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Energy emission by accelerated charges

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Anno Accademico 2017/2018

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1. Introduction

This paper analyzes the power emitted by charges accelerated on general trajectories. All the arguments discussed have been developed during the second half of the XIX century, starting from the equations of Maxwell (1865), whose general solution led to the expressions of the retarded potentials, deduced by Liénard and Wiechert in 1898 and 1900. These formulas allow to obtain the electric and magnetic fields produced by an accelerated charge; then some approximated regimes, in which these relations can be simplified, are treated.
Through the calculation of the Poynting vector, the total electromagnetic radiated power can be obtained; the two cases of linear and circular accleration are analyzed, because of their importance for the functioning of accelerators: at the beginning of the XX century, the first apparati were linear and could reach an energy of a few MeV, while nowadays circular ones are used more widely, and they can accelerate charges up to some TeV.
After this, the distribution in frequencies and angles of the radiation is studied, with particular attention to the case of an accelerated magnetic moment. The intensity distribution in angles and frequencies is generalized to the case of a continuous set of charges in motion, which is a situation of great interest for the accelerators.

2. Electric and magnetic fields

2.1. Point-like charge approximation

Consider an electric charge distribution, with electric charge and current density given by

$$\rho(\boldsymbol{x},t) = \frac{q}{4\pi r_0^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right), \tag{2.1.1a}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = \frac{q}{4\pi r_0^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\dot{\boldsymbol{x}}(t) \tag{2.1.1b}$$

where $\boldsymbol{x}(t)$ is a spatial trajectory, $\boldsymbol{x}$ is the observation point, r_0 and q are constants with the dimensions of a length and a charge. The function $f(\xi)$ describes the geometric distribution of the electric charge. Imposing conditions

$$\int_0^\infty d\xi \xi^2 f(\xi) = 1, \tag{2.1.2}$$

$$f(\xi) \to 0 \text{ for } \xi \to \infty \tag{2.1.3}$$

we are dealing with charges distributed with spherical simmetry, concentrated in a ball of radius r_0 around $\boldsymbol{x}(t)$ and moving rigidly along this trajectory. The normalization condition (2.1.2) means that q is the distribution's total charge. Quantities (2.1.1a) and (2.1.1b) satisfy the charge conservation equation

$$\frac{\partial}{\partial t}\rho(\boldsymbol{x},t) + \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{j}(\boldsymbol{x},t) = 0. \tag{2.1.4}$$

Indeed, calculating the two members of (2.1.4), it can be found

$$\begin{aligned}\frac{\partial}{\partial t}\rho(\boldsymbol{x},t) &= \frac{q}{4\pi r_0^4} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\frac{\partial}{\partial t}|\boldsymbol{x}-\boldsymbol{x}(t)| \\ &= -\frac{q}{4\pi r_0^4} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\frac{\boldsymbol{x}-\boldsymbol{x}(t)}{|\boldsymbol{x}-\boldsymbol{x}(t)|}\boldsymbol{\cdot}\dot{\boldsymbol{x}}(t)),\end{aligned} \tag{2.1.5a}$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{j}(\boldsymbol{x},t) = \frac{q}{4\pi r_0^4} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\frac{\boldsymbol{x}-\boldsymbol{x}(t)}{|\boldsymbol{x}-\boldsymbol{x}(t)|}\boldsymbol{\cdot}\dot{\boldsymbol{x}}(t). \tag{2.1.5b}$$

In (2.1.5b) it has been used the fact that $\boldsymbol{\nabla}\boldsymbol{\cdot}\dot{\boldsymbol{x}}(t) = 0$. This proves (2.1.4). In general, the distribution's size can be considered negligible compared to the distance between it and the observation point, so the charge is treatable as point-like; this is summarized by the condition

$$r_0 \ll |\boldsymbol{x}(t) - \boldsymbol{x}| \tag{2.1.6}$$

which means that the entire distribution is peaked around $\boldsymbol{x}(t)$ for every time t. Indeed, it can be shown that

$$\lim_{r_0\to 0}\frac{q}{4\pi r_0^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right) = q\delta(\boldsymbol{x}-\boldsymbol{x}(t)) \tag{2.1.7}$$

where $\delta(\boldsymbol{\xi})$ is the three dimensional delta function. Applying (2.1.7) to the charge and current density expressions, given in (2.1.1a) and (2.1.1b), it can be found

$$\rho(\boldsymbol{x}, t) = q\delta(\boldsymbol{x} - \boldsymbol{x}(t)), \tag{2.1.8a}$$

$$\boldsymbol{j}(\boldsymbol{x}, t) = q\delta(\boldsymbol{x} - \boldsymbol{x}(t))\, \dot{\boldsymbol{x}}(t). \tag{2.1.8b}$$

Equations (2.1.8a) and (2.1.8b) represent the charge density and the current density of a point-like charge, located in $\boldsymbol{x}(t)$ at time t and following this trajectory.

2.2. Computation of electric and magnetic fields

Electric and magnetic fields can be expressed in terms of the scalar and vector potentials ϕ and $\boldsymbol{A}$:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{2.2.1a}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{2.2.1b}$$

If substituting the potentials ϕ and $\boldsymbol{A}$ in the above equations with other ones of the form

$$\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{\nabla} f, \tag{2.2.2a}$$

$$\phi' = \phi + \frac{1}{c}\frac{\partial f}{\partial t} \tag{2.2.2b}$$

with f a scalar function, the electric and magnetic fields will not change: this is called gauge invariance and allows to impose the Lorentz condition

$$\frac{1}{c}\frac{\partial}{\partial t}\phi + \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \tag{2.2.3}$$

In order to find the scalar and vector potentials, it is now necessary to consider the Maxwell equations involving sources:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho, \tag{2.2.4}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}. \tag{2.2.5}$$

Substituting into (2.2.4) and (2.2.5) the expressions (2.2.1a) and (2.2.1b) and using condition (2.2.3), we find

$$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \tag{2.2.6}$$

$$\boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j}. \tag{2.2.7}$$

The solutions of (2.2.6) and (2.2.7) can be written as the sum of a particular and general solution of the homogeneous equations. In order to find the former, it is useful to divide the space occupied by the distribution in little volume elements: for the linearity of the equations, the total field is the sum of the fields produced by the charges in each of these volumes. The charge dq in each of them is a time function; if we choose the origin of the coordinates inside the volume considered and we call $\boldsymbol{r}$ the distance between the charge and the observation point, the charge density can be expressed as

$$\rho = dq(t)\delta(\boldsymbol{r}). \tag{2.2.8}$$

Inserting (2.2.8) into (2.2.6) we have

$$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi dq(t)\delta(\boldsymbol{r}) \tag{2.2.9}$$

which, for $\boldsymbol{r} \neq \boldsymbol{0}$, turns to

$$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{2.2.10}$$

ϕ exhibits spherical simmetry and is a function of $\boldsymbol{r}$ only, so we can rewrite (2.2.10) in spherical coordinates; applying the substitution $\chi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, t)r$ we find

$$\frac{\partial^2\chi}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0. \tag{2.2.11}$$

Equation (2.2.11) is the plane waves equation and has general solution of the form

$$\chi = f_1\left(t - \frac{r}{c}\right) + f_2\left(t + \frac{r}{c}\right) \tag{2.2.12}$$

where $f_1(\xi)$ and $f_2(\xi)$ are arbitrary functions. Since we are looking for the retarded solutions, we can set $f_2(\xi) = 0$. The scalar potential can be written as

$$\phi = \frac{\chi\left(t - \frac{r}{c}\right)}{r}. \tag{2.2.13}$$

For the function $\chi(\xi)$, we can choose one of those for which ϕ solves (2.1.7); we can see from (2.2.13) that $\phi \to \infty$ when $r \to 0$, so that spatial derivatives increase faster than the time ones and we can omit the latter in this limit, obtaining

$$\boldsymbol{\nabla}^2\left(\frac{\chi\left(t - \frac{r}{c}\right)}{r}\right) = -4\pi dq(t)\delta(\boldsymbol{r}). \tag{2.2.14}$$

Considering only the most singular spatial derivatives, it is possible to bring $\chi(\xi)$ out of the Laplacian operator. Recalling that, in distributional sense, $\boldsymbol{\nabla}^2(1/r) = 4\pi\delta(\boldsymbol{r})$, it follows that equation (2.2.14) has solution $\chi(t - r/c) = dq(t)$. To obtain the scalar potential it is necessary to integrate above all the origins of the coordinates of each small volume into which the distribution is divided. Using expression $dq = \rho\, dV$ and substituting in (2.2.13), we have

$$\phi(\boldsymbol{x}, t) = \int d^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t)|}\, \rho\left(\boldsymbol{x'}, t - \frac{|\boldsymbol{x} - \boldsymbol{x}(t)|}{c}\right). \tag{2.2.15}$$

Repeating analogous calculation for the vector potential, we find

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{c}\int d^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|}\, \boldsymbol{j}\left(\boldsymbol{x'}, t - \frac{|\boldsymbol{x} - \boldsymbol{x}(t)|}{c}\right). \tag{2.2.16}$$

Now we apply the above general results to the point-like charge distribution studied in the previous section 2.1. In order to simplify the calculations, we introduce the unit vector along the direction from the charge to the observation point and the velocity of the charge in units equal to c, as it can be seen in Fig. 2.2.1:

$$\boldsymbol{n}(\boldsymbol{x}, t) = \frac{\boldsymbol{x} - \boldsymbol{x}(t)}{|\boldsymbol{x} - \boldsymbol{x}(t)|}, \tag{2.2.17a}$$

$$\boldsymbol{\beta} = \frac{\dot{\boldsymbol{x}}(t)}{c}. \tag{2.2.17b}$$

Inserting expressions (2.1.8a) and (2.1.8b) in the potentials' expressions (2.2.15) and (2.2.16) and writing the charge and current density as

$$\rho(\boldsymbol{x}, t) = \int dt' \rho(\boldsymbol{x}, t')\, \delta(t' - t), \tag{2.2.18a}$$

$$\boldsymbol{j}(\boldsymbol{x}, t) = \int dt' \boldsymbol{j}(\boldsymbol{x}, t')\, \delta(t' - t) \tag{2.2.18b}$$

we find

$$\begin{aligned} \phi(\boldsymbol{x}, t) &= \int d^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} \int dt' \rho(\boldsymbol{x'}, t')\, \delta\left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t')|}{c}\right) \\ &= \int d^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} \int dt'\, q\delta(\boldsymbol{x'} - \boldsymbol{x}(t'))\, \delta\left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x'}|}{c}\right) \\ &= q \int dt' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t')|}\, \delta\left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t')|}{c}\right). \end{aligned} \tag{2.2.19a}$$

By similar calculation

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{q}{c} \int dt' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t')|} \dot{\boldsymbol{x}}(t')\, \delta\left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t')|}{c}\right). \tag{2.2.19b}$$

In order to apply again the integral definition of the Dirac delta function in (2.2.19a) and (2.2.19b), it is necessary to find the values of t' for which its argument vanishes, that is the solutions of the equation

$$t^* - t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}{c} = 0. \tag{2.2.20}$$

We can see that $t^* < t$ because $|\boldsymbol{x} - \boldsymbol{x}(t^*)|/c$ is nonnegative; the latter quantity is the amount of time a light signal needs to travel from the trajectory point $\boldsymbol{x}(t^*)$ to the observation point $\boldsymbol{x}$. Since $\beta < 1$, a spherical wavefront moving backward in time from

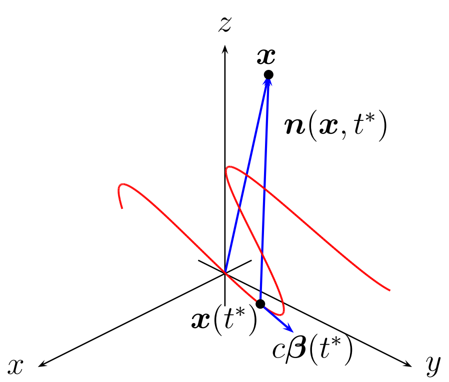

FIGURE 2.2.1. A moving charge. $\boldsymbol{x}$ is the observation point, $\boldsymbol{x}(t^*)$ the position and $c\boldsymbol{\beta}(t^*)$ the velocity of the particle at the retarded time

the observation point at time t to infinite distance would encounter the charge only once, so (2.2.20) has the only solution $t^* = t^*(\boldsymbol{x}, t)$, which is called retarded time. Recalling that the Dirac delta function has the property

$$\delta(f(t)) = \frac{\delta(t-t_0)}{|f'(t_0)|} \quad \text{if} \quad f(t_0) = 0, f'(t_0) \neq 0 \tag{2.2.21}$$

we can simplify it in expressions (2.2.19a) and (2.2.19b). We have indeed

$$\begin{aligned}\delta\left(t' - t + \frac{|\boldsymbol{x}-\boldsymbol{x}(t')|}{c}\right) &= \frac{\delta(t'-t^*(\boldsymbol{x},t))}{\left|\frac{d}{dt'}\left(t'-t+\frac{|\boldsymbol{x}-\boldsymbol{x}(t')|}{c}\right)\right|} \\ &= \frac{\delta(t'-t^*(\boldsymbol{x},t))}{1-\boldsymbol{\beta}(t')\cdot\boldsymbol{n}(\boldsymbol{x},t')}\end{aligned} \tag{2.2.22}$$

where absolute value was dropped because $|\boldsymbol{\beta}|, |\boldsymbol{n}| \leq 1$. Inserting (2.2.22) in (2.2.19a) and (2.2.19b), the following expressions for the potentials can be found:

$$\begin{aligned}\phi(\boldsymbol{x},t) &= q\int dt' \frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t')|}\frac{\delta(t'-t^*(\boldsymbol{x},t))}{1-\boldsymbol{\beta}(t')\boldsymbol{n}(\boldsymbol{x},t')} \\ &= \frac{q}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\frac{1}{1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)}\bigg|_{t^*=t^*(\boldsymbol{x},t)},\end{aligned} \tag{2.2.23a}$$

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x},t) &= \frac{q}{c}\int dt' \frac{\dot{\boldsymbol{x}}(t')}{|\boldsymbol{x}-\boldsymbol{x}(t')|}\frac{\delta(t'-t^*(\boldsymbol{x},t))}{1-\boldsymbol{\beta}(t')\cdot\boldsymbol{n}(\boldsymbol{x},t')} \\ &= \frac{q}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\frac{\dot{\boldsymbol{\beta}}(t^*)}{1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)}\bigg|_{t^*=t^*(\boldsymbol{x},t)}.\end{aligned} \tag{2.2.23b}$$

It appears that

$$\boldsymbol{A}(\boldsymbol{x},t) = \dot{\boldsymbol{\beta}}(t^*)\phi(\boldsymbol{x},t^*)\Big|_{t^*=t^*(\boldsymbol{x},t)}. \tag{2.2.24}$$

It is now possible to calculate the electric and magnetic fields emitted by the charge: to this end, we employ techniques of variational calculus. The first step is to compute the variation of ϕ under infinitesimal variations of $\boldsymbol{x}$ and t, which is

$$\begin{aligned}\delta\phi(\boldsymbol{x},t) = q\bigg[&-\frac{1}{1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2}\frac{\boldsymbol{x}-\boldsymbol{x}(t^*)}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\cdot(\delta\boldsymbol{x}-\delta\boldsymbol{x}(t^*)) \\ &+\frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\frac{\delta(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^2}\bigg].\end{aligned} \tag{2.2.25}$$

We notice further that

$$\delta\boldsymbol{x}(t^*) = \dot{\boldsymbol{x}}(t^*)\delta t^* = c\dot{\boldsymbol{\beta}}(t^*)\delta t^*. \tag{2.2.26}$$

For the second variation in (2.2.25), it is necessary to recall the expression of the variation of a unit vector $\boldsymbol{n} = \boldsymbol{r}/r$, where $r = |\boldsymbol{r}|$; this is

$$\begin{aligned}\delta\boldsymbol{n} = \delta\left(\frac{\boldsymbol{r}}{r}\right) = \frac{\delta\boldsymbol{r}}{r} + \boldsymbol{r}\delta\left(\frac{1}{r}\right) = \frac{\delta\boldsymbol{r}}{r} - \frac{\boldsymbol{r}}{r^2}\delta|\boldsymbol{r}| \\ = \frac{\delta\boldsymbol{r}}{r} - \frac{\boldsymbol{r}}{r^2}\left(\frac{\boldsymbol{r}}{r}\boldsymbol{\cdot}\delta\boldsymbol{r}\right) = \frac{\delta\boldsymbol{r}}{r} - \frac{\boldsymbol{r}\boldsymbol{r}}{r^2}\frac{\delta\boldsymbol{r}}{r} \\ = \left(\mathbb{I} - \frac{\boldsymbol{r}\boldsymbol{r}}{r^2}\right)\frac{\delta\boldsymbol{r}}{r} = (\mathbb{I} - \boldsymbol{n}\boldsymbol{n})\frac{\delta\boldsymbol{r}}{r}.\end{aligned} \tag{2.2.27}$$

where $\boldsymbol{n}\boldsymbol{n}$ denotes the dyadic product of the vector with itself, which produces a matrix, and $\mathbb{I}$ is the unit matrix. Using (2.2.27), the Leibnitz rule and (2.2.26), the second variation becomes

$$\begin{aligned}\delta(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)) &= \boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\delta\boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\delta\boldsymbol{\beta}(t^*) \\ &= \boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\frac{\mathbb{I} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\boldsymbol{\cdot}(\delta\boldsymbol{x} - \delta\boldsymbol{x}(t^*)) + \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\delta t^* \\ &= \boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\frac{\mathbb{I} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\boldsymbol{\cdot}(\delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*) + \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\delta t^*.\end{aligned} \tag{2.2.28}$$

Inserting (2.2.28) and (2.2.26) into (2.2.25), the variation of the scalar potential becomes

$$\begin{aligned}\delta\phi(\boldsymbol{x},t) = \frac{q}{1 - \boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\Bigg\{ - \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\boldsymbol{\cdot}(\delta\boldsymbol{x} - c\dot{\boldsymbol{\beta}}(t^*)\delta t^*) \\ + \frac{1}{1 - \boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)}\Bigg[\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\frac{\mathbb{I} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\boldsymbol{\cdot}(\delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*) \\ + \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\delta t^*\Bigg]\Bigg\}.\end{aligned} \tag{2.2.29}$$

To simplify this expression, it is useful to introduce

$$\alpha = 1 - \boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*), \tag{2.2.30a}$$

$$r = |\boldsymbol{x} - \boldsymbol{x}(t^*)|. \tag{2.2.30b}$$

This notation allows to rewrite the scalar potential in (2.2.13) as

$$\phi(\boldsymbol{x},t) = \frac{q}{r\alpha} \tag{2.2.31}$$

and its variation as

$$\begin{aligned}\delta\phi(\boldsymbol{x},t) = \frac{q}{r\alpha}\Bigg\{ - \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r}\boldsymbol{\cdot}\Big(\delta\boldsymbol{x} - c\dot{\boldsymbol{\beta}}(t^*)\delta t^*\Big) + \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\delta t^* \\ + \frac{1}{\alpha}\Bigg[\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\frac{\mathbb{I} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{r}\boldsymbol{\cdot}\Big(\delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*\Big)\Bigg]\Bigg\}.\end{aligned} \tag{2.2.32}$$

It is possible to obtain the gradient of ϕ recalling that

$$\boldsymbol{\nabla}\phi(\boldsymbol{x},t)\boldsymbol{\cdot}\delta\boldsymbol{x} = \delta\phi(\boldsymbol{x},t)|_{t=const}. \tag{2.2.33}$$

This means that it is necessary to set $\delta t = 0$ and to divide (2.2.32) by $\delta \boldsymbol{x}$, but in this case $\delta\phi$ is expressed in terms of δt^* and not δt: we have to find a relation between these two variations. From (2.2.20), performing calculation, it follows that

$$\begin{aligned} 0 &= \delta\left(t^* - t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}{c}\right) \\ &= \delta t^*\left(1 - \frac{\boldsymbol{x} - \boldsymbol{x}(t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\frac{\dot{\boldsymbol{x}}(t^*)}{c}\right) - \delta t + \frac{1}{c}\frac{\boldsymbol{x} - \boldsymbol{x}(t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\delta\boldsymbol{x} \\ &= \alpha\delta t^* - \delta t + \frac{1}{c}\delta\boldsymbol{x} \cdot \boldsymbol{n}(\boldsymbol{x}, t^*). \end{aligned} \tag{2.2.34}$$

Because of (2.2.34), setting $\delta t = 0$ is the same as

$$\delta t^* = -\frac{1}{\alpha c}\delta\boldsymbol{x} \cdot \boldsymbol{n}(\boldsymbol{x}, t^*). \tag{2.2.35}$$

Substituting (2.2.35) into the variation of the scalar potential (2.2.32), we get

$$\begin{aligned} \delta\phi(\boldsymbol{x}, t) = \frac{q}{\alpha r}\Bigg\{ &- \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{r} - \frac{1}{\alpha r}\boldsymbol{\beta}(t^*)\cdot(\mathbb{I} - 2\boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)) \\ &+ \frac{1}{\alpha^2 r}\boldsymbol{\beta}(t^*)\cdot(\mathbb{I} - \boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*))\cdot\boldsymbol{\beta}(t^*)\boldsymbol{n}(\boldsymbol{x}, t^*) \\ &- \frac{1}{\alpha^2 c}\boldsymbol{n}(\boldsymbol{x}, t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)\Bigg\}\cdot\delta\boldsymbol{x}. \end{aligned} \tag{2.2.36}$$

From the above expression, using (2.2.33) it is easy to find an expression of $\boldsymbol{\nabla}\phi$. In order to simplify this latter, it is necessary to work out the dyadic products:

$$\boldsymbol{\beta}(t^*)\cdot(\mathbb{I} - 2\boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)) = \boldsymbol{\beta}(t^*) - 2(1-\alpha)\boldsymbol{n}(\boldsymbol{x}, t^*), \tag{2.2.37a}$$

$$\begin{aligned} \boldsymbol{\beta}(t^*)\cdot(\mathbb{I} - \boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*))\cdot\boldsymbol{\beta}(t^*) &= \boldsymbol{\beta}(t^*)\left[\boldsymbol{\beta}(t^*) - (1-\alpha)\boldsymbol{n}(\boldsymbol{x}, t^*)\right] \\ &= |\boldsymbol{\beta}(t^*)|^2 - (1-\alpha)^2. \end{aligned} \tag{2.2.37b}$$

Substituting the relations (2.2.37) in the expression of the gradient of the scalar potential, we find

$$\begin{aligned} \boldsymbol{\nabla}\phi(\boldsymbol{x}, t^*) &= \frac{q}{\alpha r}\Bigg\{ - \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{r} - \frac{1}{\alpha r}(\boldsymbol{\beta}(t^*) - 2(1-\alpha)\boldsymbol{n}(\boldsymbol{x}, t^*)) \\ &\quad + \frac{1}{\alpha^2 r}(|\boldsymbol{\beta}(t^*)|^2 - (1-\alpha)^2)\boldsymbol{n}(\boldsymbol{x}, t^*) - \frac{1}{\alpha^2 c}\boldsymbol{n}(\boldsymbol{x}, t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x}, t^*)\Bigg\} \\ &= -\frac{q}{\alpha^3 r^2}\Bigg\{(\alpha^2 + 2\alpha(1-\alpha) + (1-\alpha)^2 - |\boldsymbol{\beta}(t^*)|^2)\boldsymbol{n}(\boldsymbol{x}, t^*) - \alpha\boldsymbol{\beta}(t^*) \\ &\quad + \frac{r}{c}\boldsymbol{n}(\boldsymbol{x}, t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x}, t^*)\Bigg\} \\ &= -\frac{q}{\alpha^3 r^2}\Bigg\{(1 - |\boldsymbol{\beta}(t^*)|^2)\boldsymbol{n}(\boldsymbol{x}, t^*) - \alpha\boldsymbol{\beta}(t^*) \\ &\quad + \frac{r}{c}\boldsymbol{n}(\boldsymbol{x}, t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x}, t^*)\Bigg\}. \end{aligned} \tag{2.2.38}$$

In order to calculate the partial derivative $\partial\boldsymbol{A}/\partial t$, we can firstly evaluate the differential of $\boldsymbol{A}$. It follows from (2.2.24)

$$\delta\boldsymbol{A} = \phi(\boldsymbol{x},t)\delta\boldsymbol{\beta}(t^*) + \boldsymbol{\beta}(t^*)\delta\phi(\boldsymbol{x},t) = \phi(\boldsymbol{x},t)\dot{\boldsymbol{\beta}}(t^*)\delta t^* + \boldsymbol{\beta}(t^*)\delta\phi(\boldsymbol{x},t). \tag{2.2.39}$$

Since we want to find the time partial derivative of the vector potential, the variation of ϕ has to be evaluated for $\delta\boldsymbol{x} = 0$. This can be done by imposing this condition in (2.2.32), which leads to

$$\begin{aligned}\delta\phi(\boldsymbol{x},t) = \frac{q}{\alpha r}\bigg\{&\frac{c}{r}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\delta t^* + \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\delta t^* \\ &+ \frac{1}{\alpha}\left[\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\frac{\mathbb{I} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{r}(-c\boldsymbol{\beta}(t^*)\delta t^*)\right]\bigg\}.\end{aligned} \tag{2.2.40}$$

The substitution of the above expression in the gradient of $\boldsymbol{A}$, given in (2.2.39), allows to find $\partial\boldsymbol{A}/\partial t^*$; since the quantity we want to find is $\partial\boldsymbol{A}/\partial t$, it is necessary to explicit the relation between the differentials of t and t^*. Setting $\delta\boldsymbol{x} = 0$ in expression (2.2.34), we find

$$\frac{\delta t^*}{\delta t} = \frac{1}{\alpha}. \tag{2.2.41}$$

Setting $\delta t = 0$, instead, brings to

$$\boldsymbol{\nabla} t^* = -\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha c}. \tag{2.2.42}$$

Making use of (2.2.40), (2.2.37b) and (2.2.41), it is now possible to work out the derivative

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{x},t) &= \frac{\delta\boldsymbol{A}(\boldsymbol{x},t)}{\delta t^*}\frac{\delta t^*}{\delta t} = \frac{1}{\alpha}\left(\phi(\boldsymbol{x},t)\dot{\boldsymbol{\beta}}(t^*) + \boldsymbol{\beta}(t^*)\frac{\delta\phi(\boldsymbol{x},t)}{\delta t^*}\right) \\ &= -\frac{q}{\alpha^3 r^2}\bigg\{-\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*) - \frac{\alpha r}{c}\boldsymbol{\beta}(t^*)\bigg[\frac{c}{r}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*) \\ &\quad - \frac{c}{\alpha r}\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\big(\mathbb{I} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\big)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*) + \frac{1}{\alpha}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\bigg]\bigg\} \\ &= -\frac{qc}{\alpha^3 r^2}\bigg\{-\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*) + (|\boldsymbol{\beta}(t^*)|^2 + \alpha - 1)\boldsymbol{\beta}(t^*) \\ &\quad - \frac{r}{c}\big(\boldsymbol{n}(\boldsymbol{x},t^*)\dot{\boldsymbol{\beta}}(t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*)\big)\bigg\}.\end{aligned} \tag{2.2.43}$$

The electric field of the point-like charge is given by (2.2.1a), using the expressions of $\boldsymbol{\nabla}\phi$ and $\partial\boldsymbol{A}/\partial t$ found in (2.2.38) and (2.2.43). This leads to

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{\alpha^3 r^2}\bigg\{&(1 - |\boldsymbol{\beta}(t^*)|^2)\Big(\boldsymbol{n}(\boldsymbol{x},t^*) - \boldsymbol{\beta}(t^*)\Big) \\ &+ \frac{r}{c}\bigg[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\Big)\Big(\boldsymbol{n}(\boldsymbol{x},t^*) - \boldsymbol{\beta}(t^*)\Big) - \dot{\boldsymbol{\beta}}(t^*) \\ &+ \Big(\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*)\Big)\dot{\boldsymbol{\beta}}(t^*)\bigg]\bigg\}.\end{aligned} \tag{2.2.44}$$

The quantity inside the square brackets in the above expression can by written as a double cross product, indeed

$$\begin{aligned}(\boldsymbol{n}-\boldsymbol{\beta})(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})-\dot{\boldsymbol{\beta}}+(\boldsymbol{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}} &= (\boldsymbol{n}-\boldsymbol{\beta})(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})-|\boldsymbol{n}|^2\dot{\boldsymbol{\beta}}+(\boldsymbol{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}} \\ &= (\boldsymbol{n}-\boldsymbol{\beta})(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})-\dot{\boldsymbol{\beta}}(\boldsymbol{n}(\boldsymbol{n}-\boldsymbol{\beta})) \\ &= \boldsymbol{n}\times\left[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right].\end{aligned} \tag{2.2.45}$$

Using the above relation and substituting back the expressions (2.2.30a) and (2.2.30b) of α and r, we find the final expression of the electric field, which is

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2}\bigg\{&(1-|\boldsymbol{\beta}(t^*)|^2)\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big) \\ &+\frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\times\left[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)\times\dot{\boldsymbol{\beta}}(t^*)\right]\bigg\}\bigg|_{t^*=t^*(\boldsymbol{x},t)}.\end{aligned} \tag{2.2.46}$$

The magnetic field of the point-like charge is given by (2.2.1b) which, making use of (2.2.24), becomes

$$\boldsymbol{B}(\boldsymbol{x},t)=\boldsymbol{\nabla}\times\Big(\phi(\boldsymbol{x},t)\boldsymbol{\beta}(t^*)\Big)=\phi(\boldsymbol{x},t)\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*)-\boldsymbol{\beta}(t^*)\times\boldsymbol{\nabla}\phi(\boldsymbol{x},t). \tag{2.2.47}$$

In order to calulate $\boldsymbol{\nabla}\times\boldsymbol{\beta}$, it is useful to consider the mixed product

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*)=\boldsymbol{u}\times\boldsymbol{\nabla}\cdot\boldsymbol{\beta}(t^*)=\boldsymbol{u}\times(\boldsymbol{\nabla}t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)=\boldsymbol{u}\cdot\boldsymbol{\nabla}t^*\times\dot{\boldsymbol{\beta}}(t^*). \tag{2.2.48}$$

Since in the previous expression $\boldsymbol{u}$ is any constant vector, we have

$$\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*)=-\dot{\boldsymbol{\beta}}(t^*)\times\boldsymbol{\nabla}t^* \tag{2.2.49}$$

where the gradient of the retarded time is given by (2.2.42). Inserting (2.2.49) and (2.2.38) in the magnetic field expression (2.2.47) and substituting back (2.2.30a) inside the curly brackets, we obtain

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x},t) &= \frac{q}{\alpha r}\dot{\boldsymbol{\beta}}(t^*)\times\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha c}+\boldsymbol{\beta}(t^*)\times\frac{q}{\alpha^3 r^2}\bigg\{(1-|\boldsymbol{\beta}(t^*)|^2)\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad -\alpha\boldsymbol{\beta}(t^*)+\frac{r}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\bigg\} \\ &= -\frac{q}{\alpha^3 r^2}\boldsymbol{n}(\boldsymbol{x},t^*)\times\bigg\{\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*)+(1-|\boldsymbol{\beta}(t^*)|^2)\boldsymbol{\beta}(t^*) \\ &\quad +\frac{r}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\cdot\boldsymbol{\beta}(t^*)\bigg\} \\ &= -\frac{q}{\alpha^3 r^2}\boldsymbol{n}(\boldsymbol{x},t^*)\times\bigg\{\boldsymbol{\beta}(t^*)(1-|\boldsymbol{\beta}(t^*)|^2) \\ &\quad +\frac{r}{c}\bigg[\Big(1-\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\boldsymbol{\beta}(t^*)\Big)\dot{\boldsymbol{\beta}}(t^*)+\Big(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Big)\boldsymbol{\beta}(t^*)\bigg]\bigg\}.\end{aligned} \tag{2.2.50}$$

The factors inside the curly brackets in the above expression are all cross multiplied with $\boldsymbol{n}(\boldsymbol{x},t)$, so we can add whatever quantity proportional to this unit vector inside the brackets without changing the result. Using this expedient and substituting back the expressions of α and r, which are given by (2.2.30a) and (2.2.30b), we find

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x},t) = &\frac{q}{\Big(1-\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)\Big)^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2}\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\times\Bigg\{(1-|\boldsymbol{\beta}(t^*)|^2)\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)+\frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}{c}\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\times\Big[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)\times\dot{\boldsymbol{\beta}}(t^*)\Big]\Bigg\}\Bigg|_{t^*=t^*(\boldsymbol{x},t)}.\end{aligned} \tag{2.2.51}$$

In the previous expression, the quantity inside the curly brackets has been written as a double cross product, as with the electric field in (2.2.46). It is easy to see that

$$\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{n}(\boldsymbol{x},t^*)\Big|_{t^*=t^*(\boldsymbol{x},t)}\times\boldsymbol{E}(\boldsymbol{x},t) \tag{2.2.52}$$

and that the two fields are always perpendicular to each other, as

$$\boldsymbol{B}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{E}(\boldsymbol{x},t) = 0. \tag{2.2.53}$$

The electromagnetic waves associated to these fields move along the direction of $\boldsymbol{n}(\boldsymbol{x},t^*)$, which is different from the one of $\boldsymbol{n}(\boldsymbol{x},t)$: the potentials in the observation point $\boldsymbol{x}$ at time t do not dipend on the position of the charge at the same moment $\boldsymbol{x}(t)$, but on its position $\boldsymbol{x}(t^*)$ when the fields where emitted. This is a consequence of the finite propagation velocity of the electromagnetic waves.

2.3. Approximations and regimes

In certain regimes, it is convenient to use approximated expressions of the electric and magnetic fields. The **non-relativistic regime** occurs when

$$|\boldsymbol{\beta}(t^*)| \ll 1. \tag{2.3.1}$$

The following approximations can be made:

$$1-\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*) \approx 1, \tag{2.3.2a}$$

$$(1-|\boldsymbol{\beta}(t^*)|^2)\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big) \approx \boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*) \approx \boldsymbol{n}(\boldsymbol{x},t^*). \tag{2.3.2b}$$

Using these relations, the electric field of the point-like charge, given by (2.2.46), becomes

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) \approx &\frac{q}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2}\Bigg\{\boldsymbol{n}(\boldsymbol{x},t^*) \\ &+\frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\times\Big(\boldsymbol{n}(\boldsymbol{x},t^*)\times\dot{\boldsymbol{\beta}}(t^*)\Big)\Bigg\}\Bigg|_{t^*=t^*(\boldsymbol{x},t)}.\end{aligned} \tag{2.3.3}$$

In this regime, the cross products in expression (2.2.51) of the magnetic field can be approximated as

$$\begin{aligned}\boldsymbol{n}\times\left\{\boldsymbol{n}\times\left[\left(\boldsymbol{n}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\} &= \left\{\left[\left(\boldsymbol{n}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\times\boldsymbol{n}\right\}\times\boldsymbol{n} \\ &= \left\{\boldsymbol{n}\times\left(\dot{\boldsymbol{\beta}}\times\boldsymbol{n}\right)-\boldsymbol{\beta}\times\left(\dot{\boldsymbol{\beta}}\times\boldsymbol{n}\right)\right\}\times\boldsymbol{n} \\ &= \left(1-\boldsymbol{\beta}\cdot\boldsymbol{n}\right)\dot{\boldsymbol{\beta}}\times\boldsymbol{n}\approx\dot{\boldsymbol{\beta}}\times\boldsymbol{n}.\end{aligned} \tag{2.3.4}$$

Furthermore

$$\boldsymbol{n}\times\left\{(1-|\boldsymbol{\beta}|^2)\left(\boldsymbol{n}-\boldsymbol{\beta}\right)\right\} = (1-|\boldsymbol{\beta}|^2)\boldsymbol{\beta}\times\boldsymbol{n}\approx\boldsymbol{\beta}\times\boldsymbol{n}. \tag{2.3.5}$$

Inserting (2.3.4) and (2.3.5) into (2.2.51), we obtain a non-relativistic expression for the magnetic field, which is

$$\boldsymbol{B}(\boldsymbol{x},t)\approx\frac{q}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2}\left\{\boldsymbol{\beta}(t^*)+\frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}{c}\dot{\boldsymbol{\beta}}(t^*)\right\}\times\boldsymbol{n}(\boldsymbol{x},t^*)\Bigg|_{t^*=t^*(\boldsymbol{x},t)}. \tag{2.3.6}$$

The **quasistatic regime** occurs when, in the time the fields need to travel from the particle to the observation point, the charge moves very little compared to the the distance covered by the fields. This means that the variation in each derivative of $\boldsymbol{x}(t)$ is smaller than the previous one, as

$$|\boldsymbol{x}-\boldsymbol{x}(t)|\gg\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{c}|\dot{\boldsymbol{x}}(t)|\gg\left[\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|^2}{c}\right]^2|\ddot{\boldsymbol{x}}(t)|\gg\dots \tag{2.3.7}$$

The above condition of the quasistatic regime implies the non-relativistic one, but also holds when the charge trajectory and the observation point are close to each other. As the Taylor series of $|\boldsymbol{x}-\boldsymbol{x}(t^*)|$ is rapidly convergent, the derivatives of $\boldsymbol{x}(t)$ vary little in the lapse $t-t^*$, so the effect of retardation is negligible. Hence we can consider

$$\boldsymbol{\beta}(t^*)\approx\boldsymbol{\beta}(t), \tag{2.3.8a}$$

$$\dot{\boldsymbol{\beta}}(t^*)\approx\dot{\boldsymbol{\beta}}(t). \tag{2.3.8b}$$

The quasistatic expressions for the fields follow from the non-relativistic ones by carrying out the limit $|\boldsymbol{x}-\boldsymbol{x}(t^*)|/c\to 0$ and applying conditions (2.3.8). This leads to

$$\boldsymbol{E}(\boldsymbol{x},t)\approx\frac{q}{|\boldsymbol{x}-\boldsymbol{x}(t)|^2}\boldsymbol{n}(\boldsymbol{x},t), \tag{2.3.9a}$$

$$\boldsymbol{B}(\boldsymbol{x},t)\approx\frac{q}{|\boldsymbol{x}-\boldsymbol{x}(t)|^2}\boldsymbol{\beta}(t)\times\boldsymbol{n}(\boldsymbol{x},t). \tag{2.3.9b}$$

It appears that the quasistatic electric field has the expression of the Coulombian field produced by a stationary charge.

The **radiation regime** occurs when the charge motion is strongly accelerated or when the distance between the spatial region where the charge acclerates and the observation point is very large. This is summarized by the condition

$$|\dot{\boldsymbol{x}}(t)|\ll\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{c}|\ddot{\boldsymbol{x}}(t)|. \tag{2.3.10}$$

This means that we can neglige the first term of the sum inside the brackets in the expressions (2.2.46) and (2.2.51) of the fields, because it is small respect to the triple product. In this way we obtain

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) &= \frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2}\frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}{c}\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad\times\left[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)\times\dot{\boldsymbol{\beta}}(t^*)\right]\Bigg|_{t^*=t^*(\boldsymbol{x},t)} \\ &= \frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{c|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad\times\left[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)\times\dot{\boldsymbol{\beta}}(t^*)\right]\Bigg|_{t^*=t^*(\boldsymbol{x},t)},\end{aligned} \tag{2.3.11a}$$

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x},t) &= \frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2}\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad\times\left\{\frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\times\left[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)\times\dot{\boldsymbol{\beta}}(t^*)\right]\right\}\Bigg|_{t^*=t^*(\boldsymbol{x},t)} \\ &= \frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{c|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad\times\left\{\boldsymbol{n}(\boldsymbol{x},t^*)\times\left[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)\times\dot{\boldsymbol{\beta}}(t^*)\right]\right\}\Bigg|_{t^*=t^*(\boldsymbol{x},t)}.\end{aligned} \tag{2.3.11b}$$

In this regime, the two fields satisfy

$$\boldsymbol{n}(\boldsymbol{x},t^*)\Big|_{t^*=t^*(\boldsymbol{x},t)}\cdot\boldsymbol{E}(\boldsymbol{x},t)=0, \tag{2.3.12a}$$

$$\boldsymbol{n}(\boldsymbol{x},t^*)\Big|_{t^*=t^*(\boldsymbol{x},t)}\cdot\boldsymbol{B}(\boldsymbol{x},t)=0 \tag{2.3.12b}$$

while being perpendicular to each other, as it is stated by (2.2.53). In module, they go like $1/|\boldsymbol{x}-\boldsymbol{x}(t^*)|$, so the energy they carry is proportional to $1/|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2$ and the energy flow is constant, since the surface of the sphere on which the energy is distributed goes like $|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2$. In the next sessions, we will always assume the radiation regime.

3. Poynting vector and power emission

3.1. Poynting vector

The Poynting vector field is defined as

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B}. \tag{3.1.1}$$

Reacalling the relation (2.2.52) among the fields and the relations (2.3.12), we find that the Poynting vector for a moving charge in the radiation regime is

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{x},t) &= \frac{c}{4\pi}\boldsymbol{E}(\boldsymbol{x},t)\times[\boldsymbol{n}(\boldsymbol{x},t^*)\times\boldsymbol{E}(\boldsymbol{x},t)]\Big|_{t^*=t^*(\boldsymbol{x},t)} \\ &= \frac{c}{4\pi}\Big[\boldsymbol{n}(\boldsymbol{x},t^*)|\boldsymbol{E}(\boldsymbol{x},t)|^2 - \boldsymbol{E}(\boldsymbol{x},t)\Big(\boldsymbol{E}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\Big)\Big]\Big|_{t^*=t^*(\boldsymbol{x},t)} \\ &= \frac{c}{4\pi}|\boldsymbol{E}(\boldsymbol{x},t)|^2\boldsymbol{n}(\boldsymbol{x},t^*)\Big|_{t^*=t^*(\boldsymbol{x},t)}. \end{aligned} \tag{3.1.2}$$

This vector represents the electromagnetic current density, which is the amount of energy that flows through a unitary area surface transverse to the energy flow in a unitary time; it is directioned like the energy flow. As a consequence of the retardation effect, this vector points towards $\boldsymbol{n}(\boldsymbol{x},t^*)$: the space around the charge at time t is not influenced by the particle at the same time, but by an earlier configuration of it, so it is convenient to express the Poynting vector in terms of the retarded time t^*, which is the time of the particle. In this way, t^* becomes an independent time variable, while t can be obtained by $t = t(\boldsymbol{x},t^*)$, that is the inverse function of $t^* = t^*(\boldsymbol{x},t)$. In this way, using (2.2.41), we obtain the Poynting vector in the frame of the charge, which is

$$\boldsymbol{S}^*(\boldsymbol{x},t^*) = \boldsymbol{S}(\boldsymbol{x},t)\Big|_{t=t(\boldsymbol{x},t^*)}\frac{\partial t}{\partial t^*}(\boldsymbol{x},t^*) = \boldsymbol{S}(\boldsymbol{x},t)\Big|_{t=t(\boldsymbol{x},t^*)}\Big(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\Big). \tag{3.1.3}$$

3.2. Total radiated power

It is evident from the previous sections that a moving charge generates an electromagnetic field, so it must convert part of its kinetic energy into electromagnetic one. The energy emitted by a moving charge in the point $\boldsymbol{x}$ at time t^* through a solid angle δo along the direction $\boldsymbol{n}(\boldsymbol{x},t^*)$ during δt^* is

$$\delta W = \boldsymbol{S}^*(\boldsymbol{x},t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2\delta o\,\delta t^* \tag{3.2.1}$$

where $|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2\delta o$ is the area of the surface element subtended by δo around $\boldsymbol{x}$. The power emitted per unit solid angle is

$$\frac{d}{do}P^*(\boldsymbol{x},t^*) = \frac{\delta W}{\delta o\,\delta t^*} \tag{3.2.2}$$

and is called differential radiated power. Substituting (3.1.2) and (3.1.3) inside the above expression, we obtain

$$\begin{aligned}\frac{d}{do}P^*(\boldsymbol{x},t^*) &= \boldsymbol{S}(\boldsymbol{x},t)\Big|_{t=t(\boldsymbol{x},t^*)}\Big(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\Big)\boldsymbol{n}(\boldsymbol{x},t^*)|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2 \\ &= \frac{c}{4\pi}|\boldsymbol{E}(\boldsymbol{x},t)|^2\Big|_{t=t(\boldsymbol{x},t^*)}\Big(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\Big)|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2.\end{aligned}\tag{3.2.3}$$

The electric field can be approximated as in the radiation regime: using (2.3.11a) we find

$$\begin{aligned}\frac{d}{do}P^*(\boldsymbol{x},t^*) &= \frac{q^2}{4\pi c}\frac{1}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^5}\Big\{\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad\times\Big[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)\times\dot{\boldsymbol{\beta}}(t^*)\Big]\Big|_{t=t(\boldsymbol{x},t^*)}\Big\}^2 \\ &= \frac{q^2}{4\pi c}\frac{1}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^5}\Big\{\Big[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\Big)\times\dot{\boldsymbol{\beta}}(t^*)\Big]^2 \\ &\quad-\Big[\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\Big(\boldsymbol{\beta}(t^*)\times\dot{\boldsymbol{\beta}}(t^*)\Big)\Big]^2\Big\}\end{aligned}\tag{3.2.4}$$

where it has been considered that $\boldsymbol{n}\cdot(\boldsymbol{n}\times\dot{\boldsymbol{\beta}})=0$. The differential radiated power, given by the above expression, depends on $\boldsymbol{x}$ only through $\boldsymbol{n}(\boldsymbol{x},t^*)$, so it is possible to write

$$\frac{d}{do}P^*(\boldsymbol{x},t^*) = \frac{d}{do}P^*(\boldsymbol{n}(\boldsymbol{x},t^*),t^*).\tag{3.2.5}$$

The total radiated power is the integral of (3.2.5) over every possible direction of emission, that is each unit vector starting from the charge position $\boldsymbol{x}(t^*)$. Developing calculation, we find

$$\begin{aligned}P^*(t^*) &= \int d^2\boldsymbol{n}\frac{d}{do}P^*(\boldsymbol{n},t^*) \\ &= \frac{q^2}{4\pi c}\int d^2\boldsymbol{n}\frac{1}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^5}\Big\{\Big[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\Big]^2-\Big[\boldsymbol{n}(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\Big]^2\Big\}.\end{aligned}\tag{3.2.6}$$

The quantity inside the curly brackets can be computed as

$$\begin{aligned}\Big[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\Big]^2-\Big[\boldsymbol{n}\cdot(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\Big]^2 &= (\boldsymbol{n}\times\dot{\boldsymbol{\beta}})^2+(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2 \\ &\quad-2(\boldsymbol{n}\times\dot{\boldsymbol{\beta}})\cdot(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})-\Big[\boldsymbol{n}\cdot(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\Big]^2 \\ &= \dot{\boldsymbol{\beta}}\cdot(\mathbb{I}-\boldsymbol{n}\boldsymbol{n})\cdot\dot{\boldsymbol{\beta}}+(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\cdot(\mathbb{I}-\boldsymbol{n}\boldsymbol{n})\cdot(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})-2\boldsymbol{n}\Big(\dot{\boldsymbol{\beta}}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\Big).\end{aligned}\tag{3.2.7}$$

In this way, the integral given in (3.2.6) becomes

$$
\begin{aligned}
P^*(t^*) = \frac{q^2}{4\pi c}\Big\{ & \dot{\boldsymbol{\beta}}\boldsymbol{\cdot}\int d^2\boldsymbol{n}\frac{\mathbb{I}-\boldsymbol{n}\boldsymbol{n}}{(1-\boldsymbol{\beta}\boldsymbol{n})^5}\boldsymbol{\cdot}\dot{\boldsymbol{\beta}} + (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\boldsymbol{\cdot}\int d^2\boldsymbol{n}\frac{\mathbb{I}-\boldsymbol{n}\boldsymbol{n}}{(1-\boldsymbol{\beta}\boldsymbol{n})^5}\boldsymbol{\cdot}(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}) \\
& - 2\int d^2\boldsymbol{n}\frac{\boldsymbol{n}}{(1-\boldsymbol{\beta}\boldsymbol{n})^5}\boldsymbol{\cdot}\Big(\dot{\boldsymbol{\beta}}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\Big)\Big\} \\
= \frac{q^2}{4\pi c}\Big\{ & \dot{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{I}_1(\boldsymbol{\beta})\boldsymbol{\cdot}\dot{\boldsymbol{\beta}} + (\dot{\boldsymbol{\beta}})\boldsymbol{\cdot}\boldsymbol{I}_1(\boldsymbol{\beta})\boldsymbol{\cdot}(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}) - 2\boldsymbol{I}_2(\boldsymbol{\beta})\boldsymbol{\cdot}\Big(\dot{\boldsymbol{\beta}}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\Big)\Big\}
\end{aligned}
\tag{3.2.8}
$$

where it has been made the substitution

$$
\begin{aligned}
\boldsymbol{I}_1(\boldsymbol{\beta}) = \int d^2\boldsymbol{n}\frac{\mathbb{I}-\boldsymbol{n}\boldsymbol{n}}{(1-\boldsymbol{\beta}\boldsymbol{n})^5} &= \int d^2\boldsymbol{n}\frac{\mathbb{I}}{(1-\boldsymbol{\beta}\boldsymbol{n})^5} \\
&- \frac{1}{12}\boldsymbol{\nabla}_{\boldsymbol{\beta}}\boldsymbol{\nabla}_{\boldsymbol{\beta}}\int d^2\boldsymbol{n}\frac{1}{(1-\boldsymbol{\beta}\boldsymbol{n})^3},
\end{aligned}
\tag{3.2.9a}
$$

$$
\boldsymbol{I}_2(\boldsymbol{\beta}) = \int d^2\boldsymbol{n}\frac{\boldsymbol{n}}{(1-\boldsymbol{\beta}\boldsymbol{n})^5} = \frac{1}{4}\boldsymbol{\nabla}_{\boldsymbol{\beta}}\int d^2\boldsymbol{n}\frac{1}{(1-\boldsymbol{\beta}\boldsymbol{n})^4}. \tag{3.2.9b}
$$

It is easy to see that the above expressions lead to the calculation of the integrals

$$
J_p(\boldsymbol{\beta}) = \int d^2\boldsymbol{n}\frac{1}{(1-\boldsymbol{\beta}\boldsymbol{n})^p} \tag{3.2.10}
$$

where p, in our case, takes the values 3, 4, 5. J_p can be calculated by passing to polar coordinates, with polar axis oriented as $\boldsymbol{\beta}$, as it is shown in Fig. 3.2.1. In this way we obtain

$$
\begin{aligned}
J_p(\boldsymbol{\beta}) &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{1}{(1-|\boldsymbol{\beta}|\cos\theta)^p} \\
&= 2\pi\int_{-1}^{1} d\xi \frac{1}{(1-|\boldsymbol{\beta}\xi|)^p} \quad \textit{where} \quad \xi = \cos\theta \\
&= \frac{2\pi}{|\boldsymbol{\beta}|}\frac{1}{p-1}\left(\frac{1}{(1-|\boldsymbol{\beta}|)^{p-1}} - \frac{1}{(1+|\boldsymbol{\beta}|)^{p-1}}\right).
\end{aligned}
\tag{3.2.11}
$$

It is now possible to evaluate the integrals (3.2.9a) and (3.2.9b). Indeed

$$
\boldsymbol{I}_2(\boldsymbol{\beta}) = \frac{1}{4}\boldsymbol{\nabla}_{\boldsymbol{\beta}}J_4(\boldsymbol{\beta}) = \frac{4\pi}{3}\frac{5+\boldsymbol{\beta}^2}{(1-|\boldsymbol{\beta}|^2)^4}\boldsymbol{\beta}, \tag{3.2.12a}
$$

$$
\boldsymbol{I}_1(\boldsymbol{\beta}) = J_5(\boldsymbol{\beta})\mathbb{I} - \frac{1}{12}\boldsymbol{\nabla}_{\boldsymbol{\beta}}\boldsymbol{\nabla}_{\boldsymbol{\beta}}J_3(\boldsymbol{\beta}) = \frac{8\pi}{3}\frac{(1+2\boldsymbol{\beta}^2)\mathbb{I} - 3\boldsymbol{\beta}\boldsymbol{\beta}}{(1-|\boldsymbol{\beta}|^2)^4}. \tag{3.2.12b}
$$

In the above calculations, the following relations have been used:

$$
\boldsymbol{\nabla}_{\boldsymbol{\beta}}f(|\boldsymbol{\beta}|) = f'(|\boldsymbol{\beta}|)\frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}, \tag{3.2.13a}
$$

$$
\boldsymbol{\nabla}_{\boldsymbol{\beta}}\boldsymbol{\nabla}_{\boldsymbol{\beta}}f(|\boldsymbol{\beta}|) = f''(|\boldsymbol{\beta}|)\frac{\boldsymbol{\beta}\boldsymbol{\beta}}{|\boldsymbol{\beta}|^2} + \frac{f'(|\boldsymbol{\beta}|)}{|\boldsymbol{\beta}|}\left(\mathbb{I} - \frac{\boldsymbol{\beta}\boldsymbol{\beta}}{|\boldsymbol{\beta}|^2}\right) \tag{3.2.13b}
$$

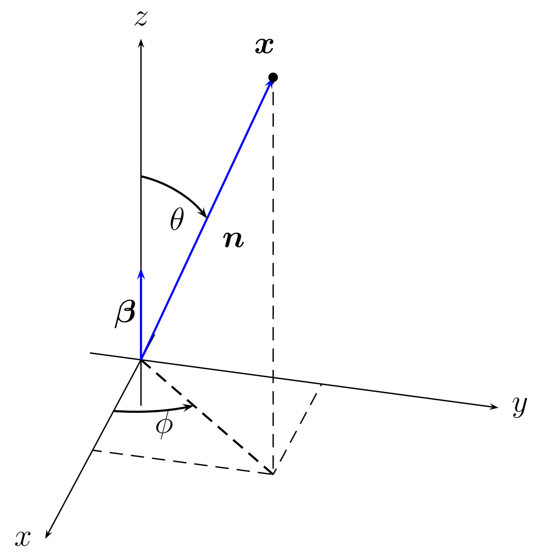

FIGURE 3.2.1. Polar coordinates. $\boldsymbol{x}$ is the observation point, $\boldsymbol{\beta}$ the velocity of the charge in units of c

where $f(\xi)$ is a generic function. Substituting back these expressions in the total radiated power relation, given by (3.2.8), we find

$$P^*(t^*) = \frac{2q^2}{3c}\frac{1}{(1-\boldsymbol{\beta}(t^*))^3}\left\{\dot{\boldsymbol{\beta}}(t^*)^2 - \Big(\boldsymbol{\beta}(t^*)\times\dot{\boldsymbol{\beta}}(t^*)\Big)^2\right\}. \tag{3.2.14}$$

The total radiated power can be written also in terms of the charge's acceleration components along the directions which are parallel and perpendicular to its speed. These are

$$\dot{\boldsymbol{\beta}}_{||} = (\dot{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}}, \tag{3.2.15a}$$

$$\dot{\boldsymbol{\beta}}_{\perp} = \dot{\boldsymbol{\beta}} - \dot{\boldsymbol{\beta}}_{||} = \dot{\boldsymbol{\beta}} - \dot{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}. \tag{3.2.15b}$$

Using the above expressions, the quantity inside the curly brackets of (3.2.14) becomes

$$\begin{aligned}\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2 &= (\dot{\boldsymbol{\beta}}_{||} + \dot{\boldsymbol{\beta}}_{\perp})^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}_{\perp})^2 \\ &= \dot{\boldsymbol{\beta}}_{||}^2 + (1-\boldsymbol{\beta}^2)\dot{\boldsymbol{\beta}}_{\perp}^2\end{aligned} \tag{3.2.16}$$

where we have used the facts that $\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}_{||} = 0$ and $\boldsymbol{\beta}\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}_{\perp} = 0$. Substituting the previous relation into (3.2.14), we find for the total radiated power

$$P^*(t^*) = \frac{2q^2}{3c}\frac{1}{(1-\boldsymbol{\beta}(t^*)^2)^3}\left\{\dot{\boldsymbol{\beta}}_{||}(t^*)^2 + \Big(1-\boldsymbol{\beta}(t^*)^2\Big)\dot{\boldsymbol{\beta}}_{\perp}(t^*))\right\}. \tag{3.2.17}$$

The total radiated power can be written also in terms of the relativistic momentum of the charge, which has the form

$$\boldsymbol{p}(t^*) = \frac{mc\boldsymbol{\beta}(t^*)}{\sqrt{1-\boldsymbol{\beta}(t^*)^2}}. \tag{3.2.18}$$

The components of $\dot{\boldsymbol{p}}$ which are parallel and perpendicular to the charge speed are

$$\dot{\boldsymbol{p}}_{||} = \dot{\boldsymbol{p}}\boldsymbol{\cdot}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}, \tag{3.2.19a}$$

$$\dot{\boldsymbol{p}}_\perp = \dot{\boldsymbol{p}} - \dot{\boldsymbol{p}}_{||} = \dot{\boldsymbol{p}} - \dot{\boldsymbol{p}}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}. \tag{3.2.19b}$$

The expressions (3.2.15) can be used in the calculation of the derivative respect to t^* of (3.2.18), which is

$$\dot{\boldsymbol{p}} = mc\left\{\frac{\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}^2)^{1/2}} + \frac{\dot{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{\beta}\boldsymbol{\beta}}{(1-\boldsymbol{\beta}^2)^{3/2}}\right\} = mc\left\{\frac{\dot{\boldsymbol{\beta}}_{||}+\dot{\boldsymbol{\beta}}_\perp}{(1-\boldsymbol{\beta}^2)^{1/2}} + \frac{|\boldsymbol{\beta}|^2\dot{\boldsymbol{\beta}}_{||}}{(1-\boldsymbol{\beta}^2)^{3/2}}\right\} \tag{3.2.20}$$

$$= mc\left\{\frac{\dot{\boldsymbol{\beta}}_\perp}{(1-\boldsymbol{\beta}^2)^{1/2}} + \frac{\dot{\boldsymbol{\beta}}_{||}}{(1-\boldsymbol{\beta}^2)^{3/2}}\right\}.$$

The above relation shows the form of the components of the charge momentum, that are

$$\dot{\boldsymbol{p}}_{||} = \frac{mc\dot{\boldsymbol{\beta}}_{||}}{(1-\boldsymbol{\beta}^2)^{3/2}}, \tag{3.2.21a}$$

$$\dot{\boldsymbol{p}}_\perp = \frac{mc\dot{\boldsymbol{\beta}}_\perp}{(1-\boldsymbol{\beta}^2)^{1/2}}. \tag{3.2.21b}$$

Inverting these expressions, the relations among the components of the velocity and the momentum of the charge can be found; inserting them in (3.2.17) we find

$$P^*(t^*) = \frac{2q^2}{3m^2c^3}\left(\dot{\boldsymbol{p}}_{||}^2 + \frac{\dot{\boldsymbol{p}}^2}{1-\boldsymbol{\beta}^2}\right). \tag{3.2.22}$$

According to (3.2.14), a longitudinal acceleration $\dot{\boldsymbol{\beta}}_{||}$ causes an energy loss in electromagnetic radiation which is $1/(1-\boldsymbol{\beta}^2)$ times larger than the one produced by a trasversal acceleration $\dot{\boldsymbol{\beta}}_\perp$ of the same magnitude. According to (3.2.22), conversely, a longitudinal force $\dot{\boldsymbol{p}}_{||}$ involves an energy loss $(1-\boldsymbol{\beta}^2)$ times smaller than a transversal force of the same magnitude. Even if this seems contraddictory, the two statements are analogous to each other, because the mass of the particle depends on its velocity, in accord with the relation

$$m(t^*) = \frac{m_0}{\sqrt{1-\boldsymbol{\beta}^2}}. \tag{3.2.23}$$

4. Linear and circular accelerators

4.1. Linear accelerator

We shall now consider a point-like charge q moving on a linear trajectory $\boldsymbol{x}(t)$, like in Fig. 4.1.1; the origin time is chosen such that in $t = 0$ the particle goes through $\boldsymbol{x} = 0$. We can write

$$\boldsymbol{x}(t) = s(t)\boldsymbol{e} \tag{4.1.1}$$

where $s(t)$ is the ascissa of the particle and $\boldsymbol{e}$ is the unit vector along the direction of the motion. It follows that

$$\boldsymbol{\beta} = \beta\boldsymbol{e}, \tag{4.1.2a}$$

$$\dot{\boldsymbol{\beta}} = \dot{\beta}\boldsymbol{e}. \tag{4.1.2b}$$

As a consequence

$$\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} = \boldsymbol{0}. \tag{4.1.3}$$

These relations allow us to simplify some of the terms appearing in the expression (3.2.4) of the differential radiated power. We have

$$1 - \boldsymbol{\beta}\cdot\boldsymbol{n} = 1 - \beta\boldsymbol{n}\cdot\boldsymbol{e}, \tag{4.1.4a}$$

$$\left[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]^2 = (\boldsymbol{n}\times\dot{\boldsymbol{\beta}})^2 = \dot{\beta}^2(\boldsymbol{n}\times\boldsymbol{e})^2. \tag{4.1.4b}$$

In this way, (3.2.4) reduces to

$$\frac{d}{do}P^*(\boldsymbol{x},t^*) = \frac{q^2}{4\pi c}\frac{\dot{\boldsymbol{\beta}}(t^*)^2(\boldsymbol{n}(\boldsymbol{x},t^*)\times\boldsymbol{e})^2}{(1-\beta(t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\boldsymbol{e})^5} \tag{4.1.5}$$

where $\beta(t) = \dot{s}(t)/c$. Introducing the angle $\theta(\boldsymbol{x},t^*)$ between $\boldsymbol{n}(\boldsymbol{x},t^*)$ and $\boldsymbol{e}$, the above expression can be written as

$$\frac{d}{do}P^*(\theta,t^*) = \frac{q^2}{4\pi c}\frac{\dot{\beta}(t^*)^2\sin^2\theta}{(1-\beta(t^*)\cos\theta)^5}. \tag{4.1.6}$$

The angular dependence in the differential radiated power is given by the function

$$f(\theta,\beta) = \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{4.1.7}$$

where β is an independent variable such that $0 \le \beta < 1$, and $0 \le \theta < \pi$. This function shows two minima for $\theta = 0, \pi$ and a maximum for $\theta = \theta(\beta)$, where

$$\theta(\beta) = \pm\arccos\left(\frac{-1+\sqrt{1+15\beta^2}}{3\beta}\right). \tag{4.1.8}$$

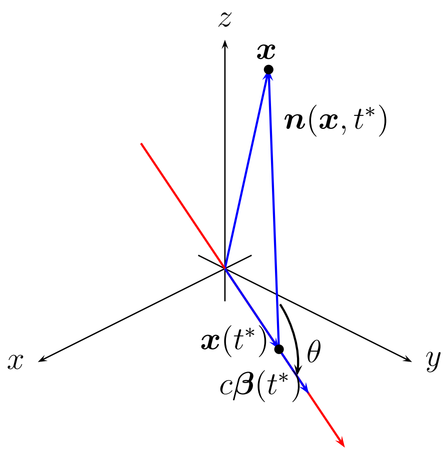

FIGURE 4.1.1. A linear accelerator

The other zero of $\partial f/\partial\theta$ is not physically accettable because it diverges to $-\infty$ for $\beta \to 0$, so it has been discarded. In these points, the function takes the values

$$f_{min} = f(0,\beta) = f(\pi,\beta) = 0, \tag{4.1.9a}$$

$$f_{max} = f\big(\theta(\beta),\beta\big) = \frac{-1-3\beta^2+\sqrt{1+15\beta^2}}{\big(4-\sqrt{1+15\beta^2}\big)^5}\frac{54}{\beta^2}. \tag{4.1.9b}$$

These results can be simplyfied in the non-relativistic regime $\beta \to 0$ and in the relativistic limit $\beta \to 1$. Using taylor expansions, we can write

$$\cos\theta(\beta) = \frac{-1+\sqrt{1+15\beta^2}}{3\beta} = O(\beta) \quad \text{for} \quad \beta \to 0, \tag{4.1.10a}$$

$$\begin{aligned}\cos\theta(\beta) &= 1-\frac{1-\beta}{4}+O((1-\beta)^2) = 1-\frac{1-\beta^2}{4(1+\beta)}+O((1-\beta)^2)\\ &= 1-\frac{1-\beta^2}{8}+O((1-\beta)^2) \quad \text{for} \quad \beta \to 1.\end{aligned} \tag{4.1.10b}$$

Given that $\arccos x \approx \pi/2$ as $x \to 0$ and $\arccos(x+1) \approx \sqrt{-2x}$ as $x \to 0^-$, we find for $\theta(\beta)$

$$\theta(\beta) \approx \frac{\pi}{2} \quad \text{for} \quad \beta \to 0^+, \tag{4.1.11a}$$

$$\theta(\beta) \approx \frac{\sqrt{1-\beta^2}}{2} \quad \text{for} \quad \beta \to 1^-. \tag{4.1.11b}$$

The above approximations lead to

$$f_{max}(\beta) \approx 1 \quad \text{for} \quad \beta \to 0, \tag{4.1.12a}$$

$$f_{max}(\beta) \approx \frac{1}{4}\left(\frac{8}{5}\right)^{1/5}\frac{1}{(1-\beta^2)^4} \quad \text{for} \quad \beta \to 1. \tag{4.1.12b}$$

In the last relation, the following Taylor expansions, in the variable β^2 around $\beta^2 = 1$, have been used:

$$4 - \sqrt{1 + 15\beta^2} = \frac{15}{8}(1 - \beta^2) + O((1 - \beta^2)^2), \tag{4.1.13a}$$

$$-1 - 3\beta^2 + \sqrt{1 + 15\beta^2} = \frac{9}{8}(1 - \beta^2) + O((1 - \beta^2)^2). \tag{4.1.13b}$$

The total radiated power follows from (3.2.14), recalling (4.1.3); its expression is

$$P^*(t^*) = \frac{2q^2}{3c} \frac{\dot{\boldsymbol{\beta}}(t^*)^2}{(1 - \beta(t^*)^2)^3}. \tag{4.1.14}$$

Angular plots of the function $f(\theta, \beta)$, which determines the angular reliance in the differential radiated power, are shown in Fig. 4.1.2.

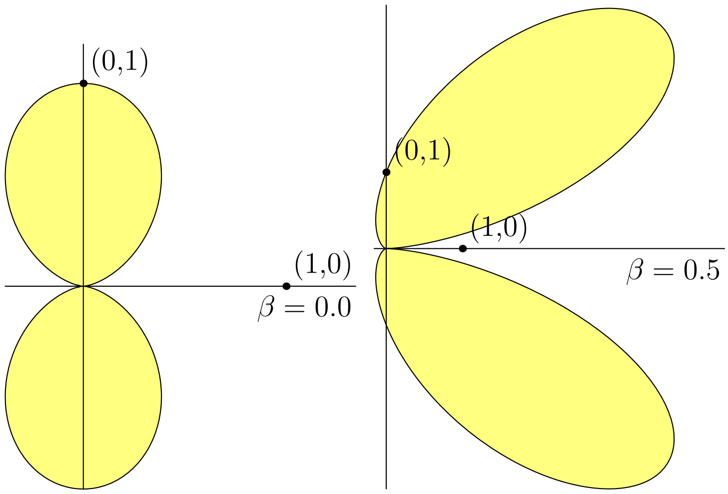

FIGURE 4.1.2. Angular plots of $f(\theta, \beta)$, with $\beta = 0.0, 0.5$

4.2. Circular accelerator

Consider a charge q moving along a circular orbit $\boldsymbol{x}(t)$ of radius r_0 with constant angular velocity $\boldsymbol{\omega}$, as it is shown in Fig. 4.2.1; it is convenient to set the origin of the frame in the center of the circumference. This case will be studied using cylindrical coordinates, with the vertical axis along the direction of the angular velocity, as

$$\boldsymbol{\omega} = \omega \boldsymbol{e}_z. \tag{4.2.1}$$

The trajectory of the charge can be written as

$$\boldsymbol{x}(t) = r_0 \boldsymbol{e}_\rho(\omega t) \tag{4.2.2}$$

and its velocity is given by

$$\boldsymbol{\beta} = \frac{r_0 \dot{\boldsymbol{e}}_\rho}{c}. \tag{4.2.3}$$

The following relations take place between the unit vectors:

$$\dot{\boldsymbol{e}}_\rho = \omega \boldsymbol{e}_\phi, \tag{4.2.4a}$$

$$\dot{\boldsymbol{e}}_\phi = -\omega \boldsymbol{e}_\rho, \tag{4.2.4b}$$

$$\boldsymbol{e}_z \times \boldsymbol{e}_\phi = -\boldsymbol{e}_\rho, \tag{4.2.4c}$$

$$\boldsymbol{e}_\phi \times \boldsymbol{e}_\rho = -\boldsymbol{e}_z. \tag{4.2.4d}$$

Defining $\beta = \omega r_0/c$ and using (4.2.4a), the velocity of the charge can be written as

$$\boldsymbol{\beta} = \beta \boldsymbol{e}_\phi. \tag{4.2.5}$$

Using (4.2.4b) instead, the acceleration of the charge becomes

$$\dot{\boldsymbol{\beta}} = -\omega\beta \boldsymbol{e}_\rho. \tag{4.2.6}$$

The relations (4.2.4c) and (4.2.4d) lead to

$$\dot{\boldsymbol{\beta}} = \omega\beta(\boldsymbol{e}_z \times \boldsymbol{e}_\phi) = \boldsymbol{\omega} \times \boldsymbol{\beta}, \tag{4.2.7a}$$

$$\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = \beta \boldsymbol{e}_\phi \times (-\omega\beta)\boldsymbol{e}_\rho = \omega\beta^2 \boldsymbol{e}_z = \beta^2 \boldsymbol{\omega}. \tag{4.2.7b}$$

In order to obtain the differential radiated power using (3.2.4), using (4.2.7) we know that

$$1 - \boldsymbol{\beta} \cdot \boldsymbol{n} = 1 - \beta \boldsymbol{n} \cdot \boldsymbol{e}_\phi \tag{4.2.8}$$

$$\begin{aligned}
\Big[(\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\Big]^2 - \Big[\boldsymbol{n} \cdot \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}\Big]^2 &= \big[\boldsymbol{n} \times (\boldsymbol{\omega} \times \boldsymbol{\beta}) - \beta^2 \boldsymbol{\omega}\big]^2 - \big[\beta^2 \boldsymbol{n} \cdot \boldsymbol{\omega}\big]^2 \\
&= \big[(\boldsymbol{n} \cdot \boldsymbol{\beta} - \beta^2)\boldsymbol{\omega} - \boldsymbol{n} \cdot \boldsymbol{\omega} \boldsymbol{\beta}\big]^2 - \beta^4 (\boldsymbol{n} \cdot \boldsymbol{\omega})^2 \\
&= (\boldsymbol{n} \cdot \boldsymbol{\beta} - \beta^2)^2 \omega^2 + (\boldsymbol{n} \cdot \boldsymbol{\omega})^2 \beta^2 \\
&\qquad - 2(\boldsymbol{n} \cdot \boldsymbol{\beta} - \beta^2)\boldsymbol{n} \cdot \boldsymbol{\omega} \boldsymbol{\omega} \cdot \boldsymbol{\beta} - \beta^4 (\boldsymbol{n} \cdot \boldsymbol{\omega})^2.
\end{aligned} \tag{4.2.9}$$

Since $\boldsymbol{\omega} \cdot \boldsymbol{\beta} = \omega\beta \boldsymbol{e}_z \cdot \boldsymbol{e}_\phi = 0$ and using the definitions (4.2.1) and (4.2.3) of $\boldsymbol{\omega}$ and $\boldsymbol{\beta}$, the above relation becomes

$$\begin{aligned}
\Big[(\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\Big]^2 - \Big[\boldsymbol{n} \cdot \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}\Big]^2 &= \\
&= \big[(\boldsymbol{n} \cdot \boldsymbol{e}_\phi - \beta)^2 + (1 - \beta^2)(\boldsymbol{n} \cdot \boldsymbol{e}_z)^2\big] \beta^2 \omega^2 \\
&\qquad + \beta^2 \omega^2 (1 - \beta \boldsymbol{n} \cdot \boldsymbol{e}_\phi)^2 - \beta^2 \omega^2 (1 - \beta \boldsymbol{n} \cdot \boldsymbol{e}_\phi)^2 \\
&= \beta^2 \omega^2 \Big[(1 - \beta \boldsymbol{n} \cdot \boldsymbol{e}_\phi)^2 - (1 - \beta^2)\Big(1 - (\boldsymbol{n} \cdot \boldsymbol{e}_\phi)^2 - (\boldsymbol{n} \cdot \boldsymbol{e}_z)^2\Big)\Big].
\end{aligned} \tag{4.2.10}$$

Using the fact that $(\boldsymbol{n} \cdot \boldsymbol{e}_\rho)^2 + (\boldsymbol{n} \cdot \boldsymbol{e}_\phi)^2 + (\boldsymbol{n} \cdot \boldsymbol{e}_z)^2 = |\boldsymbol{n}|^2 = 1$, expression (4.2.10) becomes

$$\Big[(\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\Big]^2 - \Big[\boldsymbol{n} \cdot \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}\Big]^2 = \beta^2 \omega^2 \big[(1 - \beta \boldsymbol{n} \cdot \boldsymbol{e}_\phi)^2 - (1 - \beta^2)(\boldsymbol{n} \cdot \boldsymbol{e}_\rho)^2\big]. \tag{4.2.11}$$

Inserting (4.2.8) and (4.2.11) in (3.2.4), we find for the differental radiated power

$$\frac{d}{do} P^*(\boldsymbol{x}, t^*) = \frac{(q\omega\beta)^2}{4\pi c} \frac{(1 - \beta \boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \boldsymbol{e}_\phi(\omega t^*))^2 - (1 - \beta^2)(\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \boldsymbol{e}_\rho(\omega t^*))^2}{(1 - \beta \boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \boldsymbol{e}_\phi(\omega t^*))^5}. \tag{4.2.12}$$

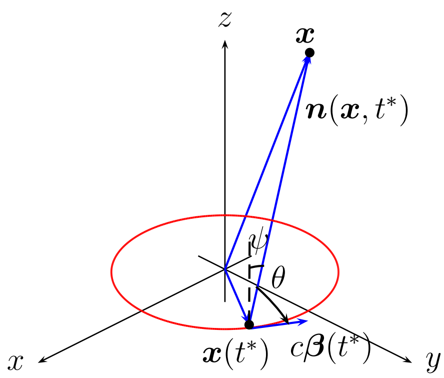

FIGURE 4.2.1. A circular acclerator

The previous expression depends on $\boldsymbol{x}$ and t^* by two angles: the one between $\boldsymbol{n}(\boldsymbol{x}, t^*)$ and $\boldsymbol{e}_\phi(\omega t^*)$, which we denote with $\theta(\boldsymbol{x}, t^*)$, and the one between $\boldsymbol{n}(\boldsymbol{x}, t^*)$ and $\boldsymbol{e}_z$, which we denote with $\psi(\boldsymbol{x}, t^*)$. Using these angles, the differential radiated power becomes

$$\frac{d}{do}P^*(\theta, \psi, t^*) = \frac{(q\omega\beta)^2}{4\pi c}\frac{(1-\beta\cos\theta)^2 - (1-\beta^2)[\sin^2\psi - \cos^2\theta]}{(1-\beta\cos\theta)^5} \tag{4.2.13}$$

where $0 \le \beta < 1$ and $0 \le \theta, \psi < \pi$. The angular dependance in (4.2.13) is worked out by the function

$$f(\theta, \psi; \beta) = \frac{(1-\beta\cos\theta)^2 - (1-\beta^2)[\sin^2\psi - \cos^2\theta]}{(1-\beta\cos\theta)^5}. \tag{4.2.14}$$

For a fixed value of ψ, this function shows two maxima for $\theta = 0, \pi$ and a minimum for $\theta = \theta(\psi; \beta)$, where

$$\theta(\psi; \beta) = \pm \arccos\left\{\frac{1-\beta^2}{\beta}\left[\frac{1}{1-\beta^2} - \frac{4}{3} + \frac{4}{3}\sqrt{1 - \frac{15(1-\beta^2\sin^2\psi)}{16(1-\beta^2)}}\right]\right\}. \tag{4.2.15}$$

The other root of $\partial f/\partial\theta$ has to be rejected because it diverges in the limit $\beta \to 0$. Setting in these relations $\psi = \pi/2$, we can find the angular trend of the differential radiated power in the direction perpendicular to the circular trajectory. This leads to

$$\theta(\pi/2; \beta) = \pm \arccos\beta. \tag{4.2.16}$$

The function $f(\theta, \pi/2; \beta)$, evaluated at the stationary points, takes the values

$$f(0, \pi/2; \beta) = \frac{1}{(1-\beta)^3}, \tag{4.2.17a}$$

$$f(\theta(\pi/2; \beta), \pi/2; \beta) = 0, \tag{4.2.17b}$$

$$f(\pi, \pi/2; \beta) = \frac{1}{(1+\beta)^3}. \tag{4.2.17c}$$

This means that the radiation reaches the highest value in the direction of the charge's motion, while it is null in the direction of $\theta(\pi/2; \beta)$. In order to evaluate the total radiated power for this kind of motion, it is necessary to simplify the expression in the curly brackets of (3.2.14). Using relations (4.2.7) we find

$$\begin{aligned}|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2 &= |\boldsymbol{\omega}\times\boldsymbol{\beta}|^2 - |\beta^2\boldsymbol{\omega}|^2 \\ &= |\boldsymbol{\omega}|^2|\boldsymbol{\beta}|^2 - (\boldsymbol{\omega}\cdot\boldsymbol{\beta})^2 - |\beta^2\boldsymbol{\omega}|^2 = \omega^2\beta^2(1-\beta^2)\end{aligned} \tag{4.2.18}$$

where in the last step it has been used the fact that $\boldsymbol{\omega}\cdot\boldsymbol{\beta} = 0$. In this way, the total radiated power becomes

$$P^*(t^*) = \frac{2(q\omega\beta)^2}{3c}\frac{1}{(1-\beta^2)^2}. \tag{4.2.19}$$

Angular plots of the function $f(\theta,\psi;\beta)$, which determines the angular dependence in the differential radiated power, are shown in Fig. 4.2.2.

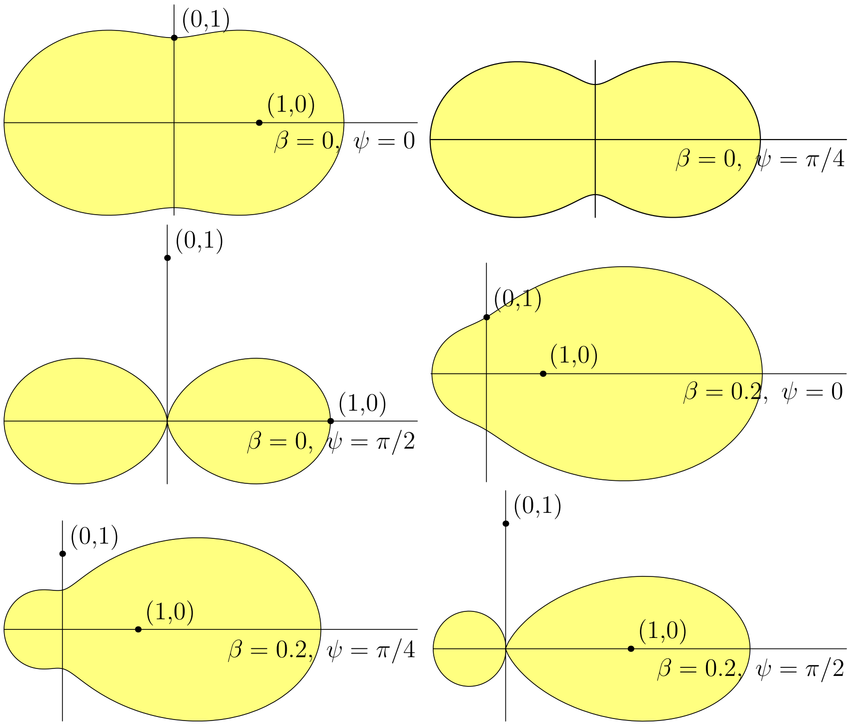

FIGURE 4.2.2. Angular plots of $f(\theta,\psi;\beta)$, for different values of β and ψ

5. Distribution in frequencies and angles of the radiated energy

5.1. Radiation from a moving electric charge

In the following calculations, quantities will be expressed in the time t of the observer, contrarily to the previous chapters, because we want to consider the frequency spectrum in terms of the frequencies of the observer. The general form of the power radiated per unit solid angle comes from (3.2.3): using the relation (2.2.41) among the observer's and the particle's times, we find

$$\frac{d}{do}P(\boldsymbol{x},t) = \frac{d}{do}P^*(\boldsymbol{x},t^*)\frac{\partial}{\partial t}t^*(\boldsymbol{x},t) = \frac{c}{4\pi}|\boldsymbol{E}(\boldsymbol{x},t)|^2\bigg|_{t=t(\boldsymbol{x},t^*)}|\boldsymbol{x}-\boldsymbol{x}(t^*)|^2. \tag{5.1.1}$$

We shall consider an acceleration occurring only for a finite interval of time, so that the total radiated power is finite. Furthermore, we suppose the radiation regime, which means that the charge accelerates in a region of linear size that is small compared to its distance from the observation point, so it subtends a small solid angle element. The total energy radiated per unit solid angle is

$$\frac{d}{do}W(\boldsymbol{n}) = \int_{-\infty}^{+\infty} dt\left|\frac{d}{do}P(\boldsymbol{x},t)\right| = \int_{-\infty}^{+\infty} dt|\boldsymbol{A}(t)|^2. \tag{5.1.2}$$

where

$$\boldsymbol{A}(t) = \sqrt{\frac{c}{4\pi}}\,|\boldsymbol{E}(\boldsymbol{x},t)|\,\bigg|_{t=t(\boldsymbol{x},t^*)}|\boldsymbol{x}-\boldsymbol{x}(t^*)| \tag{5.1.3}$$

Using Fourier transforms, this can be expressed in the frequency spectrum. We introduce the Fourier transform of $\boldsymbol{A}(t)$

$$\boldsymbol{A}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\boldsymbol{A}(t)e^{-i\omega t} \tag{5.1.4a}$$

and its inverse

$$\boldsymbol{A}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\boldsymbol{A}(\omega)e^{i\omega t}. \tag{5.1.4b}$$

Then (5.1.2) can be written as

$$\begin{aligned}\frac{dW}{do} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} d\omega\int_{-\infty}^{+\infty} d\omega'\boldsymbol{A}^*(\omega')\cdot\boldsymbol{A}(\omega)e^{i(\omega-\omega')t} \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\int_{-\infty}^{+\infty} d\omega'\boldsymbol{A}^*(\omega')\cdot\boldsymbol{A}(\omega)\int_{-\infty}^{+\infty} dt\, e^{i(\omega-\omega')t} \\ &= \int_{-\infty}^{+\infty} d\omega\int_{-\infty}^{+\infty} d\omega'\boldsymbol{A}^*(\omega')\cdot\boldsymbol{A}(\omega)\delta(\omega-\omega') = \int_{-\infty}^{+\infty} d\omega|\boldsymbol{A}(\omega)|^2.\end{aligned} \tag{5.1.5}$$

Since the sign of the frequency has no physical meaning, we can integrate over the positive ones only. $\boldsymbol{A}(\omega)$ is real, so we have $\boldsymbol{A}(-\omega) = \boldsymbol{A}^*(\omega)$ and

$$\frac{d}{do}W(\boldsymbol{n}) = \int_0^{+\infty} d\omega\, |\boldsymbol{A}(\omega)|^2 + \int_{-\infty}^{0} d\omega\, |\boldsymbol{A}(\omega)|^2 = 2\int_0^{+\infty} d\omega\, |\boldsymbol{A}(\omega)|^2. \tag{5.1.6}$$

The quantity dW/do is related to the energy radiated per unit solid angle and per unit frequency interval $d^2I(\omega, \boldsymbol{n})/d\omega\, do$, as

$$\frac{d}{do}W(\boldsymbol{n}) = \int_0^{+\infty} d\omega\, \frac{d^2I(\omega,\boldsymbol{n})}{d\omega\, do}. \tag{5.1.7}$$

Comparing (5.1.6) and (5.1.7), we see that the latter defines a quantity which relates the behaviour of the radiated energy to its frequency spectrum. Indeed

$$\frac{d^2I(\omega,\boldsymbol{n})}{d\omega\, do} = 2|\boldsymbol{A}(\omega)|^2 \tag{5.1.8}$$

where $\omega > 0$. Using the expression (2.3.11a) of the electric field of an accelerated charge in radiation regime, it is possible to find a general expression for the energy radiated per unit solid angle and per unit frequency interval in terms of an integral over the trajectory of the particle. Firstly, we have

$$\boldsymbol{A}(t) = \frac{1}{\sqrt{4\pi c}} \frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3} \boldsymbol{n}(\boldsymbol{x},t^*)\times[(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)]\Big|_{t^*=t^*(\boldsymbol{x},t)}. \tag{5.1.9}$$

The Fourier transform of the above expression, which is defined by (5.1.4a), is

$$\boldsymbol{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{+\infty} dt^*\, e^{-i\omega\left(t^* + \frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}{c}\right)} \frac{\boldsymbol{n}(\boldsymbol{x},t^*)\times[(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)]}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^2}\Bigg|_{t^*=t^*(\boldsymbol{x},t)} \tag{5.1.10}$$

where the variable of integration has been changed from t to t^*, using (2.2.41). The radiation regime implies the observation point to be far from the region where the charge is accelerated, so $\boldsymbol{n}(\boldsymbol{x},t^*) \approx \boldsymbol{n}$ and it will be considered constant in time. Furthermore, the distance between the trajectory and the observation point can be approximated as

$$|\boldsymbol{x} - \boldsymbol{x}(t^*)| \approx x - \boldsymbol{n}\cdot\boldsymbol{x}(t^*). \tag{5.1.11}$$

Using the last expression, we find that, apart from an overall phase factor, (5.1.10) becomes

$$\boldsymbol{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{+\infty} dt^*\, e^{-i\omega\left(t^* - \frac{\boldsymbol{n}\cdot\boldsymbol{x}(t^*)}{c}\right)} \frac{\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)]}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n})^2}. \tag{5.1.12}$$

The energy radiated per unit solid angle and per unit frequency interval, accordingly to (5.1.8), yields to

$$\frac{d^2I(\omega,\boldsymbol{n})}{d\omega\, do} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty} dt^*\, e^{-i\omega\left(t^* - \frac{\boldsymbol{n}\cdot\boldsymbol{x}(t^*)}{c}\right)}\, \frac{\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)]}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n})^2}\right|^2. \tag{5.1.13}$$

For a specified motion $\boldsymbol{x}(t)$, by computing $\boldsymbol{\beta}(t^*)$ and $\dot{\boldsymbol{\beta}}(t^*)$, the integral can be evaluated as a function of ω and $\boldsymbol{n}$. If more than one charge is involved in the accelerated motion, the single amplitude in the above expression would be replaced by a coherent sum of amplitudes $\boldsymbol{A}_j(\omega)$, one for each particle.
In expression (5.1.13), the time interval of integration is confined to times during which the acceleration is different from $\mathbf{0}$, but a simpler relation can be found through integration by parts in (5.1.12). It is easy to see that the integrand of this expression, excluding the exponential, is a perfect differential, as

$$\frac{\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^2}=\frac{d}{dt}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right]. \tag{5.1.14}$$

Considering this, an integration by parts in (5.1.12) leads to the intensity distribution

$$\frac{d^2I(\omega,\boldsymbol{n})}{d\omega\, do}=\frac{q^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{+\infty}dt^*\,\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})\,e^{-i\omega\left(t^*-\frac{\boldsymbol{n}\cdot\boldsymbol{x}(t^*)}{c}\right)}\right|^2. \tag{5.1.15}$$

Integration by parts yields an expression with contributions also from the time intervals during which, while acceleration in vanished, velocity is not. This may lead to potential's divergencies at large values of $|t|$, but these can be controlled introducing a time cutoff $e^{-\epsilon|t|}$ and taking the limit $\epsilon\to 0$ at the end.
For a number of charges q_j in accelerated motion, (5.1.15) still stands upon the replacement

$$q\boldsymbol{\beta}(t^*)e^{-i\frac{\omega}{c}\boldsymbol{n}\cdot\boldsymbol{x}(t^*)}\to\sum_{j=1}^{N}q_j\boldsymbol{\beta}_j(t^*)e^{-i\frac{\omega}{c}\boldsymbol{n}\cdot\boldsymbol{x}_j(t^*)}. \tag{5.1.16}$$

In the limit of a continuous distribution of charge in motion, the sum becomes an integral over the total current density $\boldsymbol{J}(\boldsymbol{x},t)$, so the substitution takes the form

$$q\boldsymbol{\beta}(t^*)e^{-i\frac{\omega}{c}\boldsymbol{n}\cdot\boldsymbol{x}(t^*)}\to\frac{1}{c}\int d^3x\,\boldsymbol{J}(\boldsymbol{x},t^*)e^{-i\frac{\omega}{c}\boldsymbol{n}\cdot\boldsymbol{x}}. \tag{5.1.17}$$

In this way, the intensity distribution becomes

$$\frac{d^2I(\omega,\boldsymbol{n})}{d\omega\, do}=\frac{\omega^2}{4\pi^2c^3}\left|\int dt^*\int d^3x\,\boldsymbol{n}\times\left[\boldsymbol{n}\times\boldsymbol{J}(\boldsymbol{x},t^*)\right]e^{-i\omega\left(t^*-\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c}\right)}\right|^2. \tag{5.1.18}$$

5.2. Radiation from a moving magnetic moment

A magnetization density $\boldsymbol{M}(\boldsymbol{x},t)$ is equivalent to a current density

$$\boldsymbol{J}_M=c\boldsymbol{\nabla}\times\boldsymbol{M}. \tag{5.2.1}$$

This is not, however, the only contribution to the current density, because a moving magnetization has an associated electric polarization. The effective source current for a moving magnetic movement comes from the Ampère-Maxwell equation and is

$$\boldsymbol{J}_{eff}=c\boldsymbol{\nabla}\times\boldsymbol{M}+\frac{\partial\boldsymbol{P}}{\partial t} \tag{5.2.2}$$

where $\boldsymbol{P}$ is the associated electric polarization density. Substituting into (5.1.18) and integrating by parts, we find

$$\begin{aligned}\frac{d^2I(\omega,\boldsymbol{n})}{d\omega\, do} &= \frac{\omega^2}{4\pi^2c^3}\left|\int dt^* \int d^3x\, \boldsymbol{n}\times\Big[\boldsymbol{n}\times\Big(c\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{x},t^*)\right. \qquad (5.2.3)\\ &\left.+\frac{\partial}{\partial t^*}\boldsymbol{P}(\boldsymbol{x},t^*)\Big)\Big]e^{-i\omega\left(t^*-\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c}\right)}\right|^2\\ &= \frac{\omega^2}{4\pi^2c^3}\left|\int dt^*\boldsymbol{n}\times\int d^3x\Big[c\Big(\boldsymbol{\nabla}(\boldsymbol{n}\cdot\boldsymbol{M}(\boldsymbol{x},t^*))-(\boldsymbol{n}\cdot\boldsymbol{\nabla})\boldsymbol{M}(\boldsymbol{x},t^*)\Big)\right.\\ &\left.+\boldsymbol{n}\times\frac{\partial}{\partial t}\boldsymbol{P}(\boldsymbol{x},t^*)\Big]e^{i\omega\left(t^*-\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c}\right)}\right|^2\\ &= \frac{\omega^2}{4\pi^2c^3}\left|\int dt^*\boldsymbol{n}\times\int d^3x\,\Big\{c\Big[\boldsymbol{M}(\boldsymbol{x},t^*)(\boldsymbol{n}\cdot\boldsymbol{\nabla})-(\boldsymbol{n}\cdot\boldsymbol{M}(\boldsymbol{x},t^*))\boldsymbol{\nabla}\Big]\right.\\ &\left.-(\boldsymbol{n}\times\boldsymbol{P}(\boldsymbol{x},t^*))\boldsymbol{\nabla}\Big\}e^{i\omega\left(t^*-\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c}\right)}\right|^2\\ &= \frac{\omega^2}{4\pi^2c^3}\left|-i\omega\int dt^*\int d^3x\,\boldsymbol{n}\times[\boldsymbol{n}\times(\boldsymbol{M}(\boldsymbol{x},t^*)\times\boldsymbol{n})\right.\\ &\left.+\boldsymbol{n}\times\boldsymbol{P}(\boldsymbol{x},t^*)]e^{i\omega\left(t^*-\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c}\right)}\right|^2\\ &= \frac{\omega^4}{4\pi^2c^3}\left|\int dt^*\int d^3x\,\boldsymbol{n}\times\left[\boldsymbol{M}(\boldsymbol{x},t^*)+\boldsymbol{n}\times\boldsymbol{P}(\boldsymbol{x},t^*)\right]e^{-i\omega\left(t^*-\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c}\right)}\right|^2.\end{aligned}$$

For a point-like magnetic moment $\boldsymbol{\mu}(t)$ placed in $\boldsymbol{x}(t)$, the magnetization is

$$\boldsymbol{M}(\boldsymbol{x},t)=\boldsymbol{\mu}(t)\,\delta(\boldsymbol{x}-\boldsymbol{x}(t)) \qquad (5.2.4)$$

If the particle shows only a magnetic moment in its rest frame K, it will appear in another frame K', where its velocity is $\boldsymbol{\beta}$ in units of c, as possessing a magnetic moment and an electric dipole moment

$$\boldsymbol{p}=\boldsymbol{\beta}\times\boldsymbol{\mu} \qquad (5.2.5)$$

where $\boldsymbol{\mu}$ is the magnetic moment measured in K'. Therefore, the electric polarization density is given by

$$\boldsymbol{P}(\boldsymbol{x},t)=\boldsymbol{\beta}(t)\times\boldsymbol{\mu}(t)\delta(\boldsymbol{x}-\boldsymbol{x}(t)). \qquad (5.2.6)$$

Substituting (5.2.4) and (5.2.6) in (5.2.3), we find the energy radiated per unit solid angle per unit frequency integral for a moving magnetic moment, which is

$$\frac{d^2I(\omega,\boldsymbol{n})}{d\omega\, do}=\frac{\omega^4}{4\pi^2c^3}\left|\int dt^*\,\boldsymbol{n}\times\left[\boldsymbol{\mu}(t^*)+\boldsymbol{n}\times(\boldsymbol{\beta}(t^*)\times\boldsymbol{\mu}(t^*))\right]e^{-i\omega\left(t^*-\frac{\boldsymbol{n}\cdot\boldsymbol{x}(t^*)}{c}\right)}\right|^2. \qquad (5.2.7)$$

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