leftrightarrow k_2, \nu, c$. Thereby, carrying out some proper simplifications, we will have:

$$\begin{aligned} I^{\mu\nu}_{abc}(k_1, k_1) =& i\mu^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \omega_{abc} tr\Big[ -\frac{(\not{l} - \not{k}_2)\gamma^5\gamma^\mu \not{l}\gamma^\nu}{(l-k_2)^2 l^2} - \frac{\gamma^5(\not{l} + \not{k}_1)\gamma^\mu \not{l}\gamma^\nu}{(l+k_1)^2 l^2} + \\ &+ \underbrace{2\frac{(\not{l} - \not{k}_2)\gamma^5(\hat{\not{l}} + \hat{\not{k}}_1)(\not{l} + \not{k}_1)\gamma^\mu \not{l}\gamma^\nu}{(l-k_2)^2(l+k_1)^2 l^2}}_{\Delta^{\mu\nu}(k_1,k_2)} \Big] + (k_1, \mu, b \leftrightarrow k_2, \nu, c) \end{aligned} \tag{A.25}$$

If we perform the change of variable $l' = l - k_2$ in the first addend

$$\begin{aligned} I^{\mu\nu}_{abc}(k_1, k_1) =& i\mu^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \omega_{abc} tr\Big[ -\frac{\not{l}\gamma^5\gamma^\mu(\not{l} + \not{k}_2)\gamma^\nu}{l^2(l+k_2)^2} - \frac{\gamma^5(\not{l} + \not{k}_1)\gamma^\mu \not{l}\gamma^\nu}{(l+k_1)^2 l^2} + \\ &+ \Delta^{\mu\nu}(k_1, k_2) \Big] + (k_1, \mu, b \leftrightarrow k_2, \nu, c) \end{aligned} \tag{A.26}$$

we can use the trace cyclicity and $\{\hat{\gamma}^\mu, \gamma^5\} = 2\hat{\gamma}^\mu\gamma_5$ once more, in the second contribution, to write:

$$\begin{aligned} I^{\mu\nu}_{abc}(k_1, k_1) =& i\mu^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \omega_{abc} tr\Big[ \underbrace{-\frac{\not{l}\gamma^5\gamma^\mu(\not{l} + \not{k}_2)\gamma^\nu}{l^2(l+k_2)^2} + \frac{\not{l}\gamma^5\gamma^\nu(\not{l} + \not{k}_1)\gamma^\mu}{(l+k_1)^2 l^2}}_{A^{\mu\nu}(k_1,k_2)} + \\ & \underbrace{-2\frac{\not{l}\gamma^5\hat{\gamma}^\nu(\not{l} + \not{k}_1)\gamma^\mu}{(l+k_1)^2 l^2}}_{\Pi^{\mu\nu}(k_1,k_2)} + \Delta^{\mu\nu}(k_1, k_2) \Big] + (k_1, \mu, b \leftrightarrow k_2, \nu, c) \end{aligned} \tag{A.27}$$

The term $A^{\mu\nu}(k_1, k_2)$ is manifestly antisymmetric for the exchange of $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. Hence, by adding it to its counterpart $(k_1, \mu, b \leftrightarrow k_2, \nu, c)$, we observe

$$\begin{aligned} &\omega^{abc} tr[A^{\mu\nu}(k_1, k_2)] + \omega^{acb} tr[A^{\nu\mu}(k_2, k_1)] = \omega^{abc} tr[A^{\mu\nu}(k_1, k_2)] - \omega^{acb} tr[A^{\mu\nu}(k_1, k_2)] = \\ &= (\omega^{abc} - \omega^{acb}) tr[A^{\mu\nu}(k_1, k_2)] = tr[\mathcal{X}^a[\tau^b, \tau^c]] tr[A^{\mu\nu}(k_1, k_2)] = f^{bce} tr[\mathcal{X}^a \tau^e] tr[A^{\mu\nu}(k_1, k_2)] = \\ &= f^{bce} tr[\mathcal{X}^a] \underbrace{tr[\tau^e]}_{=0} tr[A^{\mu\nu}(k_1, k_2)] = 0 \end{aligned} \tag{A.28}$$

because infinitesimal generators of unitary groups have null traces. But this is true if $\mathcal{X}^a$ and $\tau^e$ belong to different internal symmetry groups, allowing us to split the trace in the second-to-last passage. In case $\mathcal{X}^a = \tau^a$, the result is unchanged:

$$\begin{aligned} &\omega^{abc} tr[A^{\mu\nu}(k_1, k_2)] + \omega^{acb} tr[A^{\nu\mu}(k_2, k_1)] = f^{bce} tr[\tau^a \tau^e] tr[A^{\mu\nu}(k_1, k_2)] = \\ &= \frac{f^{bce}}{2} tr[[\tau^a, \tau^e]] tr[A^{\mu\nu}(k_1, k_2)] = \frac{f^{bce}}{2} f^{aed} tr[\tau^d] tr[A^{\mu\nu}(k_1, k_2)] = 0 \end{aligned} \tag{A.29}$$

At the same time, the contribution $\Pi^{\mu\nu}(k_1, k_2)$ presents the following numerator:

$$\begin{aligned} tr[\not{l}\gamma^5\hat{\gamma}^\nu(\not{l} + \not{k}_1)\gamma^\mu] &= tr[\gamma^5\hat{\gamma}^\nu\gamma^\alpha\gamma^\mu\gamma^\beta](l+k_1)_\alpha l_\beta = \hat{g}^\nu{}_\rho tr[\gamma^5\gamma^\rho\gamma^\alpha\gamma^\mu\gamma^\beta](l+k_1)_\alpha l_\beta = \\ &= -4i\hat{g}^\nu{}_\rho \epsilon^{\rho\alpha\mu\beta}(l+k_1)_\alpha l_\beta = 0 \end{aligned} \tag{A.30}$$

owning to the contraction property of the $(d-4)$-metric tensor and the four dimensional Levi-Civita symbol.

Taking that into account, we are left with:

$$I^{\mu\nu}_{abc}(k_1,k_1) = -\,2i\mu^{4-2\omega}\omega_{abc}\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{tr[(\not{l}-\not{k}_2)\gamma^5(\hat{\not{l}}+\hat{\not{k}}_1)(\not{l}+\not{k}_1)\gamma^\mu \not{l}\gamma^\nu]}{(l-k_2)^2(l+k_1)^2 l^2} + (k_1,\mu,b \leftrightarrow k_2,\nu,c) \tag{A.31}$$

The denominator can be disentangled by means of Feynman parameters:

$$\begin{aligned}\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{1}{(l-k_2)^2(l+k_1)^2 l^2} &= \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\int_0^1 dxdydz\frac{2\delta(x+y+z-1)}{[x(l-k_2)^2+y(l+k_1)^2+zl^2]^3} = \\ = \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\int_0^1 dxdy&\frac{2}{[(l-xk_2-yk_1)^2-x(x-1)k_2^2-y(y-1)k_1^2]^3}\end{aligned} \tag{A.32}$$

Actually, one should carefully notice that:

$$(l-xk_2-yk_1)^2 = \hat{l}^2 + (\bar{l}-xk_2-yk_1)^2 \tag{A.33}$$

because of the four dimensional nature of the external momenta. But from expression (A.32), we can easily extract the condition legitimating a Wick rotation. As usual, we can analytically continue our integral into euclidean space, provided that $\Delta = x(x-1)k_2^2 + y(y-1)k_1^2 > 0$ in the interval $x,y \in [0,1]$. That is ensured for $k_1^2, k_2^2 > 0$. By looking at (A.10), the euclidean extension will be given by:

$$\begin{aligned}&\bar{\varepsilon}^r_{b\mu}(k_1)\bar{\varepsilon}^s_{c\nu}(k_2)(k_1+k_2)_\lambda\widetilde{\mathcal{M}}^{abc}_{\lambda\mu\nu}(k_1,k_1,k_1+k_2) = \\ &= -4\mu^{4-2\omega}\omega^{abc}\bar{\varepsilon}^r_{b\mu}(k_1)\bar{\varepsilon}^s_{c\nu}(k_2)\int\frac{d^{2\omega}l_e}{(2\pi)^{2\omega}}\int_0^1 dxdy\frac{tr[(\not{l}-\not{k}_2)\gamma_{e5}(\hat{\not{l}}+\hat{\not{k}}_1)(\not{l}+\not{k}_1)\gamma_{e\mu}\not{l}\gamma_{e\nu}]}{[(l-xk_2-yk_1)^2+x(x-1)k_2^2+y(y-1)k_1^2]^3}+ \\ &\quad + (k_1,\mu,b\leftrightarrow k_2,\nu,c)\end{aligned} \tag{A.34}$$

where we remind that $\gamma_{e5} = -\gamma^5$, $\gamma_{e0} = \gamma^0$ and $\gamma_{ek} = -i\gamma^k$. The polarization vectors have been extended to euclidean space, too, as $\bar{\varepsilon}^r_{b\mu} = (\bar{\varepsilon}^r_{b0} = -i\varepsilon^r_{b0}, \varepsilon^r_{bk})$: therefore, $\gamma_e \cdot \bar{\varepsilon}^r_b = -i\not{\varepsilon}^r_b$. Nonetheless, we are going to drop any euclidean subscript from now on, not to weight the notation: after all, we will remain in euclidean space until the end of our calculation.

We should proceed by performing a shift redefinition of $\bar{l}'_\mu = (\bar{l}-xk_2-yk_1)_\mu$. But in order to do that, we have to analyse the structure of the numerator, noticing that $\hat{\not{k}}_1 = \hat{\not{k}}_2 = 0$ (because we are contracting a $(d-4)$-dimensional tensor with a 4-dimensional one):

$$\begin{aligned}tr[(\not{l}-\not{k}_2)\gamma_5\hat{\not{l}}(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu] &= tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho]\hat{l}_\alpha(l+k_1)_\beta l_\delta(l-k_2)_\rho = \\ = tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho]&(\hat{l}_\alpha l_\beta l_\delta l_\rho + \hat{l}_\alpha k_{1\beta} l_\delta l_\rho - \hat{l}_\alpha l_\beta l_\delta k_{2\rho} - \hat{l}_\alpha k_{1\beta} l_\delta k_{2\rho})\end{aligned} \tag{A.35}$$

But the second and third terms will not contribute, because they lead to an integration over an odd number of variables. The quartic term in $l$ can be shown to vanish by simple algebraic passages. Indeed, by projecting the trace over the symmetric part in $\alpha$ and $\beta$ indeces:

$$\begin{aligned}&tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho]\hat{l}_\alpha l_\beta l_\delta l_\rho = \hat{\delta}_{\alpha\zeta} tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho] l_\zeta l_\beta l_\delta l_\rho = \\ &= tr[\gamma_5\hat{\gamma}_\zeta\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho] l_\zeta l_\beta l_\delta l_\rho = \frac{1}{2}tr[\gamma_5\{\hat{\gamma}_\zeta,\gamma_\beta\}\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho] l_\zeta l_\beta l_\delta l_\rho = \\ &= \hat{\delta}_{\zeta\beta} tr[\gamma_5\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho] l_\zeta l_\beta l_\delta l_\rho = 4\hat{\delta}_{\zeta\beta}\epsilon_{\mu\delta\nu\rho} l_\zeta l_\beta l_\delta l_\rho = 4\epsilon_{\mu\delta\nu\rho}\hat{l}^2 l_\delta l_\rho = 0\end{aligned} \tag{A.36}$$

because we are contracting a symmetric tensor $l_\delta l_\rho$ with an antisymmetric one in the same couple of indeces.

We have almost complete this lengthy procedure of reduction of the initial integral. Indeed, we have:

$$\begin{aligned} I^{abc}_{\mu\nu}(k_1,k_1) &= \omega^{abc}\int_0^1 dxdy\int\frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{-4\mu^{4-2\omega}tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho]\hat{l}_\alpha k_{1\beta}(\hat{l}_\delta+\bar{l}_\delta)k_{2\rho}}{[(l-xk_2-yk_1)^2+x(x-1)k_2^2+y(y-1)k_1^2]^3}+ \\ &+(k_1,\mu,b\leftrightarrow k_2,\nu,c)= \\ =\mu^{4-2\omega}&tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho]\int_0^1 dxdy\Bigg[\int\frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{-4\hat{l}_\alpha\hat{l}_\delta k_{1\beta}k_{2\rho}\omega^{abc}}{[(l-xk_2-yk_1)^2+x(x-1)k_2^2+y(y-1)k_1^2]^3}+ \\ -4&\underbrace{\int\frac{d^{2\omega-4}\hat{l}}{(2\pi)^{2\omega-4}}\hat{l}_\alpha\int\frac{d^4\bar{l}}{(2\pi)^4}\bar{l}_\delta\frac{k_{1\beta}k_{2\rho}\omega^{abc}}{[(l-xk_2-yk_1)^2+x(x-1)k_2^2+y(y-1)k_1^2]^3}}_{=0}\Bigg]+(k_1,\mu,b\leftrightarrow k_2,\nu,c) \end{aligned} \tag{A.37}$$

where the last term disappears, due to the two separate odd integrations in 4 and $(d-4)$ spaces. With this last suppression, we have reached the core of the anomaly. We can extract the tensor structure of the integral and perform the long-awaited change of variable in $\bar{l}$:

$$\begin{aligned} I'^{abc}_{\mu\nu}(k_1,k_1) &= -4\mu^{4-2\omega}\omega^{abc}\frac{\hat{\delta}_{\alpha\delta}}{2\omega-4}\int_0^1 dxdy\int\frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho]\hat{l}^2k_{1\beta}k_{2\rho}}{[(l-xk_2-yk_1)^2+\Delta]^3}= \\ &= -4\mu^{4-2\omega}\omega^{abc}\frac{\hat{\delta}_{\alpha\delta}}{2\omega-4}\int_0^1 dxdy\int\frac{d^{2\omega-4}\hat{l}}{(2\pi)^{2\omega-4}}\int\frac{d^4\bar{l}}{(2\pi)^4}\frac{tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho]\hat{l}^2k_{1\beta}k_{2\rho}}{[(\bar{l}-xk_2-yk_1)^2+\hat{l}^2+\Delta]^3}= \\ &= -4\mu^{4-2\omega\omega^{abc}}\frac{\hat{\delta}_{\alpha\delta}}{2\omega-4}k_{1\beta}k_{2\rho}\int_0^1 dxdy\int\frac{d^{2\omega-4}\hat{l}}{(2\pi)^{2\omega-4}}\hat{l}^2\int\frac{d^4\bar{l}}{(2\pi)^4}\frac{tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho]}{[\bar{l}^2+(\hat{l}^2+\Delta)]^3} \end{aligned} \tag{A.38}$$

The last surviving trace can be simplified once for all:

$$\begin{aligned} \hat{\delta}_{\alpha\delta}tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\mu\gamma_\delta\gamma_\nu\gamma_\rho] &= -\hat{\delta}_{\alpha\delta}tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\delta\gamma_\mu\gamma_\nu\gamma_\rho]+\hat{\delta}_{\alpha\delta}\delta_{\mu\delta}tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\nu\gamma_\rho]= \\ &= \hat{\delta}_{\alpha\delta}tr[\gamma_5\hat{\gamma}_\alpha\gamma_\delta\gamma_\beta\gamma_\mu\gamma_\nu\gamma_\rho]+\hat{\delta}_{\alpha\delta}\delta_{\mu\delta}tr[\gamma_5\hat{\gamma}_\alpha\gamma_\beta\gamma_\nu\gamma_\rho]-\hat{\delta}_{\alpha\delta}\delta_{\beta\delta}tr[\gamma_5\hat{\gamma}_\alpha\gamma_\mu\gamma_\nu\gamma_\rho]= \\ &= \hat{\delta}_{\alpha\delta}tr[\gamma_5\hat{\gamma}_\alpha\hat{\gamma}_\delta\gamma_\beta\gamma_\mu\gamma_\nu\gamma_\rho]+\hat{\delta}_{\sigma\delta}\delta_{\mu\delta}tr[\gamma_5\gamma_\sigma\gamma_\beta\gamma_\nu\gamma_\rho]-\hat{\delta}_{\sigma\delta}\delta_{\beta\delta}tr[\gamma_5\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\rho]= \\ &= (2\omega-4)tr[\gamma_5\gamma_\beta\gamma_\mu\gamma_\nu\gamma_\rho]+\underbrace{\hat{\delta}_{\sigma\delta}\delta_{\mu\delta}\epsilon_{\sigma\beta\nu\rho}}_{=0}-\underbrace{\hat{\delta}_{\sigma\delta}\delta_{\beta\delta}\epsilon_{\sigma\mu\nu\rho}}_{=0}=4(2\omega-4)\epsilon_{\beta\mu\nu\rho} \end{aligned} \tag{A.39}$$

The contribution of the $\bar{l}$ integration can be derived through a well posed Mellin transform:

$$\begin{aligned} \int\frac{d^4\bar{l}}{(2\pi)^4}\frac{1}{[\bar{l}^2+(\hat{l}^2+\Delta)]^3} &= \frac{1}{\Gamma(3)}\int\frac{d^4\bar{l}}{(2\pi)^4}\int_0^{+\infty}dtt^2e^{-t[\bar{l}^2+(\hat{l}^2+\Delta)]}= \\ =\frac{1}{\Gamma(3)}\frac{1}{16\pi^2}\int_0^{+\infty}dtt^2\frac{e^{-t(\hat{l}^2+\Delta)}}{t^2} &= \frac{1}{32\pi^2}\frac{1}{[\hat{l}^2+\Delta]^3} \end{aligned} \tag{A.40}$$

Plugging this partial result in (A.38), we can further develop our expression:

$$
\begin{aligned}
I'^{abc}_{\mu\nu}(k_1,k_1) &= -\frac{\mu^{4-2\omega}}{8\pi^2}\omega^{abc}(4\epsilon_{\beta\mu\nu\rho})k_{1\beta}k_{2\rho}\int_0^1 dxdy\int\frac{d^{2\omega-4}\hat{l}}{(2\pi)^{2\omega-4}}\frac{\hat{l}^2}{[\hat{l}^2+\Delta]^3} = \\
&= -\frac{\mu^{4-2\omega}}{2\pi^2}\omega^{abc}\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}\int_0^1 dxdy\int\frac{d^{2\omega-4}\hat{l}}{(2\pi)^{2\omega-4}}\frac{\hat{l}^2}{\Gamma(3)}\int_0^{+\infty} d\tau\tau^2 e^{-\tau(\hat{l}^2+\Delta)} = \\
&= -\frac{\mu^{4-2\omega}}{4\pi^2}\omega^{abc}\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}\int_0^1 dxdy\int_0^{+\infty} d\tau\tau^2 e^{-\tau\Delta}\left(-\frac{d}{d\tau}\right)\int\frac{d^{2\omega-4}\hat{l}}{(2\pi)^{2\omega-4}}e^{-\tau\hat{l}^2} = \\
&= \frac{\mu^{4-2\omega}}{4\pi^2}\frac{\omega^{abc}}{(2\pi)^{2\omega-4}}\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}\int_0^1 dxdy\int_0^{+\infty} d\tau\tau^2 e^{-\tau\Delta}\left(\frac{d}{d\tau}\right)\left(\frac{\pi}{\tau}\right)^{\omega-2} = \\
&= \frac{\mu^{4-2\omega}}{4\pi^{4-\omega}}\frac{(2-\omega)\omega^{abc}}{(2\pi)^{2\omega-4}}\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}\int_0^1 dxdy\int_0^{+\infty} d\tau\tau^{3-\omega}e^{-\tau\Delta} = \\
&= \frac{\mu^{4-2\omega}}{4\pi^{4-\omega}}\frac{(2-\omega)\omega^{abc}}{(2\pi)^{2\omega-4}}\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}\Gamma(2-\omega)\int_0^1 dxdy\Delta^{\omega-4} = \\
&\xrightarrow{\omega\to 2}\quad \frac{\omega^{abc}}{4\pi^2}(2-\omega)\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}\left[\frac{1}{2-\omega}+\dots\right]\int_0^1 dxdy
\end{aligned}
\tag{A.41}
$$

Therefore, the $\Gamma(2-\omega)$ expansion for $\omega \to 2$ exactly simplifies the vanishing $(2-\omega)$ overall factor. If we add the contribution $I''_{\mu\nu}(k_1,k_1)$, obtained from the previous result by exchanging $k_1, \mu, b \leftrightarrow k_2, \nu, c$, we could state:

$$
\begin{aligned}
I^{abc}_{\mu\nu}(k_1,k_1) &= I'^{abc}_{\mu\nu}(k_1,k_1) + I''^{abc}_{\mu\nu}(k_1,k_1) = \frac{1}{4\pi^2}\left[\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}\omega^{abc} + \epsilon_{\beta\nu\mu\rho}k_{2\beta}k_{1\rho}\omega^{acb}\right] = \\
&= \frac{1}{4\pi^2}\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}(\omega^{abc}+\omega^{acb}) = \frac{1}{4\pi^2}\epsilon_{\beta\mu\nu\rho}k_{1\beta}k_{2\rho}d^{abc}
\end{aligned}
\tag{A.42}
$$

Now substituing this outcome in the starting matrix element (A.10), we end up with:

$$
\langle k_1, r; k_2, s|\partial_\lambda J_5^{a\lambda}(z)|0\rangle = (-i)^2\frac{g^2}{4\pi^2}d^{abc}\epsilon_{\mu\nu\beta\rho}\varepsilon_r^{b\mu}(k_1)\varepsilon_s^{c\nu}(k_2)k_1^\beta k_2^\rho e^{i(k_1+k_2)z} + \mathcal{O}(g^2) \tag{A.43}
$$

where we moved back to Minkowski space, as the extra factor $-i$ betrays. However, to have a better insight of the right-hand side of this expression, let us consider the following matrix element: $\langle k_1, r; k_2, s|\mathcal{G}^b_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_c(z)|0\rangle$. We are developing a perturbative approach, retaining only lowest order terms in $g$. Therefore, we can just take into account the linearized version of $\mathcal{G}^b_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_c(z)$, where the four-vector potential is treated as a free field. Hence, at zero order in $g$, the normal modes expansion will hold true:

$$
A^\nu_b(z) = \int Dp\varepsilon^\nu_{rb}(p)(g_r(p)e^{-ipz} + g^\dagger_r(p)e^{ipz}) \tag{A.44}
$$

in which $Dp$ is the usual Lorentz invariant measure. Once said that, with a final effort we can draw up better our matrix element as:

$$
\begin{aligned}
&\langle k_1, r; k_2, s|\mathcal{G}^b_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}_c(z)|0\rangle = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\langle k_1, r; k_2, s|\mathcal{G}^{\mu\nu}_b\mathcal{G}^{\rho\sigma}_c(z)|0\rangle = \\
&= \frac{\epsilon_{\mu\nu\rho\sigma}}{2}\langle k_1, r; k_2, s|(\partial^\mu A^\nu_b\partial^\rho A^\sigma_c + \partial^\nu A^\mu_b\partial^\sigma A^\rho_c - \partial^\mu A^\nu_b\partial^\sigma A^\rho_c - \partial^\nu A^\mu_b\partial^\rho A^\sigma_c) + \mathcal{O}(g)|0\rangle = \\
&= 2\epsilon_{\mu\nu\rho\sigma}\langle k_1, r; k_2, s|\partial^\mu A^\nu_b\partial^\rho A^\sigma_c|0\rangle = \\
&= 2\epsilon_{\mu\nu\rho\sigma}\int DpDq\langle k_1, r; k_2, s|\partial^\mu[\varepsilon^\nu_{Ab}(p)g^\dagger_A(p)e^{ipz} + h.c.]\partial^\rho[\varepsilon^\sigma_{Bc}(q)g^\dagger_B(q)e^{iqz} + h.c.]|0\rangle = \\
&= -2\epsilon_{\mu\nu\rho\sigma}\int DpDq\varepsilon^\nu_{Ab}(p)\varepsilon^\sigma_{Bc}(q)p^\mu q^\rho\langle 0|[g_r(k_1), g^\dagger_A(p)][g_s(k_2), g^\dagger_B(q)]|0\rangle e^{i(p+q)z} + \\
&\quad - 2\epsilon_{\mu\nu\rho\sigma}\int DpDq\varepsilon^\nu_{Ab}(p)\varepsilon^\sigma_{Bc}(q)p^\mu q^\rho\langle 0|[g_s(k_2), g^\dagger_A(p)][g_r(k_1), g^\dagger_B(q)]|0\rangle e^{i(p+q)z} = \\
&= -2\epsilon_{\mu\nu\rho\sigma}\int d\mathbf{p}d\mathbf{q}\varepsilon^\nu_{Ab}(p)\varepsilon^\sigma_{Bc}(q)p^\mu q^\rho\eta_{rA}\delta^3(\mathbf{k}_1 - \mathbf{p})\eta_{sB}\delta^3(\mathbf{k}_2 - \mathbf{q})e^{i(p+q)z} + \\
&\quad - 2\epsilon_{\mu\nu\rho\sigma}\int d\mathbf{p}d\mathbf{q}\varepsilon^\nu_{Ab}(p)\varepsilon^\sigma_{Bc}(q)p^\mu q^\rho\eta_{sA}\delta^3(\mathbf{k}_2 - \mathbf{p})\eta_{rB}\delta^3(\mathbf{k}_1 - \mathbf{q})e^{i(p+q)z} = \\
&= -2\epsilon_{\mu\nu\rho\sigma}(\varepsilon^\nu_{rb}(k_1)\varepsilon^\sigma_{sc}(k_2)k^\mu_1 k^\rho_2 + \varepsilon^\nu_{sb}(k_2)\varepsilon^\sigma_{rc}(k_1)k^\mu_2 k^\rho_1)e^{i(k_1+k_2)z} = \\
&= -2\epsilon_{\mu\nu\rho\sigma}(\varepsilon^\nu_{rb}(k_1)\varepsilon^\sigma_{sc}(k_2) + \varepsilon^\nu_{rc}(k_1)\varepsilon^\sigma_{sb}(k_2))k^\mu_1 k^\rho_2 e^{i(k_1+k_2)z}
\end{aligned}
\tag{A.45}
$$

In the third line, we set $\langle k_1, r; k_2, s|\mathcal{O}(g)|0\rangle = 0$, because there is no higher order term in $g$ quadratic in the gauge four-potential. In the last passage we simply rearraged indeces, by employing the antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$. Nonetheless, the two addends in brackets will contribute in the same way, owing to the complete symmetry of the $d^{abc}$ factor. As a consequence, we will end up with the equality:

$$
\begin{aligned}
&\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\langle k_1, r; k_2, s|\mathcal{G}^{\mu\nu}_b\mathcal{G}^{\rho\sigma}_c(z)|0\rangle = -4\epsilon_{\mu\nu\rho\sigma}\varepsilon^\nu_{rb}(k_1)\varepsilon^\sigma_{sc}(k_2)k^\mu_1 k^\rho_2 e^{i(k_1+k_2)z} \\
&\frac{1}{8}\epsilon_{\mu\nu\beta\rho}\langle k_1, r; k_2, s|\mathcal{G}^{\mu\nu}_b\mathcal{G}^{\beta\rho}_c(z)|0\rangle = \epsilon_{\mu\nu\beta\rho}\varepsilon^\mu_{rb}(k_1)\varepsilon^\nu_{sc}(k_2)k^\beta_1 k^\rho_2 e^{i(k_1+k_2)z}
\end{aligned}
\tag{A.46}
$$

in which we reorganized and renamed indeces, in order to easily individuate how to rewrite the right-hand side of (A.43). This final remark lets us enunciate the Adler-Bardeen theorem:

$$
\langle k_1, r; k_2, s|\partial_\lambda J^{a\lambda}_5(z)|0\rangle = -\frac{g^2}{32\pi^2}d^{abc}\epsilon_{\mu\nu\beta\rho}\langle k_1, r; k_2, s|\mathcal{G}^{\mu\nu}_b\mathcal{G}^{\beta\rho}_c(z)|0\rangle + \mathcal{O}(g^2) \tag{A.47}
$$

where the additional factor one-half with respect to (1.64) is due to the anticommutator embedded in $d^{abc}$. Clearly, relation (A.47) can be obtained from (1.64) by acting with two functional derivatives, e.g $\frac{\delta}{\delta A^b_\mu}$, while keeping treating the gauge four-potential classically. We know that this formula is exact and it does not need higher order corrections, that we implicitly understood in $\mathcal{O}(g^2)$. Nevertheless, by using a procedure based on Feynman diagrams and perturbative expansion, it is formidably challenging to envisage how all perturbative corrections could vanish. Moreover, we derived the anomalous conservation law of a chiral fermion current as a formula valid for matrix elements. But how can we say that, for a non-abelian gauge symmetry, the four-divergence of $J^{a\lambda}_5(z)$ exactly equals the combination $\mathcal{G}_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}$ and not just its linear part? If even non-linear terms contributed, the matrix element $\langle k_1, r; k_2, s; k_3, t|\partial_\lambda J^{a\lambda}_5(z)|0\rangle$ would not vanish, too, by means of anomalous box diagrams [70]. The non-perturbative approach will extensively answer these questions.

## A.2 The $\zeta$-function approach

We have already pointed out in the first chapter of this work how divergences in determinants of differential operators are encoded in the ill-defined expression $\Pi_j \lambda_j$, which grows without control. An extremely useful instrument to regularize this kind of formulae is the $\zeta$-function procedure [15, 71]. Here the regularization is enclosed in the $\zeta$-function itself, which can be analytically extended to a meromorphic function in the whole complex plane (or to a holomorphic one in $\mathbb{C} - \{1\}$). Luckily, we can relate our functional determinants to the $\zeta$-function, to benefit from all of its properties.

Let us suppose we are given an elliptic differential operator $A$ of order $m > 0$, that means an operator

$$A\psi(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha \psi(x) \quad | \quad \forall v \in \mathbb{R}^n,\ v \neq 0 \ \Rightarrow \sum_{|\alpha|=m} a_\alpha(x) v^\alpha \neq 0 \tag{A.48}$$

That requirement simply makes $A$ positive semi-definite. We further assume $A$ is invertible and defined on a manifold without boundary $M$ of dimension $d$. If $A$ is hermitian or normal (i.e $AA^\dagger = A^\dagger A$), we can define the $\zeta$-function associated to $A$ as

$$\zeta(s, A) = Tr[A^{-s}] = \sum_j \lambda_j^{-s} \tag{A.49}$$

which converges for $Re[s] > d/m$. But we anticipated how $\zeta(s)$ can be continued on $\mathbb{C}$ as a function of the complex variable $s$: in particular, it will have a well-defined behavior on the imaginary axis $Re[s] = 0$. As a matter of fact, we can write

$$\frac{d}{ds}\zeta(s, A) = \frac{d}{ds}\left(\sum_j \lambda_j^{-s}\right) = -\sum_j (log\lambda_j)\lambda_j^{-s} \tag{A.50}$$

that, evaluted at $s = 0$, lets us claim

$$detA = exp\left\{-\frac{d}{ds}\zeta(s, A)\bigg|_{s=0}\right\} \tag{A.51}$$

Actually, formula (A.49) can be tweaked, so that we are able to associated a $\zeta$-function also for non-normal operators [15]. If we introduce the integral kernel $K(A, x, y)$ of $A$

$$A\psi(x) = \int d^4z K(A, x, z)\psi(z) \tag{A.52}$$

we could generalize (A.49) as

$$\zeta(s, A) = Tr[A^{-s}] = \sum_k \int d^4x \int d^4y u_k^*(x) K(A^{-s}, x, z) u_k(y) = \int d^4x K(A^{-s}, x, x) \tag{A.53}$$

in which $\{u_k(x)\}$ is the set of eigenfunctions of $A$ and where we employed the formal completeness relation $\sum_k u_k^*(x)u_k(y) = \delta^4(x - y)$. But the previous result can be used as a starting definition of $\zeta$, without any reference to the eigenvalues of $A$. Among the two remaining requirements on $A$ of ellipticity and invertibility, the last one can be dropped out, too, with some further considerations [72]. Nonetheless, we will not need to be that general.

Indeed, if we consider our case of interest of chapter one, we can recall (1.58), giving the effects of a chiral transformation on the fermionic determinant

$$det||i\bar{\not{D}}'/\mu|| = \mathcal{J}^{-1} det||e^{i\bar{\gamma}_5\alpha(x)}(i\bar{\not{D}}/\mu)e^{i\bar{\gamma}_5\alpha(x)}|| \tag{A.54}$$

In this context we had an fermionic field, interacting with an abelian four-potential. $\bar{\not{D}} = \bar{\not{\partial}} - ig\bar{\not{A}}$ is the covariant derivative in the euclidean formulation, where $\bar{\gamma}_\mu = \bar{\gamma}_\mu^\dagger$. $\mu$ is instead an arbitrary constant with mass dimensions, that we need to work with adimensional operators. The factor $\mathcal{J}$ is the jacobian of the chiral rotation, which can be express by means of (1.61)

$$log\mathcal{J}^{-1} = \frac{d\zeta}{ds}\left(0, e^{i\bar{\gamma}_5\delta\alpha(x)}\frac{i\bar{\not{D}}}{\mu}e^{i\bar{\gamma}_5\delta\alpha(x)}\right) - \frac{d\zeta}{ds}\left(0, \frac{i\bar{\not{D}}}{\mu}\right) \tag{A.55}$$

If we now consider that, in euclidean formalism, $(i\bar{\not{D}})^\dagger = i\bar{\not{D}}$, we will have $(i\bar{\not{D}})^\dagger i\bar{\not{D}} = (i\bar{\not{D}})^2$. By taking into account (A.51) and that $logdetA = \frac{1}{2}logdetA^2$, we will end up with an useful form of (A.55):

$$log\mathcal{J}^{-1} = \frac{1}{2}\frac{d\zeta}{ds}\left(0, e^{i\bar{\gamma}_5\delta\alpha(x)}\frac{-\bar{\not{D}}^2}{\mu^2}e^{i\bar{\gamma}_5\delta\alpha(x)}\right) - \frac{1}{2}\frac{d\zeta}{ds}\left(0, \frac{-\bar{\not{D}}^2}{\mu^2}\right) \tag{A.56}$$

The advantage of having introduced $\bar{\not{D}}^2$ is that we can deal now with a hermitian, positive semi-definite operator. We will make use of the first property later. Nonetheless, $\bar{\not{D}}^2$ still admits null eigenvalues, which can jeopardise the definition of $\zeta$, that is based on $A^{-s}$. But primarily, let us manipulate the previous formula a bit more. Employing the identity (A.49) and setting $\Delta = -\bar{\not{D}}^2/\mu^2$, we have

$$\begin{aligned} log\mathcal{J}^{-1} &= \frac{1}{2}\frac{d}{ds}\left\{Tr[\Delta + \delta_\alpha\Delta]^{-s} - Tr[\Delta]\right\}\Bigg|_{s=0} = \frac{1}{2}\frac{d}{ds}\left\{(-s)Tr[\Delta^{-s-1}\delta_\alpha\Delta]\right\}\Bigg|_{s=0} = \\ &= -\frac{1}{2}Tr[\Delta^{-s-1}\delta_\alpha\Delta]\Bigg|_{s=0} \end{aligned} \tag{A.57}$$

We can correctly determine $\delta_\alpha\Delta$ keeping in mind that:

$$e^{i\bar{\gamma}_5\delta\alpha(x)}\frac{i\bar{\not{D}}}{\mu}e^{i\bar{\gamma}_5\delta\alpha(x)} \approx \frac{i\bar{\not{D}}}{\mu}\underbrace{-\left(\bar{\gamma}_5\delta\alpha(x)\frac{\bar{\not{D}}}{\mu} + \frac{\bar{\not{D}}}{\mu}\bar{\gamma}_5\delta\alpha(x)\right)}_{\delta_\alpha(i\bar{\not{D}}/\mu)} \tag{A.58}$$

and so

$$\begin{aligned} \delta_\alpha\Delta = \delta_\alpha\left(\frac{-\bar{\not{D}}^2}{\mu}\right) &= \frac{i\bar{\not{D}}}{\mu}\delta_\alpha\left(\frac{i\bar{\not{D}}}{\mu}\right) + \delta_\alpha\left(\frac{i\bar{\not{D}}}{\mu}\right)\frac{i\bar{\not{D}}}{\mu} = \\ &= -i\left(\frac{\bar{\not{D}}}{\mu}\right)^2\bar{\gamma}_5\delta\alpha(x) - 2i\frac{\bar{\not{D}}}{\mu}\bar{\gamma}_5\delta\alpha(x)\frac{\bar{\not{D}}}{\mu} - i\bar{\gamma}_5\delta\alpha(x)\left(\frac{\bar{\not{D}}}{\mu}\right)^2 \end{aligned} \tag{A.59}$$

Therefore, we can write

$$\begin{aligned} log\mathcal{J}^{-1} &= -i\frac{1}{2}Tr\left[\left(\frac{\bar{\not{D}}^2}{\mu^2}\right)^{-s-1}\left(\left(\frac{\bar{\not{D}}}{\mu}\right)^2\bar{\gamma}_5\delta\alpha(x) + 2\frac{\bar{\not{D}}}{\mu}\bar{\gamma}_5\delta\alpha(x)\frac{\bar{\not{D}}}{\mu} + \bar{\gamma}_5\delta\alpha(x)\left(\frac{\bar{\not{D}}}{\mu}\right)^2\right)\right]\Bigg|_{s=0} = \\ &= -2iTr\left[\left(\frac{\bar{\not{D}}}{\mu}\right)^{-2s}\bar{\gamma}_5\delta\alpha(x)\right]\Bigg|_{s=0} = -2itr\left[tr_x\left[\delta\alpha(x)\left(\frac{-\bar{\not{D}}^2}{\mu}\right)^{-s}\right]\bar{\gamma}_5\right]\Bigg|_{s=0} \end{aligned} \tag{A.60}$$

where in the next-to-last passage we made use of the trace cyclicity property, while in the last one we split the total trace operation $Tr$ into the matrix $tr$ and integral $tr_x$ trace for matter of convenience. We also added an extra factor $(-1)^{-s}$, that can not affect the result at $s = 0$, but it will prove useful later. At this point, we are just missing a more explicit form of $tr_x[-\delta\alpha(x)\bar{\not{D}}^2/\mu]^{-s}$, which is equal to the $\zeta(s)$ of $-\bar{\not{D}}^2/\mu^2$, considering how $\delta\alpha(x)$ is just a multiplicative function.

For this purpose, we can return to (A.53) and consider an operator $A$ hermitian and strictly positive. That allows us to claim

$$\begin{aligned}\zeta(s,A) &= \int d^4x K(A^{-s},x,x) = \sum_n \int d^4x \lambda_n^{-s} u_n^*(x) u_n(x) = \\ &= \frac{1}{\Gamma(s)} \sum_n \int d^4x \int_0^{+\infty} dt t^{s-1} e^{-\lambda_n t} u_n^*(x) u_n(x) = \frac{1}{\Gamma(s)} \int d^4x \int_0^{+\infty} dt t^{s-1} H_A(t,x,x)\end{aligned} \tag{A.61}$$

where we relate the integral kernel of $A^{-s}$ to the heat kernel of $A$ by means of a Mellin transform, which holds because of the property of $A$ of being positive. But if we now assume the last expression of (A.61) as the definition of the $\zeta$-function of $A$, we will be able to deal with hermitian positive semi-definite operators, which is precisely what we need.

We remind here how the heat kernel of a hermitian, positive semi-definite operator is defined as [73]

$$H_A(t,x,y) = \sum_n e^{-\lambda_n t} u_n^*(x) u_n(y) \tag{A.62}$$

with $t$ an adimensional parameter. The heat kernel $H_A(t,x,y)$ satisfies the heat equation

$$\left(\frac{\partial}{\partial t} + A\right) H_A(t,x,y) = 0 \tag{A.63}$$

with the initial condition $H_A(0,x,y) = \delta^4(x-y)$, owning to the completeness of $\{u_k(x)\}$. A mathematical result states that, for small $t$, the heat kernel expansion holds true

$$H_A(t,x,y) = \frac{1}{16\pi^2 t^2} e^{-\frac{(x-y)^2}{4\nu^2 t}} \sum_{n=0}^{+\infty} a_n(x,y) \nu^{n-2} t^n \qquad t \ll 1 \tag{A.64}$$

where $a_n(x,y)$ are the Seeley-De Witt coefficients and $\nu$ an arbitrary constant, necessary to adjust dimensions. The initial condition on $H_A(0,x,y)$ automatically implies $a_0(x,y) = 1$. Plugging the heat kernel expansion in (A.61), we get

$$\zeta(s,A) = \frac{1}{\Gamma(s)} \int d^4x \left( \frac{1}{16\pi^2} \sum_{n=0}^{+\infty} a_n(x) \int_0^{\epsilon} dt t^{s-3+n} + \int_{\epsilon}^{+\infty} dt t^{s-1} h_A(t,x,x) \right) \tag{A.65}$$

in which $a_n(x,x) \equiv a_n(x)$. We restrict the $t$ integration by above with an arbitrary small $\epsilon$ and we represent the unknown part of the solution of the heat equation for large $t$ with $h_A(t,x,y)$. We are ultimately interested in the value of $\zeta$ in zero. Therefore, remembering how $\frac{1}{\Gamma(s)} \sim s + \mathcal{O}(s)$ for $s \to 0$, we will have

$$\begin{aligned}\zeta(s,A) &= \frac{1}{16\pi^2} \frac{1}{\Gamma(s)} \int d^4x \sum_{n=0}^{+\infty} a_n(x) \frac{\epsilon^{s-2+n}}{s-2+n} + \underbrace{(s+\mathcal{O}(s)) \int d^4x \int_{\epsilon}^{+\infty} dt t^{s-1} h_A(t,x,x)}_{\to 0} = \\ &\approx \frac{1}{16\pi^2}(s+\mathcal{O}(s)) \int d^4x \sum_{n=0}^{+\infty} a_n(x) \frac{\epsilon^{s-2+n}}{s-2+n} \xrightarrow{s\to 0} \frac{1}{16\pi^2} \int d^4x a_2(x)\end{aligned} \tag{A.66}$$

This result allows us to rewrite (A.60) as

$$log\mathcal{J}^{-1} = -\frac{i}{8\pi^2}\int d^4x_E tr\Big[a_2(x)\bar{\gamma}_5\delta\alpha(x)\Big] \tag{A.67}$$

where $a_2(x)$ is the second Seeley-De Witt coefficient of the heat kernel expansion, associated to the operator $-\bar{\not{\mathcal{D}}}^2/\mu^2$ [71, 74]. Equation (A.63) and (A.64) will happen to be

$$\frac{\partial}{\partial\tau}H_A(t,x,y) = \bar{\not{\mathcal{D}}}^2 H_A(t,x,y) \qquad H_A(\tau,x,y) = \frac{1}{16\pi^2}e^{-\frac{(x-y)^2}{4(\mu\nu)^2\tau}}\sum_{n=0}^{+\infty}a_n(x,y)(\mu\nu)^{2n-4}\tau^{n-2} \tag{A.68}$$

in which we include the dimensionnful parameter $\tau = t/\mu^2$. $\nu$ has length dimensions, that, in natural unit, are the inverse of mass dimensions, typical of $\mu$. Making use of the arbitrariness in the choice of these two contants, we can set $\nu = 1/\mu$, so that the unknown factors disappear from our description. We will use these two relations in order to assess $a_2(x)$.

We can start evalutating

$$\begin{aligned}\frac{\partial}{\partial\tau}H_A(t,x,y) &= \frac{e^{-\frac{(x-y)^2}{4\tau}}}{16\pi^2}\sum_{n=0}^{+\infty}\Big[a_n(x,y)(n-2)\tau^{n-3} + \frac{(x-y)^2}{4\tau^2}a_n(x,y)\tau^{n-2}\Big] = \\ &= \frac{e^{-\frac{(x-y)^2}{4\tau}}}{16\pi^2}\sum_{n=0}^{+\infty}\Big[a_{n-1}(x,y)(n-3) + \frac{(x-y)^2}{4}a_n(x,y)\Big]\tau^{n-4}\end{aligned} \tag{A.69}$$

clearly using the fact that $a_{-n} = 0$ for $n \in \mathbb{N}$. If we now consider

$$\bar{\not{\mathcal{D}}}^2 = \bar{\gamma}_\mu\bar{\gamma}_\nu\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\nu = \frac{\{\bar{\gamma}_\mu,\bar{\gamma}_\nu\}}{2}\frac{\{\bar{\mathcal{D}}_\mu,\bar{\mathcal{D}}_\nu\}}{2} + \frac{[\bar{\gamma}_\mu,\bar{\gamma}_\nu]}{2}\frac{[\bar{\mathcal{D}}_\mu,\bar{\mathcal{D}}_\nu]}{2} = \underbrace{\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\mu}_{D} + \underbrace{\frac{[\bar{\gamma}_\mu,\bar{\gamma}_\nu]W_{\mu\nu}}{4}}_{Y} \tag{A.70}$$

where $W_{\mu\nu} = [\bar{\mathcal{D}}_\mu,\bar{\mathcal{D}}_\nu]$, we will be able to compute:

$$\begin{aligned}\bar{\not{\mathcal{D}}}H_A(t,x,y) &= \frac{e^{-\frac{(x-y)^2}{4\tau}}}{16\pi^2}\sum_{n=0}^{+\infty}\Big[\bar{\not{\mathcal{D}}}a_n(x,y)\tau^{n-2} - \frac{(x-y)_\nu\not{\partial}(x-y)_\nu}{2\tau}a_n(x,y)\tau^{n-2}\Big] = \\ &= \frac{e^{-\frac{(x-y)^2}{4\tau}}}{16\pi^2}\sum_{n=0}^{+\infty}\Big[\bar{\not{\mathcal{D}}}a_{n-1}(x,y) - \frac{(x-y)_\nu\not{\partial}(x-y)_\nu}{2}a_n(x,y)\Big]\tau^{n-3}\end{aligned} \tag{A.71}$$

$$\begin{aligned}\bar{\not{\mathcal{D}}}^2 H_A(t,x,y) = \frac{e^{-\frac{(x-y)^2}{4\tau}}}{16\pi^2}\sum_{n=0}^{+\infty}\Big[&\bar{\not{\mathcal{D}}}^2 a_{n-1}(x,y)\tau^{n-3} - \bar{\not{\mathcal{D}}}a_{n-1}(x,y)\frac{(x-y)_\nu\not{\partial}(x-y)_\nu}{2}\tau^{n-4} + \\ &- \frac{\not{\partial}((x-y)_\nu\not{\partial}(x-y)_\nu)}{2\tau}a_n(x,y)\tau^{n-2} - \frac{(x-y)_\nu\not{\partial}(x-y)_\nu}{2\tau}\bar{\not{\mathcal{D}}}a_n(x,y)\tau^{n-2} + \\ &+ \frac{((x-y)_\nu\not{\partial}(x-y)_\nu)^2}{4}a_n(x,y)\tau^{n-4}\Big]\end{aligned} \tag{A.72}$$

Taking into account that

$$\not{\partial}[(x-y)_\nu\not{\partial}(x-y)_\nu] = \not{\partial}[(x-y)_\nu\gamma_\nu] = 4 \qquad \text{and} \qquad ((x-y)_\nu\not{\partial}(x-y)_\nu)^2 = (x-y)^2 \tag{A.73}$$

the previous equation can be simplified as:

$$\bar{\not{\mathcal{D}}}^2 H_A(t,x,y) = \frac{e^{-\frac{(x-y)^2}{4\tau}}}{16\pi^2} \sum_{n=0}^{+\infty} \Big[ \bar{\not{\mathcal{D}}}^2 a_{n-2}(x,y) - (x-y)_\nu \bar{\mathcal{D}}_\nu a_{n-1}(x,y) + \\ - 2a_{n-1}(x,y) + \frac{(x-y)^2}{4} a_n(x,y) \Big] \tau^{n-4} \tag{A.74}$$

As a consequence, the heat equation in (A.68) will give back the recurrence relation

$$\bar{\not{\mathcal{D}}}^2 a_{n-2}(x,y) - (x-y)_\nu \bar{\mathcal{D}}_\nu a_{n-1}(x,y) - 2a_{n-1}(x,y) = a_{n-1}(x,y)(n-3) \\ \Rightarrow \qquad \bar{\not{\mathcal{D}}}^2 a_{\bar{n}-1}(x,y) = \bar{n} a_{\bar{n}}(x,y) + (x-y)_\nu \bar{\mathcal{D}}_\nu a_{\bar{n}}(x,y) \tag{A.75}$$

Our final task is to sort this relation out up to the second Seeley-De Witt coefficient.

Let us firstly take (A.75) for $\bar{n} = 0$, with $a_{-1} = 0$ and $a_0 = 1$, and derive it for $\bar{\mathcal{D}}_\mu$:

$$\bar{\not{\mathcal{D}}}^2 a_{-1}(x,y) = 0 \cdot a_0(x,y) + (x-y)_\nu \bar{\mathcal{D}}^\nu a_0(x,y) \\ \Rightarrow \qquad \bar{\mathcal{D}}_\mu[(x-y)_\nu \bar{\mathcal{D}}^\nu a_0(x,y)] = 0 \qquad \xrightarrow{x\to y} \qquad \bar{\mathcal{D}}_\nu a_0(x) = 0 \tag{A.76}$$

which automatically implies

$$\bar{\mathcal{D}}_\mu \bar{\mathcal{D}}_\nu a_0(x) = 0 \qquad \Rightarrow \qquad \delta_{\mu\nu} \bar{\mathcal{D}}_\mu \bar{\mathcal{D}}_\nu a_0(x) = 0 \qquad \Rightarrow \qquad D a_0(x) = 0 \tag{A.77}$$

Thereby, evalutating (A.75) for $\bar{n} = 1$, we will obtain

$$a_1(x) = \bar{\not{\mathcal{D}}}^2 a_0(x) = (D+Y) a_0(x) = Y \tag{A.78}$$

Considering again (A.75) in $\bar{n} = 1$, before appliying the limit $x \to y$, we can derive it for $\bar{\mathcal{D}}_\mu$ twice:

$$\bar{\not{\mathcal{D}}}^2 a_0(x,y) = a_1(x,y) + (x-y)_\nu \bar{\mathcal{D}}_\nu a_1(x,y) \\ \Rightarrow \qquad \bar{\mathcal{D}}_\mu \bar{\not{\mathcal{D}}}^2 a_0(x,y) = 2\bar{\mathcal{D}}_\mu a_1(x,y) + (x-y)_\nu \partial_\mu \bar{\mathcal{D}}_\nu a_1(x,y) \\ \Rightarrow \qquad D\bar{\not{\mathcal{D}}}^2 a_0(x,y) = 3Da_1(x,y) + (x-y)_\nu D\bar{\mathcal{D}}_\nu a_1(x,y) \\ \xrightarrow{x\to y} \qquad Da_1(x) = \frac{1}{3} D\bar{\not{\mathcal{D}}}^2 a_0(x) \tag{A.79}$$

The relation giving the second coefficient we are interested in will be:

$$\bar{\not{\mathcal{D}}}^2 a_1(x) = 2a_2(x) \qquad \Rightarrow \qquad a_2(x) = \frac{1}{2}(D+Y)a_1(x) = \frac{1}{6} D\bar{\not{\mathcal{D}}}^2 a_0(x) + \frac{1}{2} Y a_1(x) \\ \Rightarrow \qquad a_2(x) = (D+Y)a_1(x) = \frac{1}{6} DDa_0(x) + \frac{1}{6} DY + \frac{1}{2} Y^2 \tag{A.80}$$

so that we are solely left with the calculation of $DDa_0(x)$. To achieve that, we need to derive (A.76) not once, but four times:

$$(x-y)_\nu \bar{\mathcal{D}}_\nu a_0(x,y) = 0 \qquad \Rightarrow \qquad \bar{\mathcal{D}}_\alpha a_0(x,y) + (x-y)_\nu \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\nu a_0(x,y) = 0 \\ \Rightarrow \qquad \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha a_0(x,y) + \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\beta a_0(x,y) + (x-y)_\nu \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\nu a_0(x,y) = 0 \\ \Rightarrow \qquad \bar{\mathcal{D}}_\gamma \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha a_0(x,y) + \bar{\mathcal{D}}_\gamma \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\beta a_0(x,y) + \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\gamma a_0(x,y) + (x-y)_\nu \bar{\mathcal{D}}_\gamma \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\nu a_0(x,y) = 0 \\ \Rightarrow \qquad \bar{\mathcal{D}}_\delta \bar{\mathcal{D}}_\gamma \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha a_0(x,y) + \bar{\mathcal{D}}_\delta \bar{\mathcal{D}}_\gamma \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\beta a_0(x,y) + \bar{\mathcal{D}}_\delta \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\gamma a_0(x,y) + \bar{\mathcal{D}}_\gamma \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\delta a_0(x,y) + \\ + (x-y)_\nu \bar{\mathcal{D}}_\delta \bar{\mathcal{D}}_\gamma \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\nu a_0(x,y) = 0 \tag{A.81}$$

Evalutating the last equation on the diagonal $x = y$ and performing two separate contractions, we will generate:

$$\begin{aligned}
&\delta_{\delta\gamma}\delta_{\beta\alpha}(\bar{\mathcal{D}}_\delta\bar{\mathcal{D}}_\gamma\bar{\mathcal{D}}_\beta\bar{\mathcal{D}}_\alpha + \bar{\mathcal{D}}_\delta\bar{\mathcal{D}}_\gamma\bar{\mathcal{D}}_\alpha\bar{\mathcal{D}}_\beta + \bar{\mathcal{D}}_\delta\bar{\mathcal{D}}_\beta\bar{\mathcal{D}}_\alpha\bar{\mathcal{D}}_\gamma + \bar{\mathcal{D}}_\gamma\bar{\mathcal{D}}_\beta\bar{\mathcal{D}}_\alpha\bar{\mathcal{D}}_\delta)a_0(x) = 0 \\
&\Rightarrow \qquad (DD + \bar{\mathcal{D}}_\mu D\bar{\mathcal{D}}_\mu)a_0(x) = 0 \\
&\delta_{\delta\beta}\delta_{\gamma\alpha}(\bar{\mathcal{D}}_\delta\bar{\mathcal{D}}_\gamma\bar{\mathcal{D}}_\beta\bar{\mathcal{D}}_\alpha + \bar{\mathcal{D}}_\delta\bar{\mathcal{D}}_\gamma\bar{\mathcal{D}}_\alpha\bar{\mathcal{D}}_\beta + \bar{\mathcal{D}}_\delta\bar{\mathcal{D}}_\beta\bar{\mathcal{D}}_\alpha\bar{\mathcal{D}}_\gamma + \bar{\mathcal{D}}_\gamma\bar{\mathcal{D}}_\beta\bar{\mathcal{D}}_\alpha\bar{\mathcal{D}}_\delta)a_0(x) = 0 \\
&\Rightarrow \qquad (2\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\nu\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\nu + \underbrace{DD + \bar{\mathcal{D}}_\mu D\bar{\mathcal{D}}_\mu}_{=0})a_0(x) = 0
\end{aligned} \tag{A.82}$$

Then, we can exchange two derivatives of the last result

$$\begin{aligned}
&\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\nu\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\nu a_0(x) = 0 \qquad \Rightarrow \qquad (\bar{\mathcal{D}}_\nu\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\nu + [\bar{\mathcal{D}}_\mu, \bar{\mathcal{D}}_\nu]\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\nu)a_0(x) = 0 \\
&\Rightarrow \qquad \bar{\mathcal{D}}_\nu D\bar{\mathcal{D}}_\nu a_0(x) = -\frac{1}{2}[\bar{\mathcal{D}}_\mu, \bar{\mathcal{D}}_\nu][\bar{\mathcal{D}}_\mu, \bar{\mathcal{D}}_\nu] = -\frac{1}{2}W_{\mu\nu}W_{\mu\nu}
\end{aligned} \tag{A.83}$$

From the first contraction in (A.82), we can finally derive the desired equality:

$$DDa_0(x) = -\bar{\mathcal{D}}_\mu D\bar{\mathcal{D}}_\mu a_0(x) = +\frac{1}{2}W_{\mu\nu}W_{\mu\nu} \tag{A.84}$$

As a consequence, plugging in (A.80) we get

$$a_2(x) = \frac{1}{12}W_{\mu\nu}W_{\mu\nu} + \frac{1}{6}DY + \frac{1}{2}Y^2 \tag{A.85}$$

Remembering the definitions in (A.70), we see that $D$ does not contain gamma matrices, whereas $Y$ has two of them. Hence, substituing (A.85) inside (A.67) and keeping in mind the properties

$$\begin{aligned}
&tr[\bar{\gamma}_5] = 0 \\
&tr[(\text{odd number of } \bar{\gamma}_\mu s)\bar{\gamma}_5] = 0 \\
&tr[\bar{\gamma}_\mu\bar{\gamma}_\nu\bar{\gamma}_5] = 0
\end{aligned}$$

the only contribution surviving the trace operation will be the third one. We will end up with:

$$log\mathcal{J}^{-1} = -\frac{i}{16\pi^2}\int d^4x_E tr\Big[Y^2\bar{\gamma}_5\delta\alpha(x)\Big] \tag{A.86}$$

It worth noticing how, up to now, we did not explicitly use the fact that our chiral transformation was abelian. If this is the case, we can simplify the foregoing formula as

$$log\mathcal{J}^{-1} = -\frac{i}{16\pi^2}\int d^4x_E\frac{1}{4}tr[\bar{\gamma}_5\bar{\gamma}_\mu\bar{\gamma}_\nu\bar{\gamma}_\alpha\bar{\gamma}_\beta]W_{\mu\nu}W_{\alpha\beta}\delta\alpha(x) = -\frac{i}{16\pi^2}\int d^4x_E\epsilon_{\mu\nu\alpha\beta}W_{\mu\nu}W_{\alpha\beta}\delta\alpha(x) \tag{A.87}$$

where we also used $tr[\bar{\gamma}_5\bar{\gamma}_\mu\bar{\gamma}_\nu\bar{\gamma}_\alpha\bar{\gamma}_\beta] = 4\epsilon_{\mu\nu\alpha\beta}$. By way of conclusion, we just have to move back to euclidean space. As already seen in this work, we will have a factor $i$ coming from the space-time measure and a $-i$ from the singled-out derivative with rescpect to $x_4$ in $W_{\mu\nu}W_{\alpha\beta}$. Moreover, there will be a $-1$, stemming from the Levi-Civita symbol and an extra $i^2 = -1$, that we need to recover the definition $\mathcal{F}_{\mu\nu} = i[\mathcal{D}_\mu, \mathcal{D}_\nu]/g$. By doing that, we are able to piece together the bedrock of this work, brought up in the first chapter:

$$log\mathcal{J}^{-1} = -\frac{ig^2}{16\pi^2}\int d^4x\epsilon_{\mu\nu\alpha\beta}\mathcal{F}^{\mu\nu}\mathcal{F}^{\alpha\beta}\delta\alpha(x) \tag{A.88}$$

The generalization of the previous expression to non-abelian transformation is straightforward, if we simply separate in $tr[Y^2\bar{\gamma}_5\delta\alpha(x)]$ traces carried out over spinor and internal indeces. That will modify the preceding formula as

$$log\mathcal{J}^{-1} = -\frac{ig^2}{32\pi^2}\int d^4x\epsilon_{\mu\nu\alpha\beta}\mathcal{F}_a^{\mu\nu}\mathcal{F}_b^{\alpha\beta}tr[\tau^c\{T^a, T^c\}]\delta\alpha^c(x) \tag{A.89}$$

where the fundamental coefficient $d^{cab} = tr[\tau^c\{T^a, T^c\}]$ emerges.

Therefore, we were able to derive here the anomalous contribution in a completely non perturbative way, making use of the extremely general method of $\zeta$-function regularization. Nonetheless, a by far widespread technique to assess anomalies is the Fujikawa one. But reference [15] underscores an important point: Fujikawa procedure is very useful in different contexts, but it fails when applied to more general situations, where, for example, fermions are coupled to axial vectors or pseudoscalars. Thereby, the two methods coincide when the jacobian factor is associated to a hermitian operator, but some discepancies emerge for non-hermitian ones (which are a key ingredient in different theories, such as the Weinberg-Salam model). That lets us understand how the $\zeta$-function regularization still remains the most powerful and reliable instrument to handle quadratic path integrals that we have at disposal.

# Bibliography


[1] Michael E. Peskin, Daniel V. Schroeder *An Introduction to Quantum Field Theory*, Perseus Books Publishing, 1995

[2] Joseph Schechter *Introduction to Effective Lagrangians for QCD* arXiv:hep-ph/0112205v1 (14 Dec 2001)

[3] M. Gell-Mann, M. Levy, Nuovo Cimento 16, 705 (1960)

[4] Gilberto Colangelo, Gino Isidori *An introduction to CHPT* arXiv:hep-ph/0101264v1 (23 Jan 2001)

[5] Fu-Guang Cao *Determination of the $\eta - \eta'$ mixing angle* arXiv:1202.6075v1 [hep-ph] (27 Feb 2012)

[6] S. Weinberg *The $U(1)$ problem*, Phys. Rev. D11, 3583 (1975)

[7] G. t Hooft *How instantons solve the $U(1)$ problem*, Phys. Rev. Lett. 142, No. 6 (1986) 357-387

[8] C. Patrignani et al. (Particle Data Group), Chin. Phys. C, 40, 100001 (2016) and 2017 update

[9] Mark Srednicki *Quantum Field Theory*, Cambrige University Press, 2007

[10] Hilmar Forkel *A Primer on Instantons in QCD* arXiv:hep-ph/0009136v2 (23 Nov 2002)

[11] N. A. McDougall *The strong CP problem and invariance under large gauge transformations* KEK-TH-372, KEK Preprint 93170, November 1993

[12] Lewis H. Ryder *Quantum Field Theory*, Cambrige Univeristy Press, 1996

[13] K. Fujikawa, Phys. Rev. Lett. 42 (1979), 1195; Phys. Rev. D 21 (1980), 2848; Phys. Rev. D. 22 (1980), 1499(E)

[14] S. W. Hawking, Comm. Math. Phys. 55 (1977), 133

[15] R.E. Gamboa Saravi, M.A. Muschietti, F.A. Schaposnik, J.E. Solomin *Chiral symmetry and functional integral* Annals of Phys. 157 (1984) 360

[16] N.H. Christ, C. Dawson, T. Izubuchi, C. Jung, Q. Liu, R.D. Mawhinney, C.T. Sachrajda, A. Soni, R. Zhou *The $\eta$ and $\eta'$ mesons from Lattice QCD* arXiv:1002.2999v2 [hep-lat] (20 Oct 2010)

[17] Maxim Pospelov, Adam Ritz *Electric dipole moments as probes of new physics* arXiv:hep-ph/0504231v2 (16 Sep 2005)

[18] R.D. Peccei, Helen R. Quinn *CP Conservation in the Presence of Pseudoparticles*, Phys. Rev. Lett 38, 1440 (1977)

[19] C. Vafa, E. Witten *Parity Conservation in Quantum Chromodynamics* Phys. Rev. Lett. 53, 535 (1984)

[20] Peter W. Graham, Surjeet Rajendran *New Observables for Direct Detection of Axion Dark Matter* arXiv:hep-ph/1306.6088v2 (3 Sep 2013)

[21] C. P. Burgess, Guy D. Moore *The Standard Model: A Primer* McGill University Montreal (2006)

[22] Kei Yagyu *Higgs sectors with exotic scalar fields* arXiv:1304.6338v1 (23 Apr 2013)

[23] R.D. Peccei, Helen R. Quinn *Constraints imposed by CP conservation in the presence of pseudoparticles* Phys. Rev. D 19, 1791 (1977)

[24] Michael Dine, Willy Fischler, Mark Srednicki *A simple solution to the strong CP problem with a harmless axion* Phys. Lett. B 104, 199 (1981)

[25] A.R. Zhitnitsky, Soviet Journal of Nuclear Physics 31, 260 (1980)

[26] Jihn E. Kim *Weak-Interaction Singlet and Strong CP Invariance* Phys. Rev. Lett. 43, 103 (1979)

[27] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov *Can confinement ensure natural CP invariance of strong interactions?* Nucl. Phys. B 166, 493 (1980)

[28] Domènec Espriu, Federico Mescia, Albert Renau *Axion-Higgs interplay in the two Higgs-doublet model*, arXiv:1503.02953v2 [hep-ph] (12 Nov 2015)

[29] R.R.Volkas, A.J. Davies, G.C. Joshi *Naturalness of the invisible axion model* Phys. Lett. B, Volume 215, number 1 (8 December 1988)

[30] Giovanni Grilli di Cortona, Edward Hardy, Javier Pardo Vega, Giovanni Villadoro *The QCD axion, precisely*, arXiv:1511.02867v2 [hep-ph] (8 Jan 2016)

[31] Robert Foot, Archil Kobakhidze, Kristian L. McDonald, Raymond R. Volkas *Poincar Protection for a Natural Electroweak Scale* arxiv:1310.0223v1 [hep-ph] (1 Oct 2013)

[32] 't Hooft *Naturalness, chiral symmetry and spontaneous chiral symmetry breaking* Lecture III of series of classes given at the Cargse Summer Inst., Aug. 26 - Sept. 8, 1979, published in 'Recent developments in gauge theories'. Ed. by G. 't Hooft et al., (Plenum Press, New York NY 1980)

[33] John F. Gunion, Howard E. Haber *The CP-conserving two-Higgs-doublet model: the approach to the decoupling limit* arXiv:hep-ph/0207010v5 (26 Jan 2003)

[34] M. Peskin and T. Takeuchi, Phys. Rev. Lett. 65, 1964 (1990)

[35] P.Ciafaloni, D. Espriu *The Effective Lagrangian of the Two Higgs Doublet Model* arxiv:hep-ph/9612383v1 (17 Dec 1996)

[36] M. Baak, J. Cuth, J. Hallec, A. Hoecker, R. Kogler, K. Monig, M. Schott, J. Stelzer, Eur. Phys. J. C (2014) 74:3046 [arXiv:1407.3792v1 [hep-ph]]

[37] David J. E. Marsh *Axion Cosmology* arXiv:1510.07633v2 [astro-ph.CO] (15 Jun 2016)

[38] Pierre Sikivie *Axion Cosmology* arXiv:astro-ph/0610440v2 (10 Nov 2006)

[39] Eleonora Di Valentino, Elena Giusarma, Massimiliano Lattanzi, Olga Mena, Alessandro Melchiorri, Joseph Silk *Cosmological Axion and neutrino mass constraints from Planck 2015 temperature and polarization data* arXiv:1507.08665v1 [astro-ph.CO] (30 Jul 2015)

[40] Kibble, T. W. B. *Topology of cosmic domains and strings* J. Phys. A: Math. Gen. 9: 1387 (1976)

[41] Kibble, T. W. B. *Some implications of a cosmological phase transition* Phys. Rep. 67: 183 (1980)

[42] A. Vilenkin, E. P. S. Shellard *Cosmic Strings and Other Topological Defects* Cambridge University Press, 1995

[43] Igor G. Irastorza, Javier Redondo *New experimental approaches in the search for axion-like particles* arXiv:1801.08127v1 [hep-ph] (24 Jan 2018)

[44] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D98, 030001 (2018)

[45] Pavel Fileviez Prez, Hiren H. Patel *The Electroweak Vacuum Angle* arXiv:hep-ph/1402.6340v2 (1 Apr 2014)

[46] N. Cabibbo, E. C. Swallow, R. Winston, Ann. Rev. Nucl. Part. Sci. 53, 39 (2003)

[47] Howard Georgi, David B. Kaplan, Lisa Randall *Manifesting the invisible axion at low energies* Physics Letter, Volume 169B, number 1 (20 Mar 1986)

[48] Claude Itzykson, Jean-Bernard Zuber *Quantum field theory* Dover publications, 2013

[49] Jihn E. Kim, Gianpaolo Carosi *Axion and Strong CP Problem* arXiv:0807.3125v2 [hep-ph] (3 Mar 2009)

[50] Daniele S. M. Alves, Neal Weiner *A viable QCD axion in the MeV mass range* arXiv:1710.03764v1 [hep-ph] (10 Oct 2017)

[51] S.A. Larin *The renormalization of the axial anomaly in dimensional regularization* arXiv:hep-ph/9302240v1 (9 Feb 1993)

[52] Luca Di Luzio, Federico Mescia, Enrico Nardi, Paolo Panci, Robert Ziegler *The Astrophobic Axion* arXiv:hep-ph/1712.04940 (13 Dec 2017)

[53] Adrian Ayala, Inma Domínguez, Maurizio Giannotti, Alessandro Mirizzi, Oscar Straniero *Revisiting the bound on axion-photon coupling from Globular Clusters* arxiv:1406.6053v2 [astro-ph.RH] (14 Oct 2014)

[54] Steen Hannestad, Alessandro Mirizzi, Georg Raffelt *New cosmological mass limit on thermal relic axions* arXiv:hep-hp/0504059v1 (7 Apr 2005)

[55] Maria Archidiacono, Steen Hannestad, Alessandro Mirizzi, Georg Raffelt, Yvonne Y.Y. Wong *Axion hot dark matter bounds after Planck* arxiv:1307.0615v2 [astro-ph.CO] (21 Oct 2013)

[56] Sanghyeon Chang, Kiwoon Choi *Hadronic axion window and the big-bang nucleosynthesis* arXiv:hep-ph/9306216v1 (3 Jun 1993)

[57] J. Isern, E. GarcaBerro, S. Torres, S. Cataln *Axion cosmology and the cooling of white dwarf stars* arxiv:0806.2807v2 [astro-ph] (12 Jan 2009)

[58] Maurizio Giannotti, Igor G. Irastorza, Javier Redondo, Andreas Ringwald, Kenichi Saikawa *Stellar Recipes for Axion Hunters* arXiv:1708.02111v1 [hep-ph] (7 Aug 2017)

[59] Avik Paul, Debasish Majumdar, Kamakshya Prasad Modak *Neutron Star Cooling via Axion Emission by Nucleon-Nucleon Axion Bremsstrahlung* arXiv:1801.07928v2 [hep-ph] (30 Jan 2018)

[60] Nabarun Chakrabarty, Ujjal Kumar Dey, Biswarup Mukhopadhyaya *High-scale validity of a two-Higgs doublet scenario: a study including LHC data* arxiv:1407.2145v2 [hep-ph] (6 Jan 2015)

[61] Chien-Yi Chen, S. Dawson *Exploring Two Higgs Doublet Models Through Higgs Production* arXiv:1301.0309v2 [hep-ph] (4 Mar 2013)

[62] G. C. Branco, P. M. Ferreira, L. Lavoura, M. N. Rebelo, Marc Sher, Joo P. Silva *Theory and phenomenology of two-Higgs-doublet models* arxiv:1106.0034v3 (19 Dec 2011)

[63] R. Donghia on behalf of the Mu2e Calorimeter group *The Mu2e experiment at Fermilab* arXiv:1606.05559v1 [physics.ins-det] (16 Jun 2016)

[64] Jonathan L. Feng, Takeo Moroi, Hitoshi Murayama, Erhard Schnapka *Third Generation Familons, B Factories and Neutrino Cosmology* arxiv:hep-ph/9709411v3 (17 Feb 1998)

[65] Alejandro Celis, Javier Fuentes-Martn, Hugo Serdio *A class of invisible axion models with FCNCs at tree level* arXiv:1410.6218v2 [hep-ph] (6 Jan 2015)

[66] S. Adler et al. (E787, E949) *Measurement of the* $K^+ \to \pi^+\nu\bar{\nu}$ *branching ratio* Phys. Rev. D77, 052003 (2008), arXiv:0709.1000 [hep-ex]

[67] R. Ammar et al. (CLEO) *Search for the familon via* $B^\pm \to \pi^\pm X^0$, $B^\pm \to K^\pm X^0$ *and* $B^\pm \to K^0_S X^0$ *decays* Phys. Rev. Lett. 87, 271801 (2001), arXiv:hep-ex/0106038 [hep-ex]

[68] Alexander A. Andrianov, A. Bassetto, R. Soldati *Further remarks on quantization of massive chiral electrodynamics in four-dimensions* Phys.Rev. D44 (1991) 2602-2603

[69] P. Breitenlohner, D. Maison *Dimensional renormalization and action principle* Commun. Math. Phys. 52, 11 (1977)

[70] Adel Bilal *Lectures on Anomalies* arXiv:0802.0634v1 [hep-th] (5 Feb 2008)

[71] A. Bassetto, P. Giacconi, L. Griguolo, R. Soldati *Consistent and covariant gauge anomalies as solutions of the same extended algebraic problem* Phys.Lett. B251 (1990) 266-272 (1990)

[72] R.E. Gamboa Saravi, M.A. Muschietti, J.E. Solomin Comm. Math. Phys. 89 (1983) 363

[73] C. Adam, R.A. Bertlmann, P. Hofer *Overview on the anomaly and Schwinger term in two-dimensional QED* Riv.Nuovo Cim. 16N8 (1993) 1-52

[74] A. P. Balachandran, G. Marmo, V. P. Nair, C. G. Trahern *Nonperturbative proof of the non-abelian anomalies* Phys. Rev. D 25 (1982), 2713