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$\mathcal{N} = 2$ GAUGE THEORIES FROM THE ODE/IM CORRESPONDENCE PERSPECTIVE

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Abstract

Nel presente elaborato di tesi, il sottoscritto D. Gregori ha sviluppato un metodo ricorsivo molto efficiente, per calcolare gli integrali del moto locali delle teorie conformi quantistiche. Tale calcolo è stato svolto nel contesto della corrispondenza ODE/IM, tra certe equazioni di Schrödinger e i modelli integrabili conformi. A tal fine, è stato sfruttato il semplice comportamento ricorsivo dei cosiddetti polinomi di Gelfand-Dikii, grazie anche a una precedente dimostrazione di D. Fioravanti, concernente l'espansione asintotica, per grande energia, della funzione d'onda. Inoltre, il sottoscritto D. Gregori, ha adattato tale dimostrazione alla cosiddetta espansione WKB (nota anche come approssimazione WKB), cioè per piccola constante di Planck, ottenendo così una ricorsione efficiente anche per questo noto metodo di soluzione delle equazioni di Schrödinger con singolarità. Questo nuovo approccio all'approssimazione WKB ha permesso di dimostrare una congettura di W. He a Y. Miao, nel contesto delle teorie di gauge con $\mathcal{N} = 2$ supersimmetrie (SUSY). Inoltre, nel presente elaborato, è stato parzialmente completato un lavoro, molto incompleto e pubblicato postumo, di Al. B. Zamolodchikov, riguardo alla costruzione della corrispondenza ODE/IM per il modello di Liouville, ovvero per carica centrale $c \geq 25$. L'equazione della ODE/IM da utilizzare, in questo caso, è l'equazione di Mathieu Generalizzata, così battezzata da Al. B. Zamolodchikov. Per il caso $c = 25$ "autoduale", tale equazione si riduce all'equazione di Mathieu e questo permette una diretta connessione con le teorie di gauge $\mathcal{N} = 2$ SUSY. In questo caso autoduale, il sottoscritto D. Gregori ha dimostrato una congettura di Al. B. Zamolodchikov, secondo cui la funzione T di Baxter delle teorie integrabili coincide con il coseno dell'indice di Floquet dell'equazione di Mathieu; dove quest'ultimo è proporzionale alla deformazione quantistica (nel limite di Nekrasov-Shatashvili) del ciclo classico di Seiberg-Witten, delle teorie di gauge con $\mathcal{N} = 2$ SUSY.

Contents

Introduction

An integrable system is a physical system with an *infinite number of integrals of motion.* In integrable *quantum* field theory, since there are infinite degrees of freedom, the Liouville theorem does not hold and an exact solution is not always possible. However, in *conformal quantum* field theory, as a consequence of Virasoro symmetry, it is possible to calculate almost all quantities of physical interest and in particular, the correlation functions. In this sense, conformal quantum field theory can be exactly solved.[7][8] The integrals of motion can be expressed as the coefficients, for large rapidity θ, of the asymptotic expansions of the Baxter's $Q(\theta)$ and $T(\theta)$ functions. These functions satisfy many *functional relations*, which give important characterizations of the T and Q functions for a certain integrable model. For the Liouville model (which is very useful also in string theory), we write, perhaps, the most important of such functional relations, the *TQ relation*, as

$$T(\theta)Q(\theta) = Q(\theta + i\pi p) + Q(\theta - i\pi p) \qquad \Big(p = \frac{b}{b+1/b}\Big) \tag{0.0.1}$$

where b is the characteristic Liouville parameter. The Baxter's $Q(\theta)$ function can be also expressed as the solution of a *nonlinear integral equation* (NLIE),[3] which, in certain limit,[4] reduces to a linear integral equation. Before the discovery of the ODE/IM correspondence, such NLIE were the most efficient technique to calculate the integrals of motion.[6][5]

However, today, an alternative way to characterize the Baxter's Q and T functions is the so-called ODE/IM correspondence, which means "ordinary differential equations / integrable models" correspondence,[14].[13] In fact there exist some Schrödinger equations, whose eigenfunction is, roughly speaking, the Baxter's Q function of integrability. More precisely, the wronskians calculated with the solutions of such Schrödinger equations, satisfy all functional relations and the nonlinear integral equation of conformal integrability, at least after some "a posteriori" identification between the parameters of the equation and the conformal parameters.[14] The ODE/IM correspondence has been proved useful also in a wider context, for example to rigorously prove the reality of the energies in $\mathcal{PT}$-symmetric quantum mechanics.[15] In other words, thanks to the ODE/IM correspondence, we know that there exist *non-hermitian Hamiltonians* $\hat{H} \neq \hat{H}^\dagger$ *which anyway have real eigenvalues* $\lambda = \lambda^*$, hence such systems are physical. Recently, C. Bender conjectured that $\mathcal{PT}$ symmetric quantum field theory might have a very important role for the physics *beyond the standard model.*

In this thesis, I developed a very effective technique for a systematic calculation of the local integrals of motion I_{2n-1} of integrability, both for the minimal models and for the Liouville model. For my results, it has been proved equally important both the ODE/IM correspondence and a *yet unpublished result of D. Fioravanti*, regarding the equivalence of Gelfand-Dikii polynomials and "large energy" (for large rapidity θ) expansion modes for the Schrödinger eigenfunction[18][19] . Moreover, *I adapted Fioravanti's rigorous equivalence proof* to the standard, well-known, *WKB expansion* (for small Planck constant $\hbar$). Thus, after various modifications, I found that also the standard WKB expansion contributions, at all orders, can be simply connected with Gelfand-Dikii polynomials. The power of Gelfand-Dikii polynomials lies in the fact that they have very simple *markovian* recursive properties. In fact, for the standard WKB expansion, the calculation of $n+1$-th mode S_{n+1} requires the knowledge of *all the* n *preceding modes* S_k*, for* $k = -1, 0, 1, 2..., n$, as is evident from the recursive equation:

$$S_{n+1} = -\frac{1}{2\sqrt{q}}\Big(\sum_{m=0}^{n-1} S_m S_{n-1-m} + S_n'\Big) \qquad S_{-1}(x) = \sqrt{q(x)} = \sqrt{V(x) - E} \tag{0.0.2}$$

(where E is the eigenvalue and $V(x)$ is the potential of the Schrödinger equation). Instead, the *equivalent* Gelfand Dikii recursive equation is *markovian*, that is, to calculate the $n+1$-th mode T_{n+1}, it is sufficient the knowledge of *only the precedent n-th T_n mode*, as is evident from the equation I found:

$$\boxed{-T'_{n+1} = -\frac{1}{4q}T'''_n + \frac{3}{8}\frac{q'}{q^2}T''_n + \Big(\frac{3}{8}\frac{q''}{q^2} - \frac{9}{16}\frac{q'^2}{q^3}\Big)T'_n + \Big(\frac{1}{8}\frac{q'''}{q^2} - \frac{9}{16}\frac{q''q'}{q^3} + \frac{15}{32}\frac{q'^3}{q^4}\Big)T_n\ .} \tag{0.0.3}$$

Apart from the general interest of such general markovian WKB expansion for the Schrödinger equation, I applied this result to obtain a *rigorous proof to a conjecture of He and Miao*[34], in the context of $\mathcal{N}=2$ pure gauge theory. In that theory, the partition function receives both a perturbative and a non-perturbative istantonic contribution. In $\mathbb{R}^4$ Minkowsky space, (because of the infinite volume), the instantons give infinite contribution. Hence, to calculate the non-perturbative part of the partition function, Nekrasov devised a technique which involves *spacetime deformation*, through two curvature (complex) parameters ϵ_1 and ϵ_2. The Nekrasov partition function can be written in the classical limit $\epsilon_1, \epsilon_2 \to 0$, in terms of the Seiberg Witten prepotential $\mathcal{F}_{SW}$

$$Z_{Nek}(\epsilon_1,\epsilon_2) = \exp\Big\{-\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_{SW} + O(\epsilon_1,\epsilon_2)\Big\} \tag{0.0.4}$$

or in terms of Nekrasov-Shatashvili prepotential, in the homonimous limit $\epsilon_2 \to 0$

$$Z_{Nek}(\epsilon_1,\epsilon_2) = \exp\Big\{-\frac{1}{\epsilon_2}\mathcal{F}_{NS}(\epsilon_1) + O(\epsilon_2)\Big\} \tag{0.0.5}$$

The Nekrasov-Shatashvili quantum prepotential $\mathcal{F}_{NS}(\epsilon_1)$ can be obtained[33] from the knowledge of the *Floquet index ν*, relative to the *Mathieu Schrödinger equation*

$$\boxed{\frac{\epsilon_1^2}{2}\frac{d^2}{dz^2}\psi(z) + [u - \cos 2z]\psi(z) = 0} \tag{0.0.6}$$

with eigenvalue u and ϵ_1 as Planck constant. W. He and Y. Miao conjectured[34] and I rigorously proved that there exist, for the WKB expansion modes of the Floquet index ν expansion modes, relatively simple differential operators in the eigenvalue u, which deliver the n-th mode of expansion starting from the zero order. In other words, He and Miao operators allow to calculate the whole ϵ_1 contribution, $\mathcal{F}_{NS}(\epsilon_1)$, to the Nekrasov partition function, in term of *only the classical Seiberg Witten potential $\mathcal{F}_{SW}$*. The energy eigenvalue u can then be expressed through the instanton part of the prepotential in terms of the Matone's relation in the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$.

$$u = -\frac{\partial \mathcal{F}_{inst}}{\partial \ln q} \tag{0.0.7}$$

This thesis is largely based on an unfinished work of the late scholar Al. B. Zamolodhikov [1]. V. Bazhanov,[1] in september 2011, held a seminar at Bologna, about such unfinished work. In this thesis, I partially completed Zamolodchikov's draft, which is very sketchy and full of confusing typos. Zamolodchikov's intention was to set up the ODE/IM correspondence for the Liouville model (for central charge $c \geq 25$). The ODE/IM equation which he used for the ODE/IM construction is the *Generalized Mathieu equation*, which reduces to the Mathieu equation (0.0.6) in the "self dual" Liouville case (for the $b=1$ value of the Liouville parameter, or for which $p=1/2$). In such self dual case, considering *numerical computations*,[2] Al. Zamolodchikov conjectured what I dubbed

[1] We thank P. Dorey for informing us of the existence of such publication.[2]

"fundamental Zamolodchikov's relation", between the Floquet index ν of the Mathieu equation (0.0.6) and the Baxter's T function

$$\boxed{2\cos 2\pi\nu(\theta) = T(\theta) = \frac{Q(\theta + i\pi/2)}{Q(\theta)} + \frac{Q(\theta - i\pi/2)}{Q(\theta)} \qquad (b=1)} \tag{0.0.8}$$

I devised: (i) an *exact proof* of Zamolodchikov's relation, using the Floquet theorem for Mathieu equation;[24] (ii) an *asymptotic proof*, considering the large energy expansion (large rapidity θ) at all infinite orders, using of the aforementioned efficient method I devised for calculating the local integrals of motion I_{2n-1} for $b = 1$. I and D. Fioravanti interpreted Zamolodchikov's fundamental relation (0.0.8) in terms of deformed (even in the quantum Nekrasov-Shatashvili limit $\epsilon_2 \to 0$) Seiberg Witten cycles, that is ϵ_1-expansion modes of the Floquet index ν of (0.0.6), thereby finding a very suggestive link between $\mathcal{N} = 2$ SUSY gauge theory and the fundamental TQ relation (0.0.1) of integrable conformal field theory. Moreover, a deeper interpretation is still under investigation.

Daniele Gregori

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Part I

Introductive integrability notions

1 Integrable structure of conformal field theory

1.1 Virasoro algebra and local integrals of motion

Conformal field theories are characterized by conformal symmetry. The generator of this symmetry is the energy momentum tensor, whose mode expansion (Fourier series) is

$$T(u) = -\frac{c}{24} + \sum_{n=-\infty}^{\infty} L_{-n} e^{inu} \tag{1.1.1}$$

where c stands for the central charge. The variable u is interpreted as the complex coordinate of a two dimensional cylinder of circumference 2π. This definition is consistent with the choice of periodic boundary conditions $T(u+2\pi) = T(u)$.

The modes of expansion of the energy momentum tensor satisfy the Virasoro algebra Vir

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{0,n+m} \tag{1.1.2}$$

Let $UVir$ be the algebra generated by all powers and derivatives of the energy momentum tensor $T(u)$. It can be shown that this algebra contains[7] an infinite dimensional abelian subalgebra spanned by the *local integrals of motion* $I_{2n-1} \in UVir$, with $n = 1, 2, ...$

$$\boxed{I_{2n-1} = \int_0^{2\pi} \frac{du}{2\pi} T_{2n}(u)} \tag{1.1.3}$$

The first few densities T_{2n} are

$$T_2(u) = T(u)\,, \qquad T_4(u) =: T^2(u):\,, \qquad T_6(u) =: T^3(u): + \frac{c+12}{12} : (T'(u))^2 : \tag{1.1.4}$$

but their general expression is not known. Nevertheless, they can be uniquely determined[7] by the requirements of *commutativity*

$$\boxed{[I_{2n-1}, I_{2m-1}] = 0} \tag{1.1.5}$$

and *spin assignment*

$$\oint_{\mathcal{C}} \frac{dw}{2\pi i}(w-u)\mathcal{T}(T(w)T_{2n}(u)) = 2nT_{2n}(u) \tag{1.1.6}$$

where $\mathcal{T}$ denotes chronological ordering.

The *physical space* of states $\mathcal{H}_{\text{phys}}$ is embedded[7] into the tensor product of the left and right chiral subspaces $\mathcal{H}_{\text{chiral}} \otimes \bar{\mathcal{H}}_{\text{phys}}$. Each *chiral space* is built up from a collection of highest weight Virasoro modules

$$\mathcal{H}_{\text{chiral}} = \oplus_a \mathcal{V}_{\Delta_a} \tag{1.1.7}$$

The highest weight state $|\Delta_a\rangle \in \mathcal{V}_{\Delta_a}$ satisfies

$$L_0|\Delta_a\rangle = \Delta_a|\Delta_a\rangle\,, \qquad L_n|\Delta_a\rangle = 0\,, \quad n > 0 \tag{1.1.8}$$

1.2 Classical limit

The approach used by the foundational work[7] can be regarded as an instance of the Quantum Inverse Scattering Method and can be thought as the quantum version of the Korteweg de-Vries problem. More precisely the Korteweg de-Vries problem is obtained as ithe *classical limit* $c \to -\infty$ (see (1.3.5)), through the substitution

$$T(u) \to -\frac{c}{6}U(u)\,, \qquad [\,,\,] \to \frac{6\pi}{ic}\{\,,\,\} \qquad \text{for } c \to -\infty \tag{1.2.1}$$

where $U(u) = U(u+2\pi)$ is the KdV potential. Correspondingly, the Virasoro algebra (1.1.2) reduces to the Poisson bracket algebra

$$\{U(u), U(v)\} = 2(U(u) + U(v))\delta'(u-v) + \delta'''(u-v) \tag{1.2.2}$$

and the *classical* local integrals of motion $I^{(cl)}_{2n-1}$, calculated by (1.2.1) commute in the Poisson bracket sense

$$\{I^{(cl)}_{2n-1}, I^{(cl)}_{2m-1}\} = 0 \tag{1.2.3}$$

Many quantities and relations of conformal quantum integrability have a classical analogue.[7] However, we stop here our exposition of this "correspondence principle", because we don't need it so much.

1.3 Baxter's $\mathbf{T}_j$ operators

1.3.1 Feigin-Fuchs free field representation of the Virasoro algebra

In this subsection we still follow.[7] According to the *Feigin-Fuchs free field representation of* Vir, the energy momentum tensor can be defined as[2]

$$-\beta^2 T(u) =: \varphi'(u)^2 : +(1-\beta^2)\varphi''(u) + \frac{\beta^2}{24} \tag{1.3.1}$$

where φ is a free field (chiral boson) operator

$$\varphi(u) = iQ + iPu + \sum_{n\neq 0} \frac{a_{-n}}{n} e^{inu} \tag{1.3.2}$$

and the parameter β is related to the central charge as

$$\boxed{\begin{aligned} c &= 13 - 6(\beta^2 + \beta^{-2}) && (1.3.3)\\ \beta &= \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}} && (1.3.4)\end{aligned}}$$

The operators P, Q, a_{-n} are defined to satisfy the *Heisenberg algebra*

$$[Q,P] = \frac{i}{2}\beta^2 \qquad [a_n, a_m] = \frac{n}{2}\beta^2\delta_{n+m,0} \tag{1.3.5}$$

It is evident that the classical limit corresponds to $\beta^2 \to 0$ (which, by (1.3.3), is consistent with $c \to -\infty$). We note that the free chiral boson field φ is quasi periodic: $\varphi(u+2\pi) = \varphi(u) + 2\pi iP$. In the following the vertex operators of the field φ will be used

$$V_\pm(u) \equiv : e^{\pm 2\varphi(u)} :\equiv \exp\Big(\pm 2\sum_{n=1}^{\infty} \frac{a_{-n}}{n} e^{inu}\Big) \exp(\pm 2i(Q+Pu)) \exp\Big(\mp 2\sum_{n=1}^{\infty} \frac{a_n}{n} e^{-inu}\Big) \tag{1.3.6}$$

The Fock space $\mathcal{F}_p$ corresponds to the highest weight module over the Heisenberg algebra (1.3.5) with highest weight (or vacuum) state $|p\rangle$

$$P|p\rangle = p|p\rangle\,; \qquad a_n|p\rangle = 0 \quad \text{for} \quad n > 0 \tag{1.3.7}$$

For any p and β, it can be shown that the Fock space $\mathcal{F}_p$, thus defined, is isomorphic to the highest weight Virasoro module $\mathcal{V}_\Delta$ with highest weight given by

$$\boxed{\Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}} \tag{1.3.8}$$

[2] The Feigin-Fuchs epresentation of $T(u)$ is just a quantum version of the Miura transform for $U(u)$.[7]

1.3.2 "Continuous transfer operators" $\mathbf{T}_j(\lambda)$ and local integrals of motion

Let E, F, H be the canonical generating elements of the quantum universal enveloping algebra $U_q(sl(2))$, which therefore satisfy[8]

$$[H,E] = 2E\,, \qquad [H,F] = -2F\,, \qquad [E,F] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{1.3.9}$$

where the important parameter q is function of β^2

$$\boxed{q = e^{i\pi\beta^2}} \tag{1.3.10}$$

Let π_j be an irreducible $2j+1$ dimensional matrix representation of this quantum algebra. The "quantum monodromy matrices" $\mathbf{L}_j$ are defined as[7]

$$\mathbf{L}_j(\lambda) = \pi_j \Big\{ e^{i\pi PH} \mathcal{P} \exp\Big[\lambda \int_0^{2\pi} du\, (V_-(u) q^{\frac{H}{2}} E + V_+(u) q^{-\frac{H}{2}} F)\Big]\Big\} \tag{1.3.11}$$

where $V_\pm$ are the vertex operators already defined (1.3.6); λ has the "role" of spectral parameter[3]; $\mathcal{P}$ denotes path ordering, that is

$$\mathbf{L}_j = \pi_j \Big\{ e^{i\pi PH} \sum_{k=0}^{\infty} \lambda^k \int_{2\pi \geq u_1 \geq u_2 \geq \ldots \geq u_k \geq 0} K(u_1)K(u_2)\cdots K(u_k) du_1 du_2 \ldots du_k \Big\} \tag{1.3.12}$$

$$\text{with} \quad K(u) = V_-(u) q^{\frac{H}{2}} E + V_+(u) q^{-\frac{H}{2}} F \tag{1.3.13}$$

For our purposes, it is important to note that the integrals (1.3.11) are convergent only for

$$\boxed{-\infty < c < -2\,, \qquad \Longleftrightarrow \qquad 0 < \beta^2 < \frac{1}{2}} \tag{1.3.14}$$

since the operator product expansion of the vertex operators is

$$V_+(u)V_-(u') = (u-u')^{-2\beta^2}[1 + O(u-u')]\,, \qquad u - u' \to 0 \tag{1.3.15}$$

This range can be extended by analytic continuation, since the functions "of interest" (see below) are considered to be entire.[8] Bazhanov, Lukyanov and A. Zamolodchikov, in the foundational works we are following,[7],[8],[9] did not pursue such an extension. However, this is just what Al. Zamolodchikov began to do in his unfinished draft,[2] which we are going to partially complete [4].
The so called *T operators*[5] are defined as[7]

$$\mathbf{T}_j(\lambda) : \mathcal{V}_\Delta \to \mathcal{V}_\Delta\,, \tag{1.3.16}$$

$$\mathbf{T}_j(\lambda) = tr_{\pi_j}\Big(\mathbf{L}_j\Big) \tag{1.3.17}$$

of course $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ for the representation of the quantum algebra $U_q(sl(2))$. The $\mathbf{T}_j(\lambda)$ operators can be written more explicitly as

$$\boxed{\mathbf{T}_j(\lambda) = \mathrm{tr}_{\pi_j}\Big[e^{2\pi i P H_j} \mathcal{P} \exp\Big\{\lambda \int_0^{2\pi} \Big[V_-(u) q^{\frac{H_j}{2}} E_j + V_+(u) q^{-\frac{H_j}{2}} F_j\Big]\Big\}\Big]} \tag{1.3.18}$$

[3] To be precise, λ is the spectral parameter of a Schrödinger equation with KdV potential, but *only in the classical limit.*[7] Moreover, in the context of ODE-IM correspondence,[14] this parameter is proportional to the spectral parameter of the ODE-IM equation: $\lambda \propto E$ (see (2.4.3)). However, we emphasize that *the ODE-IM equation is unphysical.*

[4] We greatly thank P. Dorey, for he informed D. Fioravanti of the existence of a posthomous publication[2] of this unfinished work of Al. Zamolodchikov.

[5] The T operators are continuous version of Baxter's transfer matrices[8]

Most importantly, the operators $\mathbf{T}_j(\lambda)$ *commute between themselves*

$$\boxed{[\mathbf{T}_j(\lambda), \mathbf{T}_j(\lambda')] = 0} \tag{1.3.19}$$

The commutative property is an immediate consequence of the *Quantum Yang-Baxter equation* for the $\mathbf{L}_j$ operators[9]

$$\mathbf{R}_{jj'}(\lambda\mu^{-1})(\mathbf{L}_j(\lambda) \otimes 1)(1 \otimes \mathbf{L}_{j'}(\mu)) = (1 \otimes \mathbf{L}_{j'}(\mu))(\mathbf{L}_j(\lambda) \otimes 1)\mathbf{R}_{j,j'}(\lambda\mu^{-1}) \tag{1.3.20}$$

where $\mathbf{R}_{j,j'}(\lambda\mu^{-1})$ is a trigonometric solution of the Yang-Baxter equation which act on the space $\pi_j \otimes \pi_{j'}$

$\mathbf{T}(\lambda)$ is an entire function of λ^2 with an essential singularity at $\lambda^2 \to -\infty$. *The coefficients of its asymptotic expansion around infinity are the local integral of motion* $\mathbf{I}_{2n-1}$ *(with a normalization constant* C_n*)*

$$\boxed{\log \mathbf{T}(\lambda) \ \simeq m\lambda^{\frac{1}{1-\beta^2}}\mathbf{I} - \sum_{n=1}^{\infty} C_n \lambda^{\frac{1-2n}{1-\beta^2}} \mathbf{I}_{2n-1} \qquad \lambda^2 \to -\infty} \tag{1.3.21}$$

where the constants m and C_n are defined as[8]

$$C_n = \frac{\sqrt{\pi}}{n!(1-\beta^2)}(\beta^2)^n \frac{\Gamma(\frac{2n-1}{2-2\beta^2})}{\Gamma(1+\frac{2n-1}{2\beta^{-2}-2})}\Big(\Gamma(1-\beta^2)\Big)^{-\frac{2n-1}{1-\beta^2}} \tag{1.3.22}$$

$$m = \frac{2\sqrt{\pi}\Gamma(\frac{1}{2}-\frac{\xi}{2})}{\Gamma(1-\frac{\xi}{2})}\Big(\Gamma(1-\beta^2)\Big)^{1+\xi} \tag{1.3.23}$$

1.3.3 Non local integrals of motion and fusion relations

For the first nontrivial[6] T operator, for $j = 1/2$, it is common use to drop the subscript

$$\mathbf{T}_{\frac{1}{2}}(\lambda) \coloneqq \mathbf{T}(\lambda) \tag{1.3.24}$$

T can be calculated evaluating the traces in the definition (1.3.17). Thus one gets a power series expansion around the point $\lambda^2 = 0$

$$\boxed{\mathbf{T}(\lambda) = 2\cos(2\pi P) + \sum_{n=1}^{\infty} \lambda^{2n} \mathbf{G}_n} \tag{1.3.25}$$

where the coefficients $\mathbf{G}_n$ the *non local integrals of motion*. The $\mathbf{G}_n$ operators commute among themselves and with the local integrals of motion $\mathbf{I}_{2n-1}$

$$[\mathbf{G}_n, \mathbf{G}_m] = 0 \tag{1.3.26}$$

$$[\mathbf{G}_n, \mathbf{I}_{2m-1}] = 0 \tag{1.3.27}$$

The higher spin operators $\mathbf{T}_j(\lambda)$ also admit a power series expansion, around $\lambda^2 = 0$, with coefficients *algebraically dependent* from the non local integrals of motion $\mathbf{G}_n$. Such interdependence between the $\mathbf{T}_j(\lambda)$ operators can be expressed through the *fusion relations*

$$\boxed{\mathbf{T}_j(q^{\frac{1}{2}}\lambda)\mathbf{T}_j(q^{-\frac{1}{2}}\lambda) = 1 + \mathbf{T}_{j-\frac{1}{2}}(\lambda)\mathbf{T}_{j+\frac{1}{2}}(\lambda)} \tag{1.3.28}$$

These relations are identical to the functional relations obeyed by the commuting transfer-matrices of the integrable XXZ model. For this reason, *the* T *operators* $\mathbf{T}_j$ *are considered the continuous field theory versions of the lattice transfer matrices.*

[6] In fact, $\mathbf{T_0}(\lambda) = \mathbf{I}$.

1.4 Baxter's Q operators

1.4.1 Construction of the A operators

In this section we shall follow.[8] The $\mathbf{A}$ operators, which will be shown to be strictly related to the $\mathbf{Q}$ operators, are constructed similarly to the $\mathbf{T}_j$ operators. The main difference is the substitution of the algebra $U_q(sl(2))$ with the *q-oscillator algebra*, which is defined by the commutation relations of its generators $\mathcal{E}_+$, $\mathcal{E}_-$, $\mathcal{H}$

$$q\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q-q^{-1}}\,, \qquad [\mathcal{H},\mathcal{E}_\pm] = \pm 2\mathcal{E}_\pm \tag{1.4.1}$$

One might choose any representation ρ of $U_q(sl(2))$ such that the following trace exists for $\Im p > 0$.

$$Z(p) = tr_\rho[e^{2\pi i p\mathcal{H}}] \tag{1.4.2}$$

The $\mathbf{A}$ operators are then defined in analogy with the $\mathbf{T}_j$ operators.

$$\boxed{\mathbf{A}_\pm(\lambda) \simeq \frac{1}{Z(\pm P)} tr_\rho\Big\{e^{\pm 2i\pi P\mathcal{H}}\mathcal{P}\exp\Big[\lambda\int_0^{2\pi} du\,(V_-(u)q^{\pm\frac{\mathcal{H}}{2}}\mathcal{E}_\pm + V_+(u)q^{\mp\frac{\mathcal{H}}{2}}\mathcal{E}_\mp)\Big]\Big\}} \tag{1.4.3}$$

Still similarly with what happened for the $\mathbf{T}_j$ operators (1.3.25), if we were to evaluates the traces in the definition of the $\mathbf{A}_\pm$ operators, we would find power series expansion around $\lambda^2 = 0$[8]

$$\mathbf{A}_\pm(\lambda) = 1 + \lambda^2 g_1\mathbf{G}_1 + \lambda^4\Big[g_2\mathbf{G}_2 + g_{11}\mathbf{G}_1^2\Big] + \lambda^6\Big[g_3\mathbf{G}_3 + g_{12}\mathbf{G}_1\mathbf{G}_2 + g_{111}\mathbf{G}_1^3\Big] + ... \tag{1.4.4}$$

with coefficients which are *polynomials of the non local integrals of motion* $\mathbf{G}_n$ (the "subcoefficients" $g_{ij..}$ are functions of β^2 and P). Define, for convenience, the alternative spectral parameter y as

$$y = \beta^{-2}\Gamma(1-\beta^2)\lambda \tag{1.4.5}$$

It is also convenient to define an alternative set of nonlocal integrals of motion $\mathbf{H}_n$, by the expansion of the logarithm of $\mathbf{A}_+$

$$\log \mathbf{A}_+(\lambda) = -\sum_{n=1}^{\infty} y^{2n}\mathbf{H}_n \tag{1.4.6}$$

Such alternative nonlocal integrals of motion $\mathbf{H}_n$, clearly, must be algebraically dependent with the previously defined nonlocal integrals of motion $\mathbf{G}_n$.[8]

We define also the *dual integrals of motion*, by letting $\beta^2 \to \beta^{-2}$ and $\varphi \to \beta^{-2}\varphi$. The local integrals of motion are not affected by this transformation,

$$\beta^2 \to \frac{1}{\beta^2} \quad \varphi \to \beta^{-2}\varphi \tag{1.4.7}$$

$$\mathbf{I}_{2n-1} \to \mathbf{I}_{2n-1} \tag{1.4.8}$$

however the nonlocal integrals of motion are, so that

$$\beta^2 \to \frac{1}{\beta^2} \quad \varphi \to \beta^{-2}\varphi \tag{1.4.9}$$

$$\mathbf{H}_n \to \tilde{\mathbf{H}}_n \tag{1.4.10}$$

1.4.2 Q operators and their properties

The $\mathbf{Q}$ operators are defined simply as the $\mathbf{A}$ operators, multiplied by a certain power of the spectral parameter

$$\mathbf{Q}_{\pm}(\lambda) : \mathcal{F}_p \to \mathcal{F}_p \tag{1.4.11}$$

$$\mathbf{Q}_{\pm}(\lambda) = \lambda^{\pm 2P/\beta^2}\mathbf{A}_{\pm}(\lambda) \tag{1.4.12}$$

The $\mathbf{Q}$ operators enjoy the following properties (for their proof we refer to[9])

1. The $\mathbf{Q}_{\pm}$ operators commute among themselves and with the operators $\mathbf{T}_j$

$$\boxed{[\mathbf{Q}_{\pm}(\lambda), \mathbf{Q}_{\pm}(\lambda')] = [\mathbf{Q}_{\pm}(\lambda), \mathbf{T}_j(\lambda')] = 0} \tag{1.4.13}$$

2. The $\mathbf{Q}_{\pm}$ operators satisfy the *Baxter functional relation*, or *TQ relation*

$$\boxed{\mathbf{T}(\lambda)\mathbf{Q}_{\pm}(\lambda) = \mathbf{Q}_{\pm}(q\lambda) + \mathbf{Q}_{\pm}(q^{-1}\lambda)} \tag{1.4.14}$$

   Hence, any eigenvalue $Q(\lambda)$ of $\mathbf{Q}(\lambda)$ (or, equivalently any eigenvalue $A(\lambda)$ of $\mathbf{A}_{+}(\lambda)$) satisfies the TQ functional relation

$$T(\lambda)Q(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda) \tag{1.4.15}$$

$$T(\lambda)A(\lambda) = e^{2\pi i p}A(q\lambda) + e^{-2\pi i p}A(q^{-1}\lambda) \tag{1.4.16}$$

3. The $\mathbf{T}_j$ operators can be expressed in terms of the $\mathbf{Q}_{\pm}$ operators[7]

$$\boxed{2i\sin(2\pi P)\mathbf{T}_j(\lambda) = \mathbf{Q}_{+}(q^{j+\frac{1}{2}}\lambda)\mathbf{Q}_{-}(q^{-j-\frac{1}{2}}\lambda) - \mathbf{Q}_{+}(q^{-j-\frac{1}{2}}\lambda)\mathbf{Q}_{-}(q^{j+\frac{1}{2}}\lambda)} \tag{1.4.17}$$

   For the particular case of $j = 0$, this equation is called *quantum wronskian*

$$2i\sin(2\pi P) = \mathbf{Q}_{+}(q^{\frac{1}{2}}\lambda)\mathbf{Q}_{-}(q^{-\frac{1}{2}}\lambda) - \mathbf{Q}_{+}(q^{-\frac{1}{2}}\lambda)\mathbf{Q}_{-}(q^{\frac{1}{2}}\lambda) \tag{1.4.18}$$

   while, for $j \geq 1/2$ the name "fused quantum wronskians" is sometimes used.

Actually, in their work,[8] Bazhanov, Lukyanov, Zamolodchikov made a further assumption

$$\boxed{\log \mathbf{A}_{\pm}(\lambda) \sim \bar{M}(-\lambda^2)^{\frac{1}{2-2\beta^2}}\,, \qquad \lambda^2 \to -\infty} \tag{1.4.19}$$

where $\bar{M}$ is the constant

$$\bar{M} = \frac{\Gamma(\frac{1}{2(1-\beta^2)} - \frac{1}{2})\Gamma(1 - \frac{1}{2(1-\beta^2)})}{\sqrt{\pi}}\Big(\Gamma(1-\beta^2)\Big)^{\frac{1}{1-\beta^2}} \tag{1.4.20}$$

They proved this leading asymptotic behaviour only in the case $p = N/2$, when N is an integer.[8]

1.5 Non-linear integral equation and generating functions for the integral of motion

The eigenvalues $A(\lambda)$ and $T(\lambda)$ of the $\mathbf{A}$ and $\mathbf{T}_j$ operators, for $0 < \beta^2 < \frac{1}{2}$ and $\Im p = 0$, enjoy the following properties

[7] This property can be interpreted as an indication that the $\mathbf{Q}$ operators are *more fundamental* than the $\mathbf{T}_j$ operators.[8]

1. The functions $A(\lambda)$ and $T(\lambda)$ are *entire* functions of λ^2 (considered as complex variable)

2. The *zeros* of the function $A(\lambda)$ are either real or occur in complex conjugated pairs. There are only a finite number of complex or real negative zeroes and real zeroes accumulate toward $+\infty$. For the *vacuum* eigenvalues functions $A^{\text{vac}}(\lambda)$ the zeroes are all real and if $2p > -\beta^2$ they are all positive.

3. The *leading asymptotic* behaviour of $A(\lambda)$ for large λ^2 is

$$\log A_{\pm}(\lambda) \sim \bar{M}(-\lambda^2)^{\frac{1}{2-2\beta^2}} \,, \qquad \lambda^2 \to -\infty \tag{1.5.1}$$

If $0 < \beta^2 < \frac{1}{2}$, an entire function with asymptotic behaviour such as (1.5.1), is completely determined by its zeroes λ_k^2. In fact, it can be expressed as a convergent infinite product over its zeros.[25]

$$A(\lambda) = \prod_{k=0}^{\infty}\left(1 - \frac{\lambda^2}{\lambda_k^2}\right) \tag{1.5.2}$$

where the normalization condition

$$A(0) = 1 \tag{1.5.3}$$

is understood.

Now, define the function $a(\lambda)$ as

$$\boxed{a(\lambda) = e^{4\pi i p}\frac{A(q\lambda)}{A(q^{-1}\lambda)}} \tag{1.5.4}$$

setting $\lambda^2 = \lambda_k^2$, the Baxter's TQ relation (1.4.16) induces the following Bethe-ansatz type equations for the positions of the zeroes

$$\boxed{a(\lambda_k) = -1} \tag{1.5.5}$$

which are an infinite set of transcendental equations of the zeroes λ_k^2. However (1.5.5) can be transformed in a non-linear integral equation[8]

$$\begin{cases} i\log a(\theta) & = -\frac{2\pi p}{\beta^2} + 2m_0\cos\frac{\pi\xi}{2}\; e^{\theta} + i\sum_a' \log S(\theta-\theta_a) - 2G * \Im \log\left(1 + a(\theta - i0)\right) \\ a(\theta_a) & = -1 \end{cases} \tag{1.5.6}$$

where the new parameters θ and ξ are defined by

$$\boxed{\beta^2 = \frac{\xi}{1+\xi}\,, \qquad \lambda = e^{\frac{\theta}{1+\xi}}\,;} \tag{1.5.7}$$

while the subscript a indicates the zeros λ_a of $A(\lambda)$ lying outside the positive real axis of λ^2; the $*$ denotes the convolution and the function of the rapidity $S(\theta)$ and $G(\theta)$ are defined by

$$S(\theta) = \exp\left\{-i\int_0^{\infty}\frac{d\nu}{\nu}\sin(\nu\theta)\frac{\sinh\left(\pi\nu\frac{(1+\xi)}{2}\right)}{\cosh\frac{\pi\nu}{2}\sinh\frac{\pi\nu\xi}{2}}\right\} \tag{1.5.8}$$

$$G(\theta) = \delta(\theta) + \frac{1}{2\pi i}\frac{d}{d\theta}\log S(\theta) \tag{1.5.9}$$

In the following, we will somehow loosely use indifferently the parameters β^2, ξ and the - yet to be introduced, cf. (2.4.1) - parameter $M = 1/\xi$. Thus it is important to keep in mind their reciprocal relations, such as

$$1-\beta^2 = \frac{1}{1+\xi} \tag{1.5.10}$$

$$\xi = \frac{\beta^2}{1-\beta^2} \tag{1.5.11}$$

Given a solution $a(\lambda)$ of the Destri-de Vega equation, the eigenvalue $A(\lambda)$ can be calculated as

$$\log A(\lambda) = -i \int_{C_\nu} d\nu \, \frac{g(\nu)}{\cosh \frac{\pi\nu}{2} \sinh \frac{\pi\nu\xi}{2}} (-\lambda^2)^{i\nu \frac{1+\xi}{2}} \tag{1.5.12}$$

wth the integration contour along the vertical line $\Im\nu = -1-\epsilon$ and $g(\nu)$ defined as

$$g(\nu) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \Im\log\left(1+a(\theta-i0)\right) e^{-i\nu\theta} \tag{1.5.13}$$

In the limit $p \to \infty$, the Destri de-Vega equation simplifies and becomes a linear equation[8]

$$-\frac{\pi p}{\beta^2} + m_0 \cos\frac{\pi\xi}{2} e^{\theta} - \int_{-\infty}^{B(p)} \frac{d\theta'}{2\pi i} \, \partial_\theta \log S(\theta-\theta') \Im\log\Big(1+a^{\mathrm{vac}}(\theta')\Big) = 0 \tag{1.5.14}$$

at least provided one assumes

$$B(p) = \frac{1+\xi}{2} \log \lambda_0^2 \sim \mathrm{const}\log p\,, \qquad p \to \infty \tag{1.5.15}$$

$$\lambda_0^2 = \min_k \lambda_k^2 \tag{1.5.16}$$

In this limit $p \to \infty$, the eigenvalue $A^{\mathrm{vac}}(\lambda)$ is given by

$$\log A^{(\mathrm{vac})}(\lambda)\Big|_{p\to\infty} \sim -\frac{p}{2\pi^{3/2}\xi} \int_{C_\nu} \frac{d\nu}{\nu^2} \Gamma\Big(1-i\nu\frac{1+\xi}{2}\Big)\Gamma\Big(1+i\nu\frac{\xi}{2}\Big)\Gamma\Big(-\frac{1}{2}+i\frac{\nu}{2}\Big) e^{i\delta\nu} \Big(-\frac{\lambda^2}{\lambda_0^2}\Big)^{i\nu\frac{1+\xi}{2}} \tag{1.5.17}$$

$$= -\sum_{n=1}^{\infty} \lambda^{2n} \beta^{-4n} \Big(\Gamma(1-\beta^2)\Big)^{2n} H_n^{\mathrm{vac}}\Big|_{p\to\infty} \tag{1.5.18}$$

The second equality can be obtained by the method of residues (poles of the Gamma functions), closing the integration contour at infinity in the lower half plane $\Im\nu < -1$. From this calculation one can also obtain the constant $\bar{M}$ which characterizes the asymptotic behaviour at $\lambda^2 \to -\infty$

$$\bar{M} = \frac{\Gamma(\frac{\xi}{2})\Gamma(\frac{1}{2}-\frac{\xi}{2})}{\sqrt{\pi}} \Big(\Gamma(1-\beta^2)\Big)^{1+\xi} \tag{1.5.19}$$

The large λ behaviour of $A^{\mathrm{vac}}(\lambda)$ is similarly calculated by the method of residues by closing the integral (1.5.17) in the upper half plane $\Im\nu \geq -1$. However, in this case the contribution over the large circle diverges; consequently, we get only an *asymptotic* expansion for $\lambda^2 \to -\infty$. Note that

$$\log A^{(\mathrm{vac})}(\lambda) = -\sum_{n=1}^{\infty} y^{2n} H_n^{\mathrm{vac}} \tag{1.5.20}$$

While the asymptotic expansion of $A^{(\mathrm{vac})}$ for $\lambda^2 \to \infty$ has the form

$$\boxed{\begin{aligned} A^{+(\mathrm{vac})} \simeq{}& (\beta^2)^{-\frac{2p}{\beta^2}} \Big(\frac{2p}{\beta^2}\Big)^{2p(\frac{1}{\beta^2}-1)} (-y^2)^{-\frac{p}{\beta^2}} \exp\Big\{\sum_{n=0}^{\infty} B_n (-y^2)^{\frac{1-2n}{2-2\beta^2}} I_{2n-1}^{\mathrm{vac}}\Big\} \times \\ &\times \exp\Big\{-\sum_{n=1}^{\infty} (-1)^n (-y^2)^{-\frac{n}{\beta^2}} \tilde{H}_n^{\mathrm{vac}}\Big\} \end{aligned}} \tag{1.5.21}$$

The normalization constants B_n are given by

$$B_n = \frac{(-1)^{n+1}}{2\sqrt{\pi}(1-\beta^2)n!}\Gamma\Big(\frac{2n-1}{2-2\beta^2}\Big)\Gamma\Big(\frac{2n-1}{2-2\beta^{-2}}\Big)(\beta^2)^{n+\frac{2n-1}{\beta^2-1}} \tag{1.5.22}$$

Here the *dual non local (alternative) integrals of motion* $\tilde{H}_n$ have appeared. The asymptotic expansion (1.5.21) was derived in the limit $p \to \infty$, but[8] conjectured and then proved[9] that it is always valid.

1.6 Asymptotic expansion of the TQ relation

It is important to observe that, in our calculation, we are not going to find any $\tilde{H}_n$ coefficient. This is a consequence of the fact that, following[14]and,[16] we will stay in the range $\beta^2 < 1/2$ (or $M > 1$, cf. (2.4.1)).

$$\boxed{\beta^2 < \frac{1}{2}} \tag{1.6.1}$$

Therefore, for each integer n, for $y \to \infty$, the power $\frac{1-2n}{2-2\beta^2}$ of y is dominant over the power $-\frac{n}{\beta^2}$; thus, *at each asymptotic order, the nonlocal contribution will be suppressed.* Hence, we can list the asymptotic expansions of $A^{+\text{vac}}$ and $Q^{+\text{vac}}$ around infinity as follows

$$A^{+\text{vac}}(\lambda) = \mathcal{N}(\beta,p)(\lambda)^{-2p\beta^{-2}} \exp\left\{\sum_{n=0}^{\infty} B_n (-1)^{\frac{1-2n}{2-2\beta^2}} (\beta^2)^{-\frac{1-2n}{1-\beta^2}} \Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}} \lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\text{vac})}\right\} \tag{1.6.2}$$

$$Q^{+\text{vac}}(\lambda) = \mathcal{N}(\beta,p) \exp\left\{\sum_{n=0}^{\infty} B_n (-1)^{\frac{1-2n}{2-2\beta^2}} (\beta^2)^{-\frac{1-2n}{1-\beta^2}} \Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}} \lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\text{vac})}\right\} \tag{1.6.3}$$

where we defined, for the "normalization function" $\mathcal{N}(\beta,p)$ (which is constant only with respect to λ)

$$\mathcal{N}(\beta,p) = \left(\frac{2p}{e}\right)^{2p(1-\beta^{-2})} (-\Gamma(1-\beta^2))^{-2p\beta^{-2}} \tag{1.6.4}$$

If we define, for simplicity, the alternative normalization constants $\tilde{B}_n$ for the local integrals of the motion

$$\tilde{B}_n = B_n (-1)^{\frac{1-2n}{2-2\beta^2}} (\beta^2)^{-\frac{1-2n}{1-\beta^2}} \Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}} \tag{1.6.5}$$

we can write the expansion for Q more simply as

$$\boxed{\begin{aligned} Q^{+\text{vac}}(\lambda) &= \left(\frac{2p}{e}\right)^{2p(1-\beta^{-2})} (-\Gamma(1-\beta^2))^{-2p\beta^{-2}} \exp\left\{\sum_{n=0}^{\infty} \tilde{B}_n \lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\text{vac})}\right\} && (1.6.6) \\ &= \left(\frac{2p}{e}\right)^{2p(1-\beta^{-2})} (-\Gamma(1-\beta^2))^{-2p\beta^{-2}} \exp\left\{\sum_{n=0}^{\infty} \tilde{B}_n e^{\theta(1-2n)} I_{2n-1}^{(\text{vac})}\right\} && (1.6.7) \end{aligned}}$$

The usual[8] normalization constants B_n are now given, with also some manipulation to make simpler comparison with the normalization constants C_n of the T eigenvalue expansion[8]

$$\begin{aligned} B_n &= \frac{(-1)^{n+1}}{2\sqrt{\pi}(1-\beta^2)n!} \Gamma\left(\frac{2n-1}{2-2\beta^2}\right) \Gamma\left(\frac{2n-1}{2-2\beta^{-2}}\right) (\beta^2)^{\frac{n\beta^2+n-1}{\beta^2-1}} \\ &= -\frac{(-1)^n}{2\sqrt{\pi}(1-\beta^2)n!} \Gamma\left(\frac{2n-1}{2-2\beta^2}\right) \Gamma\left(\frac{2n-1}{2-2\beta^{-2}}\right) (\beta^2)^{n+\frac{2n-1}{\beta^2-1}} \end{aligned} \tag{1.6.9}$$

The alternative normalization constants $\tilde{B}_n$ then are

$$\boxed{\tilde{B}_n = \frac{(-1)^{n+1}}{2\sqrt{\pi}(1-\beta^2)n!} \Gamma\left(\frac{2n-1}{2-2\beta^2}\right) \Gamma\left(\frac{2n-1}{2-2\beta^{-2}}\right) (\beta^2)^n (-1)^{\frac{1-2n}{2-2\beta^2}} \Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}}} \tag{1.6.10}$$

Our aim is to write the asymptotic expansions of the Q and T eigenvalues in terms of the same parameters in order to compare them. Define the operator $\Lambda(\lambda)$ through the TQ relation

$$\mathbb{T}(\lambda) = \Lambda(q^{\frac{1}{2}}\lambda) + \Lambda^{-1}(q^{-\frac{1}{2}}\lambda) \tag{1.6.11}$$

[8] We think that formula (3.29) of[8] for B_n has a typo, so we multiply it by β^2

$$(\beta^2)^{n+\frac{2n-\beta^2}{\beta^2-1}} \to (\beta^2)^{n+\frac{2n-1}{\beta^2-1}} \tag{1.6.8}$$

which means

$$\Lambda(q^{\frac{1}{2}}\lambda) = \frac{Q(q\lambda)}{Q(\lambda)} \tag{1.6.12}$$

In a certain Stokes sector of λ only one of the two terms of the sum will dominate

$$\boxed{T(\lambda) \sim \Lambda(q^{\frac{1}{2}}\lambda) = \frac{Q(q\lambda)}{Q(\lambda)}} \tag{1.6.13}$$

this simplification is what enables us to collate the asymptotic series of T and A.

$$\begin{aligned}
\log\Lambda(\lambda) &\simeq im(-\lambda)^{\frac{1}{1-\beta^2}} - i\sum_{n=1}^{\infty}(-1)^n C_n (-1)^{\frac{1-2n}{2-2\beta^2}}\lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\mathrm{vac})} && (1.6.14)\\
&\simeq im(-\lambda)^{\frac{1}{1-\beta^2}} + \sum_{n=1}^{\infty}\tilde{C}_n(\lambda)^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\mathrm{vac})} && (1.6.15)
\end{aligned}$$

probably there is another error in (4.14) of:[8] $-\lambda \to \lambda$

The usual[8] normalization constants C_n are now given and suitably rewritten.

$$\begin{aligned}
C_n &= \frac{\sqrt{\pi}}{n!(1-\beta^2)}(\beta^2)^n \frac{\Gamma(\frac{2n-1}{2-2\beta^2})}{\Gamma(1+\frac{2n-1}{2\beta^{-2}-2})}\Big(\Gamma(1-\beta^2)\Big)^{-\frac{2n-1}{1-\beta^2}} && (1.6.16)\\
&= -\frac{\sin\Big(\pi(n-\frac{1}{2})\frac{1}{M}\Big)}{\sqrt{\pi}n!(1-\beta^2)}(\beta^2)^n\Gamma\Big(\frac{2n-1}{2-2\beta^2}\Big)\Gamma\Big(\frac{2n-1}{2-2\beta^{-2}}\Big)\Big(\Gamma(1-\beta^2)\Big)^{-\frac{2n-1}{1-\beta^2}}
\end{aligned}$$

we specify also the leading coefficient m here, for completeness

$$m = \frac{2\sqrt{\pi}\Gamma(\frac{1}{2}-\frac{\xi}{2})}{\Gamma(1-\frac{\xi}{2})}\Big(\Gamma(1-\beta^2)\Big)^{1+\xi} \tag{1.6.17}$$

For these calculations, it is convenient to define a new constant $\mathcal{D}_n$

$$\mathcal{D}_n = \frac{(-1)^n}{\sqrt{\pi}\ (1-\beta^2)n!}(-1)^{\frac{1-2n}{1-\beta^2}}\beta^{2n}\Gamma(\frac{2n-1}{2-2\beta^2})\Gamma(\frac{2n-1}{2-2\beta^{-2}})\Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}} \tag{1.6.18}$$

The n-th term of the $Q^+(\lambda)$ expansion writes then

$$\begin{aligned}
&\frac{(-1)^{n+1}}{2\sqrt{\pi}(1-\beta^2)n!}\Gamma\Big(\frac{2n-1}{2-2\beta^2}\Big)\Gamma\Big(\frac{2n-1}{2-2\beta^{-2}}\Big)(\beta^2)^n\Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}}(-1)^{\frac{1-2n}{2-2\beta^2}}\lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\mathrm{vac})}\\
&= \sum_{n=0}^{\infty}\frac{(-1)}{2}\mathcal{D}_n\lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\mathrm{vac})}
\end{aligned}$$

while the n-th term of the $\Lambda(\lambda)$ expansion is

$$\begin{aligned}
&\frac{i(-1)^n\sin\Big(\pi(n-\frac{1}{2})\frac{1}{M}\Big)}{\sqrt{\pi}(1-\beta^2)n!}\Gamma\Big(\frac{2n-1}{2-2\beta^2}\Big)\Gamma\Big(\frac{2n-1}{2-2\beta^{-2}}\Big)\Big(\Gamma(1-\beta^2)\Big)^{-\frac{2n-1}{1-\beta^2}}(\beta^2)^n(-1)^{\frac{1-2n}{2-2\beta^2}}\lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\mathrm{vac})}\\
&= \sum_{n=0}^{\infty} i\sin\Big(\pi(n-\frac{1}{2})\frac{1}{M}\Big)\mathcal{D}_n\lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}^{(\mathrm{vac})}
\end{aligned}$$

The parameter $q = e^{i\pi\beta^2}$ has a key role, so we write the powers of it we need

$$\begin{aligned}
q^{\frac{1-2n}{1-\beta^2}} &= e^{i\pi(1-2n)\frac{\beta^2}{1-\beta^2}} = e^{-i\pi(2n-1)\xi}\\
\sqrt{q}^{\frac{1-2n}{1-\beta^2}} &= e^{-i\pi(n-1/2)\xi}
\end{aligned}$$

We make the consistency check that the TQ relation (1.6.13) be identically satisfied

$$\frac{(-1)}{2}\Big(e^{-i\pi(2n-1)\xi}-1\Big)\mathcal{D}_n = ie^{-i\pi(n-1/2)\xi}\sin\Big(\pi(n-\frac{1}{2})\xi\Big)\mathcal{D}_n$$
$$1-e^{-i\pi(2n-1)\xi} = 1-e^{-i\pi(2n-1)\xi}$$

We can finally find the relation between the normalization constants

$$\boxed{\begin{aligned} C_n &= (-1)^n 2\sin\Big(\pi(n-\frac{1}{2})\xi\Big)\Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}}(\beta^{-2})^{\frac{1-2n}{1-\beta^2}}B_n \qquad (1.6.19)\\ \tilde{C}_n &= 2\sin\Big(\pi(n-\frac{1}{2})\xi\Big)(-1)^{n-\frac{1-2n}{2-2\beta^2}}\tilde{B}_n \qquad (1.6.20)\end{aligned}}$$

We are particular interested to the $\beta^2=-1$ case

$$\begin{aligned}\tilde{C}_n &= 2\sin\Big(\pi(\frac{n}{2}-\frac{1}{4})\Big)(-1)^{n-\frac{1-2n}{4}}\tilde{B}_n\\ &= -i\Big(e^{\frac{i\pi n}{2}-\frac{i\pi}{4}}-e^{-\frac{i\pi n}{2}+\frac{i\pi}{4}}\Big)e^{i\pi n}e^{\frac{i\pi}{4}(2n-1)}\tilde{B}_n\\ &= i(-1)^{n+1}[-i(-1)^n-1]\tilde{B}_n\end{aligned}$$

In conclusion, for $\beta^2=-1$

$$\boxed{\tilde{C}_n = [-1+i(-1)^n]\tilde{B}_n \qquad (\beta^2=-1)} \qquad (1.6.21)$$

a fact which will be repeately used later.

2 "Ordinary differential equations - integrable models" correspondence

2.1 The canonical Schödinger equation of ODE-IM

Consider the spectral problem associated with the Scrödinger operator on the half line $x \in (0, \infty)$[14]

$$\boxed{\left(-\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2}\right)\psi_k(x) = E_k \psi_k(x)} \tag{2.1.1}$$

where $2M$ is a real number larger than -2. This equation was the first ODE-IM equation to be discovered: the present final form (2.1.1) is due to the foundational works of Dorey and Tateo,[12][14] and of Bazhonov, Lukyanov, Zamolodhikov.[13] We note that the "spectrum" is discrete, because the potential is confining.[12]

The point 0 is a regular singularity point,[27] the roots of the characteristic polynomial being $l+1$ and $-l$. The fundamental pair of solutions can be expanded around zero, in an ordinary *power series*, as

$$\psi^+ = x^{l+1} \sum_{n=0}^{\infty} u_n^+ x^n \simeq x^{l+1} \qquad x \to 0 \tag{2.1.2}$$

$$\psi^- = x^{-l} \sum_{n=0}^{\infty} u_n^- x^n \simeq x^{-l} \qquad x \to 0 \tag{2.1.3}$$

with $u_0^\pm = 1$. We notice that (2.1.1) is invariant under the action of the discrete symmetry $\hat{\Lambda}$ defined as[13]

$$\boxed{\hat{\Lambda}: \quad x \to x\,, \quad E \to E \quad l \to -1-l} \tag{2.1.4}$$

the application of this symmetry to ψ^+ transform it in the other solution ψ^- and viceversa.

The point at infinity is a regular singularity of rank $2[M]+4$, where $[M]$ denotes the integer part of M. Hence, we cannot expand in an ordinary series around ∞, but just in an *asymptotic series*.[23]

Following the treatment of,[14] we now restrict to the region $M > 1$, but extend x to the whole complex plane. Then equation (2.1.1) has a *entire* solution $y(x, E, l)$ with asymptotic representation

$$y^- \sim x^{-M/2} \exp\left\{-\frac{1}{M+1} x^{M+1}\right\} \qquad \Re x \to +\infty \tag{2.1.5}$$

where x tends to infinity in any closed subsector of the sector

$$|\arg x| < \frac{\pi}{2M+2} \tag{2.1.6}$$

A *Stokes line*? is a ray of the complex plane where the asymptotic behavior of the formal solutions changes character, from an exponentially decreasing behaviour to an exponential increasing behaviour[9]. Each fundamental pair of a second order equation such as (2.1.1), includes both an exponentially decreasing and an exponentially increasing solution, called also *subdominant* and *dominant* solutions. The *Stokes sectors* are now simply defined as the regions of the complex plane comprised between the Stokes lines.[23] Note that the Stokes sectors are determined just by the *leading order* of the solution.

[9]For example, if the asymptotic representation of a function is of the form $f(x)e^{\pm g(x)}$, the Stokes line can be taken as the set of zeros x_0 of the real part of $g(x)$: $\Re g(x_0) = 0$.

We can say that equation (2.1.6) defines a Stokes sector in which the solution y^- is subdominant. In general, equation (2.1.1) has the Stokes sectors $\mathcal{S}_k$

$$\left|\arg x - \frac{2k\pi}{2M+2}\right| < \frac{\pi}{2M+2}\,, \qquad k \in \mathbf{Z} \tag{2.1.7}$$

where k is an integer. Note that y^- is subdominant in $\mathcal{S}_0$, while it is dominant in $\mathcal{S}_{\pm 1}$.
We can now introduce another symmetry $\hat{\Omega}$ of equation (2.1.1). Its action can be expressed as[14]

$$\boxed{\hat{\Omega}: \quad x \to \omega^{-1} x\,, \quad E \to \omega^2 E \quad l \to l} \tag{2.1.8}$$

with the parameter ω suitably defined as the $2M+2$-th root of unity.

$$\boxed{\omega = e^{\frac{i\pi}{M+1}} \qquad \omega^{2M+2} = 1} \tag{2.1.9}$$

Note the similarity with the definition (1.3.10) of the q parameter in integrability.
Of course[27] these symmetries, even if they leave the equation (2.1.1) invariant, have a nontrivial action on the solution. In particular, the $\hat{\Omega}$ symmetry exchange a dominant and a subdominant solution (passively interpreted, in a fixed Stokes sector). The symmetries can thus be used to generate other solutions. In fact, define[14]

$$y_k = \omega^{k/2} y^-(\omega^{-k} x, \omega^{2k} E, l) \tag{2.1.10}$$

y_k is subdominant in $\mathcal{S}_k$,while it is dominant in $\mathcal{S}_{k\pm 1}$.
Clearly, there exists a solution y^+ which, in $\mathcal{S}_0$, is asymptotically represented as

$$y^+ \sim x^{-M/2} \exp\left\{\frac{1}{M+1} x^{M+1}\right\} \qquad \Re x \to +\infty \tag{2.1.11}$$

and therefore forms a fundamental pair with y^-. In a the Stokes sector $\mathcal{S}_0$, we can certainly *linearly* expand any solution $\phi(x)$ in the basis of the two fundamental *asymptotic* solutions y^- and y^+. In the other Stokes sectors $\mathcal{S}_k$, "$\phi(x)$ will be *represented asymptotically* by a *linear combination* of the two *formal* solutions; *but the coefficients may vary from sector to sector*"[10]. This is called *Stokes phenomenon*, by the name of his discoverer.
In the following, we shall use mainly the solution y^- and therefore we drop the $-$ apex and write for this solution just y.

2.2 Construction of C and D functions

We consider now the Stokes sector $\mathcal{S}_0$. Being aware of the Stokes phenomenon we expand the solution y_{k-1} in the basis $\{y_k, y_{k+1}\}$

$$y_{k-1}(x,E,l) = C_k(E,l) y_k(x,E,l) + \tilde{C}_k y_{k+1}(x,E,l) \qquad y \to \infty \tag{2.2.1}$$

The coefficients of expansion C_k and $\tilde{C}_k$ are called *Stokes multipliers* and are different in different Stoked sectors. To calculate the Stokes multipliers we must use the wronskian (by (2.1.1) independent from x). The multipliers $\tilde{C}_k$ are all equal to -1

$$\tilde{C}_k = -\frac{W[y_{k-1}, y_k]}{W[y_k, y_{k+1}]} = -1 \tag{2.2.2}$$

[10]We used exactly prof. Erdelyi's words in his book,[23] because this apparent "violation" of the fundamental theorems of linear algebra, might rise skepticism. We note, also, that an asymptotic representation of a function is defined[23] as a whole *equivalence class* of functions, not as a uniquely determined function.

because $W_{k_1+1,k_2+1}(E,l) = W_{k_1,k_2}(\omega^2 E,l)$ and $W_{0,1}(E,l) = 2i$, with obvious subscript notation. The non trivial Stokes multiplier is the other

$$\boxed{C_k(E,l) = \frac{W[y_{k-1}, y_{k+1}]}{W[y_{k-1}, y_k]}} \tag{2.2.3}$$

In particular the C_0 Stokes multiplier, from which we drop the subscript 0, is

$$C(E,l) = \frac{W[y_{-1}, y_1]}{2i} \tag{2.2.4}$$

being $W[y_0, y_1] = 2i$.
The relation (2.2.1) becomes

$$C(E,l)y(x,E,l) = \omega^{-1/2}y(\omega x, \omega^{-2}E,l) + \omega^{1/2}y(\omega^{-1}x, \omega^2 E,l) \qquad x \to \infty \tag{2.2.5}$$

In analogy to (2.1.10) define the "shifted" solutions around zero

$$\psi_k^{\pm}(x,E,l) = \omega^{k/2}\psi^{\pm}(\omega^{-k}x, \omega^{2k}E,l) \tag{2.2.6}$$

We take the wronskian of both sides of (2.2.5), defining

$$\boxed{D^{\mp}(E,l) = W[y(x,E,l), \psi^{\pm}(x,E,l)]} \tag{2.2.7}$$

the Stokes relation (2.2.1) becomes

$$\boxed{C(E,l)D^{\mp}(E,l) = \omega^{\mp(1/2+l)}D^{\mp}(\omega^{-2}E,l) + \omega^{\pm(1/2+l)}D^{\mp}(\omega^2 E,l)} \tag{2.2.8}$$

2.3 Analiticity properties and uniqueness

The function $D^-(E,l)$ has the following analyticity properties, the proof of which can be found in Dorey and Tateo's foundational work on the ODE-IM.[14]

1. $C(E,l)$ and $D^-(E,l)$ are entire functions of E

2. if $l \in \mathbb{R}$ and $l > -1/2$, then the zeros of $D^-(E,l)$ all lie on the positive real axis of the complex E-plane

3. if $M > 1$ then $D^-(E,l)$ has the large energy asymptotic

$$\log D^-(E,l) \sim \frac{a_0}{2}(-E)^{\frac{M+1}{2M}}, \qquad |E| \to \infty, \quad |\arg(-E)| < \pi \tag{2.3.1}$$

   with leading coefficient a_0

$$\frac{a_0}{2} = \int_0^{\infty} dt\,[(t^{2M}+1)^{1/2} - t^M] = -\frac{1}{2\sqrt{\pi}}\Gamma(-\frac{1}{2} - \frac{1}{2M})\Gamma(1+\frac{1}{2M}) \tag{2.3.2}$$

4. if $E = 0$, the normalization is

$$D^-(0,l) = \frac{1}{\sqrt{\pi}}\Gamma(1+\frac{2l+1}{2M+2})(2M+2)^{\frac{2l+1}{2M+2}+\frac{1}{2}} \tag{2.3.3}$$

These four properties are exactly the properties (but the normalization was chosen differently (1.5.3)) of the A^+ eigenvalue listed in subsection 1.5.

The other function $C(E, l)$, on the other hand, can be determined by the Stokes relation (2.2.8) and the following zero property[14]

- if $-1 - M/2 < l < M/2$ then the zeros of C *don't* lie on the positive real axis

Following the the Bazhanov, Lukyanov, A. Zamolodchikov[8] construction of integrable conformal field theory (summarized in section 1), all these properties permit to characterize the functions $C(E, l)$ and $D^-(E, l)$ uniquely. In fact, it can be shown, in complex analysis, that if $M < 1$, D^- can be written as an infinite product over its zeros

$$D^-(E, l) = D^-(0, l) \prod_{n=0}^{\infty} \left(1 - \frac{E}{E_n}\right) \tag{2.3.4}$$

(if $M \leq 1$ a more complicated prefactor is needed and the asymptotic density of zeros necessitates modifications to ensure convergence). Given the zeroes, the residual ambiguity is only in the normalization $D^-(0, l)$. D^- is also called *spectral determinant*. Introduce the function (cf. (1.5.4))

$$d(E, l) = \omega^{2l+1} \frac{D^-(\omega^2 E, l)}{D^-(\omega^{-2} E, l)} \tag{2.3.5}$$

It is evident, from (2.2.8), that the points at which $d = -1$ correspond to the zeros of C and D^-. The function

$$f(\theta, l) = \log\, d(\nu^{-2} e^{\theta/\mu}, l) \tag{2.3.6}$$

thus solves the Destri de Vega equation[14]

$$\begin{aligned} f(\theta, l) = i\pi\Big(l + \frac{1}{2}\Big) - i\cos\big(\frac{\pi}{2M}\big) a_0 \nu^{-2\mu} e^{\theta} + \int_{-\infty - i\epsilon}^{\infty - i\epsilon} d\theta'\, \varphi(\theta - \theta') \log\big(1 + e^{f(\theta', l)}\big) \\ - \int_{-\infty + i\epsilon}^{\infty + i\epsilon} d\theta'\, \varphi(\theta - \theta') \log\big(1 + e^{-f(\theta', l)}\big) \end{aligned} \tag{2.3.7}$$

with

$$\varphi(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik\theta} \sinh\left(\frac{\pi}{2}(\xi - 1)k\right)}{2\cosh\left(\frac{\pi}{2}k\right) \sinh\left(\frac{\pi}{2}\xi k\right)}\,, \qquad \xi = \frac{1}{M} \tag{2.3.8}$$

Dorey and Tateo assumed[14] that the solution to this equation is unique.

Given the solution f of the Destri de Vega equation, the zeroes of C and D^- are determined uniquely and the spectral determinant D^- is fixed up to an overall constant, which is given by property (2.3.3). *Since C was already given by* (2.2.4), *D^- and C has been determined.*

2.4 ODE/IM correspondence

We now make explicit the relation between integrable conformal field theory and the quantities introduced in relation to the Scrödinger equation (2.1.1). We list on the left side of the equalities the integrable CFT parameters and on the right side the Scrödinger equation quantities

$$\beta^2 = \frac{1}{M+1} \tag{2.4.1}$$

$$p = \frac{1}{2}\frac{(l + \frac{1}{2})}{M+1} \tag{2.4.2}$$

$$\lambda = \nu E^{1/2}\,, \qquad \nu = \frac{(2\beta^2)^{1-\beta^2}}{\Gamma(1-\beta^2)} \tag{2.4.3}$$

remember that $\frac{1}{1-\beta^2}=\frac{M+1}{M}$. As a consequence of uniqueness, we can also identify

$$\boxed{\begin{aligned} A_{\pm}(\lambda,p)\Big|_{\beta^2} &= \alpha^{\mp}D^{\mp}\Big(\Big(\frac{\lambda}{\nu}\Big)^2,\frac{2p}{\beta^2}-\frac{1}{2}\Big)\Big|_{M=\beta^{-2}-1} && (2.4.4)\\ Q_{\pm}(\lambda) &= \lambda^{\pm 2P/\beta^2}\alpha^{\mp}D^{\mp}\Big(\Big(\frac{\lambda}{\nu}\Big)^2,\frac{2p}{\beta^2}-\frac{1}{2}\Big)|_{M=\beta^{-2}-1} && (2.4.5)\\ T(\lambda,p)\Big|_{\beta^2} &= C\Big(\Big(\frac{\lambda}{\nu}\Big)^2,\frac{2p}{\beta^2}-\frac{1}{2}\Big)\Big|_{M=\beta^{-2}-1} && (2.4.6)\end{aligned}}$$

where $\alpha^{\mp}=\Big(D^{\mp}(0,\frac{2p}{\beta^2}-\frac{1}{2})\Big)^{-1}$.With these identifications, the Stokes relation becomes exactly the TQ relation ($q^{l+1/2}=e^{2\pi ip}=\omega^{l+1/2}$), the asymptotics of $A^+\equiv D^-$ match and the normalization (2.3.3) of D^- correctly induces normalization to unity (1.5.3) of A^+. We note that using $\sqrt{E}$ in the place of $\lambda=e^{\theta(1-\beta^2)}$ amounts just to a shift on θ as

$$\boxed{\lambda\to\sqrt{E}\quad\Longleftrightarrow\quad\theta\to\theta-\ln(2\beta^2)+\frac{1}{(1-\beta^2)}\ln\Gamma(1-\beta^2)} \tag{2.4.7}$$

Consider the expansion of y in the basis of the shifted of $\{\psi^+,\psi^-\}$ (see subsection 2.6 for further comments)

$$(2l+1)y(x,E,l)=D^-(E,l)\psi^-(x,E,l)-D^+(\omega E,l)\psi^+(x,E,l) \tag{2.4.8}$$

f we consider the limit $x\to0$ for this equation, we see that the contribution of D^+ is suppressed because $\psi^+\to0$; while the contribution of D^- is dominant, because is multiplied by the divergent function $\psi_-\sim x^{-l}$. Thus, because the wronskian $D^-(E,l)=W[y(x,E,l),\psi^+(x,E,l)]$ is an entire function, it can be calculated just by multiplying this expression by x^l

$$\boxed{D^-(E,l)=\lim_{x\to0}\Big[(2l+1)x^l y(x,E,l)\Big]} \tag{2.4.9}$$

All our calculations will be based upon this or similar identifications between the Baxter's Q function (2.4.5) and the solution of the ODE-IM Schrödinger equation calculated at some point.

2.5 Fusion relations

The "fused" $\mathbf{T}$ operators $\mathbf{T}_j$ ($j=0,1/2,1,3/2,...$) were introduced in integrable conformal field theory from the fusion relation (1.3.28), which we report here

$$\mathbf{T}_j(q^{\frac{1}{2}}\lambda)\mathbf{T}_j(q^{-\frac{1}{2}}\lambda)=1+\mathbf{T}_{j-\frac{1}{2}}(\lambda)\mathbf{T}_{j+\frac{1}{2}}(\lambda) \tag{2.5.1}$$

The $\mathbf{T}_j$ operators also satisfy relations of the TQ form

$$\mathbf{T}(\lambda)\mathbf{T}_j(q^{j+1/2}\lambda)=\mathbf{T}_{j-1/2}(q^{j+1}\lambda)+\mathbf{T}_{j+1/2}(q^j\lambda) \tag{2.5.2}$$

We now want to establish the analogues of these relations for the ordinary differential equation (2.1.1), through suitable definition of the T_j eigenvalues of the $\mathbf{T}_j$ operators.
We already obtained the $j=1/2$ case, i.e. the TQ relation, by considering the Stokes relation (2.2.1) for *adjacent* Stokes sectors. It seems reasonable to find a general Stokes relation for the general Stokes sectors, namely

$$y_{k-1}=C_k^{(m)}y_{k+m-1}+\tilde{C}_k^{(m)}y_{k+m} \tag{2.5.3}$$

where we introduced generalized Stokes multipliers

$$C_k^{(m)} = \frac{1}{2i} W_{k-1,k+m} \,, \qquad \tilde{C}_k^{(m)} = -\frac{1}{2i} W_{k-1,k+m-1} \tag{2.5.4}$$

Considering now a fundamental pair of solutions, the four Stokes multipliers can be collected in a matrix $\mathbf{C}_k^{(m)}$

$$\begin{pmatrix} y_{k-1} \\ y_k \end{pmatrix} = \mathbf{C}_k^{(m)} \begin{pmatrix} y_{k+m-1} \\ y_{k+m} \end{pmatrix} , \qquad \mathbf{C}_k^{(m)} = \begin{pmatrix} C_k^{(m)} & \tilde{C}_k^{(m)} \\ C_{k+1}^{(m-1)} & \tilde{C}_{k+1}^{(m-1)} \end{pmatrix} \tag{2.5.5}$$

It is easily proved that

$$\mathbf{C}_k^{(m)}(E,l) = \mathbf{C}_{k-1}^{(m)}(\omega^2 E, l) \tag{2.5.6}$$

The matrices $\mathbf{C}_k^{(m)}$ satisfy a sort of "transitivity" property

$$\mathbf{C}_k^{(m)} \mathbf{C}_{k+m}^{(n)} = \mathbf{C}_k^{(m+n)} \tag{2.5.7}$$

which corresponds to the requirement that the change from the basis $\{y_{k+m+n-1}, y_{k+m+n}\}$ to the basis $\{y_{k+m-1}, y_{k+m}\}$, followed by the change from $\{y_{k+m-1}, y_{k+m}\}$ to $\{y_{k-1}, y_k\}$, should be equivalent to the unique change from $\{y_{k+m+n-1}, y_{k+m+n}\}$ to $\{y_{k-1}, y_k\}$. For the particular case of $m = 1$, we get

$$C_k C_{k+1}^{(n)} - C_{k+2}^{(n-1)} = C_k^{(n+1)} \tag{2.5.8}$$

and another similar equation for the tilted multipliers $\tilde{C}_k$.
Now we define the candidate for the $T_{n/2}$ eigenvalues

$$C^{(n)}(E) = C_0^{(n)}(\omega^{-n+1} E) \tag{2.5.9}$$

then, considering property (2.5.6), the transitivity relation (2.5.8) can be written as

$$C(E) C^{(n)}(\omega^{n+1} E) = C^{(n-1)}(\omega^{n+2} E) + C^{(n+1)}(\omega^n E) \tag{2.5.10}$$

which matches precisely the fusion relation (2.5.2), provided the $T_{n/2}$ eigenvalues is defined as

$$\boxed{T_{n/2}(\nu E^{1/2}) = C^{(n)}(E) = \frac{1}{2i} W_{-1,n}(\omega^{-n+1} E)} \tag{2.5.11}$$

Taking instead $n = -m$ in the transitivity relation (2.5.7) and noting that $\mathbf{C}_k^{(0)} = \mathbf{I}$ we obtain also the fusion relations

$$\boxed{C^{(m-1)}(\omega^{-1} E) C^{(m-1)}(\omega E) = 1 + C^{(m)}(E) C^{(m-2)}(E)} \tag{2.5.12}$$

2.6 Fused quantum wronskians

The pair of functions $\{\psi^-, \psi^+\}$, defined as the solutions around zero, can be used as basis "almost everywhere in l".[11] Consider in particular the expansion of y_k in the basis of the shifted versions of this solutions (2.2.6)

$$(2l+1) y_k(x,E,l) = D^-(\omega^{2k} E, l) \psi_k^-(x,E,l) - D^+(\omega^{2k} E, l) \psi_k^+(x,E,l) \tag{2.6.1}$$

[11]For example, this is not true at $l = -1/2$, when the two functions coincide. More generally, since ψ^- was defined making the analytic continuation $l \to -l-1$ on ψ^+, there may be points at which poles are encountered. At such points a regularization is needed which may spoil the functional independence.

Taking the wronskian of y_{-1} and y_n expressed though the previous equation and taking into account the properties

$$W[\psi_k^+, \psi_p^+] = W[\psi_k^-, \psi_p^-] = 0 \tag{2.6.2}$$
$$W[\psi_p^-, \psi_k^+] = (2l+1)\omega^{(k-p)(l+1/2)} \tag{2.6.3}$$

we find an expression for $C^{(n)}(E)$ which turns out to be the fused quantum wronskians relation of integrability (1.4.17)

$$\boxed{\begin{aligned}(4l+2)iC^{(n)}(E) = {} & \omega^{(n+1)(l+1/2)}D^-(\omega^{n+1}E, l)D^+(\omega^{-n-1}E, l) \\ & - \omega^{-(n+1)(l+1/2)}D^-(\omega^{-n-1}E, l)D^+(\omega^{n+1}E, l)\end{aligned}} \tag{2.6.4, 2.6.5}$$

2.7 Singular potentials and duality

Consider the usual ODE (2.1.1)

$$\left[-\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2} - E\right]y(x) = 0 \tag{2.7.1}$$

and apply to it a sequence of transformations, the first of which is the Langer transform

$$x = e^z \qquad y(x) = e^{z/2}\psi(z) \tag{2.7.2}$$
$$\left[-\frac{d^2}{dz^2} + e^{2z(M+1)} + (l+\frac{1}{2})^2 - Ee^{2z}\right]\psi(z) = 0 \tag{2.7.3}$$

then apply a dilation and translation to z

$$z = \frac{1}{M+1}z' + \log\frac{M+1}{\sqrt{E}} \tag{2.7.4}$$
$$x = e^{\frac{z'}{M+1}}\frac{M+1}{\sqrt{E}} \tag{2.7.5}$$
$$\left[-\frac{d^2}{dz'^2} - e^{\frac{2z'}{M+1}} - \tilde{E}e^{2z'} + (\tilde{l}+\frac{1}{2})^2\right]\psi(z') = 0 \tag{2.7.6}$$

where we define new energy $\tilde{E}$ and quantum number $\tilde{l}$

$$\tilde{E} = -\frac{(M+1)^{2M}}{E^{M+1}} \tag{2.7.7}$$
$$\tilde{l} = \frac{l-\frac{M}{2}}{M+1} \qquad \text{or} \quad l+\frac{1}{2} = (\tilde{l}+\frac{1}{2})(M+1) \tag{2.7.8}$$

Now we apply an inverse Langer transfom

$$v = e^{z'} \qquad \psi(z') = v^{-1/2}\tilde{y}(v) \tag{2.7.9}$$
$$x = \frac{M+1}{\sqrt{E}}v^{\frac{1}{M+1}} \tag{2.7.10}$$
$$\psi(z') = \frac{\sqrt{M+1}}{\sqrt[4]{E}}v^{-\frac{1}{2(M+1)}}y(\frac{M+1}{\sqrt{E}}v^{\frac{1}{M+1}}) \tag{2.7.11}$$
$$\tilde{y}(v) = \frac{\sqrt[4]{E}}{\sqrt{M+1}}v^{\frac{M}{2(M+1)}}y(\frac{M+1}{\sqrt{E}}v^{\frac{1}{M+1}}) \tag{2.7.12}$$

After this last transformation the equation becomes

$$\boxed{\left[-\frac{d^2}{dv^2} - v^{2\tilde{M}} - \tilde{E} + \frac{\tilde{l}(\tilde{l}+1)}{v^2}\right]\tilde{y}(v) = 0} \tag{2.7.13}$$

where we defined a new parameter

$$\boxed{\tilde{M} = -\frac{M}{M+1}} \tag{2.7.14}$$

In terms of β^2

$$\boxed{\tilde{\beta}^2 = \frac{1}{\beta^2}} \tag{2.7.15}$$

the old range $M > 0$, $0 < \beta^2 < 1$ is mapped into

$$\boxed{-1 < \tilde{M} < 0\,, \qquad \tilde{\beta}^2 > 1} \tag{2.7.16}$$

but note the sign of the "M-term" is reversed (minus).
In particular, *the harmonic oscillator Schrödinger equation - $M = 1$ is mapped into the Coloumb potential Schrödinger equation $\tilde{M} = -1/2$ by the transformation* (2.7.10) (2.7.12)

2.7.1 Developments for the Liouville model

We make a comment concerning Dorey and Tateo's duality consideration of section 2.7.
Note that if, as it indeed happens, the solution $y(x, E)$ of the original problem (2.1.1) is analytic in the whole complex plane of E but not at infinity, the solution of the dual problem $y(w, \tilde{E})$ (2.7.12) cannot be such, because the map between E and $\tilde{E}$ (2.7.7) is not analytic for $E = 0$. Nevertheless, $\tilde{E}$ is analytic for $E \to \infty$. *Thus the entireness problem can be solved simply considering the zero energy of the original problem as the infinite energy of the dual problem and viceversa.*
One can now consider a new range for $\tilde{M}$ in (2.7.14)

$$-2 < \tilde{M} < -1 \tag{2.7.17}$$

which is inversely mapped in the range

$$M < -1 \tag{2.7.18}$$

which corresponds to the Liouville model.
However, in the range $M < -1$ both energies, for the original and the dual problem are analytic in the whole complex plane, because zero is not anymore a singularity. In the corresponding $M < -1$ there are two irregular singularities in y.

2.8 Y system and TBA

The fusion relations permit a purely algebraic derivation of the TBA equations.[16] For generic $\beta^2 > 0$, the TBA leads to an infinite system of coupled integral equations, the *Takahashi-Suzuki system*, which is somewhat complicated. *However, at rational values of β^2, it truncates to a finite system of integral equations.*
For example, at

$$\beta^2 = \frac{1}{N}\,, \quad \text{or} \quad \beta^2 = \frac{2}{N} \qquad N = 1, 2, 3, ... \tag{2.8.1}$$

(which correspond respectively to M integer or half integer $M = -1/2, 0, 1/2, 1, ...$) the fused quantum wronskian expression (1.4.17) induces an additional relation[12]

$$T_{\frac{N}{2}}(\theta) = 2\cos\left(\pi(l + \frac{1}{2})\right) + T_{\frac{N}{2}-1}(\theta) \tag{2.8.2}$$

[12] Remember that $\lambda = \exp\frac{M\theta}{1+M}$

which closes the fusion relations (1.3.28)

$$T_{\frac{1}{2}}(\theta)T_j(\theta+\frac{i\pi(2j+1)}{2M})=T_{j-\frac{1}{2}}(\theta+\frac{i\pi(2j+2)}{2M})+T_j(\theta+\frac{i\pi(2j)}{2M}) \tag{2.8.3}$$

within a finite number of functions $T_j(\theta)$ with $j=\frac{1}{2},1,\frac{3}{2},...,\frac{N}{2}-1$.
For this purpose, introduce the functions

$$\boxed{\begin{aligned} &Y_j(\theta)=T_{j-\frac{1}{2}}(\theta)T_{j+\frac{1}{2}}(\theta) \qquad j=\frac{1}{2},1,\frac{3}{2},...,\frac{N}{2}-1 && (2.8.4)\\ &Y_0(\theta)=0 && (2.8.5)\\ &\bar{Y}(\theta)=T_{\frac{N}{2}-1}(\theta) && (2.8.6)\end{aligned}}$$

then associated to the fusion relations and the truncations (2.8.2) there is a closed system of functional equations, known as *Y-system*

$$Y_j(\theta+\frac{i\pi}{2M})Y_j(\theta-\frac{i\pi}{2M})=(1+Y_{j-\frac{1}{2}}(\theta))(1+Y_{j+\frac{1}{2}}(\theta))\,, \quad j=\frac{1}{2},1,...,\frac{N}{2}-\frac{3}{2} \tag{2.8.7}$$

$$Y_{\frac{N}{2}-1}(\theta+\frac{i\pi}{2M})Y_{\frac{N}{2}-1}(\theta-\frac{i\pi}{2M})=(1+Y_{\frac{N-3}{2}}(\theta))(1+e^{\pi i(l+\frac{1}{2}}\bar{Y}(\theta))(1+e^{-\pi i(l+\frac{1}{2})}\bar{Y}(\theta)) \tag{2.8.8}$$

$$\bar{Y}(\theta+\frac{i\pi}{2M})\bar{Y}(\theta-\frac{i\pi}{2M})=1+Y_{\frac{N}{2}-1}(\theta) \tag{2.8.9}$$

This Y-system is of the D_N type in the TBA framework,[21] that is, they refer to *Sine-Gordon scattering theory at the reflection-less points* $\theta^2=8\pi/n$. Thus, *the Y-system can be transformed in the TBA integral equations.*

3 Thermodynamic Bethe ansatz for Sinh-Gordon model

3.1 Generalities on Sinh-Gordon model

The two dimensional Sinh-Gordon model is defined via the euclidean action[20]

$$\boxed{A_{\text{Shg}} = \int d^2x \left[\frac{1}{4\pi}(\partial_a\phi)^2 + 2\mu\cosh(2b\phi)\right]} \tag{3.1.1}$$

where b is a dimensionless parameter and μ is a coupling constant (assumed real positive). For the Sinh-Gordon and Liouville models, the following parameters are usually defined

$$Q = b + \frac{1}{b} \tag{3.1.2}$$

$$p = \frac{b}{Q} = \frac{b^2}{b^2+1} \tag{3.1.3}$$

$$a = 1 - 2p = \frac{b^2-1}{b^2+1} \tag{3.1.4}$$

The coupling constant μ determines the scale of the model and *it is related to the mass* precisely as

$$\pi\mu\frac{\Gamma(b^2)}{\Gamma(1-b^2)} = [mZ(p)]^{2+2b^2} \tag{3.1.5}$$

where $Z(p)$ is defined as

$$Z(p) = \frac{1}{8\sqrt{\pi}}p^p(1-p)^{1-p}\Gamma\Big(\frac{1-p}{2}\Big)\Gamma\Big(\frac{p}{2}\Big) \tag{3.1.6}$$

The Sinh-Gordon model is integrable, even if, by (3.1.5) *it is not conformal.* The spectrum consists of only one neutral particle subjected to factorized scattering with two particle amplitude, which, as function of the *rapidity* θ, is expressed by

$$S(\theta) = \frac{\sinh\theta - i\sin\pi p}{\sinh\theta + i\sin\pi p} \tag{3.1.7}$$

Evidently, *scattering theory is invariant under the symmetry* $p \to 1-p$, *which corresponds to* $b \to 1/b$ *and* $a \to -a$. Thanks to this symmetry it is sufficient to restrict to the region $0 < b^2 \leq 1$ (i.e. $0 < p \leq 1/2$, $0 \leq a < 1$). However, in order for the whole theory to be invariant, it is also necessary that the scale μ also transforms to a scale $\tilde{\mu}$, defined implicitly by

$$\left(\pi\mu\frac{\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{1/b} = \left(\pi\tilde{\mu}\frac{\Gamma(1/b^2)}{\Gamma(1-1/b^2)}\right)^{b} \tag{3.1.8}$$

3.2 Thermodynamic Bethe ansatz

3.2.1 TBA integral equation for the Sinh-Gordon model

Consider now the Sinh-Gordon model placed on a circle of finite circumference R. We can study the finite size Sinh-Gordon model through a non-linear integral equation[20] known as thermodynamic Bethe ansatz (TBA), whose solution is the so-called *pseudoenergy* $\varepsilon(\theta)$

$$\boxed{\varepsilon = mR\cosh\theta - \varphi * \log\left(1+e^{-\varepsilon}\right)} \tag{3.2.1}$$

where $*$ denotes convolution on the whole real axis of θ and the kernel $\varphi(\theta)$ is related to the scattering amplitude (3.1.7) as

$$\varphi(\theta) = -\frac{i}{2\pi}\frac{d}{d\theta}\log S(\theta) \tag{3.2.2}$$

$$= \frac{1}{2\pi}\frac{4\sin\pi p\cosh\theta}{\cosh 2\theta - \cos 2\pi p} \tag{3.2.3}$$

which, by use of simple hyperbolic trigonometric identities, can be reduced to

$$\boxed{\varphi(\theta) = \frac{1}{2\pi}\left(\frac{1}{\cosh\left(\theta + i\pi a/2\right)} + \frac{1}{\cosh\left(\theta - i\pi a/2\right)}\right)} \tag{3.2.4}$$

From this latter expression, we conclude that the function $\varepsilon(\theta)$ is even and analytic in the strip $|\Im\theta| < -\pi a/2 + \pi/2$.

3.2.2 Formal link between Sinh-Gordon and Liouville TBA and leading asymptotics

Consider the right hand side of the TBA equation (3.2.1). Given Euler formula for $\cosh\theta$, asymptotically for $\Re\theta \to +\infty$ we can write

$$mR\cosh\theta \sim \frac{mR}{2}e^{\theta} \qquad \Re\theta \to +\infty \tag{3.2.5}$$

Now, we define a shifted rapidity θ' and include the divergence only in the shift parameter Λ

$$\theta' = \theta - \Lambda\,, \qquad \Lambda \sim \Re\theta \to +\infty \tag{3.2.6}$$

Thus, the new rapidity θ' is finite and we can define a new mass by

$$M = me^{\Lambda} \qquad \Lambda \to +\infty\,, \qquad m \to 0 \tag{3.2.7}$$

which can actually be set equal to an arbitrary constant, because for the Liouville *conformal* model the mass can be considered infinitesimal

$$\boxed{m \to 0} \tag{3.2.8}$$

In particular, Zamolodchikov[2] defined

$$\boxed{MR = 2\pi} \tag{3.2.9}$$

In conclusion, we can say that, *the Liouville TBA and other $\Re\theta \to +\infty$ asymptotic relations, are formally obtained from the corresponding Sinh-Gordon relations by discarding one of the exponentials of hyperbolic function in the asymptotic terms and by setting, formally, $mR \to 2\pi$.* In particular, Liouville TBA is written as[2]

$$\boxed{\varepsilon = \pi e^{\theta} - \varphi * \log\left(1 + e^{-\varepsilon}\right)} \tag{3.2.10}$$

Note that φ is the very same kernel (3.2.4) as in the Sinh-Gordon model, because the only term of the TBA dominant in the limit $\Re\theta \to +\infty$ is the forcing term, not the convolution.

3.3 Y system

3.3.1 Y system and universality

Define the important function $Y(\theta)$

$$\boxed{Y(\theta) = e^{-\varepsilon(\theta)}} \tag{3.3.1}$$

which is called simply *Y function*. The Y function is even and, for $\Re\theta \to +\infty$ is asymptotically represented as

$$Y(\theta) \sim \exp\left(-\frac{mR}{2}e^{\theta}\right) \qquad \Re\theta \to +\infty \tag{3.3.2}$$

The *Y system* for this function is the following functional equation (to be proven below)

$$\boxed{Y(\theta + i\pi/2)Y(\theta - i\pi/2) = \Big(1 + Y(\theta + ia\pi/2)\Big)\Big(1 + Y(\theta - ia\pi/2)\Big)} \tag{3.3.3}$$

Physically, Liouville theory is very different from Sinh-Gordon theory. First of all, Sinh-Gordon theory is massive (the mass is given in (3.1.5)), while Liouville theory is massless and therefore conformal. As a consequence, for instance, Liouville three-point correlation functions are known exactly (by the so-called DOZZ formula[30]), while their determination is much more difficult in Sinh-Gordon theory.

However, the Y-systems for these two theories are the very same. In fact, *it is a general feature of integrable models that, even if the massive and massless cases have actually different TBA equations, the Y system is always the same.* In particular, the *driving term* is the only formal difference between Sinh-Gordon and Liouville TBA equations. However, given some Y system, all the related TBA equations not only differ for the forcing term, but also for the physical states to which they are referred to. We always consider the void but, In fact, the excited states have different TBA equations which correspond to the one and the same Y system[21]

3.3.2 The TBA equation corresponds to a unique Y system

We now show that the TBA equation (3.2.1) entails the validity of the Y-system, given the definition of the Y function in terms of the TBA solution $\varepsilon(\theta)$. In fact, we shift the argument of $\varepsilon(\theta)$ by $\pm i\pi/2$

$$\epsilon(\theta + i\pi/2) = imR\sinh\theta - \int_{-\infty}^{\infty} d\theta'\,\varphi(\theta + i\pi/2 - \theta')\log\left(1 + e^{-\varepsilon(\theta')}\right)$$
$$\epsilon(\theta - i\pi/2) = -imR\sinh\theta - \int_{-\infty}^{\infty} d\theta'\,\varphi(\theta - i\pi/2 - \theta')\log\left(1 + e^{-\varepsilon(\theta')}\right)$$

then sum the two equations, using the definition of Y

$$\begin{aligned}\log\left[Y(\theta + i\pi/2)Y(\theta - i\pi/2)\right] = &\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi i}\left(\frac{1}{\sinh(\theta + i\pi a/2 - \theta')} + \frac{1}{\sinh(\theta - i\pi a/2 - \theta')}\right)\log\left(1 + Y(\theta')\right)\\ &+\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi i}\left(\frac{-1}{\sinh(\theta + i\pi a/2 - \theta')} + \frac{-1}{\sinh(\theta - i\pi a/2 - \theta')}\right)\log\left(1 + Y(\theta')\right)\end{aligned}$$

Note that the result is not zero, because the integrals are singular in $\theta = \theta' \mp i\pi a/2$. The Y function is, on the real axis, also real and positive. By continuity, we expect an infinitesimal strip $\Im\theta < \epsilon$ where $1 + Y(\theta)$ is analytic. In such a strip, the whole integrand is analytic, except at $\theta = \theta' \mp i\pi a/2$, where the pole is simple. We can the encircle this singularity from above, for the first integral and from below, for the second integral. Therefore, we have to calculate

$$\oint_{C_\epsilon}\frac{d\theta'}{2\pi i}\,\frac{\log\left(1 + Y(\theta')\right)}{\theta - \theta' \pm i\pi a/2} = 1 + Y(\theta \mp i\pi a/2)$$

directly by Cauchy integral formula, with C_ϵ a rectangular contour, horizontally infinitesimal (where the $\sinh(\theta - \theta' \pm ia\pi/2)$ function can be linearly approximated). Exponentiating, we obtains finally the Y-system (3.3.3)

$$Y(\theta + i\pi/2)Y(\theta - i\pi/2) = \Big[1 + Y(\theta + ia\pi/2)\Big]\Big[1 + Y(\theta - ia\pi/2)\Big]$$

3.4 X system

3.4.1 From the X function to the X system

Al. Zamolodchikov[20] defined the X function as

$$\boxed{X(\theta) = \exp\left\{-\frac{mR}{2\sin\pi p}\cosh\theta + \int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\frac{\log\left(1+Y(\theta')\right)}{\cosh\left(\theta-\theta'\right)}\right\}} \tag{3.4.1}$$

The X function is analytic in the strip $|\Im\theta| < \pi/2$ and in this strip is asymptotically represented as

$$X(\theta) \sim \exp\left(-\frac{mR}{4\sin\pi p}e^{\theta}\right) \qquad \Re\theta \to +\infty \tag{3.4.2}$$

Now we prove that this definition implies the relation

$$\boxed{X(\theta + i\pi/2)X(\theta - i\pi/2) = 1 + Y(\theta)} \tag{3.4.3}$$

In fact, by the definition (3.4.1)

$$\begin{aligned} X(\theta+i\pi/2)X(\theta-i\pi/2) &= \exp\left\{\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\log\left(1+Y(\theta')\right)\left[\frac{1}{\cosh\left(\theta+i\pi/2-\theta'\right)}+\frac{1}{\cosh\left(\theta-i\pi/2-\theta'\right)}\right]\right\} \\ &= \exp\left\{\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi i}\frac{\log\left(1+Y(\theta')\right)}{\sinh\left(\theta-\theta'\right)} + \int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi i}\frac{\log\left(1+Y(\theta')\right)}{-\sinh\left(\theta-\theta'\right)}\right\} \end{aligned}$$

Note that the result is not zero, because the integrals are singular in $\theta = \theta'$. The Y function is, on the real axis, also real and positive. By continuity, we expect an infinitesimal strip $\Im\theta < \epsilon$ where $1 + Y(\theta)$ is analytic. In such a strip, the whole integrand is analytic, except at $\theta = \theta'$, where the pole is simple. We can the encircle this singularity from above, for the first integral and from below, for the second integral. Therefore, we have to calculate

$$X(\theta+i\pi/2)X(\theta-i\pi/2) = \exp\left\{\oint_{C_\epsilon}\frac{d\theta'}{2\pi i}\frac{\log\left(1+Y(\theta')\right)}{\theta-\theta'}\right\} = 1+Y(\theta)$$

directly by Cauchy integral formula, with C_ϵ a rectangular infinitesimal contour around $\theta = \theta'$ (where the $\sinh(\theta - \theta')$ function can be linearly approximated).
Consider an equivalent definition of the Y function

$$\boxed{Y(\theta) = X(\theta + ia\pi/2)X(\theta - ia\pi/2)} \tag{3.4.4}$$

In fact, assume the validity of the TBA equation (3.2.1), then, given the definition (3.4.1) of X, take the logarithm of this relation. Set also $c_0 = -mR/2\sin\pi p$. We get:

$$\begin{aligned} \log Y(\theta) = &\left[c_0\cosh\left(\theta+ia\pi/2\right)+\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\log\left(1+e^{-\varepsilon(\theta')}\right)}{\cosh\left(\theta+ia\pi/2-\theta'\right)}\right] + \left[c_0\cosh\left(\theta-ia\pi/2\right)\right. \\ &\left. + \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\log\left(1+e^{-\varepsilon(\theta')}\right)}{\cosh\left(\theta-ia\pi/2-\theta'\right)}\right] \\ &= 2c_0\cosh\theta\cos\left(a\pi/2\right)+\int_{-\infty}^{\infty}d\theta'\,\varphi(\theta-\theta')\log\left(1+e^{-\varepsilon(\theta')}\right) \end{aligned}$$

Now because $c_0 = -mR/2\sin\pi p$ and $a = 1 - 2p$

$$2c_0\cosh\theta\cos\left(a\pi/2\right) = -mR/2\sin\pi p\cos\left(1-2p\right)\pi/2 = -mR$$

we obtain the right hand side of the Sinh-Gordon TBA equation (3.2.1)

$$\log Y(\theta) = mR\cosh\theta - \int_{-\infty}^{\infty} d\theta' \, \varphi(\theta-\theta') \log\left(1+e^{-\varepsilon(\theta')}\right)$$

We conclude that, given the TBA and the X function, the definition (3.4.4) for the Y function is equivalent to(3.3.1).
It is immediate that, from the combination of (3.4.4) and (3.4.3), the so called X *system* holds

$$\boxed{X(\theta+i\pi/2)X(\theta-i\pi/2) = 1 + X(\theta+ia\pi/2)X(\theta-ia\pi/2)} \tag{3.4.5}$$

which is a functional equation for X.
It is commonly assumed that the X system is equivalent to the Y-system (3.3.3). To be precisely, the Y system is obtained by combining the above X system with the same X system with the rapidity θ shifted by $-\pi$ (using also the relations (3.4.3) and (3.4.4))[13].

$$\begin{aligned}
&\Big[X(\theta+ia\pi/2)X(\theta-ia\pi/2)\Big]\Big[X(\theta+ia\pi/2-i\pi)X(\theta-ia\pi/2-i\pi)\Big] \\
&= \Big[X(\theta+i\pi/2)X(\theta-i\pi/2)-1\Big]\Big[X(\theta-i\pi/2)X(\theta-3i\pi/2)-1\Big]
\end{aligned}$$

However, given the Y system, the validity of the X system is *not rigourously necessary*, even if it is always assumed that there is an *equivalence* between the X system and the Y system.

3.4.2 Inverse procedure

Given the X or Y system,[14] assuming only the knowledge of (3.4.3) and (3.4.4), it is possible to obtain Al. Zamolodchikov's definition (3.4.1). However, by this inverse procedure X is not uniquely determined, that is, *the boundary conditions must also be fixed.*
We use a theorem of,[22] which we report here

Theorem 1. *Let ξ be a function such that its Fourier transform $\hat{\xi}$ belongs to L^1. Define another function χ by*

$$\chi(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi(\theta')}{\cosh(\theta-\theta')} d\theta' \tag{3.4.6}$$

then χ is bounded and analytic in the strip $|\Im\theta| < \frac{\pi}{2}$ and its boundary functions satisfy

$$\chi(\theta+i\pi/2) + \chi(\theta-i\pi/2) = \xi(\theta) \tag{3.4.7}$$

for real θ. Conversely if ξ is bounded and analytic in the strip $|\Im\theta| < \frac{\pi}{2}$ and if (3.4.6) *then so does* (3.4.7)

For both the Sinh-Gordon and Liouville model, set $\xi(\theta) = \log(1+Y(\theta))$. Then by (**??**)

$$\hat{X}(\theta) = \exp\int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \frac{\log(1+Y(\theta'))}{\cosh(\theta-\theta')} \tag{3.4.8}$$

(the hat over X means that this is a temporary definition).
We observe that *there is freedom to add to* ξ a so called "zero-mode function" ϕ, defined as a solution of the homogeneous equation

$$\boxed{\phi(\theta+i\pi/2) + \phi(\theta-i\pi/2) = 0} \tag{3.4.9}$$

[13] To get the shifts as in (3.3.3), a final shift of $+\pi/2$ is needed

[14] Concerning the equivalence of the X and Y system, see the comments above

which implies

$$\Big(\chi(\theta+i\pi/2)-\phi(\theta+i\pi/2)\Big)+\Big(\chi(\theta-i\pi/2)-\phi(\theta-i\pi/2)\Big)=\xi(\theta)$$

A possible zero mode function is $\cosh\theta$ or $\exp\theta$. Thus, *the most general expression for ξ is*

$$\boxed{\chi(\theta)=\phi(\theta)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi(\theta')}{\cosh(\theta-\theta')}d\theta'} \tag{3.4.10}$$

Now, for the Sinh-Gordon model, we fix the zero mode of (3.4.9), following the conventions of.[20]

$$\boxed{\phi(\theta)=-\frac{mR}{2\sin\pi p}\cosh\theta \qquad \text{(SINH-GORDON)}} \tag{3.4.11}$$

so that the X function matches the previous definition (3.4.1)

$$X(\theta)=\exp\left\{-\frac{mR}{2\sin\pi p}\cosh\theta+\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\frac{\log(1+Y(\theta'))}{\cosh(\theta-\theta')}\right\} \tag{3.4.12}$$

Instead, for the Liouville model we we fix the zero mode of (3.4.9), following the conventions of[2]

$$\boxed{\phi(\theta)=-\frac{\pi}{2\sin\pi p}e^{\theta} \qquad \text{(LIOUVILLE)}} \tag{3.4.13}$$

so that the X function matches the definition of[2]

$$X(\theta)=\exp\left\{-\frac{\pi}{2\sin\pi p}e^{\theta}+\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\frac{\log(1+Y(\theta'))}{\cosh(\theta-\theta')}\right\} \tag{3.4.14}$$

Thus, we have proven that the direct and inverse procedure of Al. Zamolodchikov are only if we fix the boundary conditions, i.e. the zero modes of (3.4.9), which corresponds to the $\Re\theta\to+\infty$ asymptotics for the X function.

For the $\Re\theta\to-\infty$ asymptotics, for the Sinh-Gordon model there is no difference, as all functions constructed from the TBA solution $\varepsilon(\theta)$ are even. However $\varepsilon(\theta)$ is not even for the Liouville model. Accordingly, numerical calculations of Zamolodchikov[2] were consistent with the boundary condition

$$\boxed{X(\theta)\simeq\exp\Big\{2PQ\theta+\text{const.}\Big\} \qquad \theta\to-\infty \qquad \text{(LIOUVILLE)}} \tag{3.4.15}$$

3.5 Integrability

Lukyanov[29] found the same expression (3.4.1) for $X(\theta)$ in a rather different context. He argued that the Baxter's Q function for the Sinh-Gordon model has an asymptotic expansion without non-local integrals of motion

$$\boxed{\log Q(\theta)=-B_0e^{\theta}-\sum_{n=1}^{\infty}B_nI_{2n-1}e^{-(2n-1)\theta}} \tag{3.5.1}$$

because, differently from the Sine-Gordon model, in the Sinh-Gordon model there is *no soliton sector*. Lukyanov then gave numerical evidence that the coefficient of this expansion, which therefore are *only the local integrals of motion*, can be expressed in terms of the pseudoenergy $\varepsilon(\theta)$, that is the solution of the TBA equation (3.2.1). More precisely,

$$B_n\mathbb{I}_{2n-1}=(-1)^n\int_{-\infty}^{\infty}\frac{d\theta}{\pi}e^{(2n-1)\theta}\log(1+e^{-\epsilon(\theta)}) \tag{3.5.2}$$

$$B_n=\frac{\Gamma\big(\frac{(2n-1)b}{2Q}\big)\Gamma\big(\frac{2n-1}{2bQ}\big)}{2\sqrt{\pi}n!Q}\left[\frac{m\Gamma\big(\frac{b}{2Q}\big)\Gamma\big(\frac{1}{2Qb}\big)}{8Q\sqrt{\pi}}\right]^{1-2n} \tag{3.5.3}$$

In fact, this entails the same expression (3.4.1) for $X(\theta)$.

$$\log X(\theta) = \log Q(\theta) = -\frac{mR}{4\sin\frac{\pi b}{Q}} e^{\theta} + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\log(1+e^{-\epsilon(\theta)})}{\cosh(\theta-\theta')} \tag{3.5.4}$$

Al. Zamolodchikov[20] defined[15] the T function $T(\theta)$ and its dual $\tilde{T}(\theta)$ via the TQ relations

$$\boxed{\begin{aligned} T(\theta) &= \frac{X(\theta+i\pi p)+X(\theta-i\pi p)}{X(\theta)} \\ \tilde{T}(\theta) &= \frac{X(\theta+i\pi(1-p))+X(\theta-i\pi(1-p))}{X(\theta)} \end{aligned}} \qquad \begin{aligned} (3.5.5) \\ (3.5.6) \end{aligned}$$

he proved them also to be periodic

$$T(\theta + i\pi(1-p)) = T(\theta) \tag{3.5.7}$$

$$\tilde{T}(\theta + i\pi p) = \tilde{T}(\theta) \tag{3.5.8}$$

we note also that $p = b/Q$ and $1-p = 1/bQ$.
We discuss asymptotic behavior for $X(\theta)$ and $T(\theta)$ at $\Re\theta \to +\infty$. As in,[20] consider the leading behaviour of $X(\theta)$ in the strip $|\Im\theta| < \pi/2$

$$X(\theta) \sim e^{-B_0 e^{\theta}} \qquad B_0 = \frac{mR}{4\sin\pi p} \tag{3.5.9}$$

To get, from the TQ relations, the asymptotic behaviour of T, assume θ and p are such that we can also write

$$X(\theta \pm i\pi p) \sim e^{-B_0 e^{\pm i\pi p}}$$

then

$$\begin{aligned} \frac{X(\theta\pm i\pi p)}{X(\theta)} &\sim e^{B_0 e^{\theta} - B_0 e^{\theta\pm i\pi p}} = \exp\left\{\frac{mR}{4\sin\pi p} e^{\theta}(1-\cos\pi p \mp i\sin\pi p)\right\} \\ &= \exp\left\{\frac{mR}{8\sin\frac{\pi p}{2}\cos\frac{\pi p}{2}} e^{\theta}\left(1 - 2\cos^2\frac{\pi p}{2} + 1 \mp 2i\sin\frac{\pi p}{2}\cos\frac{\pi p}{2}\right)\right\} \\ &= \exp\left\{\frac{mR}{4\sin\frac{\pi p}{2}\cos\frac{\pi p}{2}} e^{\theta}\left(\sin^2\frac{\pi p}{2} \mp i\sin\frac{\pi p}{2}\cos\frac{\pi p}{2}\right)\right\} \\ &= \exp\left\{\frac{mR}{4\cos\frac{\pi p}{2}} e^{\theta}\left(\sin\frac{\pi p}{2} \mp i\cos\frac{\pi p}{2}\right)\right\} \\ &= \exp\left\{\frac{mR}{4}\tan\frac{\pi p}{2} e^{\theta} \mp i\frac{mR}{4} e^{\theta}\right\} \end{aligned}$$

From the TQ relation, we can now calculate the asymptotic behaviour of $T(\theta)$, at least on the Stokes line $\Im\theta = 0$, where neither of the exponentials of the sum prevails

$$\boxed{T(\theta) \sim 2\exp\left(\frac{mR}{4}\tan(\pi p/2)e^{\theta}\right)\cos\left(\frac{mR}{4}e^{\theta}\right) \qquad \Re\theta \to +\infty \qquad \Im\theta = 0} \tag{3.5.10}$$

Now, if we are not on the real axis, only one of the two addenda of the TQ relation will prevail, depending on the θ-Stokes sector we are considering. For simplicity, consider the self-case $b = 1$

[15] Actually he proved the subsequent relations for X considering this quantity as the continuum limit of the solution of the Hirota difference equation[20]

$(p = \frac{1}{2})$

$$
\begin{aligned}
T(\theta) &\sim \exp\Big(\frac{mR}{4}(1+i)e^{\theta}\Big) + \exp\Big(\frac{mR}{4}(1-i)e^{\theta}\Big) \\
&= \exp\Big(\frac{mR}{4}\sqrt{2}e^{\theta+i\pi/4}\Big) + \exp\Big(\frac{mR}{4}\sqrt{2}e^{\theta-i\pi/4}\Big) \\
&\sim \begin{cases} \exp\Big(\frac{mR}{4}\sqrt{2}e^{\theta-i\pi/4}\Big) & 0 < \Im\theta < \frac{\pi}{2} \\ \exp\Big(\frac{mR}{4}\sqrt{2}e^{\theta+i\pi/4}\Big) & -\frac{\pi}{2} < \Im\theta < 0 \end{cases}
\end{aligned}
$$

When b is not 1, by a similar argument delivers the result[20]

$$
\boxed{
\begin{aligned}
&T(\theta) \sim \exp\Big\{\frac{mR\exp(\theta - i\pi(1-p)/2)}{4\cos(\pi p/2)}\Big\} \qquad 0 < \Im\theta < \pi(1+a)/2 \\
&\tilde{T}(\theta) \sim \exp\Big\{\frac{mR\exp(\theta - i\pi p/2)}{4\sin(\pi p/2)}\Big\} \qquad 0 < \Im\theta < \pi(1-a)/2
\end{aligned}
}
\qquad
\begin{aligned}
(3.5.11) \\
(3.5.12)
\end{aligned}
$$

Part II

Gelfand-Dikii differential polynomials

4 General markovian large energy expansion

4.1 Modified Schrödinger equation

Consider the following differential equation, which is sometimes called in physics *modified Schrödinger equation*

$$\boxed{\left[-\frac{d^2}{dz^2}+u(z)+\Lambda p(z)\right]\psi(z;\Lambda)=0} \tag{4.1.1}$$

where Λ is a complex spectral parameter and $p(z)$ and $u(s)$ are "sufficiently smooth" functions. Historically, Liouville[23] discussed the asymptotic behaviour of its solutions as $\Lambda \to \infty$. To our purposes the importance of the *modified Schrödinger equation* (4.1.1) is that permits a *better set up for the ODE/IM correspondence.*

4.1.1 Bäcklund's Schrödinger form

Following Bäcklund,[11] we change variable as

$$dw=\sqrt{p(z)}dz \qquad w(z)=\int^z dz'\sqrt{p(z')} \tag{4.1.2}$$

$$\tag{4.1.3}$$

This change for the variable $z \to w$ has the effect of "separating" the spectral parameter Λ from $p(z)$

$$-\frac{d^2}{dw^2}\psi-\frac{1}{2}\frac{p'}{p^{3/2}}\frac{d}{dw}\psi+\left[\frac{u}{p}+\Lambda\right]\psi=0 \tag{4.1.4}$$

where we use the prime for the derivative in z: $'=\frac{dp}{dz}$.
If we now eliminate the first derivative, by the usual Abel transformation[27] on the wave function solution,

$$\psi(w(z))=\exp\left\{-\frac{1}{2}\int^z(dz'\sqrt{p(z')})\frac{1}{2}\frac{p'}{p^{3/2}}\right\}\chi(w(z)) \tag{4.1.5}$$

so that the two solutions are related as

$$\boxed{\psi(z)=\frac{1}{\sqrt[4]{p(z)}}\chi(w(z))} \tag{4.1.6}$$

We finally arrive to the *equation apt for the $\Lambda \to \infty$ asymptotic expansion*,

$$\boxed{\left[-\frac{d^2}{dw^2}+U(w)\right]\chi=-\Lambda\chi} \tag{4.1.7}$$

even if the new *Bäcklund potential* is rather involved

$$\boxed{U(z)=U(w(z))=\frac{1}{p}\left(u+\frac{4pp''-5p'^2}{16p^2}\right)=\frac{u}{p}+\frac{1}{4}\frac{p''}{p^2}-\frac{5}{16}\frac{p'^2}{p^3}} \tag{4.1.8}$$

4.1.2 Application to ODE-IM equations

Our goal, now, is to put both our ODE-IM equations in this form, with

$$\boxed{\Lambda=e^{2\theta} \qquad K=e^{\theta}} \tag{4.1.9}$$

where θ is the TBA rapidity.
In order to put in this form the ODE-IM minimal models Schrödinger equation (2.1.1)

$$\left[-\frac{d^2}{dx^2}+\frac{l(l+1)}{x^2}+x^{2M}-E\right]\psi(x;E)=0 \tag{4.1.10}$$

it is convenient to define[16] a new independent variable z and a fictitious gauge parameter s as follows

$$x=Cz=K^{\frac{1}{M+1}}z \tag{4.1.11}$$

$$E=C^{2M}s^{2M}=K^{\frac{2M}{1+M}}s^{2M} \tag{4.1.12}$$

with $C=e^{\frac{\theta}{M+1}}$. We observe that if we take the limit $\Lambda\to\infty$, in order for the initial (2.1.1) energy E to be finite, it is necessary also that $s\to 0$.
In any case, the initial (2.1.1) ODE-IM equation, for$\beta^2>0$, gets transformed into the Lukyanov-Zamolodchikov equation[16]

$$\boxed{\left[-\frac{d^2}{dz^2}+\frac{l(l+1)}{z^2}+\Lambda(z^{2M}-s^{2M})\right]\psi(z)=0} \tag{4.1.13}$$

We observe[16] that, whether previously (2.1.8) the action of the Ω symmetry was

$$\Omega:\quad x\to\omega x\,,\qquad E\to\omega^{-2}E\,,\qquad l\to l\,,$$

in these new variables, it would act as (at least, under the assumption that the spectral parameter Λ were fixed)

$$\Omega:\quad z\to\omega z\,,\qquad s\to\omega^{-1/M}s\,,\qquad l\to l \tag{4.1.14}$$

If we define the "Bäcklund coefficients" as[16][17]

$$\boxed{\begin{aligned} &p(z)=z^{2M}-s^{2M} && (4.1.15)\\ &u(z)=\frac{l(l+1)}{z^2}=\frac{2|k|}{z^2}\qquad 2|k|=l+\frac{1}{2} && (4.1.16)\end{aligned}}$$

we see that Lukyanov-Zamolodchikov equation (4.1.13) is in the Liouville's form (4.1.1).
Hence, for the minimal models the Bäcklund potential U is

$$\boxed{U(z)=\frac{4|k|^2-\frac{1}{4}}{z^2(z^{2M}-s^{2M})}+\frac{1}{4}\frac{z^{2M}}{z^2(z^{2M}-s^{2M})^2}-\frac{5}{16}\frac{z^{4M}}{z^2(z^{2M}-s^{2M})^3}} \tag{4.1.17}$$

We anticipate that, below (8.5.4), we are going to consider also the Al. Zamolodchikov's Generalized Mathieu equation

$$\boxed{\left[-\frac{d^2}{dy^2}+e^{2\theta}(e^{y/b}+e^{-yb})+P^2\right]\psi(y;\theta)=0} \tag{4.1.18}$$

Actually, this form is dued to D. Fioravanti, and has the advantage that the correspondence with Liouville equation (4.1.1) is immediate, setting

$$\boxed{\begin{aligned} &p(y)=e^{y/b}+e^{-yb} && (4.1.19)\\ &u(y)=P^2=\text{const.} && (4.1.20)\end{aligned}}$$

[16] This observation may be useful for further generalizations.

[17] Using $2|k|=l+\frac{1}{2}$ is not the most convenient choice for the following calculations. A more convenient notation $\lambda_D=l+\frac{1}{2}$ would had been that of,[17] but, lest confusion with the totally other large energy expansion parameter λ, we chosed to follow[16] notation.

where the *real positive* parameters b and P^2 are the standard homonimous parameter of the Liouville integrable model,[30] as we are going to explain below.
Hence, for the general ($b>0$) Liouville case, the Bäcklund potential is

$$\boxed{U(y)=\frac{P^2}{e^{y/b}+e^{-yb}}+\frac{1}{4}\frac{\frac{1}{b^2}e^{y/b}+b^2e^{-yb}}{(e^{y/b}+e^{-yb})^2}-\frac{5}{16}\frac{\frac{1}{b^2}e^{2y/b}-2e^{y/b-yb}+b^2e^{-2yb}}{(e^{y/b}+e^{-yb})^3}} \tag{4.1.21}$$

An important special case is for the self dual Liouville model, for which $b=1$. In fact, now we get the standard *modified Mathieu equation*[26]

$$-\frac{d^2}{dy^2}\psi+(2e^{2\theta}\cosh y+P^2)\psi=0 \tag{4.1.22}$$

Instead, if we consider this equation on the *imaginary axis of* y, setting $z=-iy$, we obtain the standard *Mathieu equation*[26][18]

$$\frac{d^2}{dz^2}\psi+(2e^{2\theta}\cos z+P^2)\psi=0 \tag{4.1.23}$$

In particular, for the modified Mathieu case (real axis of y)

$$p(y)=2\cosh y \tag{4.1.24}$$

$$\begin{aligned}U(y)&=\frac{P^2}{2\cosh y}+\frac{1}{4}\frac{1}{2\cosh y}-\frac{5}{16}\frac{\sinh^2 y}{\cosh^3 y}\\&=\left(\frac{P^2}{2}-\frac{1}{32}\right)\frac{1}{\cosh y}+\frac{5}{32}\frac{1}{\cosh^3 y}\end{aligned} \tag{4.1.25}$$

and for the Mathieu equation (imaginary axis of y)

$$p(z)=-2\cos z \qquad u(z)=-P^2 \tag{4.1.26}$$

$$\begin{aligned}U(z)&=\frac{1}{2\cos z}\big(P^2+\frac{1}{4}+\frac{5}{16}\tan^2 z\big)\\&=\left(P^2-\frac{1}{16}\right)\frac{1}{2\cos z}+\frac{5}{4}\frac{1}{(2\cos z)^3}\end{aligned} \tag{4.1.27}$$

We observe that all the functional relations of the ODE-IM correspondence are constructed (see section 2) through wronskians among some solutions ψ_1 and ψ_2 of the modified Schrödinger equation (4.1.1). Instead, after the Bäcklund transformation, we don't consider ψ_1 and ψ_2 anymore, but just their Bäcklund transformed χ_1 and χ_2: it seems that calculating wronskians among χ_1 and χ_2 is not correct for constructing the ODE-IM functional relations. Nevertheless, it is an elementary property of wronskians, that if the relation between any solution ψ and its transformed χ is through a *unique* function c, for every solution, as in (4.1.6)

$$\psi_i(z)=c(z)\chi_i(z) \qquad i=1,2 \qquad c(z)=p^{-1/4}(z) \tag{4.1.28}$$

then also all wronskians differ by the the same multiplicative function cc'. In fact

$$\begin{aligned}W[\chi_1,\chi_2]&=\chi_1\chi_2'-\chi_1'\chi_2=p^{1/4}\big(\frac{1}{4}\frac{p'}{p^{3/4}}\big)\Big[\psi_1\psi_2'-\psi_1'\psi_2\Big]\\&=\frac{1}{4}\frac{p'}{\sqrt{p}}W[\psi_1,\psi_2]\end{aligned} \tag{4.1.29}$$

Therefore, the ODE-IM functional relations are "form invariant" under Bäcklund transformation, as they can be modified at most by a fixed function. In this sense, the ODE-IM construction can be considered "Bäcklund invariant".

[18] It is clear that this correspondence suggested to Zamolodchikov[2] the name *Generalized Mathieu equation* for his original equation (9.2.1). However, we permit to point out that, in order to respect the standard nomenclature,[26] the most correct name would had been "Generalized Modified Mathieu equation", even if it is perhaps too lenghty.

4.2 Riccati equation for the eikonal representation

The eigenfunction solution χ of the Bäcklund equation (4.1.7) can be written in the eikonal integral representation as

$$\boxed{\begin{aligned} \chi(w;\Lambda) &= \exp\left\{\int_w^{w_0} dw' S(w';\Lambda)\right\} = \exp\left\{\int_z^{\infty} dz' \sqrt{p(z')}\, S(z';\Lambda)\right\} \\ S(w;\Lambda) &= \frac{d}{dw}\log\chi(w) \end{aligned}} \qquad \begin{aligned}(4.2.1)\\(4.2.2)\end{aligned}$$

Note the choice of $z_0 = +\infty$ ($w_0 = w(z_0)$). Considering that the spectral parameter of (4.1.7) is $-\Lambda$ (therefore in $-e^{2\theta}$, from (4.1.9); note the minus sign), the logarithmic derivative $S(w)$ can be asymptotically expanded, for $|\Lambda| \to \infty$ as[23]

$$S(z;\Lambda) \simeq \sqrt{-\Lambda} + \sum_{n=1}^{\infty} \frac{S_n(z)}{\sqrt{-\Lambda}^{\,n}} \qquad (S_0 = 0) \qquad \Lambda \to \infty \tag{4.2.3}$$

For the large energy limit $\theta \to +\infty$ it may be also convenient to define the small parameter

$$\boxed{\epsilon = \frac{1}{\sqrt{-\Lambda}} \qquad \epsilon \to 0} \tag{4.2.4}$$

so that the expansion of $S(w)$ reads

$$S(z;\Lambda) \sim \frac{1}{\epsilon} + \sum_{n=1}^{\infty} S_n(z)\epsilon^n \qquad (S_0 = 0) \qquad \epsilon \to 0 \tag{4.2.5}$$

Whatever total derivative appears in the integrand $S(z)$, we will discard it, because we intend to integrate it over the entire "space" or period. In particular, the space we will consider is:

- the real positive line of x ($0 < x < +\infty$) for equation (2.1.1)
- the entire real line of y ($-\infty < y < \infty$) for the Generalized Mathieu equation (8.5.4)
- the interval $s < z < \infty$ for the Lukyanov Zamolodhickov equation (4.1.13)

Explanations of this choices will be given below.
As is known,[27] a solution $\phi(x)$ of the general second order linear equation

$$L(\phi) = \phi'' + a_1(x)\phi' + a_2(x)\phi = 0 \tag{4.2.6}$$

can be expressed in "eikonal integral form" as

$$\phi(x) = \exp\int^x dt\, p(t) \tag{4.2.7}$$

It can be shown that, ϕ is a solution of $L(\phi) = 0$, if and only if, p satisfies the first order non-linear equation

$$p' = -p^2 - a_1(x)p - a_2(x) \tag{4.2.8}$$

which is called a *Riccati equation*.
With our variables the Riccati equation reads

$$\boxed{\frac{d}{dw}S(w) = -S^2(w) + U(w) + \Lambda} \tag{4.2.9}$$

Of course, the Riccati equation has two functional independent *dominant* and *subdominant* solutions S^+, S^-, with corresponding Bäcklund solutions χ^+ and χ^-

$$S^+(x') \simeq \sqrt{-\Lambda} + \sum_{n=1}^{\infty} \frac{S_n^+}{\sqrt{-\Lambda}^n} \qquad \Lambda \to \infty \tag{4.2.10}$$

$$S^-(x') \simeq -\sqrt{-\Lambda} + \sum_{n=1}^{\infty} \frac{S_n^-}{\sqrt{-\Lambda}^n} \qquad \Lambda \to \infty \tag{4.2.11}$$

which we expanded in terms of their modes $S_n^{\pm}$. Substituting these expansions in the Riccati equation (4.2.9) we get the standard *recursion relation for the large energy expansion*[23] [19]

$$S_{n+1}^{\pm} = \mp\frac{1}{2}\left(S_n'^{\pm} + \sum_{m=1}^{n-1} S_m^{\pm} S_{n-m}^{\pm}\right) \tag{4.2.12}$$

with initial condition established by the leading asymptotic form of the Riccati equation (4.2.9)

$$S_{-1}^{\pm} = \pm 1 \tag{4.2.13}$$

We write here the first examples and refer to appendix A for further examples

$$S_1^{\pm} = \pm\frac{1}{2}U \tag{4.2.14}$$

$$S_2^{\pm} = \mp\frac{1}{2}S_1'^{\pm} = -\frac{1}{4}U' \tag{4.2.15}$$

$$S_3^{\pm} = \mp\frac{1}{2}\left(S_2'^{\ \pm} + S_1^{\pm 2}\right) = \pm\frac{1}{8}(U'' - U^2) \tag{4.2.16}$$

In appendix A, we also (trivially) prove that, comparing the dominant S^+ and subdominant S^- solution, the even modes have the same sign, while the odd modes have the opposite sign

$$\begin{aligned} S_{2n}^+ &= S_{2n}^- \qquad n \in \mathbb{N} \\ S_{2n+1}^+ &= -S_{2n+1}^- \end{aligned} \tag{4.2.17}$$

4.3 Gelfand-Dikii recursion relation

In this subsection we shall follow mainly the, yet unpublished, Fioravanti's and Fachechi's article.[18] Let us split the generic eikonal in an even and an odd part

$$S(w) = S_{\text{even}}(w) + S_{\text{odd}}(w) \tag{4.3.1}$$

$$S_{\text{odd}}(w) = \sqrt{-\Lambda}S_{-1} + \sum_{n=1}^{\infty} \frac{S_{2n-1}(w)}{\sqrt{-\Lambda}^{2n-1}} \tag{4.3.2}$$

$$S_{\text{even}}(w) = \sum_{n=1}^{\infty} \frac{S_{2n}(w)}{\sqrt{-\Lambda}^{2n}} \tag{4.3.3}$$

with the understanding that $S_{-1} = 1$ and $S_0 = 0$.
The correspondence to the previous split in the dominant and subdominant fundamental pair S^+, S^- is established by (4.2.17) and is confirmed by[36]

$$S_{\text{even}} = \frac{1}{2}(S^+ + S^-) \qquad S_{\text{odd}} = \frac{1}{2}(S^+ - S^-) \tag{4.3.4}$$

[19]The Riccati equation (4.2.9) is just one; the two recursions appearing here correspond to the fact that, after substitution of the modes expansion, we divided the Riccati equation by $S_{-1}^{\pm} = \pm 1$.

The Riccati equation can also be splitted in two equation for the even and odd part in $\sqrt{\Lambda}^{-1}$. For the odd part this equation is nothing but (4.3.6)

$$S'_{\text{odd}} + 2S_{\text{odd}}S_{\text{even}} = 0 \tag{4.3.5}$$

which entails that *the even part S_{even} is a total derivative*

$$\boxed{S_{\text{even}} = -\frac{1}{2}\frac{S'_{\text{odd}}}{S_{\text{odd}}} = -\frac{1}{2}\frac{d}{dw}(\log S_{\text{odd}})} \tag{4.3.6}$$

As a consequence, *we can neglect it if we are to integrate over a period or over the entire space*. For the even part, instead we get

$$S'_{\text{even}} + S^2_{\text{odd}} + S^2_{\text{even}} = U + \Lambda \tag{4.3.7}$$

which becomes, eliminating S_{odd}

$$-2S_{\text{odd}}S''_{\text{odd}} + 3S'^2_{\text{odd}} + 4S^4_{\text{odd}} = 4(U+\Lambda)S^2_{\text{odd}}$$

We now can define the *Gelfand Dikii function*

$$\boxed{R = \frac{1}{2S_{\text{odd}}}} \tag{4.3.8}$$

It is clear that R admits an asymptotic expansion in terms $\sqrt{-\Lambda}$ with certain modes $\bar{R}_n$

$$R(w;\Lambda) = \sum_{n=0}^{\infty} \frac{R_n(w)}{\sqrt{-\Lambda}^{2n+1}} = \frac{1}{\sqrt{-\Lambda}} \sum_{n=0}^{\infty} \frac{R_n(w)}{(-\Lambda)^n} \tag{4.3.9}$$

with initial condition $R_0 = 1/2$. In order to match the conventions of[16] (but not of the original[19]), for which the initial condition is 1, we define the alternative modes $\bar{R}_n = 2R_n$, such that $\bar{R}_0 = 1$[20]. However, the recursive equations for the modes we are going to find are the very same. Since the only difference is only the initial condition, in the following treatment, we shall drop the bar and use R_n for what should be $\bar{R}_n$. In general when we will expand at large energy we will use this convention, to match our calculations with the article of Lukyanov and A. Zamolodhikov.

The equivalent equation for R is then

$$2R''R - R'^2 - 4(U+\Lambda)R^2 + 1 = 0$$

However, it is more convenient to consider the derivative equation

$$R''' - 4(U+\Lambda)R' - 2U'R = 0 \tag{4.3.10}$$

which, however, when integrated, introduce an inconvenient arbitrary constant. We will follow the *physical prescription that the resulting function $R(w)$ must tends to zero when the independent variable tends to infinity.*, which is also consistent with the limit of $U(w)$.

$$\lim_{w\to\infty} R_n(w) = 0 \qquad \Longleftrightarrow \qquad \lim_{w\to\infty} U(w) = 0 \tag{4.3.11}$$

Expanding equation (4.3.10) in the modes R_n and integrating we finally get the Gelfand-Dikii recursion equation[18]

$$\boxed{R_{n+1}(w) = -\frac{1}{4}\frac{d^2}{dw^2}R_n + U(w)R_n(w) - \frac{1}{2}\int^w dw' \frac{dU}{dw'} R_n(w')} \tag{4.3.12}$$

[20]This different convention explains the extra divisor 2 in formula (3.49) of,[16] with respect to (4.4.1)

with the initial condition $R_0 = 1$ In derivative form[16]

$$\boxed{\frac{dR_{n+1}}{dw} = -\frac{1}{4}\frac{d^3}{dw^3}R_n + U\frac{dR_n}{dw} + \frac{1}{2}\frac{dU}{dw}R_n} \tag{4.3.13}$$

Instead of the differential equation (4.3.12) we can equivalently obtain the Gelfand Dikii polynomials recursively by the operator $\hat{\Lambda}$ defined in[16] as follows

$$R_n = \hat{\Lambda}^n \cdot \hat{1} \qquad \hat{\Lambda} = -\frac{1}{4}\partial^2 + \hat{U} - \frac{1}{2}\partial^{-1}\hat{U}' \tag{4.3.14}$$

The first Gelfand-Dikii polynomial are[19]

$$R_0[U] = 1 \tag{4.3.15}$$
$$R_1[U] = \frac{1}{2}U \tag{4.3.16}$$
$$R_2[U] = \frac{3}{8}U^2 - \frac{1}{8}U'' \tag{4.3.17}$$
$$R_3[U] = \frac{5}{16}U^3 - \frac{5}{32}U'^2 - \frac{5}{16}U''U + \frac{1}{32}U^{iv} \tag{4.3.18}$$
$$\begin{aligned} R_4[U] = &\frac{35}{128}U^4 - \frac{35}{64}UU^{'2} - \frac{35}{64}U^2U'' + \frac{21}{128}U^{''2} + \frac{14}{64}U'U''' + \frac{7}{64}UU^{(4)} \\ &- \frac{1}{128}U^{(6)} \end{aligned} \tag{4.3.19}$$
$$\dots \tag{4.3.20}$$

where the prime indicates the derivative with respect to w. The Gellfand-Dikii polynomials are *differential polynomials in the functional argument* U, that is, they are polynomials in the function U and in its derivatives, in all possible combinations[21].[19]
The leading term in U is[16]

$$R_n[U] = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}n!}U^n + O(U^{n-1}) \qquad |U| \to +\infty \tag{4.3.21}$$
$$\tag{4.3.22}$$

as we prove in appendix A
For later developments, it is important to observe that the Gelfand-Dikii recursion equation, written in the variables z of the Liouville equation (4.1.1) is

$$\boxed{\begin{aligned} \frac{dR_{n+1}}{dz} = &-\frac{1}{4}\frac{1}{p}\frac{d^3}{dz^3}R_n + \frac{3}{8}\frac{p'}{p^2}\frac{d^2}{dz^2}R_n + \left[\frac{u}{p} + \frac{3}{8}\frac{p''}{p^2} - \frac{9}{16}\frac{p^{'2}}{p^3}\right]\frac{d}{dz}R_n \\ &+ \left[\frac{1}{2}\frac{u'}{p} - \frac{1}{2}\frac{up'}{p^2} + \frac{1}{8}\frac{p'''}{p^2} - \frac{9}{16}\frac{p''p'}{p^3} + \frac{15}{32}\frac{p^{'3}}{p^4}\right]R_n \end{aligned}} \tag{4.3.23}$$

Beware that here, and only here, the prime indicates the derivative with respect to z, not to w. This equation in the variable z will be far more useful for us.

4.4 Local integrals of motion by Gelfand Dikii polynomials

While the even terms S_{2n}, being total derivatives, give null contribution; the odd terms S_{2n-1} give nontrivial contribution. However, it turns out that[18] they differ from the R_n by a total derivative

[21] We use "combination" in its proper mathematical sense, i.e. not "dispositions", because U is a simple commutative function.

and a simple numerical factor. Therefore, under integration they can be sobstituted by the R_n as

$$\boxed{S_{2n-1} = -\frac{1}{2n-1}\bar{R}_n + \partial_z\,(\text{local fields}) \qquad n \geq 1} \tag{4.4.1}$$

This equation appears indirectly in;[16] however, until the unpublished[18] there was no proof of it. Fioravanti gave a rigorous proof of (4.4.1) in;[18] we write a *strict analogue of Fioravanti's proof for the WKB case* in subsection 6.3. This equivalence is perhaps interesting in itself, from a mathematical an computational point of view. However, from the integrability perspective, (4.4.1) implies the actual claim of Lukyanov and Zamolodchikov,[16] that is, that the *local integrals of motions can be calculated as*

$$\boxed{\tilde{B}_n I_{2n-1} = -\frac{1}{2n-1}\int_0^\infty dz\,\sqrt{p(z)}\bar{R}_n[U(z)]} \tag{4.4.2}$$

The proof of this important relation relies on the identification (2.4.9) between the wronskian (which is Q) and the solution of the ODE-IM equation calculated at a certain point. This point was $x \to 0$ in (2.4.9) for the equation (2.1.1); however, now it must be $z \to s$[16] for the equation (4.1.13). We are going to partially justify this choice "a priori" in subsection 5.2; while its best justification remains the check of the correct outcome (4.4.2) in subsection 5.3, "a posteriori". For now, we limit to follow a slight suggestion of.[16] [22]

$$\boxed{Q(E,l) \propto \lim_{z\to s}\chi(z) = \exp\left\{\int_s^\infty dz'\,\sqrt{p(z')}S(z')\right\}} \tag{4.4.3}$$

Thus, apart a normalization constant, from (1.6.7) and (2.4.7)

$$\boxed{\begin{aligned} Q(E,l) &\propto \exp\left\{\sum_{n=0}^\infty e^{\theta(1-2n)}\tilde{B}_n I_{2n-1}^{(\text{vac})}\right\} \\ &= \exp\left\{\sum_{n=0}^\infty \sqrt{E}^{\frac{1-2n}{1-\beta^2}}\int_s^\infty dz\,\sqrt{p(z)}S_{2n-1}\right\} \\ &= \exp\left\{\sum_{n=0}^\infty \sqrt{E}^{\frac{1-2n}{1-\beta^2}}\frac{-1}{2n-1}\int_s^\infty dz\,\sqrt{p(z)}R_n\right\} \end{aligned}} \tag{4.4.4}$$

where the last equality is a consequence of Fiovavanti's theorem (4.4.1);

4.5 The markovian property

If one compares the recursion (4.2.12) for the standard large-energy expansion modes S_n and the recursion (4.3.13) for the *equivalent* Gelfand Dikii polynomials R_n, one can immediately note the *vast convenience* of using the latter, rather than the former. In fact, the Gelfand Dikii recursion (4.3.13) is *markovian*, that is, *the next* $n+1$*-th step depends only on the preceding* n*-th step*; in particular, it is evident that, in order to calculate R_{n+1} from *it suffices to know only* R_n. Instead, in (4.2.12), to calculate the usual mode S_{n+1} *it is necessary to know all the preceding* S_k*, for* $k = 0, 1, ..., n$. However, we remind that Fioravanti's equivalence proof,[18] shows that the equivalence (4.4.1) holds only "modulo" total derivatives, that must be cancelled, for example through integration interval over a period or over the entire space.

[22] Lukyanov and Zamolodchikov, in,[16] indicated as integration contour the standard Mellin transform contour, taking (s,∞) as the cut for the non integer powers. Namely, the contour starts from s and goes toward $+\infty$ just above the real positive axis; then around a "big circle" in clockwise direction around infinity: then from $+\infty$ backward to s. This can be justified as (i) in the physical conformal limit $s \to 0$ this interval is approximately equal to $0 < z < \infty$; (ii) s is a singularity for the potential $U(z)$ (4.1.17) which must be avoided.

5 Local integrals of motion for the minimal models

5.1 Gelfand Dikii recursion relation for coefficients

We want to solve the Gelfand Dikii recursion equation

$$\boxed{\begin{aligned}\frac{dR_{n+1}}{dz} &= -\frac{1}{4}\frac{1}{p}\frac{d^3R_n}{dz^3} + \frac{3}{8}\frac{p'}{p^2}\frac{d^2R_n}{dz^2} + \Big(\frac{4|k|^2-\frac{1}{4}}{z^2p} + \frac{3}{8}\frac{p''}{p^2} - \frac{9}{16}\frac{p'^2}{p^3}\Big)\frac{dR_n}{dz} \\ &+ \Big[-\frac{1}{2}(4|k|^2-\frac{1}{4})\Big(\frac{1}{z^3p}+\frac{p'}{z^2p^2}\Big) + \frac{1}{8}\frac{p'''}{p^2} - \frac{9}{16}\frac{p''p'}{p^3} + \frac{15}{32}\frac{p'^3}{p^4}\Big]R_n\end{aligned}} \tag{5.1.1}$$

By direct inspection of R_1, R_2, R_3 we conjecture the following form for R_n

$$\boxed{R_n(z) = \frac{1}{z^{2n}}\sum_{m=n}^{3n} a_{nm}\frac{1}{p(z)^m}} \tag{5.1.2}$$

In other words, for each n we have expanded the polynomial R_n in the function basis $z^{-2n}p^{-m}$, for $m = n, n+1, ..., 3n$.

$$R_n(z) = \sum_{m=n}^{3n} a_{nm}z^{-2n}p^{-m} = \sum_{m=n}^{3n} a^n_m f_n(z)g_m(z) \tag{5.1.3}$$

To make things simple to control, we proceed following the generalized Leibniz rule

$$R_n^{(k)} = \sum_{m=n}^{3n} a_{nm}\sum_{l=0}^{k}\binom{k}{l}f_n^{(k-l)}g_m^{(l)}$$

We list the derivatives of f_n

$$\begin{aligned}f_n(z) &= z^{-2n} \\ f_n'(z) &= -2nz^{-2n}z^{-1} \\ f_n''(z) &= 2n(2n+1)z^{-2n}z^{-2} \\ f_n'''(z) &= -(2n)(2n+1)(2n+2)z^{-2n}z^{-3}\end{aligned}$$

It is convenient to define a varible ρ as

$$z\frac{p'}{p} = \rho = 2M\Big(1+\frac{s^{2M}}{p}\Big) \tag{5.1.4}$$

$$z^2\frac{p''}{p} = (2M-1)z\frac{p'}{p} \tag{5.1.5}$$

$$z^2\frac{p''}{p} = (2M-1)\rho \tag{5.1.6}$$

$$\begin{aligned}\frac{d}{dz}\rho &= \frac{p'}{p} + z\frac{p''}{p} - z\frac{p'^2}{p^2} \\ &= z^{-1}[\rho + (2M-1)\rho - \rho^2] \\ &= z^{-1}[(2M)\rho - \rho^2]\end{aligned}$$

After this definition, we can list the derivatives of g_n

$$
\begin{aligned}
g_m(z) &= p^{-m} \\
g'_m(z) &= -z^{-1} m\rho p^{-m} \\
g''_m(z) &= z^{-2}[m(m+1)\rho^2 - m(2M-1)\rho]p^{-m} \\
g'''_m(z) &= z^{-3}\Big[-m(m+1)(m+2)\rho^3 + 3m(m+1)(2M-1)\rho^2 - m(2M-1)(2M-2)\rho\Big]p^{-m}
\end{aligned}
$$

With in mind the Leibniz rule, we compute the derivatives of R_n

$$
\begin{aligned}
\frac{dR_n}{dz} &= z^{-2n-1}\sum_{m=n}^{3n} a_{nm}[-2n - m\rho]p^{-m} && (5.1.7) \\
\frac{d^2R_n}{dz^2} &= z^{-2n-2}\sum_{m=n}^{3n} a_{nm}[2n(2n+1) + 4nm\rho + m(m+1)\rho^2 - m(2M-1)\rho]p^{-m} \\
&= z^{-2n-2}\sum_{m=n}^{3n} a_{nm}[2n(2n+1) + (4n+1-2M)m\rho + m(m+1)\rho^2]p^{-m} && (5.1.8) \\
\frac{d^3R_n}{dz^3} &= z^{-2n-3}\sum_{m=n}^{3n} a_{nm}\{-2n(2n+1)(2n+2) - 6n(2n+1)m\rho - 6n[m(m+1)\rho^2 - m(2M-1)\rho] \\
&- m(m+1)(m+2)\rho^3 + 3m(m+1)(2M-1)\rho^2 - m(2M-1)(2M-2)\rho\Big\}p^{-m} && (5.1.9) \\
&= z^{-2n-3}\sum_{m=n}^{3n} a_{nm}\Big\{-2n(2n+1)(2n+2) - \Big[6nm(2n+2-2M) + m(2M-1)(2M-2)\Big]\rho \\
&- \Big[6nm(m+1) - 3m(m+1)(2M-1)\Big]\rho^2 - m(m+1)(m+2)\rho^3\Big\}p^{-m} && (5.1.10)
\end{aligned}
$$

We express also the potential U and its derivatives in terms of the new variable ρ.

$$
\begin{aligned}
U(z) &= \frac{1}{z^2p}\left[\Big(4|k|^2 - \frac{1}{4}\Big) + \frac{1}{4}(2M-1)\rho - \frac{5}{16}\rho^2\right] \\
U'(z) &= \frac{1}{z^3p}\left[\Big(-8|k|^2 + \frac{1}{2}\Big) - \frac{1}{2}(2M-1)\rho + \frac{5}{8}\rho^2 - \Big(4|k|^2 - \frac{1}{4}\Big)\rho - \frac{1}{4}(2M-1)\rho^2 + \frac{5}{16}\rho^3\right. \\
&\left.+ \frac{2M(2M-1)}{4}\rho - \frac{2M-1}{4}\rho^2 - \frac{5}{8}(2M)\rho^2 + \frac{5}{8}\rho^3\right] \\
&= \frac{1}{z^3p}\left[\Big(-8|k|^2 + \frac{1}{2}\Big) + \Big(-4|k|^2 + \frac{3}{4} - \frac{3}{2}M + M^2\Big)\rho + \Big(\frac{9}{8} - \frac{9}{4}M\Big)\rho^2 + \frac{15}{16}\rho^3\right]
\end{aligned}
$$

The Gelfand-Dikii equation in terms can now be written as

$$
\begin{aligned}
z^2pz\frac{dR_{n+1}}{dz} &= -\frac{1}{4}z^3\frac{d^3R_n}{dz^3} + \frac{3}{8}\rho z^2\frac{d^2R_n}{dz^2} + \Big(4|k|^2 - \frac{1}{4} + \frac{3}{8}(2M-1)\rho - \frac{9}{16}\rho^2\Big)z\frac{dR_n}{dz} \\
&+ \left[\Big(-4|k|^2 + \frac{1}{4}\Big) + \Big(-2|k|^2 + \frac{3}{8} - \frac{3}{4}M + \frac{1}{2}M^2\Big)\rho + \Big(\frac{9}{16} - \frac{9}{8}M\Big)\rho^2 + \frac{15}{32}\rho^3\right]R_n
\end{aligned}
\tag{5.1.11}
$$

The Geldand-Dikii recursion from the polynomials R_n and R_{n+1} is transmitted to their coefficients $a_{n,m}$, $a_{n+1,m}$

$$\sum_{m=n}^{3n+3} a_{n+1,m} p^{-m}(-2n-2-m\rho) = \sum_{m=n}^{3n} a_{n,m} p^{-m-1}\Big[r_0(n) + r_1(n,m)\rho + r_2(n,m)\rho^2 + r_3(m)\rho^3\Big] \tag{5.1.12}$$

where we defined r_i as the coefficient of the i-th power of ρ.

$$\begin{aligned}
r_0(n) &= 2n^3 + 3n^2 + \Big(\frac{3}{2} - 2\lambda^2\Big)n + \frac{1}{4} - 4|k^2| \\
r_1(n,m) &= 3n^2 m + \frac{3}{2}n^2 + 3(-M+1)nm + \Big(M^2 - \frac{3}{2}M + \frac{3}{4} - 4|k^2|\Big)m + \Big(-\frac{3}{2}M + \frac{3}{2}\Big)n \\
&\quad + \Big(\frac{1}{2}M^2 - \frac{3}{4}M + \frac{3}{8} - 2|k^2|\Big) \\
r_2(n,m) &= \frac{3}{2}nm^2 + \Big(-\frac{3}{2}M + \frac{3}{4}\Big)m^2 + 3nm + \Big(-3M + \frac{3}{2}\Big)m + \frac{9}{8}n - \frac{9}{8}M + \frac{9}{16} \\
r_3(m) &= \frac{1}{4}m^3 + \frac{9}{8}m^2 + \frac{23}{16}m + \frac{15}{32}
\end{aligned}$$

The powers of ρ we need are

$$\begin{aligned}
\rho &= 2M + 2Ms^{2M}\frac{1}{p} \\
\rho^2 &= 4M^2 + 8M^2 s^{2M}\frac{1}{p} + 4M^2 s^{4M}\frac{1}{p^2} \\
\rho^3 &= 8M^3 + 24M^3 s^{2M}\frac{1}{p} + 24M^3 s^{4M}\frac{1}{p^2} + 8M^3 s^{6M}\frac{1}{p^3}
\end{aligned}$$

The $n+1$-th side reads

$$\begin{aligned}
&\sum_{m=n+1}^{3n+3}\Big(-2n-2-2Mm-2Mms^{2M}\frac{1}{p}\Big)a_{n+1,m}p^{-m} \\
&= \sum_{m=n+2}^{3n+3}\Big[A_0(n,m)a_{n+1,m} + A_1(n,m-1)a_{n+1,m-1}\Big]p^{-m} \\
&\qquad + A_0(n,n+1)a_{n+1,n+1}\,p^{-n-1} + A_1(n,3n+3)\,p^{-3n-4}
\end{aligned}$$

with the important definitions

$$A_0(n,m) = A_0^n(m) = -2n-2-2Mm \tag{5.1.13}$$

$$A_1(n,m) = A_1^n(m) = -2Ms^{2M}m \tag{5.1.14}$$

The n-th side reads

$$\begin{aligned}
&\sum_{m=n}^{3n}\Big(B_1(n,m)\frac{1}{p} + B_2(n,m)\frac{1}{p^2} + B_3(n,m)\frac{1}{p^3} + B_4(n,m)\frac{1}{p^4})a_{n,m}p^{-m} \\
&= \sum_{m=n+4}^{3n+1}\Big(B_1(n,m-1)a_{n,m-1} + B_2(n,m-2)a_{n,m-2} + B_3(n,m-3)a_{n,m-3} + B_4(n,m-4)a_{n,m-4}\Big)p^{-m} \\
&+ B_1(n,1)a_{n1}p^{-n-1} + \Big(B_1(n,2)a_{n2} + B_2(n,1)a_{n1}\Big)p^{-n-2} + \Big(B_1(n,3)a_{n3} + B_2(n,2)a_{n2} + B_3(n,1)a_{n1}\Big)p^{-n-3} \\
&+ \Big(B_2(n,3n)a_{n,3n} + B_3(n,3n-1)a_{n,3n-1} + B_4(n,3n-2)a_{n,3n-2}\Big)p^{-3n-2} + \Big(B_3(n,3n)a_{n,3n} \\
&+ B_4(n,3n-1)a_{n,3n-1}\Big)p^{-3n-3} + B_4(n,3n)a_{n,3n}p^{-3n-4}
\end{aligned}$$

Were we defined

$$
\begin{aligned}
B_1(n,m) &= r_0(n) + 2Mr_1(n,m) + 4M^2 r_2(n,m) + 8M^3 r_3(m) \\
&= 2M^3m^3 + 2n^3 + 6M^2nm^2 + 6Mn^2m + (3M^3+3M^2)m^2 + (3M+3)n^2 + (6M^2 \\
&+ 6M)nm + (\frac{3}{2}M^3 + 3M^2 + \frac{3}{2}M - 8|k|^2M)m + (\frac{3}{2}M^2 + 3M + \frac{3}{2} - 8|k|^2)n + (\frac{1}{4}M^3 + \frac{3}{4}M^2 \\
&+ \frac{3}{4}M + \frac{1}{4} - 4|k|^2M - 4|k|^2) \\
B_2(n,m) &= 2Ms^{2M}r_1(n,m) + 8M^2s^{2M}r_2(n,m) + 24M^3s^{2M}r_3(m) \\
&= s^{2M}\Big[6M^3m^3 + 12M^2nm^2 + 6Mn^2m + (15M^3+6M^2)m^2 + 3Mn^2 + (18M^2 + \\
&+ 6M)nm + \Big(\frac{25}{2}M^3 + 9M^2 + \frac{3}{2}M - 8|k|^2\Big)m + (6M^2+3M)n + \frac{13}{4}M^3 + 3M^2 \\
&+ \frac{3}{4}M - 4|k|^2M\Big] \\
B_3(n,m) &= 4M^2s^{4M}r_2(n,m) + 24M^3s^{4M}r_3(m) \\
&= s^{4M}\Big[6M^3m^3 + 6M^2nm^2 + (21M^3+3M^2)m^2 + 12M^2nm + (\frac{45}{2}M^3 + 6M^2)m \\
&+ \frac{9}{2}M^2n + \frac{27}{4}M^3 + \frac{9}{4}M^2\Big] \\
B_4(m) &= 8M^3s^{6M}r_3(m) \\
&= s^{6M}[2M^3m^3 + 9M^3m^2 + \frac{23}{2}M^3m + \frac{15}{4}M^3]
\end{aligned}
$$

The "core" equation is

$$
\begin{aligned}
&\sum_{m=n+4}^{3n+1}\Big[A_0(n+1,m)a_{n+1,m} + A_1(n+1,m-1)a_{n+1,m-1}\Big]p^{-m} \\
&= \sum_{m=n+4}^{3n+1}\Big(B_1(n,m-1)a_{n,m-1} + B_2(n,m-2)a_{n,m-2} + B_3(n,m-3)a_{n,m-3} + B_4(n,m-4)a_{n,m-4}\Big)p^{-m}
\end{aligned}
\tag{5.1.15}
$$

$$(5.1.16)$$

while for the "extremal" terms the following sub-equations of the previous main equation hold

$$
\begin{aligned}
A_0(n,n)a_{n+1,n}p^{-n} &= 0 \\
\Big[A_0(n,n+1)a_{n+1,n+1} + A_1(n,n)a_{n+1,n}\Big]p^{-n-1} &= B_1(n,n+1)a_{n,n+1}p^{-n-1} \\
\Big[A_0(n,n+2)a_{n+1,n+2} + A_1(n,n+1)a_{n+1,n+1}\Big]p^{-n-2} &= \Big[B_1(n,n+2)a_{n,n+2} + B_2(n,n+1)a_{n,n+1}\Big]p^{-n-2} \\
\Big[A_0(n,n+3)a_{n+1,n+3} + A_1(n,n+2)a_{n+1,n+2}\Big]p^{-n-3} &= \Big[B_1(n,n+3)a_{n,n+3} + B_2(n,n+2)a_{n,n+2} \\
&+ B_3(n,n+1)a_{n,n+1}\Big]p^{-n-3}
\end{aligned}
$$

$$
\begin{aligned}
&\Big[A_0(n,3n+2)a_{n+1,3n+2} + A_1(n,3n+1)a_{n+1,3n+1}\Big]p^{-3n-2} = \\
&\quad = \Big[B_2(n,3n)a_{n,3n} + B_3(n,3n-1)a_{n,3n-1} + B_4(n,3n-2)a_{n,3n-2}\Big]p^{-3n-2} \\
&\Big[A_0(n,3n+3)a_{n+1,3n+3} + A_1(n,3n+2)a_{n+1,3n+2}\Big]p^{-3n-3} = \\
&\quad = \Big[B_3(n,3n)a_{n,3n} + B_4(n,3n-1)a_{n,3n-1}\Big]p^{-3n-3} \\
&\Big[A_1(n,3n+3)a_{n+1,3n+3}\Big]p^{-3n-4} = \Big[B_4(n,3n)a_{n,3n}\Big]p^{-3n-4}
\end{aligned}
$$

We note that this recursion relations for the coefficients does not give explicitly the *single* coefficient $a_{n+1,m}$ in terms of the coefficients $a_{n,k}$, for some ks, but give a *linear combination* of coefficients of the polynomial R_{n+1} in terms of the coefficients of R_n. For the $n+1$-th side we must therefore consider the upper triangular matrix A^n defined as

$$A^n = \begin{bmatrix} A_1^n(n) & A_0^n(n+1) & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & A_1^n(n+1) & A_0^n(n+2) & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & A_1^n(m) & A_0^n(m+1) & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & A_1^n(m+1) & \dots & 0 & 0 \\ \vdots & \dots & & & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & A_1^n(3n+2) & A_0^n(3n+3) \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & A_1^n(3n+3) \end{bmatrix} \tag{5.1.17}$$

The first line correspond to an equation which would be identically satisfied when multiplied by $a_{n,m}$, it is not linearly independent. We have deleted it so as to be able to define a determinant, which also does not vanish

$$\begin{aligned} \det A_n &= \prod_{m=n+1}^{3n+4} A_1(n, m-1) = \prod_{j=n}^{3n+3} A_1(n, j) \\ &= (-2Ms^{2M})^{2n+4} \frac{(3n+3)!}{(n-1)!} \end{aligned} \tag{5.1.18}$$

We want to use the Cramer method to solve the linear non homogeneous system generated by the coefficients recursion relation. We schematically write this system as

$$A^n a_{n+1} = b_k \tag{5.1.19}$$

where clearly by a_{n+1} we denote the vector of all the (a priori) non null components $a_{n+1,m}$, $m = n+1, n+2, ...3n+3$ and by $b_k = b_k[a_n]$ we denote the functional of the coefficients of a_n established by the recursion relation (5.1.15)

Therefore we define the modified matrix of coefficients A_m^n whose determinant,divided by the determinant of A^n, gives us the coefficient $a_{n+1,m}$

$$A_m^n = \begin{bmatrix} A_1^n(n) & A_0^n(n+1) & 0 & \dots & b_1 & 0 & \dots & 0 & 0 \\ 0 & A_1^n(n+1) & A_0^n(n+2) & \dots & b_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & b_m & A_0^n(m+1) & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & b_{m+1} & A_1^n(m+1) & \dots & 0 & 0 \\ \vdots & \dots & & & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & b_{3n+2} & 0 & \dots & A_1^n(3n+2) & A_0^n(3n+3) \\ 0 & 0 & 0 & \dots & b_{3n+3} & 0 & \dots & 0 & A_1^n(3n+3) \end{bmatrix} \tag{5.1.20}$$

$$\det A_n^m = \prod_{j=n}^{m-1} A_1(n,j) \sum_{k=m}^{3n+3} (-1)^{k-m} b_k \Big[A_0(n,m+1)\cdots A_0(n,k) A_1(n,k+1)\cdots$$
$$\cdots A_1(3n+3)\Big]$$
$$a_{n+1,m} = \frac{\sum_{k=m}^{3n+3}(-1)^{k-m} b_k \Big[A_0(n,m+1)\cdots A_0(n,k) A_1(n,k+1)\cdots A_1(3n+3)\Big]}{\prod_{j=m}^{3n+3} A_1(n,j)}$$

This kind of finite product is simply a Pochhammer symbol (ascending factorial) or descending factorial.

$$\prod_{j=1}^{N}(1+cj) = c^N \prod_{j=1}^{N}\Big(\frac{1}{c}+j\Big)$$
$$= c^N \frac{\Gamma(\frac{1}{c}+1+N)}{\Gamma(\frac{1}{c}+1)}$$
$$= c^N \frac{\frac{1}{c}+N}{\frac{1}{c}} \frac{\Gamma(\frac{1}{c}+N)}{\Gamma(\frac{1}{c})} = c^N(1+cN)\frac{\Gamma(\frac{1}{c}+N)}{\Gamma(\frac{1}{c})}$$

In our case

$$\prod_{i=m+1}^{k} A_0(n,i) = \prod_{i=m+1}^{k}(-2n-2-2Mi)$$
$$= (-2M)^{k-m} \prod_{i=m+1}^{k}\Big(\frac{n+1}{M}+i\Big)$$
$$= (-2M)^{k-m} \frac{\Gamma(\frac{n+1}{M}+1+k)}{\Gamma(\frac{n+1}{M}+1+m)}$$

The other product is trivially reduced to elementary factorials

$$\prod_{i=k}^{l} A_1(n,i) = (-2Ms^{2M})^{l-k+1} \frac{l!}{(k-1)!}$$

We can finally write the expression for the $a_{n+1,m}$ coefficient

$$a_{n+1,m} = \frac{\sum_{k=m}^{3n+3}(-1)^{k-m} b_k \Big[(-2M)^{k-m}\frac{\Gamma(\frac{n+1}{M}+1+k)}{\Gamma(\frac{n+1}{M}+1+m)}(-2Ms^{2M})^{3n+3-k}\frac{(3n+3)!}{k!}\Big]}{(-2Ms^{2M})^{3n+4-m}\frac{(3n+3)!}{(m-1)!}}$$
$$= -\frac{1}{2M}\sum_{k=m}^{3n+3}(-1)^{k-m}(s^{2M})^{m-k-1}\frac{(m-1)!}{k!}\frac{\Gamma(\frac{n+1}{M}+1+k)}{\Gamma(\frac{n+1}{M}+1+m)} b_k$$
$$= -\frac{1}{2M}\sum_{k=m}^{3n+3}(-1)^{k-m}(s^{2M})^{m-k-1}\frac{(m-1)!}{k!}\frac{\Gamma(\frac{n+1}{M}+1+k)}{\Gamma(\frac{n+1}{M}+1+m)}\Big[B_1(n,k)a_{n,k} + B_2(n,k-1)a_{n,k-1} \tag{5.1.21}$$
$$+ B_3(n,k-2)a_{n,k-2} + B_4(n,k-3)a_{n,k-3}\Big] \tag{5.1.22}$$

where the $B_{i+1}(n,k-i)$ are polynomials up to the third degree in k and n.
We write the final formula more symmetrically, shifting $m \to m+1$

$$a_{n+1,m+1} = \frac{(-1)^m m! s^{2Mm}}{2M}\sum_{k=m+1}^{3n+3}\frac{(-1)^k}{k!}(s^{2M})^{-k}\frac{\Gamma(\frac{n+1}{M}+k+1)}{\Gamma(\frac{n+1}{M}+m+2)}\times \tag{5.1.23}$$
$$\times\Big[B_1(n,k)a_{n,k} + B_2(n,k-1)a_{n,k-1} + B_3(n,k-2)a_{n,k-2} + B_4(n,k-3)a_{n,k-3}\Big] \tag{5.1.24}$$

Define $\hat{B}_{k-1}(n,m) = s^{-2M(k-1)}B_k(n,m)$. We reorder the terms.

$$\begin{aligned}
\hat{B}_0(n,m) &= (2m^3+3m^2+\frac{3}{2}m+\frac{1}{4})M^3+(6nm^2+3m^2+6nm+3m+\frac{3}{2}n+\frac{3}{4})M^2\\
&\quad+(6n^2m+3n^2+6nm+\frac{3}{2}m+3n+\frac{3}{4})M-(2m+1)4|k|^2M-(2n+1)4|k|^2\\
&\quad+(2n^3+3n^2+\frac{3}{2}n+\frac{1}{4}) && (5.1.25)\\
\hat{B}_1(n,m) &= (6m^3+15m^2+\frac{25}{2}m+\frac{13}{4})M^3+(12nm^2+6m^2+18nm+9m+6n+3)M^2\\
&\quad+(6n^2m+3n^2+6nm+\frac{3}{2}m+3n+\frac{3}{4})M-(2m+1)4|k|^2M && (5.1.26)\\
\hat{B}_2(n,m) &= (6m^3+21m^2+\frac{45}{2}m+\frac{27}{4})M^3+(6nm^2+3m^2+12nm+6m+\frac{9}{2}n+\frac{9}{4})M^2 && (5.1.27)\\
\hat{B}_3(m) &= (2m^3+9m^2+\frac{23}{2}m+\frac{15}{4})M^3 && (5.1.28)\\
& && (5.1.29)
\end{aligned}$$

It can proved by induction that

$$\boxed{a_{n,m} = s^{2M(m-n)}\hat{a}_{n,m} \propto s^{2M(m-n)}} \tag{5.1.30}$$

where the $\hat{a}_{n,m}$ are s-indendent. We write the s-independent final formula as

$$\boxed{\begin{aligned}\hat{a}_{n+1,m+1} = \frac{(-1)^m m!}{2M}\sum_{k=m+1}^{3n+3}\frac{(-1)^k}{k!}\frac{\Gamma(\frac{n+1}{M}+k+1)}{\Gamma(\frac{n+1}{M}+m+2)}\times\times\Big[\hat{B}_0(n,k)\hat{a}_{n,k}+\hat{B}_1(n,k-1)\hat{a}_{n,k-1}\\
+\hat{B}_2(n,k-2)a_{n,k-2}+\hat{B}_3(n,k-3)\hat{a}_{n,k-3}\Big]\end{aligned}} \tag{5.1.31}$$

We rewrite this expression isolating $a_{n,k}$ in the sum

$$\frac{(-1)^{m-1}2M\hat{a}_{n+1,m}}{(m-1)!} = \sum_{k=m-3}^{3n+3}\sum_{l=k}^{k+3}\frac{(-1)^l}{l!}\Big(\frac{n+1}{M}+m+1\Big)^{l-m}B^n_{l-k+1}(k)\hat{a}_{n,k} \tag{5.1.32}$$

or

$$\frac{\Gamma(\frac{n+1}{M}+m+1)(-1)^m(-2Mm)\hat{a}_{n+1,m}}{\Gamma(m+1)} = \sum_{k=m-3}^{3n+3}\sum_{l=k}^{k+3}(-1)^l\frac{\Gamma(\frac{n+1}{M}+l+1)}{\Gamma(l+1)}B^n_{l-k+1}(k)\hat{a}_{n,k} \tag{5.1.33}$$

$$\tag{5.1.34}$$

5.1.1 Gelfand Dikki coefficients test

We now want to control that our recursive procedure (5.1.31) is correct.

We list the first Gelfand Dikii polynomials

$$
\begin{aligned}
R_1[U] &= \frac{1}{2}U \\
R_2[U] &= \frac{3}{8}U^2 - \frac{1}{8}U'' \\
R_3[U] &= \frac{5}{16}U^3 - \frac{5}{32}U'^2 - \frac{5}{16}U''U + \frac{1}{32}U^{(4)} && (5.1.35) \\
R_4[U] &= \frac{35}{128}U^4 - \frac{35}{64}UU'^2 - \frac{35}{64}U^2U'' + \frac{5}{32}U''^2 + \frac{15}{64}U'U''' + \frac{7}{64}UU^{(4)} \\
&+ \frac{1}{64}U^{(4)2} - \frac{1}{128}U^{(6)} && (5.1.36)
\end{aligned}
$$

The initial condition is just the number 1

$$R_0 = 1 \tag{5.1.37}$$

$$a_{00} = 1 \tag{5.1.38}$$

We calculate the coefficients of R_1 immediately by halving the coefficient of U

$$U(z) = \frac{1}{z^2}\left[\left(-\frac{1}{4}M^2 - \frac{1}{2}M - \frac{1}{4} + 4|k|^2\right)\frac{1}{p} + \left(-\frac{3}{2}M^2 - \frac{1}{2}M\right)\frac{s^{2M}}{p^2} + \left(-\frac{5}{4}M^2\right)\frac{s^{4M}}{p^3}\right]$$

or

$$
\begin{aligned}
R_1(z) &= \frac{1}{2}U(z) && (5.1.39) \\
a_{11} &= -\frac{1}{8}M^2 - \frac{1}{4}M - \frac{1}{8} + 2|k|^2 && (5.1.40) \\
a_{12} &= s^{2M}\left(-\frac{3}{4}M^2 - \frac{1}{4}M\right) && (5.1.41) \\
a_{13} &= s^{4M}\left(-\frac{5}{8}M^2\right) && (5.1.42)
\end{aligned}
$$

To calculate R_2, we need U^2

$$
\begin{aligned}
U^2(z) = \frac{1}{z^4}\Bigg[&\left(\frac{1}{16}M^4 + \frac{1}{4}M^3 + \frac{3}{8}M^2 + \frac{1}{4}M + 16|k|^2 - 2|k|^2M^2 - M - 2|k|^2 + \frac{1}{16}\right)\frac{1}{p^2} \\
&+ \left(\frac{3}{4}M^4 + \frac{7}{4}M^3 + \frac{5}{4}M^2 + \frac{1}{4}M - 12|k|^2M^2 - 4|k|^2M\right)\frac{s^{2M}}{p^3} + \left(\frac{23}{8}M^4 + \frac{11}{4}M^3 + \frac{7}{8}M^2 \right.\\
&\left. - \frac{5}{2}4|k|^2M^2\right)\frac{s^{4M}}{p^4} + \left(\frac{15}{4}M^4 + \frac{5}{4}M^3\right)\frac{s^{6M}}{p^5} + \frac{25}{16}M^4\frac{s^{8M}}{p^6}\Bigg]
\end{aligned}
$$

the first derivative of U

$$
\begin{aligned}
\frac{dU}{dz} = \frac{1}{z^3}\Bigg[&\left(\frac{1}{2}M^3 + \frac{3}{2}M^2 + \frac{3}{2}M + \frac{1}{2} - 8|k|^2M - 8|k|^2\right)\frac{1}{p} \\
&+ \left(+\frac{13}{2}M^3 + 6M^2 + \frac{3}{2}M - 8|k|^2M\right)\frac{s^{2M}}{p^2} + \left(\frac{27}{2}M^3 + \frac{9}{2}M^2\right)\frac{s^{4M}}{p^3} + \frac{15}{2}M^3\frac{s^{6M}}{p^4}\Bigg]
\end{aligned}
$$

and the second derivative

$$
\begin{aligned}
\frac{d^2}{dz^2}U(z) = \frac{1}{z^4}\Bigg[&\left(-M^4 - \frac{9}{2}M^3 - \frac{15}{2}M^2 - \frac{11}{2}M - \frac{3}{2} + 16|k|^2M^2 + 40|k|^2M + 24|k|^2\right)\frac{1}{p} \\
&+ \left(-27M^4 - \frac{93}{2}M^3 - 27M^2 - \frac{11}{2}M + 48M^2|k|^2 + 40M|k|^2\right)\frac{s^{2M}}{p^2} + \left(-107M^4 \right.\\
&\left. - \frac{183}{2}M^3 - \frac{39}{2}M^2 + 32M^2|k|^2\right)\frac{s^{4M}}{p^3} + \left(-141M^4 - \frac{99}{2}M^3\right)\frac{s^{6M}}{p^4} - 60M^4\frac{s^{8M}}{p^5}\Bigg]
\end{aligned}
$$

Finally

$$R_2(w) = \frac{3}{8}U^2(w) - \frac{1}{8}\frac{d^2}{dw^2}U(w) = \frac{3}{8}U^2(z) - \frac{1}{8}\Big(\frac{1}{p}\frac{d^2}{dz^2}U(z) - \frac{1}{2}\frac{p'}{p^2}\frac{d}{dz}U(z)\Big) \tag{5.1.43}$$

$$a_{21} = 0 \tag{5.1.44}$$

$$a_{22} = \frac{27}{128}M^4 + \frac{27}{32}M^3 + \frac{81}{64}M^2 + \frac{27}{32}M + 6|k|^2 - \frac{15}{4}|k|^2M^2 - \frac{15}{2}|k|^2M - \frac{15}{4}|k|^2 + \frac{27}{128} \tag{5.1.45}$$

$$a_{23} = s^{2M}\Big[\frac{145}{32}M^4 + \frac{237}{32}M^3 + \frac{135}{32}M^2 + \frac{27}{32}M - \frac{25}{2}|k|^2M^2 - \frac{15}{2}|k|^2M\Big] \tag{5.1.46}$$

$$a_{24} = s^{4M}\Big[\frac{1085}{64}M^4 + \frac{441}{32}M^3 + \frac{189}{64}M^2 - \frac{35}{4}|k|^2M^2\Big] \tag{5.1.47}$$

$$a_{25} = s^{6M}\Big[\frac{693}{32}M^4 + \frac{231}{32}M^3\Big] \tag{5.1.48}$$

$$a_{26} = s^{8M}\Big(\frac{1155}{128}M^4\Big) \tag{5.1.49}$$

We checked the recursive procedure (5.1.31) for $R_0 \to R_1$ and $R_1 \to R_2$.

5.2 Gauge s-independence and basis for integrals

We recall that, at least for $M > 1$, Dorey and Tateo identified the Q function with the Stokes coefficient (wronskian) as in (2.4.5). Then, following Lukyanov and Zamolodchikov, not so explicit, suggestion,[16] we further identified the Q function with the Bäcklund eigenfunction calculated at $z \to s$, with the understanding that $s \to 0$ (4.4.3); so that Dorey and Tateo's rigorous identification (2.4.9) is somehow imitated. In this subsection, we try to give a more support to such identification, even if we will not be completely rigorous. In particular, we are going to charaterize the general s-dependence of the integrals of the functional basis $\mathcal{I}_{n,m}$ for the Gelfand Dikii polynomials $R_n(z;s)$ and we will show that the Q function can be written in a s-independent way. We write the general Gelfand Dikii polynomials $R_n(z;s)$

$$\int_s^\infty dz\,\sqrt{p(z)}R_n(z) = \sum_{m=n}^{3n} a_{n,m}\mathcal{I}_{n,m} \tag{5.2.1}$$

where $\mathcal{I}_{n,m}$ is the integral of the functional part of the m-th Gelfand-Dikii coefficient.

$$\boxed{\mathcal{I}_{nm} = \int_s^\infty dz\, z^{-2n}(z^{2M} - s^{2M})^{-m+1/2}} \tag{5.2.2}$$

To obtain the expansion of the Q function in terms of the local integrals of motion, such wronskian must be expanded in the energy parameter of ODE-IM equation (2.1.1), rather than in the spectral parameter $\lambda = e^{\theta(1-\beta^2)}$. The two choices are related as

$$\sqrt{E} = \frac{\lambda}{\nu} = e^{\theta(1-\beta^2)}\Big(\frac{2}{\beta^2}\Big)^{1-\beta^2}\Gamma(1-\beta^2) \qquad \text{(s=1)} \tag{5.2.3}$$

However, if this was true for the original Schrödinger equation (2.1.1), it is not necessarily true in the modified Schrödinger equation (4.1.13). In fact, Lukyanov and Zamolodhikov[16] set

$$\sqrt{E} = e^{\theta(1-\beta^2)}s^{\frac{1-\beta^2}{\beta^2}} \tag{5.2.4}$$

Therefore, we cannot write directly the expansion of Q (4.4.4), which is valid only if $s = 1$

$$\log Q \sim \sum_{n=0}^{\infty}\sqrt{E}^{\frac{(1-2n)}{1-\beta^2}}\sum_{m=n}^{3n} a_{n,m}\mathcal{I}_{n,m} \qquad E \to \infty \qquad \text{(s=1)} \tag{5.2.5}$$

we must use instead $\lambda = e^{\theta(1-\beta^2)}$ as expansion parameter and then use (5.2.4)

$$\boxed{\log Q \sim \sum_{n=0}^{\infty} e^{\theta(1-2n)} \sum_{m=n}^{3n} a_{n,m}\mathcal{I}_{n,m} \qquad \Re\theta \to +\infty} \tag{5.2.6}$$

We choosed to integrate z from s to $+\infty$, instead than from 0, wth the understanding that $s \to 0$. This choice is better justified "a posteriori", after the observation of the independence of the expansion from the fictitious gauge parameter s. Indeed, we now show that the coefficient of the n-th term of the expansion is s-independent. The general s -dependence Gelfand Dikii coefficient $a_{n,m}$ is (5.1.30)

$$a_{n,m} = s^{2M(m-n)}\hat{a}_{n,m} \tag{5.2.7}$$

where $\hat{a}_{n,m}$ are the *s-independent Gelfand-Dikii coefficients.*
We now calculate the general basis integral, by which we shall find also its s dependence

$$\begin{aligned}
\mathcal{I}_{nm} &= \int_s^{\infty} dz\, z^{-2n}(z^{2M} - s^{2M})^{-m+1/2} \qquad t = \frac{1}{z}\,, \quad dz = -\frac{dt}{t^2} \\
&= \int_0^{1/s} dt\, t^{2n-2}\Big(\frac{1 - s^{2M}t^{2M}}{t^{2M}}\Big)^{-m+1/2} \\
&= \int_0^{1/s} dt\, t^{2n-2+2Mm-M}\big(1-(st)^{2M}\big)^{-m+1/2} \qquad v = (ts)^{2M} \quad dt = \frac{1}{2Ms}v^{\frac{1}{2M}-1} \\
&= \frac{1}{2M}s^{-2n-2Mm+2+M-1}\int_0^1 dv\, v^{\frac{n}{M}-\frac{1}{M}+m-\frac{1}{2}+\frac{1}{2M}-1}(1-v)^{-m+\frac{1}{2}} \\
&= \frac{1}{2M}s^{-2n+1-2Mm+M}\int_0^1 dv\, v^{(\frac{2n-1}{2M}+m-\frac{1}{2})-1}(1-v)^{(-m+\frac{3}{2})-1} \\
&= \frac{1}{2M\, s^{2n-1+M(2m-1)}}B\Big(\frac{2n-1}{2M}+\frac{2m-1}{2}, \frac{-2m+3}{2}\Big)
\end{aligned}$$

where the Euler Beta function, in general, is to be defined through its analytic continuation to all possible complex values of the parameters, by means of the integration on the Pochhammer contour (rather than on the segment $[0,1]$).[26] In conclusion

$$\boxed{\mathcal{I}_{nm} = \frac{1}{2M\, s^{2n-1+M(2m-1)}}\frac{\Gamma(\frac{2n-1}{2M}+\frac{2m-1}{2})\Gamma(\frac{-2m+3}{2})}{\Gamma(\frac{2n-1}{2M}+1)}} \tag{5.2.8}$$

We can also define the *s-independent basic integrals* $\hat{\mathcal{I}}_{nm}$ by

$$\mathcal{I}_{nm} = s^{1-2n+M(1-2m)}\hat{\mathcal{I}}_{nm} \tag{5.2.9}$$

We can thus say that, on one hand, the integral of the Gelfand-Dikii n-th polynomial R_n depends from s only through the index n (not m) as

$$\int_s^{\infty} dz\, \sqrt{p(z)}R_n(z;s) = s^{(1-2n)(M+1)}\sum_{m=n}^{3n}\hat{a}_{n,m}\hat{\mathcal{I}}_{nm} \tag{5.2.10}$$

On the other hand, the expansion parameter includes s as (4.1.12)

$$e^{\theta} = \sqrt{E}^{\frac{M+1}{M}}s^{-(M+1)} \tag{5.2.11}$$

so that the n expansion coefficient multiplies the power (5.2.4)

$$e^{\theta(1-2n)} = \sqrt{E}^{\frac{M+1}{M}(1-2n)}s^{-(M+1)(1-2n)} \tag{5.2.12}$$

We thus see the powers of s completely cancel and the expansion of Q becomes

$$\boxed{\log Q \sim \sum_{n=0}^{\infty} \sqrt{E}^{\frac{1-2n}{1-\beta^2}} \sum_{m=n}^{3n} \hat{a}_{n,m} \hat{\mathcal{I}}_{nm} \qquad E \to +\infty} \tag{5.2.13}$$

As a consequence of the s independence, we can freely fix it to any value, in particular to 0. We thus justify the identification[14] of $\psi(0)$ with Q also with the different (4.1.13) ODE-IM equation of[16]

$$Q = \chi(0) = \exp\left\{\int_0^{\infty} dz\, \sqrt{p(z)} S(z)\right\} \tag{5.2.14}$$

$$\simeq \exp\left\{\int_s^{\infty} dz\, \sqrt{p(z)} S(z)\right\} \qquad s \to 0 \tag{5.2.15}$$

5.3 Minimal models local integrals of motion

We observe that, for the leading order

$$\hat{\mathcal{I}}_{00} = \frac{1}{2M} \frac{\Gamma(-\frac{1}{2M} - \frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(-\frac{1}{2M} + 1)} = \frac{1}{2M}(-1)\frac{\Gamma(-\frac{1}{2M} - \frac{1}{2})\Gamma(\frac{1}{2M})}{\Gamma(-\frac{1}{2})} \tag{5.3.1}$$

$$= -\frac{a_0^{DT}}{2} = \frac{1}{4M} \frac{\Gamma(-\frac{1}{2M} - \frac{1}{2})\Gamma(\frac{1}{2M})}{\sqrt{\pi}} \tag{5.3.2}$$

where $+a_0^{DT}/2$ is Dorey's and Tateo's[14] leading term. In other words, *our calculations match the standard result of Dorey Tateo, save the use of an expansion parameter with the opposite sign.* Continuing the operations on the Gamma functions

$$\hat{\mathcal{I}}_{00} = \frac{1}{4M} \frac{\Gamma(-\frac{1}{2M} - \frac{1}{2})\Gamma(\frac{1}{2M})}{\sqrt{\pi}} = -\frac{1}{2M+2} \frac{\Gamma(-\frac{1}{2M} + \frac{1}{2})\Gamma(\frac{1}{2M})}{\sqrt{\pi}} \tag{5.3.3}$$

$$= -\frac{1}{(2M+2)\Gamma(1-\beta^2)^{1+\xi}} M^{BLZ} \tag{5.3.4}$$

so after Dorey and Tateo identification (2.4.3) of the expansion parameter

$$\sqrt{E}^{1+\xi} = \left(\frac{\lambda}{\nu}\right)^{1+\xi} = \lambda^{1+\xi}(2M+2)\Gamma(1-\beta^2)^{1+\xi} \tag{5.3.5}$$

we match also the standard[8] leading order, apart the opposite sign of the expansion parameter. We now manipulate a bit the all basic integrals, which, for convenience, we rewrite using the parameter $\xi = \frac{1}{M}$)

$$\hat{\mathcal{I}}_{n,m} = \frac{\xi}{2} \frac{\Gamma((n-\frac{1}{2})\xi + m - \frac{1}{2})\Gamma(-m + \frac{3}{2})}{\Gamma((n-\frac{1}{2})\xi + 1)}$$

We use the Gamma function reflection property

$$\Gamma(-m + \frac{3}{2}) = \frac{\pi}{\sin\left(\pi(m - \frac{1}{2})\right)} \frac{1}{\Gamma(m - \frac{1}{2})}$$

$$\Gamma((n - \frac{1}{2})\xi + 1) = -\frac{\pi}{\sin\left(\pi(n - \frac{1}{2})\right)} \frac{1}{\Gamma(-(n - \frac{1}{2})\xi)}$$

so that the integral becomes

$$\hat{\mathcal{I}}_{n,m} = \frac{\xi}{2}(-1)^{m-n-1} \frac{\Gamma((n-\frac{1}{2})\xi + m - \frac{1}{2})\Gamma(-(n-\frac{1}{2})\xi)}{\Gamma(m - \frac{1}{2})}$$

We also multiply and divide by two different Gamma functions, in order to obtain a common n-dependent normalization which factorize some polynomial (the ratio of those Gamma functions)

$$\hat{\mathcal{I}}_{n,m} = \frac{\xi}{2}\frac{\Gamma\Big(-(n-\frac{1}{2})\xi\Big)\Gamma\Big((n-\frac{1}{2})(\xi+1)\Big)}{\Gamma(n-\frac{1}{2})}\left[(-1)^{m-n-1}\frac{\Gamma\Big((n-\frac{1}{2})(\xi+1)+m-n\Big)}{\Gamma((n-\frac{1}{2})(\xi+1))}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-\frac{1}{2}+m-n)}\right]$$

In fact, *the two ratios of Gamma functions in the square parenthesis are polynomials*, because numerator and denominator differ by the shift of an integer. The coefficients of such polynomials can be easily expressed in terms of the standard *Stirling numbers of the first kind*, as reported in the appendix C.1. Perhaps now it is more simple to write the whole m-dependent part as two products, which is also convenient because we can write it as a *single product*

$$\begin{aligned}\hat{\mathcal{I}}_{n,m} &= \frac{\xi}{2}\frac{\Gamma\Big(-(n-\frac{1}{2})\xi\Big)\Gamma\Big((n-\frac{1}{2})(\xi+1)\Big)}{\Gamma(n-\frac{1}{2})}\left[(-1)^{m-n-1}\prod_{l=0}^{m-n-1}\frac{\Big((n-\frac{1}{2})(\xi+1)+l\Big)}{\Big(n-\frac{1}{2}+l\Big)}\right]\\ &= \frac{\xi}{2}\frac{\Gamma\Big(-(n-\frac{1}{2})\xi\Big)\Gamma\Big((n-\frac{1}{2})(\xi+1)\Big)}{\Gamma(n-\frac{1}{2})}\left[(-1)^{m-n-1}\prod_{l=0}^{m-n-1}\left(1+\frac{(n-\frac{1}{2})\xi}{n-\frac{1}{2}+l}\right)\right] \end{aligned}\tag{5.3.6}$$

Now, we recall Dorey and Tateo fundamental ODE-IM identification (2.4.3) (apart the aforementioned minus sign)

$$\lambda = e^{\theta(1-\beta^2)} = \sqrt{E}^{1-\beta^2}\frac{2^{1-\beta^2}\beta^{2(1-\beta^2)}}{\Gamma(1-\beta^2)} \tag{5.3.7}$$

$$e^{\theta} = \sqrt{E}\frac{\beta^2}{2\Gamma(1-\beta^2)^{\frac{1}{1-\beta^2}}} \tag{5.3.8}$$

Besides, by (2.4.4), the expansion must be intended in the parameter $\sqrt{E}$, not $\sqrt{\Lambda}$[23]. The correct ODE-IM expansion becomes, for $\theta \to +\infty$

$$\begin{aligned}\log Q &\sim \sum_{n=0}^{\infty}\sqrt{E}^{1-2n}\frac{\xi}{2}(-1)^{n+1}\frac{\Gamma\Big(-(n-\frac{1}{2})\xi\Big)\Gamma\Big((n-\frac{1}{2})(\xi+1)\Big)}{\Gamma(n-\frac{1}{2})}\sum_{m=n}^{3n}\hat{a}_{n,m}\left[-\frac{(-1)^m}{n-\frac{1}{2}}\prod_{l=0}^{m-n-1}\left(1+\frac{(n-\frac{1}{2})\xi}{n-\frac{1}{2}+l}\right)\right]\\ &= -\sum_{n=0}^{\infty}\sqrt{E}^{1-2n}\frac{\xi}{2}(-1)^{n+1}\frac{\Gamma\Big(-(n-\frac{1}{2})\xi\Big)\Gamma\Big((n-\frac{1}{2})(\xi+1)\Big)}{\Gamma(n+\frac{1}{2})}\sum_{m=n}^{3n}\hat{a}_{n,m}\left[(-1)^m\prod_{l=0}^{m-n-1}\left(1+\frac{(n-\frac{1}{2})\xi}{n-\frac{1}{2}+l}\right)\right]\end{aligned}$$

We use now the property

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}2^{-n}(2n-1)!! \tag{5.3.9}$$

or

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}2^{-n}(2n)!!\frac{(2n-1)!!}{(2n)!!} = \sqrt{\pi}n!\frac{(2n-1)!!}{(2n)!!} \tag{5.3.10}$$

[23] Dorey and Tateo identification[14] can be thought to amount to a shift of the rapidity θ

Hence, we continue our calculation, also noting that $\xi = \beta^2/(1-\beta^2)$ (there were typos in[8])

$$\log Q = -\sum_{n=0}^{\infty} \sqrt{E}^{1-2n} \frac{(-1)^{n+1}\beta^2}{(1-\beta^2)} \frac{\Gamma\Big(-(n-\frac{1}{2})\xi\Big)\Gamma\Big((n-\frac{1}{2})(\xi+1)\Big)}{2\sqrt{\pi}n!} \beta^{2n}\beta^{-4n} \times$$
$$\left\{\beta^{2n}\frac{(2n)!!}{(2n-1)!!}\sum_{m=n}^{3n}\hat{a}_{n,m}\left[(-1)^m \prod_{l=0}^{m-n-1}\left(1+\frac{(n-\frac{1}{2})\xi}{n-\frac{1}{2}+l}\right)\right]\right\}$$
$$= -\sum_{n=0}^{\infty} \frac{\sqrt{E}^{1-2n}\beta^{2(1-2n)}}{\Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}}} \frac{(-1)^{n+1}}{(1-\beta^2)} \frac{\Gamma\Big(-(n-\frac{1}{2})\xi\Big)\Gamma\Big((n-\frac{1}{2})(\xi+1)\Big)}{2\sqrt{\pi}n!} \beta^{2n}\Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}} \times$$
$$\left\{\beta^{2n}\frac{(2n)!!}{(2n-1)!!}\sum_{m=n}^{3n}\hat{a}_{n,m}\left[(-1)^m \prod_{l=0}^{m-n-1}\left(1+\frac{(n-\frac{1}{2})\xi}{n-\frac{1}{2}+l}\right)\right]\right\}$$
$$= -\sum_{n=0}^{\infty} e^{\theta(1-2n)}2^{1-2n} \frac{(-1)^{n+1}}{(1-\beta^2)} \frac{\Gamma\Big(-(n-\frac{1}{2})\xi\Big)\Gamma\Big((n-\frac{1}{2})(\xi+1)\Big)}{2\sqrt{\pi}n!} \beta^{2n}\Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}} \times$$
$$\left\{\beta^{2n}\frac{(2n)!!}{(2n-1)!!}\sum_{m=n}^{3n}\hat{a}_{n,m}\left[(-1)^m \prod_{l=0}^{m-n-1}\left(1+\frac{(n-\frac{1}{2})\xi}{n-\frac{1}{2}+l}\right)\right]\right\}$$
$$= \sum_{n=0}^{\infty} e^{\theta(1-2n)}\tilde{B}_n I_{2n-1}$$

where we defined the local integrals of motion as

$$\boxed{I_{2n-1} = 2^{-2n}\beta^{2n}\frac{(2n)!!}{(2n-1)!!}\sum_{m=n}^{3n}\hat{a}_{n,m}\left[(-1)^m \prod_{l=0}^{m-n-1}\left(1+\frac{(n-\frac{1}{2})\xi}{n-\frac{1}{2}+l}\right)\right]} \tag{5.3.11}$$

and the $\tilde{B}_n$ are the standard normalization constants.

$$\tilde{B}_n = \frac{(-1)^{n+1}}{2\sqrt{\pi}(1-\beta^2)}\Gamma(\frac{2n-1}{2-2\beta^2})\Gamma(\frac{2n-1}{2-2\beta^{-2}})\beta^{2(1-2n)}\Gamma(1-\beta^2)^{\frac{1-2n}{1-\beta^2}} \tag{5.3.12}$$

Apart the factor β^{2n}, the summand I_{2n-1}/β^{2n} in formula (5.3.11) is a *polynomial* in M, because the Gelfand Dikii coefficient $a_{n,m}$ is a polynomial in M, with lowest power M^{m-n}

$$\hat{a}_{n,m} = \sum_{p=m-n}^{2n}\hat{a}_{n,m,p}M^p \tag{5.3.13}$$
$$= M^{m-n}\sum_{p'=0}^{3n-m}\hat{a}_{n,m,p'+m-n}M^{p'} \tag{5.3.14}$$

and the finite product is a polynomial in M^{-1}, with highest power $M^{-(m-n)}$.

$$\prod_{l=1}^{m-n}\left(1+\frac{(n-\frac{1}{2})\frac{1}{M}}{m-\frac{1}{2}-l}\right) = \frac{1}{M^{m-n}}\prod_{l=1}^{m-n}\left(M+\frac{(n-\frac{1}{2})}{m-\frac{1}{2}-l}\right) \tag{5.3.15}$$

so that the lowest and highest power perfectly compensates

The multiplication by the factor β^{2n}, instead, *makes* (5.3.11) *a polynomial in* c, as it should be.[10]

In fact, it trivially follows from the central charge expression (1.3.3) that, for some $2n-1$-degree polynomial with M with $\mathfrak{M}_{2n-1,l}$ coefficients

$$I_{2n-1} = \frac{1}{(M+1)^n} \sum_{l=0}^{2n-1} \mathfrak{M}_{2n-1,l} M^l \tag{5.3.16}$$

$$= \frac{1}{(M+1)^n} \sum_{l=1}^{2n} \left[\sum_{q=0}^{2n-1} S_{lq} \mathfrak{M}_{2n-1,q} \right] (M+1)^l \tag{5.3.17}$$

$$= \sum_{l=0}^{n} \mathfrak{C}_{2n-1,l} c^l \tag{5.3.18}$$

where S_{lq} is some, simply determined, matrix for the basis transformation. Thus we get an n-degree polynomial in c with $\mathfrak{C}_{2n-1,l}$ coefficients.

5.3.1 Test

We control that our result for I_1 matches that of[17] 24

$$\frac{I_1}{4^{-1}\beta^2} = 2\left[a_{11} - \left(1+\frac{1}{M}\right)a_{12} + \left(1+\frac{1}{M}\right)\left(1+\frac{1}{3M}\right)a_{13} \right] \tag{5.3.19}$$

$$= 4|k|^2 - \frac{M+1}{6}$$

$$I_1 = \frac{1}{4(M+1)}\left[|k|^2 - \frac{M+1}{6} \right] \tag{5.3.20}$$

so that we match also the result of the the standard reference[10]

$$I_1 = \Delta - \frac{c}{24} \tag{5.3.21}$$

because $\Delta = (p/\beta)^2 + (c-1)/24$ and $p = |k|\beta^2$.
We control that our result for I_3 matches that of[17]

$$\frac{I_3}{4^{-2}\beta^4} = \frac{8}{3}\left[a_{11} - \left(1+\frac{1}{M}\right)a_{22} + \left(1+\frac{1}{M}\right)\left(1+\frac{3}{5M}\right)a_{23} - \left(1+\frac{1}{M}\right)\left(1+\frac{3}{5M}\right)\left(1+\frac{3}{7M}\right)a_{34} \right.$$
$$\left. + \left(1+\frac{1}{M}\right)\left(1+\frac{3}{5M}\right)\left(1+\frac{3}{7M}\right)\left(1+\frac{3}{9M}\right)a_{25} - \left(1+\frac{1}{M}\right)\left(1+\frac{3}{5M}\right)\left(1+\frac{3}{7M}\right)\left(1+\frac{3}{9M}\right)a_{26} \right] \tag{5.3.22}$$

$$= 16|k|^4 - 4|k|^2(M+1) - \frac{(M+1)(4M+3)(M-3)}{60} \tag{5.3.23}$$

$$I_3 = \frac{1}{(4M+4)^2}\left[16|k|^4 - 4|k|^2(M+1) - \frac{(M+1)(4M+3)(M-3)}{60} \right] \tag{5.3.24}$$

and with the standard reference[10]

$$I_3 = \Delta^2 - \frac{(c+2)}{12}\Delta + \frac{c(5c+22)}{2880} \tag{5.3.25}$$

[24] In their work[17] concerning the ODE-IM correspondence, Dorey, Millikan-Slater and Tateo used the parameter $\lambda_D = 4|k|^2$. However, in our work we have chosen to follow Lukyanov's and Zamolodchikov's ,[16] because the λ variable of[17] causes confusion with the $\lambda = e^{\theta(1-\beta^2)}$ of the asymptotic expansions of integrability (1.5.7)

6 General markovian WKB expansion

6.1 Riccati equation and standard WKB expansion

Consider a Schrödinger operator of the following form (for our purposes, it is convenient to use w as independent variable)

$$-\hbar^2 \frac{d^2}{dw^2}\chi(w) + q(w)\chi(w) = 0 \tag{6.1.1}$$

We represent the eigenfunction χ in terms of its logarithmic derivative S, by the eikonal representation

$$\chi(w) = \exp \int_w^{w_0} dw' S(w') \tag{6.1.2}$$

which satisfies the following Riccati equation (cf. (4.2.9))

$$\boxed{S^2(w) + \frac{dS(w)}{dw} = \frac{1}{\hbar^2} q(w)} \tag{6.1.3}$$

We now expand S asymptotically for $\hbar \to 0$ in terms of its modes $S_n(w)$

$$S(w) = \sum_{n=-1}^{\infty} S_n(w)\hbar^n \tag{6.1.4}$$

$$= \frac{\sqrt{q}}{\hbar} + \sum_{n=0}^{\infty} S_n \hbar^n \tag{6.1.5}$$

The first mode S_{-1} is trivially obtained from the leading order in $\hbar$ of the Riccati equation (6.1.3)

$$S_{-1}(w) = \sqrt{q(w)} \tag{6.1.6}$$

Note that now it is not constant (cf. (4.2.13)). Thus, S_{-1} serves as initial condition for the *WKB standard recursion* relation. The WKB recursion, for all orders in $\hbar$, is obtained substituting in the Riccati equation (6.1.3) the asymptotic expansion (6.1.4).

$$2S_{-1}S_n + \sum_{m=0}^{n-1} S_m S_{n-1-m} + S'_n = 0 \tag{6.1.7}$$

S_{n+1}, is thus determined by

$$\boxed{S_{n+1} = -\frac{1}{2\sqrt{q}}\Big(\sum_{m=0}^{n-1} S_m S_{n-1-m} + S'_n\Big)} \tag{6.1.8}$$

We report the first terms of the WKB expansion in terms of the multiplicative part of the Schrödinger operator $q(w)$

$$S_{-1} = \sqrt{q} \tag{6.1.9}$$

$$\begin{aligned} S_0 &= -\frac{1}{4}\frac{q'}{q} \\ &= -\frac{1}{4}\frac{d}{dw}\log q \end{aligned} \tag{6.1.10}$$

$$S_1 = \frac{1}{8}\frac{q''}{q^{3/2}} - \frac{5}{32}\frac{q'^2}{q^{5/2}} \tag{6.1.11}$$

$$\begin{aligned} S_2 &= -\frac{1}{16}\frac{q'''}{q^2} + \frac{9}{32}\frac{q''q'}{q^3} - \frac{15}{64}\frac{q'^3}{q^4} \\ &= -\frac{1}{2}\frac{d}{dw}\Big(\frac{S_1}{S_{-1}}\Big) \end{aligned} \tag{6.1.12}$$

$$S_3 = \frac{1}{32}\frac{q^{iv}}{q^{5/2}} - \frac{7}{32}\frac{q'''q'}{q^{7/2}} - \frac{19}{128}\frac{q''^2}{q^{7/2}} + \frac{221}{256}\frac{q''q'^2}{q^{9/2}} - \frac{1105}{2048}\frac{q'^4}{q^{11/2}} \tag{6.1.13}$$

Since the WKB recursion (6.1.8) *is very similar to the large energy recursion* (4.2.12)*, we expect that we may similarly use the Gelfand-Dikii analysis.* However, the different form of the Riccati equation (6.1.3) and the different initial condition (6.1.6), suggest that some not too trivial modifications might be needed. The aim of this section is to develop these observations, in order to obtain a *very convenient markovian* (cf. subsection 4.5) recursion relation for the WKB expansion.

6.2 Decomposition in odd and even part

This section and the next are the adaptation, for the WKB expansion, of D. Fioravanti's analysis and proof, for the large energy expansion, in his still unpublished work with A. Fachechi.[18]

We begin by separating, as usual, the even and odd part of the solution of the Riccati

$$S_{\text{even}}(w) = \sum_{n=0}^{\infty} S_{2n}\hbar^{2n} \tag{6.2.1}$$

$$S_{\text{odd}}(w) = \sum_{n=0}^{\infty} S_{2n-1}\hbar^{2n-1} \tag{6.2.2}$$

so that the Riccati equation can be splitted in two equations

$$S^2_{\text{even}}(w) + S^2_{\text{odd}}(w) + \frac{dS_{\text{even}}}{dw} = q(w) \tag{6.2.3}$$

$$2S_{\text{even}}(w)S_{\text{odd}}(w) + \frac{dS_{\text{odd}}}{dw} = 0 \tag{6.2.4}$$

The latter equation implies that the even part S_{even} is a total derivative

$$S_{\text{even}} = -\frac{1}{2}\frac{S'_{\text{odd}}}{S_{\text{odd}}} \tag{6.2.5}$$

and therefore is negligible if we integrate over a period or over the entire dominion. We substitute S_{even} in the former equation, obtaining (until differently claimed, we use the prime $'$ to indicate the w derivative)

$$-2S''_{\text{odd}}S_{\text{odd}} + 3S'^2_{\text{odd}} + 4S^4_{\text{odd}} = \frac{1}{\hbar^2}4qS^2_{\text{odd}} \tag{6.2.6}$$

Following Fioravanti,[18] we define the function R as (1/2) the algebraic inverse of S_{odd}

$$\boxed{R(w) = \frac{1}{2S_{\text{odd}}(w)}} \tag{6.2.7}$$

such function $R(w)$ expands asymptotically for $\hbar \to 0$ in terms of its modes R_n

$$R(w) = \sum_{n=0}^{\infty} R_n(w)\hbar^{2n+1} \tag{6.2.8}$$

with initial condition $R_0 = 1/2\sqrt{q}$.

We pause the usual procedure for an important observation. *The functions R_n*[25]*, integrated over a period, are not equivalent to the densities S_{2n-1}, for the WKB expansion*; they were such only for the large energy expansion. The equivalence, in fact, was rigorously proved by Fioravanti in,[18] as we already reported in (4.4.1). As a consequence, the conjecture we made in subsection 6.1, regarding

[25] For setting the problem, we use the same notation we used for the large energy expansion; however, the *resulting* Gelfand-Dikii polynomials for the WKB expansion, will not be the R_n modes. The R_n functions, are always defined as the modes of the algebraic inverse of the eikonal integrand S function.

the applicability of Gelfand-Dikii polynomials to the small $\hbar$ expansion, will evidently require some modification of Fioravanti's proof.
Continuing the previous calculations, by (6.2.6) we obtain the the function $R(w)$ satisfies the equivalent equation

$$2R''R - R^{'2} = \frac{4q}{\hbar^2}R^2 - 1$$

Still following,[18] we apply the w-derivative

$$\hbar^2 R''' = 4qR' + 2q'R \tag{6.2.9}$$

In terms of the modes R_n we obtain

$$\boxed{R'_{n+1} + \frac{q'}{2q}R_{n+1} = \frac{1}{4q}R'''_n} \tag{6.2.10}$$

This recursion is evidently markovian, in the sense the to calculate the successive term is sufficient only the precedent one. However, this recursion, which has just been obtained as the direct analogue (6.2.7) of Fioravanti's analysis of,[18] appears not at all of the Gelfand-Dikii form.[16] We obtain thus *a markovian recursion, but not for the well-known*[19] *Gelfand Dikii polynomials.* However, we are going to show that the markovian modes R_n are simply related to some markovian and Gelfand-Dikii modes.

6.2.1 Examples and heuristics

We report the first examples of the modes R_n. This will also support further developments.
The leading order in $\hbar$ is

$$R \simeq \frac{\hbar}{2S_{-1}} = \hbar\frac{1}{2\sqrt{q}}$$

So the S_{-1} is equivalent (actually, strictly equal) to

$$S_{-1} = 2qR_0 = \sqrt{q} \tag{6.2.11}$$

note a factor q correcting the usual (6.2.7) density.
We continue with the next $\hbar^3$ order, using the technique of algebraic series inversion, which is a fundamental tool of complex analysis[25]

$$\begin{aligned} R &\simeq \frac{\hbar}{2S_{-1}(1+\hbar^2 S_1/S_{-1})} = \hbar\frac{1}{2\sqrt{q}}(1-\hbar^2\frac{S_1}{S_{-1}}) \\ &= \hbar R_0 + \hbar^3\Big(-\frac{1}{16}\frac{q''}{q^{5/2}} + \frac{5}{64}\frac{q^{'2}}{q^{7/2}}\Big) \end{aligned}$$

the mode R_1 therefore is

$$R_1 = -\frac{1}{16}\frac{q''}{q^{5/2}} + \frac{5}{64}\frac{q^{'2}}{q^{7/2}} \tag{6.2.12}$$

Comparing with (6.1.11) we can establish the equivalence, as strict equality

$$S_1 = -2qR_1 \tag{6.2.13}$$

Note, again, a factor q and a different numerical coefficient correcting the usual (6.2.7) density.

We proceed to the $\hbar^5$ order

$$R \simeq \frac{\hbar}{2S_{-1}(1+\hbar^2 S_1/S_{-1}+\hbar^4 S_3/S_{-1})} = \hbar\frac{1}{2\sqrt{q}}(1-\hbar^2\frac{S_1}{S_{-1}}-\hbar^4\frac{S_3}{S_{-1}}+\hbar^4\frac{S_1^2}{S_{-1}^2})$$
$$= \hbar R_0 + \hbar^3 R_1 + \hbar^5\frac{1}{2q}(-S^3+\frac{S_1^2}{S_{-1}})$$

So, we can establish the strict equality

$$\begin{aligned}2qR_2 &= -S_3+\frac{S_1^2}{S_{-1}} = -3S_3+\frac{1}{S_{-1}}(2S_3S_{-1}+S_1^2)\\ &= -3S_3+\frac{1}{S_{-1}}(-2S_0S_2-S_2')\\ &= -3S_3-d_2\end{aligned}$$

or

$$S_3 = -\frac{2}{3}qR_2+\frac{1}{3}d_2 \tag{6.2.14}$$

where d_2 can be shown to be a total w derivative as follows

$$\begin{aligned}-d_2 &= \frac{1}{S_{-1}}(-2S_0S_2-S_2')\\ &= \frac{1}{\sqrt{q}}\Big[\frac{1}{2}\frac{q'}{q}\Big(-\frac{1}{16}\frac{q'''}{q^2}+\frac{9}{32}\frac{q'q''}{q^3}-\frac{15}{64}\frac{q'^3}{q^4}\Big)+\frac{1}{16}\frac{q^{iv}}{q^2}-\frac{13}{32}\frac{q'q'''}{q^3}-\frac{9}{32}\frac{q''^2}{q^3}+\frac{99}{64}\frac{q'^2q''}{q^4}-\frac{15}{16}\frac{q'^4}{q^5}\Big]\\ &= \frac{q^{iv}}{16q^{5/2}}-\frac{7}{16}\frac{q'q'''}{q^{7/2}}-\frac{9}{32}\frac{q''^2}{q^{7/2}}+\frac{27}{16}\frac{q''q'^2}{q^{9/2}}-\frac{135}{128}\frac{q'^4}{q^{11/2}}\\ &= \frac{1}{16}\Big(\frac{q'''}{q^{5/2}}\Big)'-\frac{9}{32}\Big(\frac{q''q'}{q^{7/2}}\Big)'+\frac{15}{64}\Big(\frac{q'^3}{q^{9/2}}\Big)'\end{aligned}$$

The expression of R_2 is

$$R_2 = \frac{1}{32\sqrt{q}}\Big[-\frac{1}{2}\frac{q^{iv}}{q^3}+\frac{7}{2}\frac{q'''q'}{q^4}+\frac{21}{8}\frac{q''^2}{q^4}-\frac{231}{16}\frac{q''q'^2}{q^5}+\frac{1155}{128}\frac{q'^4}{q^5}\Big] \tag{6.2.15}$$

6.3 Equivalence proof for the WKB integrands

In this paragraph we just adapt, to the WKB expansion, Fioravanti's rigorous proof in,[18] *made for the large energy expansion.* To make the analogy perfect, define $k^{-1} = \hbar$ and $s = S_{\text{odd}}$ as

$$s(w) = k\sqrt{q(w)}+\sum_{n=1}^{\infty}\frac{s_{2n-1}(w)}{k^{2n-1}} \tag{6.3.1}$$

We recall that the usual large energy equivalent density R is WKB expanded as

$$R(w) = \sum_{n=0}^{\infty}\frac{R_n(w)}{k^{2n+1}} \tag{6.3.2}$$

We also report in the new notation the Riccati equation (6.2.6) for the odd part of S

$$4s^4+3s'^2-2ss''-4k^2qs^2 = 0 \tag{6.3.3}$$

Now, following Fioravanti[18] derive s with respect to k

$$\frac{\partial s}{\partial k} = \sqrt{q}-\sum_{n=1}^{\infty}(2n-1)\frac{s_{2n-1}}{k^{2n}} \tag{6.3.4}$$

and define a new quantity t as

$$t = \frac{1}{k}\frac{\partial s}{\partial k} = \frac{\sqrt{q}}{k} - \sum_{n=1}^{\infty}(2n-1)\frac{s_{2n-1}}{k^{2n+1}} \tag{6.3.5}$$

With the aid of the previous heuristics, we conjecture

$$\boxed{t = 2Rq + d} \tag{6.3.6}$$

where d is a total w derivative, to be specified below.
We now prove this, again, by analogy with Fioravanti.[18] We differentiate (6.3.3) with respect to k

$$16s^3\frac{\partial s}{\partial k} + 6s'\frac{\partial s'}{\partial k} - 2\frac{\partial s}{\partial k}s'' - 2s\left(\frac{\partial s}{\partial k}\right)'' - 8kqs^2 - 8k^2qs\frac{\partial s}{\partial k} = 0$$

then divide by $2k$

$$8s^3t + 3s't' - ts'' - st'' - 4qs^2 - 4k^2qst = 0$$

and divide by s

$$8s^2t + 3\frac{s'}{s}t' - t\frac{s''}{s} - t'' - 4qs - 4k^2qt = 0$$

Now, since

$$\frac{s'}{s} = -\frac{R'}{R} \qquad \frac{s''}{s} = 2\left(\frac{R'}{R}\right)^2 - \frac{R''}{R} \tag{6.3.7}$$

we can replace s by R to obtain

$$\begin{gathered}
2\frac{1}{R^2}t - 3\frac{R'}{R}t' - 2t\left(\frac{R'}{R}\right)^2 + \frac{R''}{R}t - t'' - 2q\frac{1}{R} - 4k^2qt = 0 \\
2t - 3R'Rt' - 2R^{'2}t + RR''t - R^2t'' - 2qR - 4k^2qR^2t = 0 \\
4qk^2R^2t + 2qR = 2t - 3RR't' - 2R^{'2}t + RR''t - R^2t''
\end{gathered}$$

Recalling also equation (6.2) for R

$$2R''R - R^{'2} + 1 = 4qk^2R^2 \tag{6.3.8}$$

we can write

$$\begin{aligned}
2qR &= t - 3RR't' - R^{'2}t - RR''t - R^2t'' \\
t - 2qR &= 3RR't' + R^{'2}t + RR''t + R^{'2}t''
\end{aligned} \tag{6.3.9}$$

We note that d *really is a total derivative*

$$\boxed{\begin{aligned}
d &= 3RR't' + R^{'2}t + RR''t + R^2t'' && (6.3.10)\\
&= (tR^2)'' - (tRR')' && (6.3.11)
\end{aligned}}$$

In other words we have proven the conjecture

$$t = 2qR + d \tag{6.3.12}$$

which, in terms of the modes is

$$\sum_{n=1}^{\infty}\frac{s_{2n-1}}{k^{2n+1}} = -\frac{2}{2n-1}\sum_{n=1}^{\infty}\frac{qR_n}{k^{2n+1}} - \frac{d}{2n-1} \tag{6.3.13}$$

or (restoring the capital letter)

$$\boxed{S_{2n-1} = -\frac{2}{2n-1} W_n + \partial_w(\text{local fields}) \qquad n \geq 1} \tag{6.3.14}$$

where we defined the correct *equivalent densities*

$$\boxed{W_n = q R_n} \tag{6.3.15}$$

By analogy with Fioravanti's rigorous proof in,[18] we have thus proved that, under integration over a period or over the entire dominion, integrating the standard WKB density S_{2n-1} is equivalent to integrating the new WKB densities W_n, up to a simple numerical n dependent factor. The difference with respect to the large energy expansion of Fioravanti is, however, that the equivalent densities are not directly R_n, but require multiplication by the function q.

6.4 The markovian recursion relation and its solution

Evidently, the new densities W_n satisfy a markovian recursion relation, that is, W_{n+1} is determined *only* by W_n, without contributions from all the preceding orders. In particular, from (6.2.10) we obtain

$$qW'_{n+1} - \frac{1}{2} q' W_n = \frac{1}{4} W'''_n - \frac{3}{4}\frac{q'}{q} W''_n + \Big(-\frac{3}{4}\frac{q''}{q} + \frac{3}{2}\frac{q'^2}{q^2}\Big) W'_n + \Big(-\frac{1}{4}\frac{q'''}{q} + \frac{3}{2}\frac{q''q'}{q^2} - \frac{3}{2}\frac{q'^3}{q^3}\Big) W_n \tag{6.4.1}$$

If one now wish to have only one term on the right side, new densities T_n must be defined as

$$\boxed{T_n = \frac{W_n}{\sqrt{q}} = \sqrt{q} R_n} \tag{6.4.2}$$

which *turn out to be exactly the Gelfand-Dikii polynomials.* In fact equation (6.4.1) yields

$$\boxed{-T'_{n+1} = -\frac{1}{4q} T'''_n + \frac{3}{8}\frac{q'}{q^2} T''_n + \Big(\frac{3}{8}\frac{q''}{q^2} - \frac{9}{16}\frac{q'^2}{q^3}\Big) T'_n + \Big(\frac{1}{8}\frac{q'''}{q^2} - \frac{9}{16}\frac{q''q'}{q^3} + \frac{15}{32}\frac{q'^3}{q^4}\Big) T_n} \tag{6.4.3}$$

We emphasize that the derivatives in this section are with respect to the variable w. The Gelfand Dikii recursion equation (6.4.3) of the T_n densities, *for the WKB expansion*, written in the *Bäcklund* variable w, is of the *very same form* of the Gelfand Dikii recursion equation of the R_n densities, *for the large energy expansion* (4.3.23), written in the *non-Bäcklund* variable z. The only substantial difference is that q is used in the place p and u is set to zero[26]. In the next section we are going to clarify the very simple reason behind this correspondence.

6.4.1 Conventions and test

We now make some simple tests of the recursion.
The initial condition T_0 for the recursion must be set as in Gelfand Dikii original work.[19]

$$\boxed{T_0 = \frac{1}{2}} \tag{6.4.4}$$

With respect to the previously written large energy Gelfand Dikii polynomials R_n in the other convention, there are the following differences. First, because the zero term was not 1 but $\frac{1}{2}$, each

[26] The sign convention is another difference, because here it is opposite to that Lukyanov and Zamolodhikov,[16] but in accord with the original Gelfand Dikii[19] work.

polynomial must be multiplied by this factor (the recursion equation is linear). Second, each odd n Gelfand-Dikii polynomial R_{2n+1} must be taken with the opposite sign, because of the sign convention in (6.4.3). Third, we recall now the prime indicates derivative with respect to w, not to z.

We can thus check directly with Gelfand Dikii original work,[19] using as function U (remember $u = 0$)

$$\boxed{V(w) = \frac{1}{4}\frac{q''}{q^2} - \frac{5}{16}\frac{q'^2}{q^3}} \tag{6.4.5}$$

from (6.4.3) we obtain

$$T_1' = -\frac{1}{16}\frac{q'''}{q^2} + \frac{9}{32}\frac{q''q'}{q^3} - \frac{15}{64}\frac{q'^3}{q^4} \tag{6.4.6}$$

which can be obtained also from differentiating $2W_1$ of (6.2.13) and dividing by $2\sqrt{q}$. This can be integrated to give

$$T_1 = -\frac{1}{4}V \tag{6.4.7}$$

Continuing

$$T_2 = \frac{3}{16}V^2 - \frac{1}{16}\frac{d^2}{dw^2}V \tag{6.4.8}$$

$$= \frac{3}{16}V^2 + \frac{1}{32}\frac{q'}{q^2}\frac{d}{dz}V - \frac{1}{16}\frac{d^2}{dz^2}V \tag{6.4.9}$$

$$= \frac{1}{32}\Big[-\frac{1}{2}\frac{q^{iv}}{q^3} + \frac{7}{2}\frac{q'''q'}{q^4} + \frac{21}{8}\frac{q''^2}{q^4} - \frac{231}{16}\frac{q''q'^2}{q^5} + \frac{1155}{128}\frac{q'^4}{q^5}\Big]$$

which perfectly matches with (6.2.15)

So, without calculating S_3 from the WKB non-markovian recursion relation, which involves all the previous S_i and also without inverting the analytic series in $\hbar$, we can say that the next Gelfand-Dikii polynomial is

$$T_3 = -\frac{5}{32}V^3 + \frac{5}{64}\Big(\frac{dV}{dw}\Big)^2 + \frac{5}{32}\frac{d^2V}{dw^2}V - \frac{1}{64}\frac{d^4V}{dw^4} \tag{6.4.10}$$

We thus see that using the Gelfand Dikii polynomials of the WKB expansion makes the calculations far easier.

6.5 Simple justification for the possibility of a Gelfand-Dikii analysis

There is a simple reason which explains why the recursion of the WKB densities T_n has the very same form of the recursion of the large energy densities R_n at $u = 0$. In fact consider the generic modified Schrödinger equations

$$-\frac{d^2}{dx^2}\phi(x) + \Lambda p(x)\phi(x) = 0 \tag{6.5.1}$$

$$-\frac{d^2}{dx^2}\phi(x) + \frac{1}{\hbar^2}p(x)\phi(x) = 0 \tag{6.5.2}$$

it appears evident that if we treat Λ and $\hbar^2$ as *dumb* parameters, if they were asymptotically related as

$$\boxed{\hbar \sim \frac{1}{\sqrt{\Lambda}} \qquad \Lambda \to \infty} \tag{6.5.3}$$

(as they indeed can) the WKB asymptotic expansion of $\phi(x)$, at positive powers of $\hbar$ *must necessarily be the same* as the large energy asymptotic expansion, at negative powers of $\sqrt{\Lambda}$. as the asymptotic expansion at small reduced Planck constant (positive powers of $\hbar$).

Considering our particular equations,

$$\frac{d^2}{dz^2}\psi(z) - \Lambda p(z)\psi(z) = 0 \qquad \Lambda \to \infty \qquad \text{with } u(z) = 0 \tag{6.5.4}$$

$$\frac{d^2}{dw^2}\chi(w) - \frac{1}{\hbar^2}q(w)\chi(w) = 0 \qquad \hbar \to 0 \tag{6.5.5}$$

we can also say that the link between the two situations that *the WKB expansion of any Schrödinger equation corresponds to the large energy expansion of some modified Schrödinger equation with zero potential* $u(z)$ (cf.(4.1.1)). Substituting $p(z)$ by $q(z)$ in (4.2.1) it is clear that we must expect the function $\sqrt{q}$ in the measure of the eikonal integrals. Moreover, we expect a Bäcklund potential without the u term as in (6.4.5). The "simple justification" might end here, however, for the sake of clarity, we carry out in detail the Bäcklund change of variable we are referring to.
So, we introduce a new (second) Bäcklund variable v and eigenfunction η[27]

$$v(w(z)) = \int^w dw\,\sqrt{q(w)} = \int^z dz\,\sqrt{p(z)}\sqrt{U(z)+\Lambda} \tag{6.5.6}$$

$$\eta[\chi[\psi]] = \sqrt[4]{q}\chi = \sqrt[4]{U+\Lambda}\sqrt[4]{p}\psi \tag{6.5.7}$$

where we specified a double equality to clarify the link with the large energy situation, with old Bäcklund potential $U(z)$ (4.1.8), eigenvalue Λ and the use of $p(z)$ in the basis for the polynomials. Note that *the Bäcklund change of variable introduces the square root* $\sqrt{q}$ *in the measure of the integral.* Consequently, the new Schrödinger equation in the variable v has the Bäcklund potential

$$V = \frac{1}{4}\frac{q''}{q^2} - \frac{5}{16}\frac{q'^2}{q^3} \tag{6.5.8}$$

and as "energy" eigenvalue $\frac{1}{\hbar^2}$. Explicitly,

$$\boxed{\frac{d^2}{dv^2}\eta(v) - V(v)\eta(v) = \frac{1}{\hbar^2}\eta(v)} \tag{6.5.9}$$

Let's now introduce, as usual, the eikonal density σ for the eigenfunction

$$\eta(v) = \exp\Big\{\int^v dv'\sigma(v')\Big\} \tag{6.5.10}$$

$$= \exp\Big\{\int^w dw'\sqrt{q(w')}\sigma(w')\Big\} \tag{6.5.11}$$

which satisfies the Riccati equation

$$\sigma' + \sigma^2 = V + \frac{1}{\hbar^2} \tag{6.5.12}$$

If we decompose in odd and even part the asymptotic expansion at small $\hbar$,

$$\sigma = \sigma_e + \sigma_o \tag{6.5.13}$$

we will again find that the even part is a total derivative, therefore negligible under integration over a period or the entire space. Further, taking the inverse for the odd part

$$T = \frac{1}{2\sigma_o} \tag{6.5.14}$$

$$= \sum_{n=0}^{\infty} T_n\hbar^{2n+1} \tag{6.5.15}$$

[27]Note that we are legitimated to say that the Bäcklund transformation (6.5.6), (6.5.7) was actually a *second* Bäcklund transformation, only if our ODE-IM equations (4.1.13) and (4.1.18), were already Bäcklund transformed in the form (6.1.1).

Since, from the Riccati equation, the first $\hbar$-mode of σ is $\sigma_{-1} = 1$, we understand the reason for the initial condition

$$T_0 = \frac{1}{2} \tag{6.5.16}$$

Note that, however, we have yet to prove that all the T_n are indeed the Gelfand Dikii polynomials with V as "fake KdV" potential. Following the same procedure as in section 4, from the Riccati equation for σ_o, we find the equation for T

$$2T''T - (T')^2 - 4(V + \frac{1}{\hbar^2})T^2 + 1 = 0 \tag{6.5.17}$$

to which we can apply the v derivative, finding

$$T''' - 4(V + \frac{1}{\hbar^2})T' - 2V'T = 0 \tag{6.5.18}$$

In terms of the modes T_n of the WKB $\hbar \to 0$ asymptotic expansion, the equivalent recursion is finally found to be the Gelfand Dikii recursion[19]

$$\boxed{-\frac{dT_{n+1}}{dv} = -\frac{1}{4}\frac{d^3}{dv^3}T_n + V\frac{d}{dv}T_n + \frac{1}{2}\frac{dV}{dv}T_n} \tag{6.5.19}$$

This is already in the form of Gelfand Dikii recursion equation (4.3.12) if we use the second transformed Bäcklund variable v instead of the first transformed Bäcklund variable w as variable and further use the potential V of (6.4.5) instead of U of (4.1.8).

To make comparison clearer, we write also this recursion in the first transformed Bäcklund variable w. The Bäcklund change of variable implies

$$\begin{aligned}
\frac{d}{dv} &= \frac{1}{\sqrt{q}}\frac{d}{dw} \\
\frac{d^2}{dv^2} &= \frac{1}{q}\frac{d^2}{dw^2} - \frac{1}{2}\frac{q'}{q^2}\frac{d}{dw} \\
\frac{d^3}{dv^3} &= \frac{1}{q^{3/2}}\frac{d^3}{dw^3} - \frac{3}{2}\frac{q'}{q^{5/2}}\frac{d^2}{dw^2} + \left(-\frac{1}{2}\frac{q''}{q^{3/2}} + \frac{q'^2}{q^3}\right)\frac{d}{dw}
\end{aligned}$$

so that equation (6.5.19) becomes

$$-q\frac{d}{dw}T_{n+1} = -\frac{1}{4}\frac{d^3}{dw^3}T_n + \frac{3}{8}\frac{q'}{q}\frac{d^2}{dw^2}T_n + \left(\frac{3}{8}\frac{q''}{q} - \frac{9}{16}\frac{q'^2}{q^3}\right)\frac{d}{dw}T_n + \left(\frac{1}{8}\frac{q'''}{q} - \frac{9}{16}\frac{q''q'}{q^2} + \frac{15}{16}\frac{q'^3}{q^3}\right)T_n \tag{6.5.20}$$

exactly as expected.

6.6 Conclusive remarks

In conclusion, *thanks to a strict analogy with Davide Fioravanti's rigorous proof,*[18] *we have proven also that the WKB eikonal densities S_{2n-1} are equivalent, modulo total derivatives, to the densities* W_n defined by

$$W_n(w) = q(w)R_n(w) \tag{6.6.1}$$

which satisfies the markovian recursion relation (6.4.1). Further, we have shown that the strictly related densities T_n

$$T_n(w) = \frac{W_n(w)}{\sqrt{q(w)}} \tag{6.6.2}$$

are nothing but the standard Gelfand-Dikii polynomials,[19] *with a "fake KdV" potential given by the Bäcklund potential* related to the Bäcklund transformation (6.5.6), (6.5.7).

In other words, *thanks to,*[18] *we have drastically simplified the standard WKB approximation recursion relation*, because

1. *The recursion is markovian*, i.e. ,in order to obtain a certain mode, it is sufficient to operate only on the preceding one, disregarding all the preceding modes.

2. *The Gelfand Dikii polynomials* are equivalent, via Bäcklund transformation, to the eikonal densities for the WKB expansion.

We note that the first property also opens to the possibility of an exact solution, *at all orders*, for the Gelfand Dikii recursion, as we did in section 11.2.
The application of Gelfand-Dikii polynomials to the WKB approximation *seems an important result, but it was deduced rather simply from: (i) the knowledge of Gelfand-Dikii polynomials from;*[19] *(ii) the rigorous proof of Fioravanti in*[18]. We thus wonder whether someone else in post[19] history has already deduced our result. In any case, in the literature we examined until now, *this both powerful and simple result seems to be unknown.*

7 Proof of He-Miao conjecture for $\mathcal{N}=2$ gauge theory

7.1 Introduction

For the self dual case $b^2=-\beta^2=1$, that is $M=-2$ the generalized Mathieu equation reduces to the Mathieu equation.

$$\frac{d^2}{dx^2}\psi(x)+(P^2+2e^{2\theta}\cos x)\psi(x)=0 \tag{7.1.1}$$

Actually, the standard form for the Mathieu equation needs the rescaling $x=2z$

$$\frac{d^2}{dz^2}\psi(z)+(4P^2+8e^{2\theta}\cos 2z)\psi(z)=0 \tag{7.1.2}$$

Therefore we must also rescale by $\frac{1}{2}$ the Floquet exponent of the sections where we used the other convention for the Mathieu equation.
We relate our parameters to those of He and Miao[34] λ_{HM} and q_{HM}

$$\lambda_{HM}=4P^2 \tag{7.1.3}$$
$$q_{HM}=-4e^{2\theta}=-(2K)^2 \tag{7.1.4}$$

The following parameters instead are standard: ϵ stands for $\hbar$, while u stands for the so-called *Coulomb branch moduli.*

$$\epsilon^2=\frac{1}{q_{HM}}=-\frac{1}{4\Lambda}=-\frac{1}{4}e^{-2\theta} \tag{7.1.5}$$
$$u=\frac{\lambda_{HM}}{2q_{HM}}=-\frac{P^2}{2\Lambda}=-\frac{P^2}{2e^{2\theta}} \tag{7.1.6}$$

Dividing by $2q$, the Mathieu equation can be expressed in Schrödinger form as

$$\boxed{\frac{\epsilon^2}{2}\frac{d^2}{dz^2}\psi(z)+[u-\cos 2z]\psi(z)=0} \tag{7.1.7}$$

The solution ψ can be expanded in a WKB series with

$$\epsilon\sim\hbar\to 0 \tag{7.1.8}$$

However note that in this WKB case ϵ as defined (11.3.1)

$$\epsilon=\pm\frac{1}{2i}e^{-\theta}\to 0 \tag{7.1.9}$$

is a bit different with respect to the large energy expansion parameter (4.2.4) which we denote with the same symbol.
Therefore, *it is essential to observe that also the P^2 parameter must diverge in order for u to be finite as $q_{HM}\sim\epsilon^{-2}\to\infty$.* More precisely, the WKB approximation is valid only if

$$\boxed{\begin{aligned} P&=\pm\sqrt{-2u}e^{\theta} && (7.1.10)\\ P^2&=O(e^{2\theta})\to+\infty \qquad \text{as} \qquad \theta\to+\infty && (7.1.11)\end{aligned}}$$

Hence the wave function is asymptotically expanded for $\epsilon\to 0$ as

$$\psi(z)=\exp\left\{i\int_{z_0}^{z}dz'\,p(z')\right\} \tag{7.1.12}$$
$$=\exp\left\{i\int_{z_0}^{z}dz'\left[\frac{p_0(z')}{\epsilon}+p_1(z')+\epsilon p_2(z')+...\right]\right\} \tag{7.1.13}$$

For example,

$$p_0 = \sqrt{2(u - \cos 2z)} \tag{7.1.14}$$

$$p_1 = \frac{i}{2}(\ln p_0)' \tag{7.1.15}$$

$$p_2 = -\frac{1}{8p_0}[2(\ln p_0)'' - (\ln p_0)'^2] \tag{7.1.16}$$

He and Miao[34] *conjectured the existence of differential operators in* u*, with polynomial coefficients in* u *of the form*

$$\boxed{\begin{aligned} &\oint_{-\pi/2}^{\pi/2} p_{2n+1}(z')dz = 0 \\ &\oint_{-\pi/2}^{\pi/2} p_{2n}(z')dz = \sum_{k=0}^{n} \hat{C}_{n,k} u^k \frac{\partial^{k+n}}{\partial u^{k+n}} \oint_{-\pi/2}^{\pi/2} p_0(z')dz \end{aligned}} \tag{7.1.17–7.1.19}$$

where $n = 0, 1, 2...$ and the $\hat{C}_{n,k}$ $(k = 0, 1, ..., n)$ are *numerical coefficients* which they left unspecified (apart for $n = 1, 2, 3, 4$).

$$C_{n,k} = \hat{C}_{n,k} u^k \qquad k = 0, 1, 2, ..., n \tag{7.1.20}$$

In this section, *we give a rigorous proof of the existence and uniqueness of the He-Miao differential operators and give a general algorithm for calculating them.*
In our usual conventions the eikonal density S_n is related to that of He and Miao[34] as

$$ip_n(z) = S_{n-1}(z) \tag{7.1.21}$$

7.2 Gelfand-Dikii WKB markovian recursion for $b = 1$

The simplicity of the Gelfand Dikii recursion will be our first step for the construction of the $C_{n,k}$ He-Miao operator coefficients, so we now turn our attention to it.
First, we have to apply our previously discussed *Gelfand Dikii Markovian WKB analysis* with

$$\boxed{q(z; u) = 2\cos 2z - 2u} \tag{7.2.1}$$

$$\boxed{\cos 2z = \frac{q(z; u)}{2} + u} \tag{7.2.2}$$

From the first integrands we conjecture the general form of the integrands

$$\boxed{T_n(z; u) = \sum_{m=n}^{3n} \frac{b_{n,m}(u)}{q^m(z)}} \tag{7.2.3}$$

For this first step, *the following simple observation is crucial*

$$\boxed{\frac{1}{q^{m-1/2}} = -\frac{1}{(2m-3)!!} \frac{\partial^m}{\partial u^m} \sqrt{q}} \tag{7.2.4}$$

$$\boxed{\int_{-\pi/2}^{\pi/2} dz \sqrt{q} T_n = \sum_{m=n}^{3n} -\frac{b_{n,m}(u)}{(2m-3)!!} \int_{-\pi/2}^{\pi/2} dz \frac{\partial^m}{\partial u^m} \sqrt{q}} \tag{7.2.5}$$

The integrals for z from $-\pi/2$ to $\pi/2$ are integrals with compact support. Hence, the exchange of order of u-derivative and z integral, surely satisfies the necessary assumptions of convergence.**?**

$$\int_{-\pi/2}^{\pi/2} dz S_{2n-1}(z;u) = \frac{2}{2n-1}\sum_{m=n}^{3n}\frac{b_{n,m}(u)}{(2m-3)!!}\frac{\partial^m}{\partial u^m}\int_{-\pi/2}^{\pi/2} dz S_{-1}(z;u) \tag{7.2.6}$$

As simple as it is, *this is only a partial proof of He-Miao conjecture, because the number of terms in our operator is* $2n+1$ *rather than* $n+1$ *and the* $b_{n,m}$ *are not at all homogeneous in* u *as those of He and Miao* (see the next subsection for clarifications).
However, *this "redundant operators" anyway give the correct integrals and we can immediately write a recursion* (7.2.26) *for the coefficients of the derivatives.*
The integral of $S_{-1} = \sqrt{q}$ is reduced to the Gauss hypergeometric function[34]

$$\int_{-\pi/2}^{\pi/2} dz S_{2n-1}(z;u) = \frac{2\pi i}{2n-1}\sum_{m=n}^{3n}\frac{b_{n,m}(u)}{(2m-3)!!}\frac{\partial^m}{\partial u^m}\Big\{\sqrt{2(u+1)}\,{}_2F_1(-\frac{1}{2},\frac{1}{2};1;\frac{2}{u+1})\Big\} \tag{7.2.7}$$

by means of the well-known Euler formula[26]

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\,\frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \tag{7.2.8}$$

We can now pass to the main objective of this section: the markovian recursion relation for the coefficients $b_{n,m}$. We consider[28] the usual Gelfand Dikii recursion equation (6.4.3) for the integrated densities.

$$-T'_{n+1} = -\frac{1}{4q}T'''_n + \frac{3}{8}\frac{q'}{q^2}T''_n + \Big(\frac{3}{8}\frac{q''}{q^2} - \frac{9}{16}\frac{q'^2}{q^3}\Big)T'_n + \Big(\frac{1}{8}\frac{q'''}{q^2} - \frac{9}{16}\frac{q''q'}{q^3} + \frac{15}{32}\frac{q'^3}{q^4}\Big)T_n \tag{7.2.9}$$

Hence, on the $n+1$ side

$$\begin{aligned}\oint dz\, S_{2n+1}(z) &= -\frac{2}{2n+1}\oint dz\,\sqrt{q(z)}T_{n+1}(z) = \frac{2}{2n+1}\oint dz\,\sqrt{q(z)}\int^z d\bar{z}\,(-1)\frac{\partial T_{n+1}}{\partial \bar{z}}\\ &= \frac{2}{2n+1}\oint dz\sqrt{q(z)}\int^z d\bar{z}\sum_{m=n+1}^{3n+3} b_{n+1,m}(u)\Big[m\frac{q'(\bar{z})}{q^{m+1}(\bar{z})}\Big]\end{aligned} \tag{7.2.10}$$

While, on the n side

$$\oint dz\sqrt{q(z)}\int^z d\bar{z}\, b_{n,m}\frac{1}{q^{m+1}}\Big\{\Big[\frac{1}{4}m+\frac{1}{8}\Big]\frac{q'''}{q}$$
$$+\Big[-\frac{3}{4}m(m+1)-\frac{3}{8}m-\frac{3}{8}m-\frac{9}{16}\Big]\frac{q'q''}{q^2}+\Big[\frac{1}{4}m(m+1)(m+2)+\frac{3}{8}m(m+1)+\frac{9}{16}m+\frac{15}{32}\Big]\frac{q'^3}{q^3}\Big\}$$

Until now the calculation respected the generality of the Gelfand Dikii recursion (apart the ansatz form (7.2.3)). To proceed we must simplify the derivatives of $q(z)$ and their product for the particular case of the Mathieu equation. We start collecting the derivatives of q with respect to z

$$\begin{aligned}\frac{\partial q}{\partial z} &= -4\sin 2z\\ \frac{\partial^2 q}{\partial z^2} &= -8\cos 2z = -4q - 8u = -8(u+\frac{q}{2})\\ \frac{\partial^3 q}{\partial z^3} &= 16\sin 2z = -4q'\\ \frac{\partial^4 q}{\partial z^4} &= 16q\end{aligned}$$

[28] For further considerations (in particular, the discarding of a total derivative from recursion relation), it is important to keep in mind the double integration in which the Gelfand Dikii recursion for the density is embedded

The expression for $q^{'2}$ will also be essential below

$$q^{'2} = 16 - 16\cos^2 2z = 16\Big(1 - \frac{q^2}{4} - qu - u^2\Big)$$

$$\boxed{q^{'2} = -16[(u^2-1) + qu + \frac{q^2}{4}]} \tag{7.2.11}$$

The Gelfand Dikii recursion becomes

$$\oint dz\sqrt{q(z)}\int^z d\bar{z}\, m b_{n+1,m}\frac{1}{q^{m+1}} = \oint dz\sqrt{q(z)}\int^z d\bar{z}\, b_{n,m}\frac{1}{q^{m+1}}\Big\{\Big[\frac{1}{4}m+\frac{1}{8}\Big](-4)\frac{1}{q}$$
$$+\Big[\frac{3}{4}m^2+\frac{3}{2}m+\frac{9}{16}\Big]\frac{(4q+8u)}{q^2}+\Big[\frac{1}{4}m^3+\frac{9}{8}m^2+\frac{23}{16}m+\frac{15}{32}\Big]\frac{(-16)[u^2-1+qu]-4q^2}{q^3}\Big\}$$
$$= \oint dz\sqrt{q(z)}\int^z d\bar{z}\, b_{n,m}\Big\{\Big[-\frac{1}{8}(1+2m)^3\Big]\frac{1}{q^{m+2}}+\Big[-4u(m+\frac{1}{2})(m+1)(m+\frac{3}{2})\Big]\frac{1}{q^{m+3}}$$
$$-4(u^2-1)\Big[(m+\frac{1}{2})(m+\frac{3}{2})(m+\frac{5}{2})\Big]\frac{1}{q^{m+4}}$$

We therefore find the Gelfand Dikii recursion for the Gelfand Dikii coefficients $b_{n,m}$

$$\begin{aligned} b_{n+1,m} &= -\frac{(m-\frac{1}{2})^3}{m}b_{n,m-1} - 4u\frac{(m-\frac{3}{2})(m-1)(m-\frac{1}{2})}{m}b_{n,m-2} \\ &\quad -4(u^2-1)\frac{(m-\frac{5}{2})(m-\frac{3}{2})(m-\frac{1}{2})}{m}b_{n,m-3} \end{aligned} \tag{7.2.12}$$

or also

$$\boxed{\begin{aligned} b_{n+1,m+1} &= -\frac{(m+\frac{1}{2})^3}{m+1}b_{n,m} - 4u\frac{(m-\frac{1}{2})m(m+\frac{1}{2})}{m+1}b_{n,m-1} \\ &\quad -4(u^2-1)\frac{(m-\frac{3}{2})(m-\frac{1}{2})(m+\frac{1}{2})}{m+1}b_{n,m-2} \end{aligned}} \tag{7.2.13}$$

7.2.1 Test of the recursion relation

We tested the coefficient recursion (7.2.13) directly $T_0 \to T_1$ and $T_1 \to T_2$, using the WKB formulas for general q .In fact, we know from the previous section on the Markovian WKB that

$$T_0 = \frac{1}{2} \tag{7.2.14}$$

$$T_1 = -\frac{1}{4}U^\hbar = -\frac{1}{16}\frac{q''}{q^2} + \frac{5}{64}\frac{q^{'2}}{q^3} \tag{7.2.15}$$

$$T_2 = \frac{1}{32}\Big[-\frac{1}{2}\frac{q^{iv}}{q^3} + \frac{7}{2}\frac{q'''q'}{q^4} + \frac{21}{8}\frac{q^{''2}}{q^4} - \frac{231}{16}\frac{q''q^{'2}}{q^5} + \frac{1155}{128}\frac{q^{'4}}{q^6}\Big] \tag{7.2.16}$$

which simplified corresponds to

$$b_{00} = \frac{1}{2} \tag{7.2.17}$$

$$b_{1,1} = -\frac{1}{16} \qquad b_{1,2} = -\frac{3}{4}u \qquad b_{1,3} = -\frac{5}{4}(u^2-1) \tag{7.2.18}$$

$$\begin{aligned} &b_{2,2} = \frac{27}{256} \qquad b_{2,3} = \frac{145}{32}u \qquad b_{2,4} = \frac{1085}{32}u^2 - \frac{455}{32} \\ &b_{2,5} = \frac{693}{8}u^3 - \frac{693}{8}u \qquad b_{2,6} = \frac{1155}{16}u^4 - \frac{1155}{8}u^2 + \frac{1155}{16} \end{aligned} \tag{7.2.19}$$

7.2.2 Further examples of Gelfand-Dikii coefficients

From Wolfram Mathematica we get also the coefficients of T_3

$$b_{33} = -\frac{1125}{2048} \qquad b_{34} = -\frac{26285}{512}u \qquad b_{35} = -\frac{63}{512}\left(6905u^2 - 2133\right) \tag{7.2.20}$$

$$b_{36} = -\frac{231}{64}u\left(1513u^2 - 1063\right) \qquad b_{37} = -\frac{429}{128}(u^2-1)\left(4943u^2 - 1235\right) \tag{7.2.21}$$

$$b_{38} = -\frac{765765}{32}u\left(u^2-1\right)^2 \qquad b_{39} = -\frac{425425}{32}\left(u^2-1\right)^3 \tag{7.2.22}$$

We get also the coefficients of T_4

$$\begin{aligned}
b_{44} &= \frac{385875}{65536} \qquad b_{45} = \frac{3945753u}{4096}\\
b_{46} &= \frac{231\left(493415u^2 - 129609\right)}{4096} \qquad b_{47} = \frac{429u\left(771239u^2 - 447347\right)}{1024}\\
b_{48} &= \frac{6435\left(610843u^4 - 603310u^2 + 80667\right)}{2048} \qquad b_{49} = \frac{36465}{256}u(u^2-1)\left(44887u^2 - 23011\right)\\
b_{4,10} &= \frac{46189}{256}\left(u^2-1\right)^2\left(67117u^2 - 11875\right) \qquad b_{4,11} = \frac{780825045}{64}u\left(u^2-1\right)^3\\
b_{4,12} &= \frac{1301375075}{256}\left(u^2-1\right)^4
\end{aligned} \tag{7.2.23}$$

We calculated also the coefficients of T_5 but we don't report them.

7.2.3 "Redundant He-Miao coefficients" recursion relation

The "partially correct" coefficient operators would be

$$B_{n,m}(u) = \frac{2}{(2n-1)(2m-3)!!}b_{n,m}(u) \tag{7.2.24}$$

$$\int_{-\pi/2}^{\pi/2} dz S_{2n-1}(z;u) = \sum_{m=n}^{3n} B_{n,m}(u)\frac{\partial^m}{\partial u^m}\int_{-\pi/2}^{\pi/2} dz S_{-1}(z;u) \tag{7.2.25}$$

but, with respect to those of He and Miao,[34] they are not neither in the correct number, nor of the homogeneous form.

However, as we anticipated, for this "Redundant He-Miao coefficients" we can write a recursion relation. It just suffices to slightly correct the recursion for the Gelfand Dikii coefficients $b_{n,m}$ (7.2.12).

$$\boxed{\begin{aligned}
B_{n+1,m+1} = \frac{n-\frac{1}{2}}{n+\frac{1}{2}}\Bigg\{&-\frac{1}{2}\frac{(m+\frac{1}{2})^3}{(m+1)(m-\frac{1}{2})}B_{n,m} - u\frac{m(m+\frac{1}{2})}{(m+1)(m-\frac{3}{2})}B_{n,m-1}\\
&-\frac{1}{2}(u^2-1)\frac{m+\frac{1}{2}}{(m+1)(m-\frac{5}{2})}B_{n,m-2}\Bigg\}
\end{aligned}} \tag{7.2.26}$$

7.3 General algorithmic proof of He Miao conjecture

We now prove He-Miao conjecture in all generalities. A first step for the proof was already made in the previous section, however the "redundant He-Miao operator" (7.2.25) was not exactly that of He-Miao[34] and quite more complex. In fact, the number of derivative involved was double and they multiplied polynomial, rather than monomial coefficients in u.

We can write the general structure of the $b_{n,m}$ coefficient as

$$\boxed{b_{n,m} = \begin{cases} \sum_{k=0}^{(m-n)/2} \beta_{n,m,l}u^{2k} & \text{if } m-n \text{ is even}\\ \sum_{k=1}^{(m-n+1)/2} \beta_{n,m,l}u^{2k-1} & \text{if } m-n \text{ is odd}\end{cases}} \tag{7.3.1}$$

In particular, if $m = 2n + l$ with $l \geq 0$ the Gelfand Dikii coefficients are divisible by $(u^2-1)^l$

$$\boxed{b_{n,2n+l} = (u^2-1)^l \hat{b}_{n,2n+l} := \begin{cases} (u^2-1)^l \sum_{k=0}^{n} \beta_{n,m,l} u^{2k} & \text{if } n+l \text{ is even} \\ (u^2-1)^l \sum_{k=1}^{(2n+1)/2} \beta_{n,m,l} u^{2k-1} & \text{if } n+l \text{ is odd} \end{cases}} \tag{7.3.2}$$

We can immediately check the form (7.3.2) for $n = 1, 2, 3, 4$. In general we have proven true it by induction, using formula (7.2.12). This fact is going to be shown to be the condition for the elimination of the unwanted higher $(m > 2n)$ derivatives.
To obtain the fundamental formula of our proof, consider the following second derivative

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{q^{m-5/2}(z)}\right) = (m-\frac{5}{2})(m-\frac{3}{2})\frac{q'^2}{q^{m-1/2}(z)} - (m-\frac{3}{2})\frac{q''}{q^{m-3/2}(z)} \tag{7.3.3}$$

We simplify this expression using the usual formulas for the particular case of Mathieu equation

$$q'^2 = -16\Big[(u^2-1) + qu + \frac{q^2}{4}\Big] \tag{7.3.4}$$

$$q'' = -4q - 8u \tag{7.3.5}$$

So we get the fundamental formula of our proof

$$\frac{u^2-1}{q^{m-1/2}(z)} = \frac{1}{4}\frac{\partial^2}{\partial z^2}\left(\frac{1}{q^{m-5/2}(z)}\right) - \frac{m-2}{m-\frac{3}{2}}u\frac{1}{q^{m-1-1/2}(z)} - \frac{m-\frac{5}{2}}{4(m-\frac{3}{2})}\frac{1}{q^{m-2-1/2}(z)} \tag{7.3.6}$$

which under the integration over a period corresponds to the equivalence

$$\boxed{\frac{u^2-1}{q^{m-1/2}(z)} \doteq -\frac{m-2}{m-\frac{3}{2}}u\frac{1}{q^{m-1-1/2}(z)} - \frac{m-\frac{5}{2}}{4(m-\frac{3}{2})}\frac{1}{q^{m-2-1/2}(z)}} \tag{7.3.7}$$

Let's define for convenience the functions which perform the fundamental operation for each power $m-1$ and $m-2$

$$d_1(m) = -\frac{m-2}{m-\frac{3}{2}} \qquad D_1(m) = ud_1(m) \tag{7.3.8}$$

$$d_2(m) = -\frac{m-\frac{5}{2}}{4(m-\frac{3}{2})} \qquad D_2(m) = d_2(m) \tag{7.3.9}$$

so that the fundamental operations reads

$$\boxed{\frac{u^2-1}{q^{m-1/2}(z)} \doteq ud_1(m)\frac{1}{q^{m-1-1/2}(z)} + d_2(m)\frac{1}{q^{m-2-1/2}(z)}} \tag{7.3.10}$$

Now, consider the general (up to a trivial factor u) coefficient in (7.3.2) for $2n \leq m \leq 3n$

$$b_{n,m}(u) = (u^2-1)^{(m-2n)}\hat{b}_{n,m}(u) \tag{7.3.11}$$

$$\begin{aligned} \frac{b_{n,m}(u)}{q^{m-1/2}} &= \Big[(u^2-1)^{m-2n-1}\hat{b}_{n,m}(u)\Big]\frac{u^2-1}{q^{m-1/2}} \\ &\doteq \Big[(u^2-1)^{m-2n-1}\hat{b}_{n,m}(u)\,u\,d_1(m)\Big]\frac{1}{q^{m-1-1/2}} \\ &+ \Big[(u^2-1)^{m-2n-1}\hat{b}_{n,m}(u)\,d_2(m)\Big]\frac{1}{q^{m-2-1/2}} \end{aligned} \tag{7.3.12}$$

The result is a modification of the the two "lower" coefficients

$$b^{(2)}_{n,2n+l-1} = b^{(1)}_{n,2n+l-1} + (u^2-1)^{l-1}\hat{b}^{(2)}_{n,2n+l}(u)\,u\,d_1(2n+l) \tag{7.3.13}$$

$$b^{(1)}_{n,2n+l-2} = b_{n,2n+l-2} + (u^2-1)^{l-1}\hat{b^{(2)}}_{n,2n+l}(u)\,d_2(2n+l) \tag{7.3.14}$$

$$b^{(2)}_{n,2n+l-2} = b^{(1)}_{n,2n+l-2} + (u^2-1)^{l-2}\hat{b}^{(2)}_{n,2n+l-1}(u)\,u\,d_1(2n+l-1) \tag{7.3.15}$$

where the superscript (1) or two (2) stands for the number of transformations which have affected the coefficient, in the process starting from $m = 3n$. Of course, if we consider the process starting from $m = 3n$ our coefficient $b_{n,m}$ is actually already twice transformed and should be denoted as $b^{(2)}_{n,2n+m}$. Note that the transformation (7.3.10) guarantees that the divisibility characteristics (7.3.2) are unchanged.

It is clear that in this manner *we can eliminate all coefficients for $2n+1 \leq m \leq 3n$, the result being a double transformation of the two lowest index coefficients*, namely $b_{n,n-1}$ and $b_{n,n-2}$.

We begin to write some explicit formulas. In all our formulae, we shall always use certain coefficients $B^j_{n,m}(u)$ calculated very simply with the following "Fibonacci-like" recursion

$$B^j_{n,m} = D_1(m+j)B^{j-1}_{n,m} + (u^2-1)D_2(m+j)B^{j-2}_{n,m} \tag{7.3.16}$$

$$= ud_1(m+j)B^{j-1}_{n,m} + (u^2-1)d_2(m+j)B^{j-2}_{n,m} \tag{7.3.17}$$

and initial conditions

$$B^0_{n,m} = 1 \tag{7.3.18}$$

$$B^1_{n,m} = D_1(m+1) = ud_1(m+1) \tag{7.3.19}$$

To begin with, we need also the second transformed of the $n+1$-th Gelfand-Dikii coefficient $b_{n,2n+1}$, even if its third transformed is null.

$$b^{(2)}_{n,2n+1} = \sum_{j=0}^{n-1} B^j_{n,2n+1}\frac{b_{n,2n+1+j}}{(u^2-1)^j} \qquad b^{(2)}_{n,2n+1} = (u^2-1)\hat{b}^{(2)}_{n,2n+1} \tag{7.3.20}$$

In fact it permits us to calculate the first transformed of the $n-1$-th coefficient

$$b^{(1)}_{n,2n-1} = b_{n,2n-1} + D_2(2n+1)\frac{b^{(2)}_{n,2n+1}}{(u^2-1)} \tag{7.3.21}$$

$$= b_{n,2n-1} + \sum_{j=1}^{n} D_2(2n+1)B^{j-1}_{n,2n+1}\frac{b_{n,2n+j}}{(u^2-1)^j} \tag{7.3.22}$$

which of course is a partial result. We can directly write the second transformed of the n-th coefficient in terms of the Gelfand Dikii coefficient of higher index $2n < m \leq 3n$

$$b^{(2)}_{n,2n} = \sum_{j=0}^{n} B^j_{n,2n}\frac{b_{n,2n+j}}{(u^2-1)^j} \tag{7.3.23}$$

$$\tag{7.3.24}$$

We now expand all the remaining coefficients in the n dimensional basis

$$u^2-1\,,\;(u^2-1)^2\,,\cdots,\;(u^2-1)^n\,,\;u^n \tag{7.3.25}$$

up to the usual global factor u the expansion is (we assume n odd)

$$\begin{aligned}
b^{(2)}_{n,2n} &= \beta^{(2)}_{n,2n,n} u^n + u \sum_{l=1}^{(n-1)/2} \gamma^{(2)}_{n,2n,l} (u^2-1)^l \\
b^{(1)}_{n,2n-1} &= \beta^{(1)}_{n,2n-1,n-1} u^{n-1} + \sum_{l=1}^{(n-1)/2} \gamma^{(1)}_{n,2n-1,l} (u^2-1)^l \\
b_{n,2n-2} &= \beta_{n,2n-2,n-2} u^{n-2} + u \sum_{l=1}^{(n-1)/2-1} \gamma_{n,2n-2,l} (u^2-1)^l \\
b_{n,2n-3} &= \beta_{n,2n-3,n-3} u^{n-3} + \sum_{l=1}^{(n-1)/2-1} \gamma_{n,2n-3,l} (u^2-1)^l \\
&\dots
\end{aligned} \tag{7.3.26}$$

$$b_{n,n+k} = \beta_{n,n+k,k} u^k + u^{\delta_k} \sum_{l=1}^{(n-1)/2-[(n-k)/2]} \gamma_{n,n+k,l} (u^2-1)^l \tag{7.3.27}$$

$$\begin{aligned}
&\dots \\
b_{n,n+2} &= \beta_{n,n+2,2} u^2 \\
b_{n,n+1} &= \beta_{n,n+1,1} u \\
b_{n,n} &= \beta_{n,n,0}
\end{aligned}$$

where $\delta_k = 1$ if k is odd, whether $\delta_k = 0$ if k is even. We introduced alternative $\gamma_{n,n+k,l}$ Gelfand-Dikii subcoefficients (rather than $\beta_{n,n+k,l}$), because we changed the basis (7.3.25) for the polynomial in u. It is clear now that the same fundamental transformation (7.3.10) can be applied to the part of the coefficients $b_{n,n+k}$ ($0 \le k \le n$) proportional to some power of u^2-1.

$$b_{n,n+k} - \beta_{n,n+k,k} u^k = (u^2-1) u^{\delta_k} \sum_{l=1}^{(n-1)/2-[(n-k)/2]} \gamma_{n,n+k,l} (u^2-1)^{l-1} \tag{7.3.28}$$

$$\begin{aligned}
\Big[b_{n,n+k} - \beta_{n,n+k,k} u^k\Big] \frac{1}{q^{n+k-1/2}(z)} \doteq{}& d_1(n+k) u^{\delta_k+1} \sum_{l=1}^{(n-1)/2-[(n-k)/2]} \gamma_{n,n+k,l} (u^2-1)^{l-1} \frac{1}{q^{n+k-3/2}(z)} \\
&+ d_2(n+k) u^{\delta_k} \sum_{l=1}^{(n-1)/2-[(n-k)/2]} \gamma_{n,n+k,l} (u^2-1)^{l-1} \frac{1}{q^{n+k-5/2}(z)}
\end{aligned} \tag{7.3.29}$$

It is evident that final result is of the form

$$\begin{aligned}
b^{(3)}_{n,2n} &= \beta^{(2)}_{n,2n,n} u^n \\
b^{(3)}_{n,2n-1} &= \beta^{(3)}_{n,2n-1,n-1} u^{n-1} \\
b^{(3)}_{n,2n-2} &= \beta^{(3)}_{n,2n-2,n-2} u^{n-2} \\
b^{(3)}_{n,2n-3} &= \beta^{(3)}_{n,2n-3,n-3} u^{n-3} \\
&\dots \\
b^{(3)}_{n,n+k} &= \beta^{(3)}_{n,n+k,k} u^{k} \\
&\dots \\
b^{(3)}_{n,n+2} &= \beta^{(3)}_{n,n+2,2} u^{2} \\
b^{(2)}_{n,n+1} &= \beta^{(2)}_{n,n+1,1} u \\
b^{(1)}_{n,n} &= \beta^{(1)}_{n,n,0}
\end{aligned} \tag{7.3.30}$$

In practice, given a second transformed Gelfand-Dikii coefficient $b^{(2)}_{n,2n-l}$, with $l = 0, 1, ..., n$, we can obtain the final coefficient $c_{n,2n-l}$ by calculating it in $u^2 = 1$ and then restoring the lost u dependence by multiplication by u^{n-l}.

$$c_{n,2n-l}(u) = b^{(2)}_{n,2n-l}(u)\Big|_{u^2=1} u^{n-l} \qquad l = 0, 1, ..., n \tag{7.3.31}$$

the difference $\Delta_{n,2n-l}$ is defined by

$$\Delta_{n,2n-l} = b_{n,2n-l} - c_{n,2n-l} \tag{7.3.32}$$

The general formula for the modified is similar to the previous for $2n \le m \le 3n$, but now is recursive on the second index m because requires the nowledge of $c_{n,k}$

$$b^{(2)}_{n,2n-l} = b_{n,2n-l} + \sum_{j=1}^{n+l} B_{n,2n-l+j} \frac{\Delta_{n,2n-l+j}}{(u^2-1)^j} \qquad l = 2, 3, ..., n-1 \tag{7.3.33}$$

The "extremal" indexes $l = 0, 1, n$ are related to different equations. For $l = 0$ we use the second transformed coefficient

$$c_{n,2n} = b^{(2)}_{n,2n}(u)\Big|_{u=1} u^n \tag{7.3.34}$$

$$\Delta^{(2)}_{n,2n} = b^{(2)}_{n,2n} - c_{n,2n} \tag{7.3.35}$$

For $l = 1$ we use the first transformed coefficient

$$b^{(2)}_{n,2n-1} = b^{(1)}_{n,2n-1} + \frac{1}{u^2-1} D_1(2n)\Delta_{n,2n} \tag{7.3.36}$$

$$c_{n,2n-1} = b^{(2)}_{n,2n-1}(u)\Big|_{u=1} u^{n-1} \tag{7.3.37}$$

$$\Delta^{(1)}_{n,2n-1} = b^{(1)}_{n,2n-1} - c_{n,2n-1} \tag{7.3.38}$$

While for $l = n$ there is no contribution from the next higher coefficient (corresponding to $l = n-1$)

$$b^{(1)}_{n,n} = b_{n,n} + D_2(n+2)\frac{b^{(2)}_{n,n+2}}{u^2-1} \tag{7.3.39}$$

$$= b_{n,n} + D_2(n+2)\frac{b_{n,n+2}}{u^2-1} + \sum_{j=1}^{n-2} D_2(n+2) B_{n,n+2+j} \frac{\Delta_{n,n+2+j}}{(u^2-1)^{j+1}} \tag{7.3.40}$$

Because of (7.2.4), *the He-Miao operator coefficients are in general given by*

$$\boxed{C_{n,m} = \frac{2}{(2n-1)(2m-3)!!}\beta^{(3)}_{n,m,m-n}u^{m-n} \qquad n \leq m \leq 2n} \tag{7.3.41}$$

as they are in the exact number and of the exact u-homogeneous form. This completes the proof of He-Miao conjecture.

7.4 Examples

We report here only the first two examples of application of our procedure and proof. Further examples can be found in appendix. In all cases we checked our results with those of He and Miao[34]

7.4.1 Trivial example: T_1

The only transformation we need is that of the highest degree coefficient of R_1, that is b_{13}

$$\frac{b_{13}}{q^{5/2}} = -\frac{5}{4}\frac{(u^2-1)}{q^{5/2}} \tag{7.4.1}$$

$$\equiv -\frac{5}{4}\left[\frac{ud_1(3)}{q^{3/2}} + \frac{d_2(3)}{q^{1/2}}\right] \tag{7.4.2}$$

which implies that

$$b^{(1)}_{13} = 0 \tag{7.4.3}$$

$$b^{(1)}_{12} = b_{12} - \frac{5}{4}d_1(3)u = \frac{u}{12} \tag{7.4.4}$$

$$b^{(1)}_{11} = b_{11} - \frac{5}{4}d_2(3) = \frac{1}{24} \tag{7.4.5}$$

Formula (7.3.41) now gives the coefficients of the differential operator in u

$$C_{12} = \frac{1}{6}u \tag{7.4.6}$$

$$C_{11} = \frac{1}{12} \tag{7.4.7}$$

which are exactly those of He and Miao in their article[34]
We note that

$$b^{(1)}_{12} = c_{12} = b_{12} + \frac{b_{13}}{u^2-1}ud_1(3) \tag{7.4.8}$$

$$= \Big[\beta_{121} + \beta_{132}d_1(3)\Big]\, u \tag{7.4.9}$$

$$b^{(1)}_{11} = c_{11} = b_{110} + \frac{b_{13}}{u^2-1}d_2(3) \tag{7.4.10}$$

$$= \beta_{110} + \beta_{132}d_2(3) \tag{7.4.11}$$

7.4.2 Simple example: T_2

Let's begin with transformation of b_{26}, that is, the highest degree coefficient of R_2

$$\frac{b_{26}}{q^{11/2}} = \frac{1155}{16}(u^2-1)\frac{(u^2-1)}{q^{11/2}} \tag{7.4.12}$$

$$\equiv \frac{1155}{16}(u^2-1)\left[\frac{ud_1(6)}{q^{9/2}} + \frac{d_2(6)}{q^{7/2}}\right] \tag{7.4.13}$$

which implies that

$$b_{26}^{(1)} = 0 \tag{7.4.14}$$

$$b_{25}^{(1)} = b_{25} + \frac{1155}{16} d_1(6)(u^2-1)u = \frac{539}{24} u \left(u^2 - 1\right) \tag{7.4.15}$$

$$b_{24}^{(1)} = b_{24} + \frac{1155}{16} d_2(6)(u^2-1) = \frac{35}{192} \left(109u^2 - 1\right) \tag{7.4.16}$$

We operate now on (the transformed of) b_{25}

$$\frac{b_{25}^{(1)}}{q^{9/2}} = \frac{539}{24} u \frac{(u^2-1)}{q^{9/2}} \tag{7.4.17}$$

$$\equiv \frac{539}{24} u \left[\frac{u d_1(5)}{q^{7/2}} + \frac{d_2(5)}{q^{5/2}} \right] \tag{7.4.18}$$

which implies that

$$b_{25}^{(2)} = 0 \tag{7.4.19}$$

$$b_{24}^{(2)} = b_{24}^{1} + \frac{539}{24} u^2 d_1(5) = \frac{7}{192} \left(17u^2 - 5\right) \tag{7.4.20}$$

$$b_{23}^{(1)} = b_{23} + \frac{539}{24} u d_2(5) = \frac{25u}{48} \tag{7.4.21}$$

We end thus the elimination of the "extra coefficients" (with $m > 2n = 4$). To proceed further we operate with the fundamental operation on (the second transformed of) b_{24}, but before it is necessary to expand it in the (7.3.25)

$$\frac{b_{24}^{(2)}}{q^{7/2}} = \frac{\frac{7u^2}{16}}{q^{7/2}} + \frac{35}{192} \frac{\left(u^2 - 1\right)}{q^{7/2}} \tag{7.4.22}$$

$$\equiv \frac{\frac{7u^2}{16}}{q^{7/2}} + \frac{35}{192} u d_1(4) \frac{1}{q^{5/2}} + \frac{35}{192} d_2(4) \frac{1}{q^{3/2}} \tag{7.4.23}$$

so that

$$b_{24}^{(3)} = \frac{7u^2}{16} \tag{7.4.24}$$

$$b_{23}^{(2)} = b_{23}^{(1)} + \frac{35}{192} u d_1(4) = \frac{3u}{8} \tag{7.4.25}$$

$$b_{22}^{(1)} = b_{22} + + \frac{35}{192} d_2(4) = \frac{5}{64} \tag{7.4.26}$$

Formula (7.3.41) now gives the coefficients of the differential operator in u

$$C_{24} = \frac{1}{2^5} \frac{28}{45} u^2 \tag{7.4.27}$$

$$C_{23} = \frac{1}{2^5} \frac{8}{3} u \tag{7.4.28}$$

$$C_{22} = \frac{1}{2^5} \frac{5}{3} \tag{7.4.29}$$

which are exactly those of He and Miao in their article[34]

We now write explicit formulas.

$$b_{24}^{(2)} = b_{24} + \frac{1}{u^2-1} \Big[u d_1(5) b_{25} + d_2(6) b_{26} \Big] + \frac{1}{(u^2-1)^2} u^2 d_1(6) d_1(5) b_{26} \tag{7.4.30}$$

$$= b_{24} + \frac{1}{u^2-1} u d_1(5) b_{25} + \frac{1}{(u^2-1)^2} \Big[(u^2-1) d_2(6) + u^2 d_1(6) d_1(5) \Big] b_{26} \tag{7.4.31}$$

$$= \Big[\beta_{240} - \beta_{264} d_2(6) \Big] + \Big[\beta_{242} + \beta_{253} d_1(5) + \beta_{264} d_1(6) d_1(5) + \beta_{264} d_2(6) \Big] u^2 \tag{7.4.32}$$

(7.4.33)

So the highest corrected coefficient c_{24} is

$$c_{24} = \Big[\beta_{240} + \beta_{242} + \beta_{253} d_1(5) + \beta_{264} d_1(6) d_1(5)\Big] u^2 \tag{7.4.34}$$

$$\Delta_{24}^{(2)} = -\Big[\beta_{240} - \beta_{264} d_2(6)\Big](u^2 - 1) \tag{7.4.35}$$

Passing to the next Gelfand-Dikii coefficient b_{23}, for its first transformation we need the transformed of b_{25}

$$b_{25}^{(1)} = b_{25} + \frac{1}{u^2-1} u d_1(6) b_{26}$$

so

$$b_{23}^{(1)} = b_{23} + d_2(5) b_{25}^{(1)} = b_{23} + d_2(5)\Big[b_{25} + \frac{1}{u^2-1} u d_1(6) b_{26}\Big] \tag{7.4.36}$$

$$= b_{23} + d_2(5) b_{25} + d_1(6) d_2(5) u \frac{b_{26}}{u^2-1} \tag{7.4.37}$$

$$b_{23}^{(2)} = c_{23} = b_{23}^{(1)} + \frac{\Delta_{24}^{(2)}}{u^2-1} u d_1(4) \tag{7.4.38}$$

$$= \Big[\beta_{231} + d_2(5)\beta_{253} + d_1(6) d_2(5)\beta_{264} - d_1(4)\beta_{240} - d_2(6) d_1(4)\beta_{264}\Big] u \tag{7.4.39}$$

$$= \Big[\beta_{231} - d_1(4)\beta_{240} + d_2(5)\beta_{253} + d_1(6) d_2(5)\beta_{264} - d_2(6) d_1(4)\beta_{264}\Big] u \tag{7.4.40}$$

We arrive now to the lowest coefficient b_{22}, which is transformed only once

$$b_{22}^{(1)} = c_{22} = b_{22} + \frac{\Delta_{24}^{(2)}}{u^2-1} d_2(4) \tag{7.4.41}$$

$$= \beta_{22} - d_2(4)\beta_{240} - d_2(6) d_2(4)\beta_{264} \tag{7.4.42}$$

Part III

Liouville ODE/IM

8 Zamolodchikov's Generalized Mathieu equation

8.1 Derivation of generalized Mathieu equation

In the draft of his unfinished work,[2] Al. Zamolodchikov considered the equation (2.1.1)

$$\left\{-\frac{d^2}{dx^2}+\frac{l(l+1)}{x^2}+x^{2M}-E\right\}\phi(x)=0 \tag{8.1.1}$$

which was considered by Dorey, Tateo[12] and Bazhanov, Lukyanov, A. Zamolodchikov, to establish the ODE-IM correspondence for the *minimal models*. In particular, they considered the range $M>0$[29]. Instead, the Liouville model corresponds to the range

$$\boxed{-\infty<\beta^2<0 \qquad \Longleftrightarrow \qquad -\infty<M<-1 \qquad \text{(LIOUVILLE)}} \tag{8.1.2}$$

Then in his draft, he conjectured that to establish the ODE-IM correspondence in the Liouville model, the following differential equation must be considered

$$\boxed{\left\{-\frac{d^2}{dy^2}+\mu_+e^{y/b}+\mu_-e^{-yb}+P^2\right\}\psi(y)=0} \tag{8.1.3}$$

where $y\in\mathbb{C}$; b is the dimensionless parameter for the Liouville model (which is the same as that of the massive Sinh-Gordon model); while P parametrized the highest weight vector of the Liouville Virasoro-Heisenberg module

$$\boxed{\Delta=-P^2+\frac{Q^2}{4}} \tag{8.1.4}$$

Zamolodchikov[2] propose to call μ_+ and μ_- *coupling constants* to be determined.
Al. Zamolodchikov propose to call this (conjecturely) ODE-IM equation (8.1.3) *Generalized Mathieu equation*. Actually, a better name would be "Generalized modified-Mathieu equation", because for $b=1$, for certain choices of μ_+ and μ_-, the equation reduces to the well-known modified-Mathieu equation.[26] We now reconstruct the passages which permit to obtain this equation (8.1.3) from the corresponding ODE-IM standard (8.1.1)[30]
The central charge for the Liouville model is

$$\boxed{c=1+6Q^2=13+6\Big(b^2+\frac{1}{b^2}\Big)} \tag{8.1.5}$$

It is evident that it is obtained by (1.3.3) by the analytic continuation on the parameter β of the Sine-Gordon model[8]

$$\boxed{\beta\to ib \quad b\in\mathbb{R}^+ \qquad \text{(LIOUVILLE)}} \tag{8.1.6}$$

so as to effectively transform it in the Sinh-Gordon model.[29]
Namely, we apply a succession of transformations, the first of which is the Langer transform

$$x=e^{\hat{y}} \quad \phi(x)=e^{\hat{y}/2}\psi(\hat{y}) \tag{8.1.7}$$

[29] Actually, in detail in the case $M>1$, and studied very briefly in the "dual" case $-1<M<0$.
[30] Al. Zamolodchikov did not report this calculations in his draft,[2] we aid ourselves with Bazhanov's seminar at Bologna INFN in september 2011,[1] which also informed us of the existence of this unplished and unfinished work of Zamolodchikov.

so that the equation becomes

$$\left\{-\frac{d^2}{d\hat{y}^2}+e^{2\hat{y}/\beta^2}-Ee^{2\hat{y}}+\left(l+\frac{1}{2}\right)^2\right\}\psi(\hat{y})=0 \tag{8.1.8}$$

Now, we continue to transform, passing to the variables

$$\hat{y}=\frac{\beta}{2}\bar{y} \tag{8.1.9}$$

$$\left\{-\frac{d^2}{d\bar{y}^2}+\frac{\beta^2}{4}e^{\bar{y}/\beta}-\frac{\beta^2}{4}Ee^{\beta\bar{y}}+\frac{\beta^2}{4}\left(l+\frac{1}{2}\right)^2\right\}\psi(\bar{y})=0 \tag{8.1.10}$$

and also

$$\bar{y}=i\tilde{y} \tag{8.1.11}$$

$$\left\{+\frac{d^2}{d\tilde{y}^2}+\frac{\beta^2}{4}e^{i\tilde{y}/\beta}-\frac{\beta^2}{4}Ee^{i\beta\tilde{y}}+\frac{\beta^2}{4}\left(l+\frac{1}{2}\right)^2\right\}\psi(\tilde{y})=0 \tag{8.1.12}$$

Now we send $\beta=ib$, that is, rather than of giving real values to β, we imagine it to have imaginary values. The equation for the minimal models is thus transformed in the equation for the Liouville model. We also define a new parameter[31]

$$\boxed{P^2=-\frac{p^2}{\beta^2}=\frac{p^2}{b^2}} \tag{8.1.13}$$

or also

$$P^2=\frac{(l+\frac{1}{2})^2b^2}{4}=-\frac{(l+\frac{1}{2})^2\beta^2}{4} \tag{8.1.14}$$

$$\left\{-\frac{d^2}{d\tilde{y}^2}+\frac{b^2}{4}e^{\tilde{y}/b}-\frac{b^2}{4}Ee^{-b\tilde{y}}+P^2\right\}\psi(\tilde{y})=0 \tag{8.1.15}$$

A final change of variable and parametrization

$$y=\tilde{y}-\alpha+b\ln\frac{b^2}{4} \tag{8.1.16}$$

$$\alpha=\frac{\ln(-E)}{2b}+\frac{b^2+1}{2b}\ln\frac{b^2}{4} \tag{8.1.17}$$

according to which the energy of (8.1.1) can be expressed as

$$E=-e^{2\alpha b}\left(\frac{4}{b^2}\right)^{b^2+1} \tag{8.1.18}$$

delivers now the *generalized Mathieu equation*

$$\boxed{\left\{-\frac{d^2}{dy^2}+e^{(\alpha+y)/b}+e^{(\alpha-y)b}+P^2\right\}\psi(y)=0} \tag{8.1.19}$$

with the coupling constants of Bazhanov seminar[1]

$$\mu_+^B=e^{\alpha/b}\qquad \mu_-^B=e^{\alpha b}\qquad (\mathcal{N}=2\text{ conventions}) \tag{8.1.20}$$

[31] Note that the limit $P^2\to\infty$ can be interpreted as the limit $l+\frac{1}{2}\to\infty$ and β constant, which means also $p\to\infty$. This consideration can be useful for the WKB analysis.

Somewhat contradictorily, *Bazhanov's choice of parameters make impossible the matching with the Sinh-Gordon*[32] *TBA conventions, even if they are correct for the conventions of $\mathcal{N}=2$ gauge theory.* Anyway, we show below that a simple shift in α and y is sufficient to get the correct TBA normalization. For now, we use μ_+^B and μ_-^B, as Bazhanov[1] invited us to do.
For later considerations, we observe also that the initial variable x and the Langer's variable $\hat{y}$, can be conveniently expressed in terms of the final variable y as

$$\boxed{\begin{aligned} x &= \exp\left[-\frac{yb}{2}-\frac{\alpha b}{2}+\frac{b^2}{2}\ln\frac{b^2}{4}\right] \qquad (8.1.21)\\ \hat{y} &= -\frac{yb}{2}-\frac{\alpha b}{2}+\frac{b^2}{2}\ln\frac{b^2}{4} \qquad (8.1.22)\end{aligned}}$$

This shows that $x\to 0$ corresponds to $\Re y\to+\infty$, while $x\to+\infty$ corresponds to $\Re y\to-\infty$.

8.2 ODE-IM for the Generalized Mathieu equation

8.2.1 Study of asymptotic solutions

Still following Bazhanov,[1] we specify the two uniquely defined WKB decaying solutions of Al. Zamolodchikov's generalizes Mathieu equation (8.1.19)

$$\boxed{\begin{aligned} U_0(y) &\simeq \frac{1}{\sqrt{2b}}e^{-\frac{\alpha+y}{4b}}e^{-2be^{\frac{\alpha+y}{2b}}} \qquad \Re y\to+\infty \qquad (8.2.1)\\ V_0(y) &\simeq \frac{b}{\sqrt{2}}e^{-\frac{(\alpha-y)b}{4}}e^{-\frac{2}{b}e^{\frac{(\alpha-y)b}{2}}} \qquad \Re y\to-\infty \qquad (8.2.2)\end{aligned}}$$

The Stokes sectors of $\mathbb{C}$[26] characterizing the generalized Mathieu equation are are only defined asymptotically, for $\Re y\to+\infty$, or $\Re y\to-\infty$, respectively:

$$D_k^+ = \{y\in\mathbb{C}|(2k-1)\pi b<\Im y<(2k+1)\pi b\} \quad k\in\mathbb{Z} \qquad (8.2.3)$$
$$D_k^- = \{y\in\mathbb{C}|(2k-1)\pi/b<\Im y<(2k+1)\pi/b\} \quad k\in\mathbb{Z} \qquad (8.2.4)$$

Equation (8.1.19) has two important symmetries:

$$\boxed{\begin{aligned} \Omega_b:\quad & D_k^+\to D_{k+1}^+\,,\quad D_k^-\to D_k^- && (8.2.5)\\ & y\mapsto y+\pi ib\,,\quad \alpha\mapsto\alpha+i\pi b\,, && (8.2.6)\\ \Omega_{1/b}:\quad & D_k^-\to D_{k+1}^-,\quad D_k^+\to D_k^+ && (8.2.7)\\ & y\mapsto y-\pi i/b\,,\quad \alpha\mapsto\alpha+i\pi/b && (8.2.8)\end{aligned}}$$

which in the original coordinates of (8.1.1) are respectively

$$\Omega_\beta:\quad x\to qx\,,\quad E\to q^{-2}E\,, \qquad (8.2.9)$$
$$\Omega_{1/\beta}:\quad x\to x\,,\quad E\to E \qquad (8.2.10)$$

where, as standard, $q=e^{i\pi\beta^2}$. Note that these symmetries are exact, that is, they hold in the whole plane, not only in a neighborhood of infinity.
By using these symmetries one can automatically generate new solutions of (8.1.19)

$$U_k(y)=\Omega_b^kU_0(y), \qquad V_0(y)=\Omega_b^kV_0(y), \qquad (8.2.11)$$
$$U_0(y)=\Omega_{1/b}^kU_0(y), \qquad V_k(y)=\Omega_{1/b}^kV_0(y), \qquad (8.2.12)$$

[32]Therefore, Liouville TBA conventions, since the leading order for large rapidity is the same.

whose explicit asymptotic expression is

$$U_k(y) \sim \frac{e^{-ik\pi/2}}{\sqrt{2}b} e^{-(\alpha+y)/4b} e^{(-1)^{k+1} 2b e^{(\alpha+y)/2b}} \tag{8.2.13}$$

$$V_k(y) \sim \frac{b e^{-ik\pi/2}}{\sqrt{2}} e^{-(\alpha-y)b/4} e^{(-1)^{k+1} \frac{2}{b} e^{(\alpha-y)b/2}} \tag{8.2.14}$$

Note, however, that the application of the symmetries to the original asymptotic expression does not give us information about the subdominant terms of the dominant solutions. The subdominant dominant or dominant behaviour of the solutions can be expressed by the limits:

$$\lim_{\Re y \to +\infty} U_{2n}(y) = 0 \tag{8.2.15}$$

$$\lim_{\Re y \to -\infty} V_{2n}(y) = 0 \tag{8.2.16}$$

$$\lim_{\Re y \to +\infty} U_{2n+1}(y) = i(-1)^{n+1}\infty \tag{8.2.17}$$

$$\lim_{\Re y \to -\infty} V_{2n+1}(y) = i(-1)^{n+1}\infty \tag{8.2.18}$$

The wronskians of "nearby solutions" is readily calculated

$$W[U_{k+1}, U_k] = -\frac{i}{b^2} \tag{8.2.19}$$

$$W[V_{k+1}, V_k] = +ib^2 \tag{8.2.20}$$

$$\tag{8.2.21}$$

However, we are not allowed to calculate in this way wronskians of the type $W[U_k, U_{-k}]$. Analytic continuation of the asymptotic expression would be needed[14]

8.2.2 QQ system

Define the entire function of α

$$\boxed{\bar{X}(\alpha) = W[V_0, U_0]} \tag{8.2.22}$$

i.e. the wronskian of the WKB solution around $y \to +\infty$ and that around $y \to -\infty$ ($\Im y = 0$). That it is non-null will be shown below (8.2.33). Note that it is independent from y.
As a consequence, applying the Ω_b and $\Omega_{1/b}$ symmetries we can obtain also

$$\bar{X}(\alpha + i\pi b) = W[V_0, U_1] \tag{8.2.23}$$

$$\bar{X}(\alpha + i\pi/b) = W[V_1, U_0] \tag{8.2.24}$$

and so on.
Let us expand linearly, but asymptotically, the fundamental pair of solutions at $+\infty$ in terms of the fundamental pair at $-\infty$.

$$V_0(y) = AU_0(y) + BU_1(y) \tag{8.2.25}$$

$$V_1(y) = CU_0(y) + DU_1(y) \tag{8.2.26}$$

the *Stokes coefficients* are readily calculated taking some wronskians

$$W[V_0, U_1] = AW[U_0, U_1] = \bar{X}(\alpha + i\pi b) = \frac{i}{b^2} A$$

$$A = -ib^2 \bar{X}(\alpha + i\pi b) \tag{8.2.27}$$

$$W[V_0, U_0] = BW[U_1, U_0] = \bar{X}(\alpha) = -\frac{i}{b^2} B$$

$$B = +ib^2 \bar{X}(\alpha) \tag{8.2.28}$$

$$
\begin{aligned}
W[V_1, U_1] = CW[U_0, U_1] = \bar{X}(\alpha + i\pi b + i\pi/b) = \frac{i}{b^2} C \\
C = -ib^2 \bar{X}(\alpha + i\pi b + i\pi/b)
\end{aligned} \tag{8.2.29}
$$

$$
\begin{aligned}
W[V_1, U_0] = DW[U_1, U_0] = \bar{X}(\alpha + i\pi/b) = -\frac{i}{b^2} D \\
D = ib^2 \bar{X}(\alpha + i\pi/b)
\end{aligned} \tag{8.2.30}
$$

Rewriting the relations (8.2.25),(8.2.26) we find the *linear relations*

$$
\frac{i}{b^2} V_0(y) = \bar{X}(\alpha + i\pi b) U_0(y) - \bar{X}(\alpha) U_1(y) \tag{8.2.31}
$$

$$
\frac{i}{b^2} V_1(y) = \bar{X}(\alpha + i\pi Q) U_0(y) - \bar{X}(\alpha + i\pi/b) U_1(y) \tag{8.2.32}
$$

From the definitions

$$
\lim_{\Re y \to +\infty} U_0(y) = 0 \qquad \lim_{\Re y \to +\infty} U_1(y) = i\infty
$$

and the relation (8.2.31)

$$
\lim_{\Re y \to +\infty} V_0(y) = ib^2 \bar{X}(\alpha) \lim_{\Re y \to +\infty} U_1(y) \tag{8.2.33}
$$

so that, in modulus, $|V_0| \to \infty$ as $y \to +\infty$. In other words V_0, *defined as subdominant for* $y \to -\infty$ *is dominant at for* $y \to +\infty$. This proves the linear independence of V_0 and U_0 and therefore that $\bar{X}(\alpha)$ is non null (this is not an assumption of the foregoing operations).
We can also obtain the QQ system

$$
\boxed{\bar{X}(\alpha)\bar{X}(\alpha + i\pi Q) - \bar{X}(\alpha + i\pi b)\bar{X}(\alpha + i\pi/b) = 1} \tag{8.2.34}
$$

In fact,

$$
\begin{aligned}
\left(\frac{i}{b^2}\right)^2 W[V_0, V_1] = \frac{i}{b^2} &= W[U_0, U_1]\Big[-\bar{X}(\alpha + i\pi b)\bar{X}(\alpha + i\pi/b) + \bar{X}(\alpha)\bar{X}(\alpha + i\pi Q)\Big] \\
&= \frac{i}{b^2}\Big[-\bar{X}(\alpha + i\pi b)\bar{X}(\alpha + i\pi/b) + \bar{X}(\alpha)\bar{X}(\alpha + i\pi Q)\Big]
\end{aligned}
$$

8.2.3 TQ systems

Taking inspiration from Dorey and Tateo[14] (summarized in 2), but modifying slightly their calculations, we can construct the first TQ system with the $\Omega_{1/b}$ symmetry, which operates on the solutions around $y \to -\infty$ or $x \to +\infty$.

$$
\begin{aligned}
V &\coloneqq V_0 \\
V_k &\coloneqq \Omega_{1/b} V = V(y - k\pi i/b; \alpha + k\pi i/b)
\end{aligned}
$$

Formula (8.2.35) is the source of difference between Dorey and Tateo construction and ours, i.e. we have no prefactor as in (2.1.10). The general Stokes relation reads

$$
V_{k-1}(y; \alpha) = C_k(\alpha) V_k(y; \alpha) + \tilde{C}_k(\alpha) V_{k+1}(y; \alpha) \tag{8.2.35}
$$

$$
C_k(\alpha) = C_{k-1}(\alpha + \pi i/b) \qquad \tilde{C}_k(\alpha) = \tilde{C}_{k-1}(\alpha + \pi i/b) \tag{8.2.36}
$$

Define the general wronskian

$$W_{k_1,k_2} \coloneqq W[V_{k_1}, V_{k_2}] \tag{8.2.37}$$

$$W_{k_1+1,k_2+1}(\alpha) = W_{k_1,k_2}(\alpha + i\pi/b) \tag{8.2.38}$$

$$W_{k_1,k_2}(\alpha + i\pi b) = W_{k_1,k_2} \tag{8.2.39}$$

$$W_{01} = W[V_0, V_1] = -ib^2 \tag{8.2.40}$$

In particular,

$$C \coloneqq C_0 = \frac{W_{-1,1}}{W_{01}} = \frac{i}{b^2} W[V_{-1}, V_1]$$

$$\tilde{C}_k = -\frac{W_{k-1,k}}{W_{k,k+1}} = -1 \quad \forall k$$

remember that C corresponds to the Baxter T function (2.4.6). The Stokes relation for $k = 0$ corresponds to the TQ relation

$$V_1 = CV_0 - V_{-1}$$

$$CV_0 = V_1 + V_{-1} \tag{8.2.41}$$

The solutions around $y \to +\infty$, i.e $x = 0$

$$U^+ \coloneqq U_0 \qquad U^- \coloneqq U_1 \tag{8.2.42}$$

$$U_k^\pm \coloneqq \Omega_{1/b} U^\pm = U^\pm \tag{8.2.43}$$

$$W[V_k, U^\pm](\alpha) = W[V, U^\pm](\alpha + k\pi i/b) \tag{8.2.44}$$

permit us to define also the D function, which corresponds (2.4.4) to the A or Q function

$$D^\mp(\alpha) = W[V, U^\pm](\alpha) \tag{8.2.45}$$

But, recalling the previous definition (8.2.22), we note that $D^-(\alpha)$ coincides with $\bar{X}(\alpha)$

$$\bar{X}(\alpha) = -D^-(\alpha) \tag{8.2.46}$$

Similarly

$$\bar{X}(\alpha + i\pi b) = -D^+(\alpha) \tag{8.2.47}$$

We can therefore write in different notations the TQ relation

$$C(\alpha)D^-(\alpha) = D^-(\alpha + \pi i/b) + D^-(\alpha - \pi i/b) \tag{8.2.48}$$

or

$$\boxed{\tilde{\bar{T}}(\alpha)\bar{X}(\alpha) = \bar{X}(\alpha + \pi i/b) + \bar{X}(\alpha - \pi i/b)} \tag{8.2.49}$$

and also

$$C(\alpha)D^+(\alpha) = D^+(\alpha + \pi i/b) + D^+(\alpha - \pi i/b) \tag{8.2.50}$$

$$\tilde{\bar{T}}(\alpha)\bar{X}(\alpha + \pi ib) = \bar{X}(\alpha + \pi i/b + \pi ib) + \bar{X}(\alpha - \pi i/b + \pi ib) = \tilde{\bar{T}}(\alpha + i\pi b)\bar{X}(\alpha + i\pi b) \tag{8.2.51}$$

the last equality being due to the $i\pi b$ periodicity (8.2.39)in α of the wronskians constructed with the V_k functions.

$$\tilde{\tilde{T}}(\alpha + i\pi b) = \tilde{\tilde{T}}(\alpha) \tag{8.2.52}$$

It is important to note that, thanks to the similarity of structure[33] of the generalized Mathieu equation (8.1.19) at $y \to +\infty$ and $y \to -\infty$, we can construct the first TQ system with the Ω_b symmetry, which acts on the solutions around $y \to +\infty$, i. e. $x = 0$. We merely repeat the just done passages.

$$U := U_0 \tag{8.2.53}$$

$$U_k := \Omega_b^k U = U(y + k\pi ib; \alpha + k\pi ib) \tag{8.2.54}$$

the Stokes relation between the solutions around $x = 0$ is

$$U_{k-1}(y;\alpha) = B_k(\alpha)U_k(y;\alpha) + \tilde{B}_k(\alpha)U_{k+1}(y;\alpha) \tag{8.2.55}$$

with new Stokes coefficients B_k and $\tilde{B}_k$, which enjoy the properties

$$B_k(\alpha) = B_{k-1}(\alpha + \pi ib) \qquad \tilde{B}_k(\alpha) = \tilde{B}_{k-1}(\alpha + \pi ib) \tag{8.2.56}$$

Define the general wronskian as

$$W_{k_1,k_2} := W[U_{k_1}, U_{k_2}] \tag{8.2.57}$$

which has the property

$$W_{k_1+1,k_2+1}(\alpha) = W_{k_1,k_2}(\alpha + i\pi b) \tag{8.2.58}$$

and the periodicity

$$W_{k_1,k_2}(\alpha + i\pi/b) = W_{k_1,k_2}(\alpha) \tag{8.2.59}$$

In particular,

$$B := B_0 = \frac{W_{-1,1}}{W_{01}} \tag{8.2.60}$$

$$\tilde{B}_k = -\frac{W_{k-1,k}}{W_{k,k+1}} = -1 \quad \forall k \tag{8.2.61}$$

$$W_{01} = W[U_0, U_1] = \frac{i}{b^2} \tag{8.2.62}$$

The Stokes relation for $k = 0$ is

$$U_1 = BU_0 - U_{-1}$$

$$BU_0 = U_1 + U_{-1} \tag{8.2.63}$$

and corresponds to the TQ relation, with B playing the role of T. Because of the periodicity (8.2.59) of the wronskians constructed with the U_k functions, the T function is periodic

$$\bar{T}(\alpha) = B(\alpha) = B(\alpha + i\pi/b) = \bar{T}(\alpha + i\pi/b) \tag{8.2.64}$$

The solution around $y \to -\infty$, i.e. $x \to +\infty$ are

$$V^+ := V_0 \qquad V^- := V_1 \tag{8.2.65}$$

$$V_k^\pm := \Omega_b^k V^\pm = V^\pm \tag{8.2.66}$$

[33] At both edges of the dominion there are essential singularities.

A useful wronskian is

$$W[V^-, V^+] = ib^2 \tag{8.2.67}$$

We define the analogues of the $D^{\mp}$ functions, which have the role of $A_{\pm}$ function

$$E^{\mp}(\alpha) = -W[U, V^{\pm}](\alpha) \tag{8.2.68}$$

we observe that

$$W[U_k, V^{\pm}](\alpha) = W[U, V^{\pm}](\alpha + k\pi ib) \tag{8.2.69}$$

Recalling the definition (8.2.22) we can write the correspondence

$$E^-(\alpha) = \bar{X}(\alpha) \tag{8.2.70}$$

$$E^+(\alpha) = \bar{X}(\alpha + i\pi/b) \tag{8.2.71}$$

We can write another TQ system[34]

$$B(\alpha)E^{\mp}(\alpha) = E^{\mp}(\alpha + \pi ib) + E^{\mp}(\alpha - \pi ib) \tag{8.2.72}$$

or

$$\boxed{\bar{T}(\alpha)\bar{X}(\alpha) = \bar{X}(\alpha + \pi ib) + \bar{X}(\alpha - \pi ib)} \tag{8.2.73}$$

and also

$$\bar{T}(\alpha + i\pi/b)\bar{X}(\alpha + \pi i/b) = \bar{X}(\alpha + \pi i/b + \pi ib) + \bar{X}(\alpha - \pi i/b + \pi ib) \tag{8.2.74}$$

8.2.4 Observations

In the ordinary variable x of (8.1.1) there is only one TQ system, where the Q is the wronskian between the eigenfunctions calculated at 0 and $+\infty$ in x (see, for instance[14]). In our case there are two different TQ-system essentially because in the Langer variable y, $+\infty$ and $-\infty$ are symmetrical, that is, the eigenfunctions have analogous form. Here Q is the wronskian between the eigenfunctions calculated at $-\infty$ and $+\infty$ in y. For the minimal models (β^2 real) there is only one T operator, while for the Liouville model there are two of them. However, for the Liouville model there only one $\bar{X}$ (Q) operator, since E^- and D^- differ only by a sign; while for the minimal models there are actually two $Q_{\pm}$ operators, which are obtain through the action of the symmetry Λ (2.1.4) which sends $p \to -p$. The two symmetries used in the ODE-IM construction for the range $\beta^2 > 0$ (minimal models) are very different: Ω (2.1.8) acts on the solutions at $x \to +\infty$ only through x and Λ (2.1.4) acts on the solutions at $x \to 0$ only through l. Now, the the two symmetries used for the range $\beta^2 < 0$ (Liouville model) are very similar: both Ω_b and $\Omega_{1/b}$ act on the solutions at $y \to \pm\infty$ through y and α. In the Liouville model, $P^2 \to P^2$ under the minimal models symmetry Λ (because $l \to -l-1$ leaves $l(l+1)$ invariant). Thus,
Therefore, we can somehow justify why the symmetry $\Omega_{1/b}$ in (8.2.10) is just the trivial symmetry in the minimal models set up, that is the *identity transformation*. In fact, we have just seen that also Λ (2.1.4) is the trivial symmetry in the Liouville model set up.

[34]In fact, they are two but differ by an insignificant shift of argument.

8.2.5 Quantum wronskians

We construct the quantum wronskians again adapting[14] construction to our situation.
We can write a Stokes relation for U also using the $V^{\pm}$ basis. Recalling that $W[V^-,V^+] = ib^2$ and the definitions of $E^{\mp}$, we must write

$$ib^2U(y,\alpha) = E^-(\alpha)V^-(y,\alpha) - E^+(\alpha)V^+(y,\alpha) \tag{8.2.75}$$

which, applying the Ω_b symmetry becomes

$$ib^2U_k(y,\alpha) = E^-(\alpha+ik\pi b)V^-(y,\alpha) - E^+(\alpha+ik\pi b)V^+(y,\alpha) \tag{8.2.76}$$

because the $V^{\mp}$ functions are invariant under this symmetry. We now take the wronskian of this U_{-1} and U_n using this formula

$$\begin{aligned} -b^4W[U_{-1},U_n](\alpha) &= -ib^2\bar{X}(\alpha-i\pi b)\bar{X}(\alpha+i\pi/b+i\pi nb) + ib^2\bar{X}(\alpha+i\pi/b-i\pi b)\bar{X}(\alpha+i\pi nb) \\ W[U_{-1},U_n](\alpha) &= i\frac{1}{b^2}\bar{X}(\alpha-i\pi b)\bar{X}(\alpha+i\pi/b+i\pi nb) - i\frac{1}{b^2}\bar{X}(\alpha+i\pi/b-i\pi b)\bar{X}(\alpha+i\pi nb) \end{aligned} \tag{8.2.77}$$

Now, defining

$$C^{(n)}(\alpha) = C_0^{(n)}(\alpha+i\pi(1-n)b) = \frac{W[U_{-1},U_n](\alpha+i\pi(1-n)b)}{W[U_0,U_1]} = -ib^2W[U_{-1},U_n](\alpha+i\pi(1-n)b) \tag{8.2.78}$$

we get

$$\begin{aligned} C^{(n)}(\alpha+i\pi(n-1)b) &= \bar{X}(\alpha-i\pi b)\bar{X}(\alpha+i\pi/b+i\pi nb) - \bar{X}(\alpha+i\pi/b-i\pi b)\bar{X}(\alpha+i\pi nb) \\ C^{(n)}(\alpha) &= \bar{X}(\alpha-i\pi nb)\bar{X}(\alpha+i\pi/b+i\pi b) - \bar{X}(\alpha+i\pi/b-i\pi nb)\bar{X}(\alpha+i\pi b) \end{aligned} \tag{8.2.79}$$

In the case $n=1$

$$\boxed{\bar{T}(\alpha) = -\bar{X}(\alpha+i\pi b)\bar{X}(\alpha+i\pi/b-i\pi b) + \bar{X}(\alpha-i\pi b)\bar{X}(\alpha+i\pi/b+i\pi b)} \tag{8.2.80}$$

We note that with respect to the seminar of V. Bazhanov[1] we got the opposite sign.

8.3 Liouville integrable structure

The works of Bazhanov, Lukyanov, A. B. Zamolodchikov,[7],[8] restricted consideration on the region $-\infty < c \leq 1$. In his unfinished work,[2] Al. Zamolodchikov tried to extend their considerations to the region $c > 1$. In this subsection, we report and comment Zamolodchikov"s (somewhat non-rigorous) definitions of the *general* $\mathbf{T}_j$ and $\mathbf{Q}$ operators (not just their void eigenvalues, as we are used to do).
Zamolodchikov considered the usual chiral free boson field

$$\boxed{\phi(u) = \phi_0 + \hat{P}u + \sum_{n\neq 0}\frac{a_{-n}}{in}e^{inu}} \tag{8.3.1}$$

but defined in a such a way as to be free from the parameter b (analogue of β, cf. (1.3.1) and (1.3.5))

$$[a_m,a_n] = \frac{1}{2}m\delta_{m,n} \tag{8.3.2}$$

$$[a_m,P] = [a_m,\phi_0] = 0 \tag{8.3.3}$$

$$[\phi_0,P] = \frac{i}{2} \tag{8.3.4}$$

For any b, the eigenspace $\mathcal{B}(P)$ of the operator $\hat{P}$ with eigenvalue P is a highest weight representation of Vir with central charge and conformal weight

$$c = 1 + 6Q^2 = 13 + 6b^2 + 6\frac{1}{b^2} \tag{8.3.5}$$

$$\Delta = \frac{Q^2}{4} - P^2 \tag{8.3.6}$$

The corresponding highest weight vector is[30]

$$e^{2iP\phi_0}|p\rangle \tag{8.3.7}$$

As we observed in section 8.2.4, at variance with the minimal models case, now there are two different kind of $\mathbf{T}$ operators. In fact, Zamolodchikov[2] considered two copies of vertex operators (cf. (1.3.6)) and of other quantities of quantum integrability

$$V_{\pm b} =: \exp\left(\pm 2b\phi(u)\right): \tag{8.3.8}$$

$$V_{\pm 1/b} =: \exp\left(\pm 2\phi(u)/b\right): \tag{8.3.9}$$

Since there are two symmetries at infinity, two q parameters must also be defined (cf. (1.3.10))

$$\boxed{q = e^{i\pi b^2}\,; \qquad \tilde{q} = e^{i\pi/b^2}} \tag{8.3.10}$$

As a consequence, Zamolodchikov had to consider also two quantum algebras $U_q(sl(2))$, and $U_{\hat{q}}(sl(2))$, whose generators have the commutation relations (cf. (1.3.9))

$$[H, E_\pm] = \pm 2E_\pm\,, \qquad [E_+, E_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{8.3.11}$$

$$[\tilde{H}, \tilde{E}_\pm] = \pm 2\tilde{E}_\pm\,, \qquad [\tilde{E}_+, \tilde{E}_-] = \frac{\tilde{q}^H - \tilde{q}^{-H}}{\tilde{q} - \tilde{q}^{-1}} \tag{8.3.12}$$

Therefore, by analogy with the[8] foundational work for the minimal models, Zamolodhikov defined the two $\mathbf{T}_j$ operators as

$$\mathbf{T}_j(\lambda) = tr_{\pi_j}\left[e^{2\pi i bPH}\mathcal{P}\exp\left(\lambda\int_0^{2\pi} K(u)du\right)\right] \tag{8.3.13}$$

$$\tilde{\mathbf{T}}_j(\lambda) = tr_{\pi_j}\left[e^{2\pi i P\tilde{H}/b}\mathcal{P}\exp\left(\tilde{\lambda}\int_0^{2\pi} \tilde{K}(u)du\right)\right] \tag{8.3.14}$$

where

$$K(u) = V_{-b}(u)q^{\frac{H}{2}}E_+ + V_b(u)q^{-\frac{H}{2}}E_- \tag{8.3.15}$$

$$\tilde{K}(u) = V_{-1/b}(u)\tilde{q}^{\frac{\tilde{H}}{2}}\tilde{E}_+ + V_{1/b}(u)\tilde{q}^{-\frac{H}{2}}\tilde{E}_- \tag{8.3.16}$$

Similarly, Zamolodchikov[2] defined also two quantum oscillator algebras (cf. (1.4.1))

$$[H, \mathcal{E}_\pm] = \pm 2\mathcal{E}_\pm\,, \qquad q\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q - q^{-1}} \tag{8.3.17}$$

$$[\tilde{\mathcal{H}}, \tilde{\mathcal{E}}_\pm] = \pm 2\tilde{\mathcal{E}}_\pm\,, \qquad \tilde{q}\tilde{\mathcal{E}}_+\tilde{\mathcal{E}}_- - \tilde{q}^{-1}\tilde{\mathcal{E}}_-\tilde{\mathcal{E}}_+ = \frac{1}{\tilde{q} - \tilde{q}^{-1}} \tag{8.3.18}$$

However, he did not specify the Q operator expression relative to this algebra.
For the purpose of our ODE-IM construction, the main thing we have to note is that, for the large energy (or rapidity) expansion of the T and Q functions, there are *two different expansion parameters,*

rather than one (cf. the parameter λ in (1.3.21) and (1.6.6)). In fact, their dependence can be deduced by (8.1.18). Explicitly[2]

$$\lambda_b \propto e^{\alpha b} = e^{\theta Q b} \tag{8.3.19}$$

$$\lambda_{1/b} \propto e^{\alpha/b} = e^{\theta Q/b} \tag{8.3.20}$$

However, Zamolodchikov did not specify the normalizations. We are going to fix them through the leading order through the TBA matching.[35]
Comparing with λ in (1.5.7) we note that it coincides only with λ_b

$$\lambda = e^{\frac{\theta}{1+\xi}} = e^{\theta(1+b^2)} = e^{\theta Q b} = \lambda_b \tag{8.3.21}$$

This means that our transformations to get the generalized Mathieu equation (8.1.19) somewhat break the duality symmetry, as one can see by (8.1.18).

8.4 Minimal models analogue

We now stop our considerations on the Liouville model for an important comment: the previous equation (8.1.10) up describes minimal models if $0 < \beta^2 < 1$. If we define

$$t = \bar{y} - a + \beta \ln \frac{\beta^2}{4} \tag{8.4.1}$$

the equation becomes

$$\left\{ -\frac{d^2}{dt^2} + e^{a/\beta} e^{t/\beta} - E \frac{\beta^2}{4} \left(\frac{\beta^2}{4} \right)^{-\beta^2} e^{\beta a} e^{\beta t} + \frac{\beta^2}{4} \left(l + \frac{1}{2} \right)^2 \right\} \psi(t) = 0$$

And defining also a as

$$\tilde{\alpha} = -\frac{\ln(-E)}{2\beta} + \frac{\beta^2 - 1}{2\beta} \ln \frac{\beta^2}{4} \tag{8.4.2}$$

which means

$$E = -e^{-2\beta\tilde{\alpha}} \left(\frac{4}{\beta^2} \right)^{1-\beta^2} \tag{8.4.3}$$

Finally, we can write the new "Langer form" minimal models ODE-IM equation

$$\left\{ -\frac{d^2}{dt^2} + e^{(\tilde{\alpha}+t)/\beta} + e^{\beta(-\tilde{\alpha}+t)} - P^2 \right\} \psi(t) = 0 \tag{8.4.4}$$

Note the minus sign of P^2 and that now t in the two exponential appears always with the same sign, while now $\tilde{\alpha}$ does change sign. *Only one of the two exponentials, evidently, dominates for* $\beta \neq 1$, *but it is always the same for every* t. This seems to make impossible the construction of the following paragraph with *two TQ systems*. However, as it should be, one TQ system can be constructed.
At $\beta = 1$ this equation becomes

$$\left\{ -\frac{d^2}{dt^2} + (2e^a)e^t - P^2 \right\} \psi(t) = 0 \,, \qquad \beta = 1 \tag{8.4.5}$$

i.e. an exponential potential for the energy P^2.

[35] In particular, we shall find that *the $\bar{X}(\alpha)$ function which we used until now has an expansion in terms of $e^{\theta Q}$, even if the standard TBA function $X(\theta)$ has an expansion in $K = e^\theta$, if we define it by $X(\theta) = \bar{X}(\alpha)$ (see section 9 for further specifications).*

8.5 Modified Schrödinger form

Consider the Generalized Mathieu equation

$$\left\{-\frac{d^2}{dy^2}+e^{(\alpha+y)/b}+e^{(\alpha-y)b}+P^2\right\}\psi(y)=0$$

we search a change of variable that would put it in the form apt for the large energy expansion

$$\left\{-\frac{d^2}{dy'^2}+\Lambda p(y')+u(y')\right\}\psi(y)=0 \tag{8.5.1}$$

for some parameter Λ and functions p and u.
It suffices a simple shift of y

$$y=y'+s \qquad s=\frac{\alpha(b-1/b)}{Q}=\frac{\alpha(b-1/b)}{b+1/b} \tag{8.5.2}$$

$$2\theta=(\alpha-s)b=\frac{(\alpha+s)}{b} \tag{8.5.3}$$

the new form is

$$\boxed{\left\{-\frac{d^2}{dy'^2}+e^{2\theta}(e^{y/b}+e^{-yb})+P^2\right\}\psi(y)=0} \tag{8.5.4}$$

with

$$\boxed{\theta=\frac{\alpha}{Q}} \tag{8.5.5}$$

and $\Lambda=e^{2\theta}$, $p(y)=e^{y/b}+e^{-yb}$ and $u=P^2$.
The symmetry transformations in the new parameter θ become

$$\Omega_b: \quad \alpha\to\alpha+i\pi b \quad \Longleftrightarrow \quad \theta\to\theta+i\pi p \tag{8.5.6}$$

$$\Omega_{1/b}: \quad \alpha\to\alpha+i\pi/b \quad \Longleftrightarrow \quad \theta\to\theta+i\pi(1-p) \tag{8.5.7}$$

being $p=b/Q$ and $1-p=1/bQ$.
The same kind of transformation on the Langer form of the minimal models equation (8.4.4)

$$\left\{-\frac{d^2}{dt^2}+e^{(\tilde{\alpha}+t)/\beta}+e^{\beta(-\tilde{\alpha}+t)}-P^2\right\}\psi(y)=0 \tag{8.5.8}$$

requires the shift

$$t=y'+s \qquad s=\frac{a(1+\beta^2)}{-1+\beta^2} \tag{8.5.9}$$

so that the equation becomes

$$\left\{-\frac{d^2}{dy'^2}+e^{2\tilde{\theta}}(e^{y'/\beta}+e^{\beta y'})-P^2\right\}\psi(y')=0 \tag{8.5.10}$$

where

$$2\tilde{\theta}=\frac{a+s}{\beta}=(-a+s)\beta=\frac{2\alpha\beta}{\beta^2-1} \tag{8.5.11}$$

$$\tilde{\theta}=\alpha\frac{\beta}{\beta^2-1} \tag{8.5.12}$$

8.6 Reflection amplitude

Following Zamolodchikov[2] we want to express U_0 and U_1 in (8.2.13) as

$$U_0 \simeq \frac{1}{\sqrt{kr}} e^{-kr} \qquad \Re y \to +\infty \tag{8.6.1}$$

$$U_1 \simeq \frac{1}{\sqrt{kr}} e^{+kr} \qquad \Re y \to +\infty \tag{8.6.2}$$

with

$$kr = 2be^{\frac{\alpha+y}{2b}}$$

For example, we can set

$$k = (2b)^{Q/b} e^{\alpha/b} = (2b)^{Q/b} e^{\theta Q/b} = (2be^{\theta})^{Q/b} \tag{8.6.3}$$

$$r = (4b^2 e^{\alpha b})^{-1/2b^2} e^{y/2b} = (2b)^{-1/b^2} e^{-\alpha/2b} e^{y/2b} \tag{8.6.4}$$

We rewrite the linear relation (8.2.31) shifting $\alpha \to \alpha - i\pi b/2$ (or $\theta \to \theta - i\pi p/2$)

$$V_0(y;\theta) = -ib^2 \bar{X}(\theta + i\pi p) U_0(y;\theta) + ib^2 \bar{X}(\theta) U_1(y;\theta)$$

$$V_0(y;\theta - i\pi p/2) = -ib^2 \bar{X}(\theta + i\pi p/2) U_0(y;\theta + i\pi p/2) + ib^2 \bar{X}(\theta - i\pi p/2) U_1(y;\theta - i\pi p/2)$$

with the new parametrization

$$\begin{aligned}
U_0(y;\theta) &= \frac{\exp{(ikr)}}{\sqrt{kr}} \\
U_0(y;\theta - i\pi p/2) &= i\frac{\exp{(ikr - i\pi/4)}}{\sqrt{kr}} \\
V_0(y;\theta - i\pi p/2) &= b^2 \bar{X}(\theta + i\pi p/2) \frac{\exp{(ikr - i\pi/4)}}{\sqrt{kr}} + b^2 \bar{X}(\theta - i\pi p/2) \frac{\exp{(-ikr + i\pi 4)}}{\sqrt{kr}} \\
&= e^{i\pi/4} b^2 \bar{X}(\theta - i\pi p/2) \left[S(k) \frac{\exp{(ikr)}}{\sqrt{kr}} + \frac{\exp{(-ikr)}}{\sqrt{kr}} \right]
\end{aligned}$$

Zamolodchikov[2] defined a reflection amplitude as

$$\boxed{S(\theta) = -i \frac{\bar{X}(\theta + i\pi p/2)}{\bar{X}(\theta - i\pi p/2)}} \tag{8.6.5}$$

However, since the waves and variables are different, this is only formally a reflection amplitude, but is not exactly the standard Liouville reflection amplitude.[30]

9 Thermodynamic Bethe ansatz for Liouville model

9.1 X and T functions matching

The correct X function for the Sinh-Gordon or Liouville TBA of sections 3 (cf. (3.5.5) and (8.2.73)) is not $X(\alpha)$ defined in this section but a new function $\bar{X}(\theta)$, defined by

$$X(\theta) = \bar{X}(\alpha) \tag{9.1.1}$$

with $\alpha = \theta Q$ from (8.5.5), so that $X(\theta) = \bar{X}(\theta Q)$. The same discourse must be applied to the T function

$$T(\theta) = \bar{T}(\alpha) \tag{9.1.2}$$

In other words, one recovers the TBA constructions of the previous sections by a dilation of the rapidity θ.

For example, the *TQ relation* is correctly expressed as (cf. (8.2.73))

$$T(\theta)X(\theta) = X(\theta + i\pi p) + X(\theta - i\pi p) \tag{9.1.3}$$

simlarly, the *TQ dual relation* is matched (cf. (8.2.49))

$$\tilde{T}(\theta)X(\theta) = X(\theta + i\pi(1-p)) + X(\theta - i\pi(1-p)) \tag{9.1.4}$$

Another example is the X system for the argument α (cf. (8.2.34))

$$X(\theta + i\pi/2)X(\theta - i\pi/2) = 1 + X(\theta + ia\pi/2)X(\theta - ia\pi/2) \tag{9.1.5}$$

As a consequence, for instance, even if the standard[29] asymptotic expansion of $X(\theta)$ for Liouville is in terms of the parameter $K = e^{\theta}$

$$\boxed{\log X(\theta) = -\tilde{B}_0 e^{\theta} - \sum_{n=1}^{\infty} \tilde{B}_n I_{2n-1} e^{-(2n-1)\theta}} \tag{9.1.6}$$

we expect that the asymptotic expansion of $\bar{X}$ to be in the parameter $e^{\alpha Q}$

$$\boxed{\log \bar{X}(\alpha) = -\tilde{B}_0 e^{\alpha Q} - \sum_{n=1}^{\infty} \tilde{B}_n I_{2n-1} e^{-(2n-1)\alpha Q}} \tag{9.1.7}$$

The normalization constants are given by Lukyanov[29]

$$\tilde{B}_n = \left[\frac{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})mR}{8\sqrt{\pi}Q}\right]^{1-2n} \frac{\Gamma(\frac{(2n-1)p}{2})\Gamma(\frac{(2n-1)(1-p)}{2})}{2\sqrt{\pi}n!Q} \tag{9.1.8}$$

9.2 Matching of large energy leading order

We estimate the large μ behaviour, by the semiclassical approximation to the differential equation originally considered by Al. Zamolodchikov[2] (not Bazhanov[1])

$$\boxed{-\frac{d^2}{dy^2}\psi(y) + \left[P^2 + \mu_- e^{-yb} + \mu_+ e^{y/b}\right]\psi(y) = 0} \tag{9.2.1}$$

i.e., without making any choice of μ_+ and μ_-. In fact, as we already pointed out, Bazhanov's choice (8.1.20) is not consistent with TBA. Define, as Al. Zamolodchikov did[2]

$$\bar{p}(y) = \mu_- e^{-yb} + \mu_+ e^{y/b} \tag{9.2.2}$$

Now, at the leading order in μ_- and μ_+ (considering P^2 negligible)

$$W[V_0, U_0] \sim 2\sqrt{\bar{p}(y)}U_0(y)V_0(y) = \int_{-\infty}^{\infty} dy' \sqrt{\bar{p}(y')} \tag{9.2.3}$$

where the integral is understood as analytic continuation in b.

$$\begin{aligned}
\int_{-\infty}^{\infty} dy' \sqrt{\mu_- e^{-yb} + \mu_+ e^{y/b}} &= \mu_-^{1/2} \int_{-\infty}^{\infty} dy\, e^{-yb/2}\Big[1 + \frac{\mu_+}{\mu_-}e^{yQ}\Big]^{1/2} \qquad y = y' - \frac{1}{Q}\ln\frac{\mu_+}{\mu_-} \\
&= \mu_-^{1/2}\left[\frac{\mu_+}{\mu_-}\right]^{\frac{b}{2Q}} \int_{-\infty}^{\infty} dy'\, e^{-y'b/2}\Big[1 + e^{y'Q}\Big]^{1/2} \qquad x = e^{y'Q} \;\; dy' = \frac{1}{Q}\frac{dx}{x} \\
&= \frac{1}{Q}\mu_-^{1/2}\left[\frac{\mu_+}{\mu_-}\right]^{\frac{b}{2Q}} \int_0^{\infty} dx\, x^{-\frac{b}{2Q}-1}\Big[1 + x\Big]^{1/2} \qquad x = \frac{t}{1-t} \quad dx = \frac{dt}{(1-t)^2} \\
&= \frac{1}{Q}\mu_-^{1/2}\left[\frac{\mu_+}{\mu_-}\right]^{\frac{b}{2Q}} \int_0^{1} dt\, (1-t)^{-2} t^{-\frac{b}{2Q}-1}(1-t)^{\frac{b}{2Q}+1}(1-t)^{-1/2} \\
&= \frac{1}{Q}\mu_-^{1/2}\left[\frac{\mu_+}{\mu_-}\right]^{\frac{b}{2Q}} \int_0^{1} dt\, (1-t)^{-1/2+b/2Q-1} t^{-\frac{b}{2Q}-1} \\
&= \frac{1}{Q}\mu_-^{\frac{1-p}{2}}\mu_+^{\frac{p}{2}} \frac{\Gamma(-\frac{p}{2})\Gamma(-\frac{1-p}{2})}{\Gamma(-\frac{1}{2})}
\end{aligned} \tag{9.2.4}$$

that is, we reduced our integral to a Euler Beta function. We continue, using the reflection property of the Gamma function

$$\begin{aligned}
\int_{-\infty}^{\infty} dy' \sqrt{\mu_- e^{-yb} + \mu_+ e^{y/b}} &= -\frac{1}{Q}\mu_-^{\frac{1-p}{2}}\mu_+^{\frac{p}{2}} \frac{\Gamma(\frac{3}{2})}{\Gamma(1+\frac{p}{2})\Gamma(1+\frac{1-p}{2})} \frac{\pi}{\sin(\pi p/2)\sin(\pi(1-p)/2)} \\
&= -\frac{1}{Q}\mu_-^{\frac{1-p}{2}}\mu_+^{\frac{p}{2}} \frac{\frac{1}{2}\sqrt{\pi}\pi}{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})} \frac{4}{p(1-p)} \frac{1}{\sin(\pi p/2)\cos(\pi p/2)} \\
&= -\mu_-^{\frac{1-p}{2}}\mu_+^{\frac{p}{2}} \frac{2\sqrt{\pi}\pi Q}{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})} \frac{1}{b(Q-b)} \frac{1}{\sin(\pi p/2)\cos(\pi p/2)} \\
&= -\mu_-^{\frac{1-p}{2}}\mu_+^{\frac{p}{2}} \frac{4\sqrt{\pi}\pi Q}{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})} \frac{1}{\sin(\pi p)}
\end{aligned} \tag{9.2.5}$$

so we match Zamolodchikov's[2] result. Zamolochikov noted that, *notwithstanding the general form* (9.2.1) *of the his Generalized Mathieu equation, by a sort of gauge symmetry, all its property depend on the following combination of parameters μ_+ and μ_-.*

$$\boxed{\mu = \mu_+^b \mu_-^{1/b}} \tag{9.2.6}$$

so for the TBA matching at the leading order, comparing with (3.4.14), it must hold the equality

$$\mu^{1/2Q}\frac{4\sqrt{\pi}\pi Q}{\Gamma(\frac{b}{Q})\Gamma(\frac{1}{2bQ})}\frac{1}{\sin(\pi b/Q)} = e^{\theta}\frac{\pi}{2\sin(\pi b/Q)} \tag{9.2.7}$$

so we infer that μ must be chosen as

$$\boxed{\mu = e^{2Q\theta}\tilde{\mu}} \tag{9.2.8}$$

where $\tilde{\mu}$ is

$$\tilde{\mu} = \left[\frac{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8\sqrt{\pi}Q}\right]^{2Q} \tag{9.2.9}$$

$$:= \tilde{\mu}_+^b \tilde{\mu}_-^{1/b} \tag{9.2.10}$$

and it has been parametrized in terms of some $\tilde{\mu}_+$ and $\tilde{\mu}_-$ given by

$$\tilde{\mu}_+^B = \left[\frac{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8\sqrt{\pi}Q}\right]^{Q/b} \tag{9.2.11}$$

$$\tilde{\mu}_-^B = \left[\frac{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8\sqrt{\pi}Q}\right]^{Qb} \tag{9.2.12}$$

This parametrization is convenient for a direct match with Bazhanov's equation (8.1.19). In fact, if we make a shift of α in

$$\alpha \to \alpha + Q\ln\left[\frac{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8\sqrt{\pi}Q}\right] \tag{9.2.13}$$

$$-\frac{d^2}{dy^2}\psi(y) + \left[P^2 + \tilde{\mu}_- e^{\alpha b}e^{-yb} + \tilde{\mu}_+ e^{\alpha/b}e^{y/b}\right]\psi(y) = 0 \tag{9.2.14}$$

Alternatively, if we we consider Fioravanti's "shifted equation" (8.5.4), which is more conveniently for large energy expansion in the parameter e^{θ} (recall that $\alpha = \theta Q$) we must parametrize $\tilde{\mu}$ in an equal way, that iswith is

$$\tilde{\mu} = \tilde{\mu}_+^b \tilde{\mu}_-^{1/b} = \left[\frac{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8\sqrt{\pi}Q}\right]^{2Q} \tag{9.2.15}$$

$$\tilde{\mu}_+ = \left[\frac{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8\sqrt{\pi}Q}\right]^{Q/b} \tag{9.2.16}$$

$$\tilde{\mu}_- = \left[\frac{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8\sqrt{\pi}Q}\right]^{Qb} \tag{9.2.17}$$

This parametrization is convenient, because if we make a shift of θ in Fioravanti's equation (8.5.4)

$$\boxed{\theta \to \theta_L = \theta + \ln\left[\frac{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8\sqrt{\pi}Q}\right]} \tag{9.2.18}$$

$$\boxed{-\frac{d^2}{dy^2}\psi(y) + \left[P^2 + \tilde{\mu}^{1/Q}e^{2\theta}\left(e^{-yb} + e^{y/b}\right)\right]\psi(y) = 0} \tag{9.2.19}$$

We emphasize that *the Generalized Mathieu equation in this form, that is, after both a shift in the independent variable y and in the rapidity θ with respect to Bazhanov's form,[1] is apt for the Liouville ODE-IM analysis, because it matches the standard leading Sinh-Gordon TBA.[?] However, the previous Fioravanti's form* (8.5.4)*, after only a shift in y, is apt for the $\mathcal{N} = 2$ analysis.*

9.2.1 Liouville TBA

Liouville TBA was already, formally, obtained from Sinh-Gordon TBA in (3.2.10). Anyway, we report here that important result

$$\varepsilon = \pi e^{\theta_L} - \varphi * \log\left(1 + e^{-\varepsilon}\right) \tag{9.2.20}$$

where φ is the usual Sinh-Gordon kernel (3.2.4).
The $b = 1$ self dual case is particularly important, so we specify the TBA equation

$$\varepsilon = \pi e^{\theta_L} - 2\hat{\varphi} * \log\left(1 + e^{-\varepsilon}\right) \tag{9.2.21}$$

where we defined a new kernel

$$\hat{\varphi(\theta)} = \frac{1}{2\pi}\frac{1}{\cosh\theta_L} \tag{9.2.22}$$

Note that because $a = 0$ at $b = 1$ the former kernel (3.2.4) is twice the present

$$\varphi(\theta_L) = 2\hat{\varphi}(\theta_L) \tag{9.2.23}$$

In fact, this definition was made because the factor 2 is crucial for further developments, as we are going to show.

9.2.2 Speculations on the TBA for Q at the self dual point

We now concentrate in particular on the self dual case $b = 1$. It is important to note that, because $a = 0$ for $b = 1$

$$Y(\theta) = X^2(\theta) \tag{9.2.24}$$

So in terms of the pseudoenergy the TBA is

$$\varepsilon(\theta) = -Z_1 e^{\theta} - 2\int_{l_1} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \log\left(1 + e^{-\varepsilon(\theta')}\right) \tag{9.2.25}$$

We can consider then also a new "pseudoenergy" ε_X for X, namely

$$X(\theta) = e^{-\varepsilon_X(\theta)} \qquad \varepsilon_X(\theta) = \frac{1}{2}\varepsilon(\theta) \tag{9.2.26}$$

Noting that

$$\log(1 + X^2) = \log(1 + iX) + \log(1 - iX) \tag{9.2.27}$$

we can write

$$\varepsilon_X(\theta) = -\frac{1}{2}Z_1 e^{\theta} - \int_{l_1} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \left[\log\left(1 + e^{-\varepsilon_X(\theta') + i\pi/2}\right) + \log\left(1 + e^{-\varepsilon_X(\theta') - i\pi/2}\right)\right] \tag{9.2.28}$$

9.2.3 $\mathcal{N} = 2$ TBA and proof of a conjecture by Gaiotto

Gaiotto, in his article,[37] considered the pure $SU(2)$ gauge theory spectral curve

$$x^2 = \frac{1}{z} + \frac{u}{z^2} + \frac{1}{z^3} \tag{9.2.29}$$

which is reconducible to Seiberg-Witten spectral curve.[36] Gaiotto's noted that setting $u = 0$, for symmetry reasons his "TBA-like" integral equation for a certain $X_G(\theta)$ reduces to (I set $\epsilon_G = e^{-\theta}, \epsilon'_G = e^{-\theta'}$)

$$\log X_G(\theta) = +Z_1 e^{\theta} - 2\frac{e^{-\theta}}{\pi} \int_{l_1} \frac{d\theta' e^{-\theta'}}{e^{-2\theta'} + e^{-2\theta}} \log(1 + X_G(\theta)) \tag{9.2.30}$$

which becomes

$$\begin{aligned} \log X_G(\theta) &= +Z_1 e^{\theta} + 2\frac{1}{\pi} \int_{l_1} \frac{d\theta'}{e^{-\theta'+\theta} + e^{-\theta+\theta'}} \log(1 + X_G(\theta)) \\ &= Z_1 e^{\theta} + 2\int_{l_1} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \log(1 + X_G(\theta)) \end{aligned} \tag{9.2.31}$$

In order for this to actually be $\mathcal{N} = 2$ TBA Z_1 must be such that

$$\boxed{e^{\theta_L - \theta} Z_1 = \pi Z_1 = \pi e^{-(\theta - \theta_L)} = \pi \frac{8\sqrt{\pi}Q}{\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}} \tag{9.2.32}$$

Gaiotto reported also that numerical evidence gave suggested him the following candidate as Schrödinger operator ($u \neq 0$)

$$\boxed{-\epsilon_G^2 \partial_z^2 + \frac{1}{z} + \frac{u - \epsilon^2/4}{z^2} + \frac{1}{z^3}} \tag{9.2.33}$$

which corresponds exactly to our own with $b = 1$.

$$\epsilon^2 \partial_z^2 + \frac{1}{z} + \frac{u - \epsilon^2/4}{z^2} + \frac{1}{z^3} \tag{9.2.34}$$

Consider the generalized Mathieu equation on the imaginary axis $x = -iy$, set $\epsilon = e^{-\theta}$ and $u = P^2\epsilon^2$

$$\left\{\frac{d^2}{dx^2} + \frac{1}{\epsilon^2}\left[e^{ix/b} + e^{-ibx} + u\right]\right\}\psi(x) = 0 \tag{9.2.35}$$

Now change variable $z = e^{ix/b}$ and $\phi(z) = \sqrt{z}\psi(z)$, the equation becomes

$$\left\{-\frac{d^2}{dz^2} + \frac{1}{\epsilon^2}\frac{b^2}{z^2}\left[z + u + \frac{1}{z^{b^2}}\right] - \frac{1}{4}\right\}\phi(z) = 0$$

or

$$\boxed{\left\{-\frac{d^2}{dz^2} + \frac{1}{\epsilon^2}\left[\frac{b^2}{z} + \frac{b^2 u}{z^2} + \frac{b^2}{z^{b^2+2}}\right] - \frac{1}{4}\right\}\phi(z) = 0} \tag{9.2.36}$$

We could also choose $t = e^{-ix/b}$ and obtain

$$\left\{-\frac{d^2}{dt^2} + \frac{1}{\epsilon^2}\frac{1}{b^2 t^2}\left[t + u + \frac{1}{t^{1/b^2}}\right] - \frac{1}{4}\right\}\phi(t) = 0$$

$$\left\{-\frac{d^2}{dt^2} + \frac{1}{\epsilon^2}\left[\frac{1}{b^2 t} + \frac{u}{b^2} + \frac{1}{b^2 t^{1/b^2+2}}\right] - \frac{1}{4}\right\}\phi(t) = 0 \tag{9.2.37}$$

We note that, in the general Liouville case with parameter $b \neq 1$, the operator contains rational powers. Thanks to the duality, we can always consider $0 < b \leq 1$.
However, Gaiotto's equation should refer to Y, because of the integrand of the convolution.

$$X_G(\theta) = Y(\theta) = X^2(\theta) \tag{9.2.38}$$

10 Local integrals of motion for Liouville model

In this section we use the Gelfand Dikii polynomials to calculate the Baxter's Q function for the Liouville model (9.1.6).

$$\boxed{Q(\theta) = W[V_0, U_0](\alpha)} \tag{10.0.1}$$

where we considered rescaling $X(\theta) = \bar{X}(\alpha)$ in (9.1.1). Given the subdominant solution U_0, around $y \to +\infty$ and the subdominant solution V_0, around $y \to -\infty$, defined as

$$U_0 = \exp\left\{ -\int_y^\infty dy' \sqrt{p(y')} S(y') \right\} \tag{10.0.2}$$

$$V_0 = \exp\left\{ \int_{-\infty}^y dy' \sqrt{p(y')} S(y') \right\} \tag{10.0.3}$$

where $p(y) = e^{y/b} + e^{-yb}$, directly by Fioravanti's modified Schrödinger form (8.5.4) for the generalized Mathieu equation. Following Dorey and Tateo's suggestion (2.4.9), to calculate the Baxter's Q function as the *regularized solution* of (8.1.1) for $x \to 0$. From the transformations of variable we made (8.1.21), it is evident that $x \to 0$ in the minimal models case[14] corresponds to $y \to +\infty$ in Liouville case. Therefore, we can assume that the wronskian (10.0.1) can be calculated as

$$\boxed{Q(\theta) = \lim_{y \to -\infty} U_0(y) = \exp\left\{ \int_{-\infty}^\infty dy' \sqrt{p(y')} S(y') \right\}} \tag{10.0.4}$$

In fact, we already checked in subsection 9.2 that the leading order matches the TBA result for Q. Note that this expression coincides with $U_0 V_0$ and that there is *no need for regularization*. If this assumption is correct, by the asymptotic expansion

$$Q(\theta) \simeq \exp\left\{ \sum_{n=0}^\infty (-\Lambda)^{n-1/2} \left(-\frac{1}{2n-1} \right) \int_{-\infty}^\infty dz \sqrt{p(z')} R_n(z') \right\} \qquad \Lambda = e^{2\theta} \to \infty \tag{10.0.5}$$

we must find the local integrals of motion; thereby giving an "a posteriori proof", in subsection 10.2. Note that in (10.0.5) we took into account also Fioravanti's theorem (4.4.1). More explicitly, in order to match the leading order, as already discussed in subsection 9.2, it is necessary to shift the rapidity in the Fioravanti's form of the Generalized Mathieu equation

$$\theta \to \theta_L = \theta + \ln \frac{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})}{8\sqrt{\pi}Q} \tag{10.0.6}$$

we then must find the asymptotic expansion (9.1.6)

$$Q(\theta) \simeq \exp\left\{ -\sum_{n=0}^\infty \tilde{B}_n I_{2n-1} e^{-(2n-1)\theta} \right\} \tag{10.0.7}$$

with exactly Lukyanov's normalization constants (9.1.8).

10.1 Gelfand Dikii coefficients recursion in the Liouville case

The Gelfand-Dikii recursion[36] is obtained from the general one (4.3.23), remembering that, for the Liouville model, the "Bäcklund coefficient" $u = P^2$.

$$\begin{aligned} \frac{dR_{n+1}}{dy} &= -\frac{1}{4}\frac{1}{p}\frac{d^3 R_n}{dy^3} + \frac{3}{8}\frac{p'}{p^2}\frac{d^2 R_n}{dy^2} + \left(\frac{P^2}{p} + \frac{3}{8}\frac{p''}{p^2} - \frac{9}{16}\frac{p'^2}{p^3} \right)\frac{dR_n}{dy} \\ &+ \left(-\frac{P^2}{2}\frac{p'}{p^2} + \frac{1}{8}\frac{p'''}{p^2} - \frac{9}{16}\frac{p''p'}{p^3} + \frac{15}{32}\frac{p'^3}{p^4} \right) R_n \end{aligned} \tag{10.1.1}$$

[36] Following the conventiions of[16]

We rewrite the "Bäcklund coefficient" p

$$p(y) = e^{-yb} + e^{y/b} = e^{-yb}\rho(y)^{-1} \tag{10.1.2}$$

in terms of a convenient function ρ

$$\boxed{\rho(y) = \frac{1}{1+e^{yQ}} = e^{-yb}p^{-1}(y)} \tag{10.1.3}$$

From direct inspection (cf. (10.1.44) and (10.1.51)) of the first polynomials, we conjecture the general form for R_n

$$\begin{aligned} R_n(y) &= \sum_{m=n}^{3n} a_{nm} e^{nby}\rho(y)^m = \sum_{m=n}^{3n} a_{nm} e^{nby}(1+e^{yQ})^{-m} && (10.1.4) \\ &= \sum_{m=n}^{3n} a_{nm} e^{(n-m)by}[e^{-by}(1+e^{yQ})]^{-m} \\ &= \sum_{m=n}^{3n} a_{nm} e^{(n-m)by} p(y)^{-m} && (10.1.5) \end{aligned}$$

We list some useful expressions for derivatives and powers of p in terms of the function ρ. For the first derivative:

$$\frac{p'}{p} = \frac{1}{b} - Q\rho \tag{10.1.6}$$

In fact,

$$p' = \frac{1}{b}e^{\frac{y}{b}} - be^{-by} = \frac{1}{b}(e^{\frac{y}{b}} + e^{-by}) - Qe^{-by} = \frac{1}{b}p - Qe^{-by}$$

the following expressions are proved similarly. For the second derivative:

$$\frac{p''}{p} = \frac{1}{b^2} + (Q^2 - 2\frac{Q}{b})\frac{e^{-by}}{p} \tag{10.1.7}$$

In fact,

$$\begin{aligned} p'' &= \frac{1}{b^2}e^{\frac{y}{b}} + b^2e^{-by} = \frac{1}{b^2}(e^{\frac{y}{b}} + e^{-by}) + (b^2 - \frac{1}{b^2})e^{-by} \\ b^2 - \frac{1}{b^2} &= Q^2 - 2\frac{Q}{b} \end{aligned}$$

It is convenient to reorder the terms rewrite them through the Q parameter, both for computational clarity and for theoretical understanding. For the third derivative:

$$\frac{p'''}{p} = \frac{1}{b^3} + \Big[-Q^3 + 3Q\Big]\rho \tag{10.1.8}$$

In fact,

$$\begin{aligned} p''' &= \frac{1}{b^3}e^{\frac{y}{b}} - b^3e^{-by} = \frac{1}{b^3}(e^{\frac{y}{b}} + e^{-by}) - (b^3 + \frac{1}{b^3})e^{-by} \\ b^3 + \frac{1}{b^3} &= Q^3 - 3Q \end{aligned}$$

For our purposes, it is enough to calculate and rearrange only the first three derivatives, since the remaining quantities we need are simple products of these.

$$\begin{aligned} \frac{p'^2}{p^2} &= \frac{1}{b^2} - 2\frac{Q}{b}e^{-by}\frac{1}{p} + Q^2e^{-2by}\frac{1}{p^2} && (10.1.9) \\ \frac{p''p'}{p^2} &= \frac{1}{b^3} + \Big[\frac{Q^2}{b} - 3\frac{Q}{b^2}\Big]\rho + \Big[-Q^3 + 2\frac{Q^2}{b}\Big]\rho^2 && (10.1.10) \\ \frac{p'^3}{p^3} &= \frac{1}{b^3} - 3\frac{Q}{b^2}e^{-by}\frac{1}{p} + 3Q^2\frac{1}{b}e^{-2by}\frac{1}{p^2} - Q^3e^{-3by}\frac{1}{p^3} && (10.1.11) \end{aligned}$$

The Gelfand-Dikii recursion relation (10.1.1) becomes

$$p\frac{dR_{n+1}}{dy} = -\frac{1}{4}\frac{d^3R_n}{dy^3} + \Big[\frac{3}{8}\frac{1}{b} - \frac{3}{8}Q\rho\Big]\frac{d^2R_n}{dy^2} + \Big[P^2 - \frac{3}{16}\frac{1}{b^2} + \Big(\frac{3}{8}Q + \frac{3}{8}\frac{1}{b}\Big)Q\rho - \frac{9}{16}Q^2\rho^2\Big]\frac{dR_n}{dy}$$
$$+ \Big[-\frac{P^2}{2}\frac{1}{b} + \frac{1}{32}\frac{1}{b^3} + \Big(\frac{P^2}{2} - \frac{1}{8}Q^2 - \frac{3}{16}\frac{Q}{b} - \frac{3}{32}\frac{1}{b^2}\Big)Q\rho + \Big(\frac{9}{16}Q + \frac{9}{32}\frac{1}{b}\Big)Q^2\rho^2 - \frac{15}{32}Q^3\rho^3\Big]R_n \tag{10.1.12}$$

The derivatives of the Gelfand Dikii polynomial (10.1.5) are the trickier part of the calculation. For their computations, it is useful to express the derivative of ρ in term of ρ-itself.

$$\frac{\rho'}{\rho} = -Q + Q\rho \tag{10.1.13}$$

The first derivative of the Gelfand Dikii polynomial is

$$R_n'(y) = \sum_{m=n}^{3n} a_{nm}\Big\{e^{(n-m)by}p^{-m}[nb - mQ] + mQe^{(n-m-1)by}p^{-m-1}\Big\}$$
$$= \sum_{m=n}^{3n} a_{nm}e^{(n-m)by}p^{-m}\Big\{[nb - mQ] + mQe^{-by}p^{-1}\Big\} \tag{10.1.14}$$

The second derivative of the Gelfand Dikii polynomial is

$$R_n''(y) = \sum_{m=n}^{3n} a_{nm}\Big\{[nb - mQ]^2 e^{(n-m)by}p^{-m}$$
$$+ mQ[2nb - (2m+1)Q]e^{(n-m-1)by}p^{-m-1} + m(m+1)Q^2 e^{(n-m-2)by}p^{-m-2}\Big\} \tag{10.1.15}$$
$$= \sum_{m=n}^{3n} a_{nm}e^{(n-m)by}p^{-m}\Big\{[nb - mQ]^2 + mQ[2nb - (2m+1)Q]e^{-by}p^{-1}$$
$$+ m(m+1)Q^2 e^{-2by}p^{-2}\Big\} \tag{10.1.16}$$

The third derivative of the Gelfand Dikii polynomial is

$$R_n'''(y) = \sum_{m=n}^{3n} a_{nm}\Big\{[nb - mQ]^3 e^{(n-m)by}p^{-m}$$
$$+ \Big\{[2nmbQ - (2m+1)mQ^2][nb - (m+1)Q] + mQ[nb - mQ]^2\Big\}e^{(n-m-1)by}p^{-m-1}$$
$$+ m(m+1)Q^2\Big\{[nb - (m+1)Q] + [2nb - (2m+1)Q]\Big\}e^{(n-m-2)by}p^{-m-2}$$
$$+ m(m+1)(m+2)Q^3 e^{(n-m-3)by}p^{-m-3}\Big\} \tag{10.1.17}$$
$$= \sum_{m=n}^{3n} a_{nm}e^{(n-m+1/2)by}p^{-m}\Big\{[nb - mQ]^3$$
$$+ mQ\Big[(3m^2 + 3m + 1)Q^2 - (6nm + 3n)bQ + 3n^2b^2\Big]e^{-by}p^{-1}$$
$$+ m(m+1)Q^2[3nb - (3m+3)Q]e^{-2by}p^{-2} + m(m+1)(m+2)Q^3 e^{-3by}p^{-3}\Big\} \tag{10.1.18}$$

We now expand the n-th side of the Gelfand-Dikii recursion

$$\sum_{m=n}^{3n}\Bigg\{\Big\{-\frac{1}{4}[nb-mQ]^3-\frac{1}{4}mQ\Big[(3m^2+3m+1)Q^2-(6nm+3n)bQ+3n^2b^2\Big]\rho$$
$$-\frac{1}{4}m(m+1)Q^2[3nb-(3m+3)Q]\rho^2-\frac{1}{4}m(m+1)(m+2)Q^3\rho^3\Big\}$$
$$+\Big\{\frac{3}{8b}[nb-mQ]^2+\frac{3}{8}\frac{Q}{b}m[2nb-(2m+1)Q]\rho+\frac{3}{8b}m(m+1)Q^2\rho^2\Big\}$$
$$+\Big\{-\frac{3}{8}Q[nb-mQ]^2\rho-\frac{3}{8}mQ^2[2nb-(2m+1)Q]\rho^2-\frac{3}{8}m(m+1)Q^3\rho^3\Big\}$$
$$+\Big[P^2-\frac{3}{16}\frac{1}{b^2}+\frac{3}{8}\Big(Q^2+\frac{Q}{b}\Big)\rho-\frac{9}{16}Q^2\rho^2\Big](nb-mQ)+\Big[P^2-\frac{3}{16}\frac{1}{b^2}+\frac{3}{8}\Big(Q^2+\frac{Q}{b}\Big)\rho-\frac{9}{16}Q^2\rho^2\Big]mQ\rho$$
$$+\Big[-\frac{P^2}{2}\frac{1}{b}+\frac{1}{32}\frac{1}{b^3}+\Big(\frac{P^2}{2}-\frac{1}{8}Q^2+\frac{3}{8}-\frac{9}{16}\frac{Q}{b}+\frac{9}{32}\frac{1}{b^2}\Big)Q\rho+\Big(\frac{9}{16}Q+\frac{9}{32}\frac{1}{b}\Big)Q^2\rho^2-\frac{15}{32}Q^3\rho^3\Big]\Bigg\}a_{n,m}\rho^m e^{nby}$$

Expanding also the derivatives of the Gelfand Dikii polynomial in powers of ρ and restoring integration one finds

$$\oint dy\,\sqrt{p}\int^y dy'\sum_{m=n+1}^{3n+3}a_{n+1,m}e^{(n+1)by}\rho^m\Big\{[(n+1)b-mQ]+mQ\rho\Big\}$$
$$=\oint dy\,\sqrt{p}\int^y dy'\sum_{m=n}^{3n}a_{n,m}e^{nby}\frac{1}{p}\rho^m\Big\{B_1(n,m)+B_2(n,m)\rho+B_3(n,m)\rho^2+B_4(n,m)\rho^3\Big\} \tag{10.1.19}$$

where

$$B_1(n,m)=-\frac{1}{4}[nb-mQ]^3+\frac{3}{8b}[nb-mQ]^2+\Big(P^2-\frac{3}{16b^2}\Big)[nb-mQ]+\Big(-\frac{P^2}{2b}+\frac{1}{32b^3}\Big) \tag{10.1.20}$$
$$B_2(n,m)=Q^3\Big[-\frac{3}{4}m^3-\frac{15}{8}m^2-\frac{25}{16}m-\frac{13}{32}\Big]+Q^2b\Big[\frac{3}{2}(n+\frac{1}{2})(m+\frac{1}{2})(m+1)\Big]$$
$$+Qb^2\Big[-\frac{3}{4}(m+\frac{1}{2})(n+\frac{1}{2})^2\Big]+P^2Q(m+\frac{1}{2}) \tag{10.1.21}$$
$$B_3(n,m)=\frac{3}{4}(m+\frac{1}{2})(m+\frac{3}{2})^2Q^3-\frac{3}{4}(n+\frac{1}{2})(m+\frac{1}{2})(m+\frac{3}{2})bQ^2 \tag{10.1.22}$$
$$B_4(m)=-\frac{1}{4}(m+\frac{1}{2})(m+\frac{3}{2})(m+\frac{5}{2})Q^3 \tag{10.1.23}$$

We observe that *the k-th power of ρ ($k=0,1,2,3$) always multiplies at least the same k-th power of Q.* The proper basis function therefore appears to be $r(y)=Q\rho(y)$. However, we shall continue to use $\rho(y)$ in order not to add further complexity to our calculations.
Since

$$e^{(n+1)by}\rho^m=e^{nby}p^{-1}\rho^m$$
$$e^{nby}\rho^m=e^{nby}\rho^{m+1}$$

for trivial theorems of integration theory we can write

$$\oint dy\,\sqrt{p}\int^y dy'\sum_{m=n+1}^{3n+3}a_{n+1,m}\rho^m\Big\{[(n+1)b-mQ]+mQ\rho\Big\}$$
$$=\oint dy\,\sqrt{p}\int^y dy'\sum_{m=n}^{3n}a_{n,m}\rho^m\Big\{B_1(n,m)\rho+B_2(n,m)\rho^2+B_3(n,m)\rho^3+B_4(n,m)\rho^4\Big\} \tag{10.1.24}$$

It will be also convenient to define

$$A_0(n+1,m) = (n+1)b - mQ \tag{10.1.25}$$

$$A_1(m) = mQ \tag{10.1.26}$$

The *Gelfand Dikii recursion relation for the coefficients is*

$$A_0(n+1,m)a_{n+1,m} + A_1(m-1)a_{n+1,m-1} = B_1(n,m-1)a_{n,m-1} + B_2(n,m-2)a_{n,m-2} \tag{10.1.27}$$

$$+ B_3(n,m-3)a_{n,m-3} + B_4(m-4)a_{n,m-4} \tag{10.1.28}$$

up to null terms for the lowest and highest values of m.
We now repeat *exactly* the derivation already done in the minimal models case.
We note that this recursion relations for the coefficients does not give explicitly the *single* coefficient $a_{n+1,m}$ in terms of the coefficients $a_{n,k}$, for some ks, but give a *linear combination* of coefficients of the polynomial R_{n+1} in terms of the coefficients of R_n. For the $n+1$-th side we must therefore consider the upper triangular matrix A^n defined as

$$A^n = \begin{bmatrix} A_1(n+1) & A_0^{n+1}(n+2) & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & A_1(n+2) & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & & & & \vdots \\ 0 & 0 & \dots & A_1(m) & A_0^{n+1}(m+1) & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & A_1(m+1) & \dots & 0 & 0 \\ \vdots & \dots & & & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & A_1(3n+2) & A_0^{n+1}(3n+3) \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & A_1(3n+3) \end{bmatrix} \tag{10.1.29}$$

The first line correspond to an equation which would be identically satisfied when multiplied by $a_{n,m}$, it is not linearly independent. We have deleted it so as to be able to define a determinant, which also does not vanish

$$\det A_n = \prod_{j=n+1}^{3n+3} A_1(m) \tag{10.1.30}$$

$$= Q^{2n+3}\frac{(3n+3)!}{n!} \tag{10.1.31}$$

We want to use the Cramer method to solve the linear non homogeneous system generated by the coefficients recursion relation. We schematically write this system as

$$A^n a_{n+1} = b_k \tag{10.1.32}$$

where clearly by a_{n+1} we denote the vector of all the (a priori) non null components $a_{n+1,m}$, $m = n+1, n+2, ...3n+3$ and by $b_k = b_k[a_n]$ we denote the functional of the coefficients of a_n established by the recursion relation (5.1.15)
Therefore we define the modified matrix of coefficients A^n_m whose determinant,divided by the determinant of A^n, gives us the coefficient $a_{n+1,m}$

$$A_m^n = \begin{bmatrix} A_1(n+1) & A_0^{n+1}(n+2) & \dots & b_1 & 0 & \dots & 0 & 0 \\ 0 & A_1(n+2) & \dots & b_2 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & & & & & & \vdots \\ 0 & 0 & \dots & b_m & A_0^{n+1}(m+1) & \dots & 0 & 0 \\ 0 & 0 & \dots & b_{m+1} & A_1(m+1) & \dots & 0 & 0 \\ \vdots & \dots & & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & b_{3n+2} & 0 & \dots & A_1(3n+2) & A_0^{n+1}(3n+3) \\ 0 & 0 & \dots & b_{3n+3} & 0 & \dots & 0 & A_1(3n+3) \end{bmatrix} \tag{10.1.33}$$

$$\begin{aligned} \det A_n^m = & \prod_{j=n+1}^{m-1} A_1(j) \sum_{k=m}^{3n+3} (-1)^{k-m} b_k \Big[A_0(n+1,m+1)\cdots A_0(n+1,k) A_1(n,k+1)\cdots \\ & \cdots A_1(3n+3) \Big] \end{aligned} \tag{10.1.34}$$

$$a_{n+1,m} = \frac{\sum_{k=m}^{3n+3}(-1)^{k-m} b_k \Big[A_0(n+1,m+1)\cdots A_0(n+1,k) A_1(k+1)\cdots A_1(3n+3)\Big]}{\prod_{j=m}^{3n+3} A_1(j)} \tag{10.1.35}$$

We can finally write the expression for the $a_{n+1,m}$ coefficient

$$\begin{aligned} a_{n+1,m} &= \frac{\sum_{k=m}^{3n+3}(-1)^{k-m} b_k \Big[(-Q)^{k-m} \frac{\Gamma(-\frac{(n+1)b}{Q}+k+1)}{\Gamma(-\frac{(n+1)b}{Q}+m+1)} Q^{3n+3-k} \frac{(3n+3)!}{k!}\Big]}{Q^{3n+4-m}\frac{(3n+3)!}{(m-1)!}} \\ &= \frac{1}{Q} \sum_{k=m}^{3n+3} \frac{(m-1)!}{k!} \frac{\Gamma(-\frac{(n+1)b}{Q}+k+1)}{\Gamma(-\frac{(n+1)b}{Q}+m+1)} b_k \\ &= \frac{1}{Q} \sum_{k=m}^{3n+3} \frac{(m-1)!}{k!} \frac{\Gamma(-\frac{(n+1)b}{Q}+k+1)}{\Gamma(-\frac{(n+1)b}{Q}+m+1)} \Big[B_1(n,k)a_{n,k} + B_2(n,k-1)a_{n,k-1} \\ &+ B_3(n,k-2)a_{n,k-2} + B_4(n,k-3)a_{n,k-3}\Big] \end{aligned} \tag{10.1.36}$$

where the $B_{i+1}(n,k-i)$ are polynomials up to the third degree in k and n. With the understanding that for $k=m$ the ratio of Gamma functions is one.
We now write this recursion more symmetrically, shifting the index m to $m+1$ and redefining the B_k functions as

$$\hat{B}_{k-1}(n,m) = B_k(n,m) \qquad k = 1,2,3,4 \tag{10.1.37}$$

We write the final formula as

$$\boxed{\begin{aligned} a_{n+1,m+1} = \frac{m!}{Q} \sum_{k=m+1}^{3n+3} \frac{1}{k!} \frac{\Gamma(-\frac{(n+1)b}{Q}+k+1)}{\Gamma(-\frac{(n+1)b}{Q}+m+2)} \Big[\hat{B}_0(n,k)a_{n,k} + \hat{B}_1(n,k-1)a_{n,k-1} \\ + \hat{B}_2(n,k-2)a_{n,k-2} + \hat{B}_3(n,k-3)a_{n,k-3}\Big] \end{aligned}} \tag{10.1.38}$$

10.1.1 Test

In this paragraph we test the correctness of the recursion found for the Gelfand-Dikii coeffients (10.1.38).

The Bäcklund potential for the Liouville case is explicitly given by

$$U(y)=\frac{P^2}{e^{y/b}+e^{-yb}}+\frac{1}{4}\frac{\frac{1}{b^2}e^{y/b}+b^2e^{-yb}}{(e^{y/b}+e^{-yb})^2}-\frac{5}{16}\frac{\frac{1}{b^2}e^{2y/b}-2e^{y/b-yb}+b^2e^{-2yb}}{(e^{y/b}+e^{-yb})^3} \tag{10.1.39}$$

we now rearrange and simplify it in order to find its coefficients with respect to the Gelfand-Dikii basis

$$\begin{aligned}U(y)&=P^2\frac{e^{yb}}{1+e^{yQ}}+\frac{1}{4b^2}\frac{e^{yb}e^{yQ}}{(1+e^{yQ})^2}+\frac{1}{4}b^2\frac{e^{yb}}{(1+e^{yQ})^2}-\frac{5}{16b^2}\frac{e^{yb}e^{2yQ}}{(1+e^{yQ})^3}\\&+\frac{5}{8}\frac{e^{yb}e^{yQ}}{(1+e^{yQ})^3}-\frac{5}{16}b^2\frac{e^{yb}}{(1+e^{yQ})^3}\\&=e^{yb}\Bigg[\frac{1}{1+e^{yQ}}\Big(P^2-\frac{1}{16}\frac{1}{b^2}\Big)+\frac{1}{(1+e^{yQ})^2}\Big(\frac{1}{4}b^2+\frac{5}{8}+\frac{3}{8}\frac{1}{b^2}\Big)\\&+\frac{1}{(1+e^{yQ})^3}\Big(-\frac{5}{16}b^2-\frac{5}{8}-\frac{5}{16}\frac{1}{b^2}\Big)\Bigg]\end{aligned} \tag{10.1.40}$$

Hence, the Gelfand Dikii polynomial $R_1=1/2U$ has the coefficients

$$a_{11}=\frac{P^2}{2}-\frac{1}{32}\frac{1}{b^2} \tag{10.1.41}$$

$$a_{12}=\frac{1}{8}b^2+\frac{5}{16}+\frac{3}{16}\frac{1}{b^2} \tag{10.1.42}$$

$$a_{13}=-\frac{5}{32}Q^2 \tag{10.1.43}$$

In terms of the variable ρ, the first Gelfand-Dikii polynomial can be written succintly as

$$R_1(y)=e^{by}\Big[a_{11}\rho+a_{12}\rho^2+a_{13}\rho^3\Big] \tag{10.1.44}$$

The coefficients of R_1 have been tested ($R_0=1\to R_1$) from (10.1.38).
The second Gelfand Dikii polynomial is

$$R_2(w)=\frac{3}{8}U^2(w)-\frac{1}{8}\frac{d^2}{dw^2}U(w)=\frac{3}{8}U^2(y)-\frac{1}{8}\Big(\frac{1}{p}\frac{d^2}{dy^2}U(y)-\frac{1}{2}\frac{p'}{p^2}\frac{d}{dy}U(y)\Big) \tag{10.1.45}$$

By direct calculation we find

$$a_{26}=\frac{1155}{2048}Q^4 \tag{10.1.46}$$

$$a_{25}=-\frac{1155}{512}\frac{Q^3}{b}+\frac{231}{256}\Big(-Q^2+2\frac{Q}{b}\Big)Q^3 \tag{10.1.47}$$

$$a_{24}=\frac{7\left(b^2+1\right)^2\left(56b^4+b^2\left(184-80P^2\right)+155\right)}{1024b^4} \tag{10.1.48}$$

$$a_{23}=-\frac{\left(b^2+1\right)\left(2b^2+5\right)\left(8b^4+b^2\left(28-80P^2\right)+29\right)}{512b^4} \tag{10.1.49}$$

$$a_{22}=\frac{27}{2048}b^4-\frac{15P^2}{64b^2}+\frac{3P^4}{8} \tag{10.1.50}$$

In terms of the variable ρ, the first Gelfand-Dikii polynomial can be written written succintly as

$$R_2(y)=e^{2by}\Big[a_{22}\rho^2+a_{23}\rho^3+a_{24}\rho^4+a_{25}\rho^5+a_{26}\rho^6\Big] \tag{10.1.51}$$

Also the coefficients of R_2 given by (10.1.38) have been tested succesfully.

10.2 Liouville local integrals of motion

10.2.1 Basis for integrals

The general Gelfand Dikii polynomial R_n has been expressed in terms of the just considered coefficients $a_{n,m}$ of its expansion in the basis $e^{(n-m)by}p^{-m}$, for $(n \le m \le 3n)$.

$$R_n(y) = \sum_{m=n}^{3n} a_{n,m} e^{(n-m)by} p^{-m} = \sum_{m=n}^{3n} a_{n,m} e^{nby} \rho^m \tag{10.2.1}$$

Since the integration measure implies multiplication by the factor $\sqrt{p}$, in general, the basic integrals can be reduced to Mellin transforms by the change of variable $x = e^{yQ}$.

$$J_{mn} := \int_{-\infty}^{\infty} dy \frac{e^{yb(2n-1)/2}}{(1+e^{yQ})^{(2m-1)/2}}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} dy \frac{e^{yb(2n-1)/2}}{(1+e^{yQ})^{(2m-1)/2}} &= \int_{-\infty}^{\infty} dy \frac{e^{yQp(2n-1)/2}}{(1+e^{yQ})^{(2m-1)/2}} \\
&= \frac{1}{Q}\int_0^{\infty} dx\, x^{(n-\frac{1}{2})p-1} \frac{1}{(1+x)^{m-1/2}}
\end{aligned}$$

It is more convenient to make a change of variable $t^{-1} = 1 + x$ and reduce the Mellin transform to a Euler Beta function

$$\begin{aligned}
\int_0^{\infty} dx\, x^{(n-\frac{1}{2})p-1} \frac{1}{(1+x)^{m-1/2}} &= \int_0^1 \frac{dt}{t^2} t^{m-1/2} \Big(\frac{1-t}{t}\Big)^{(n-1/2)p-1} \\
&= \int_0^1 dt\, t^{m-1/2-(n-1/2)p-1} (1-t)^{(n-1/2)p-1} \\
&= B\Big((n-1/2)p, m-1/2-(n-1/2)p\Big) \\
&= B\Big((n-1/2)\frac{b}{Q}, m-1/2-(n-1/2)\frac{b}{Q}\Big)
\end{aligned}$$

$$\boxed{J_{mn} = \frac{1}{Q} \frac{\Gamma\Big((n-1/2)\frac{b}{Q}\Big)\Gamma\Big(m-1/2-(n-1/2)\frac{b}{Q}\Big)}{\Gamma(m-1/2)}} \tag{10.2.2}$$

10.2.2 Expansion in local charges

In order to match the leading order, as we discussed in subsection 9.2, we need to shift the rapidity

$$\theta \to \theta_L = \theta + \ln \frac{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})}{8\sqrt{\pi}Q} \tag{10.2.3}$$

As a consequence, the expansion (10.0.5) becomes, for $\theta \to +\infty$

$$\begin{aligned}
\log Q &\sim \sum_{n=0}^{\infty} (-1)^n e^{\theta(1-2n)} \left[\frac{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})}{8\sqrt{\pi}Q}\right]^{1-2n} \sum_{m=n}^{3n} \left[-\frac{1}{n-\frac{1}{2}} \frac{\Gamma(\frac{(2n-1)p}{2})\Gamma(-\frac{(2n-1)p}{2}+m-\frac{1}{2})}{2\Gamma(m-\frac{1}{2})Q} a_{n,m}\right] \\
&= (-1)^{n+1} \sum_{n=0}^{\infty} e^{\theta(1-2n)} \left[\frac{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})}{8\sqrt{\pi}Q}\right]^{1-2n} \frac{\Gamma(\frac{(2n-1)p}{2})\Gamma(\frac{(2n-1)(1-p)}{2})}{(n-\frac{1}{2})2\Gamma(n-\frac{1}{2})Q} \times \\
&\times \sum_{m=n}^{3n} \left[\frac{\Gamma(\frac{(2n-1)(1-p)}{2}+m-n)\Gamma(n-\frac{1}{2})}{\Gamma(\frac{(2n-1)(1-p)}{2})\Gamma(n-\frac{1}{2}+m-n)} a_{n,m}\right]
\end{aligned}$$

For the elementary property

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}2^{-n}(2n-1)!! \tag{10.2.4}$$

or

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}2^{-n}(2n)!!\frac{(2n-1)!!}{(2n)!!} = \sqrt{\pi}n!\frac{(2n-1)!!}{(2n)!!} \tag{10.2.5}$$

we continue obtain a the correct factorial of n, to match the conventions for the normalization constants (9.1.8). We thus continue, also multiplying and dividing by the mass parameter

$$\begin{aligned}\log Q = &-\sum_{n=0}^{\infty} e^{\theta(1-2n)}\left[\frac{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})mR}{8\sqrt{\pi}Q}\right]^{1-2n}\frac{\Gamma(\frac{(2n-1)p}{2})\Gamma(\frac{(2n-1)(1-p)}{2})}{2\sqrt{\pi}n!Q}\times\\ &\left\{(-1)^n m^{2n-1}\frac{(2n)!!}{(2n-1)!!}\frac{\Gamma(\frac{(2n-1)(1-p)}{2}+m-n)\Gamma(n-\frac{1}{2})}{\Gamma(\frac{(2n-1)(1-p)}{2})\Gamma(n-\frac{1}{2}+m-n)}a_{n,m}\right\}\\ = &-\sum_{n=0}^{\infty} e^{\theta(1-2n)}\tilde{B}_n I_{2n-1}\end{aligned}$$

where the *normalization constant perfectly matches that of Lukyanov*[29]

$$\tilde{B}_n = \frac{\Gamma(\frac{p}{2})\Gamma(\frac{1-p}{2})mR}{8\sqrt{\pi}Q}\Bigg]^{1-2n}\frac{\Gamma(\frac{(2n-1)p}{2})\Gamma(\frac{(2n-1)(1-p)}{2})}{2\sqrt{\pi}n!Q} \tag{10.2.6}$$

and the local integral of motions (or local charges) are expressed as

$$I_{2n-1} = (mR)^{2n-1}(-1)^n\frac{(2n)!!}{(2n-1)!!}\sum_{m=n}^{3n}\frac{\Gamma(\frac{2n-1}{2})}{\Gamma(\frac{2n-1}{2}+m-n)}\frac{\Gamma(\frac{(2n-1)(1-p)}{2}+m-n)}{\Gamma(\frac{(2n-1)(1-p)}{2})}a_{n,m} \tag{10.2.7}$$

$$= (mR)^{2n-1}(-1)^n\frac{(2n)!!}{(2n-1)!!}\sum_{m=n}^{3n}\prod_{l=0}^{m-n-1}\frac{\left[(n-\frac{1}{2})(1-p)+l\right]}{\left[(n-\frac{1}{2})+l\right]}a_{n,m} \tag{10.2.8}$$

$$= (mR)^{2n-1}(-1)^n\frac{(2n)!!}{(2n-1)!!}\sum_{m=n}^{3n}\left\{\prod_{l=0}^{m-n-1}\left[1-\frac{(n-\frac{1}{2})p}{n-\frac{1}{2}+l}\right]a_{n,m}\right\} \tag{10.2.9}$$

$$\tag{10.2.10}$$

The final *formula for Liouville integrals of motion is*

$$\boxed{\frac{I_{2n-1}(b;P)}{(mR)^{2n-1}} = (-1)^n\frac{(2n)!!}{(2n-1)!!}\sum_{m=n}^{3n}\left\{\prod_{l=0}^{m-n-1}\left[1-\frac{(n-\frac{1}{2})\frac{b}{Q}}{n-\frac{1}{2}+l}\right]a_{n,m}(b;P)\right\}} \tag{10.2.11}$$

10.2.3 Test

We now test the correctness of our formula (10.2.11). We calculate I_1 and I_3 and compare our results with those of[29] and.[10] For $n=1$, (10.2.11) gives

$$\frac{I_1}{mR} = -2\left\{a_{11}+\left(1-\frac{b}{Q}\right)a_{12}+\left(1-\frac{b}{Q}\right)\left(1-\frac{b}{3Q}\right)a_{13}\right\} \tag{10.2.12}$$

$$= -P^2-\frac{1}{24} = \Delta-\frac{Q^2}{4}-\frac{1}{24} = \Delta-\frac{c}{24} \tag{10.2.13}$$

For $n = 2$, (10.2.11) gives

$$\frac{I_3}{(mR)^3} = \frac{8}{3}\Bigg\{a_{22} + \Big(1 - \frac{b}{Q}\Big)a_{23} + \Big(1 - \frac{b}{Q}\Big)\Big(1 - \frac{3b}{5Q}\Big)a_{24} \tag{10.2.14}$$

$$+ \Big(1 - \frac{b}{Q}\Big)\Big(1 - \frac{3b}{5Q}\Big)\Big(1 - \frac{3b}{7Q}\Big)a_{25} + \Big(1 - \frac{b}{Q}\Big)\Big(1 - \frac{3b}{5Q}\Big)\Big(1 - \frac{3b}{7Q}\Big)\Big(1 - \frac{3b}{9Q}\Big)a_{26}\Bigg\} \tag{10.2.15}$$

$$= \frac{8}{3}\left[\frac{3}{8}P^2 + \frac{3}{32}P^2 + \frac{4b^4 + 17b^2 + 4}{2560b^2}\right] = P^4 + \frac{P^2}{4} + \frac{4b^4 + 17b^2 + 4}{960b^2} \tag{10.2.16}$$

$$= \Delta^2 - \frac{(c+2)}{12}\Delta + \frac{c(5c+22)}{2880} \tag{10.2.17}$$

10.3 Universality of Gelfand-Dikii recursion

We collate here the two Gelfand-Dikii recursions

$$\boxed{\begin{aligned} a_{n+1,m+1} &= \frac{m!}{Q}\sum_{k=m+1}^{3n+3}\frac{1}{k!}\frac{\Gamma(-\frac{(n+1)b}{Q}+k+1)}{\Gamma(-\frac{(n+1)b}{Q}+m+2)}\Big[\hat{B}_0(n,k)a_{n,k} + \hat{B}_1(n,k-1)a_{n,k-1} \\ &\quad + \hat{B}_2(n,k-2)a_{n,k-2} + \hat{B}_3(n,k-3)a_{n,k-3}\Big] \\ \hat{a}_{n+1,m+1} &= \frac{(-1)^m m!}{2M}\sum_{k=m+1}^{3n+3}\frac{(-1)^k}{k!}\frac{\Gamma(\frac{n+1}{M}+k+1)}{\Gamma(\frac{n+1}{M}+m+2)}\times \\ &\quad \times\Big[\hat{B}_0(n,k)\hat{a}_{n,k} + \hat{B}_1(n,k-1)\hat{a}_{n,k-1} + \hat{B}_2(n,k-2)\hat{a}_{n,k-2} + \hat{B}_3(n,k-3)\hat{a}_{n,k-3}\Big]\end{aligned}}$$

We note that not only the recursion is of the same form, but its terms its coefficients are of the same form. *In fact, by analytic continuation,* $\frac{1}{M} = -\frac{b}{Q}$. The *reason for this equality* is, simply put, that *both the local integrals of motion* I_{2n-1} *are the same* (if expressed in terms of the conformal parameters) and also *the normalization constant* $\tilde{B}_n$ *is the same.*

11 Proof of Zamolodchikov's fundamental relation for self-dual Liouville

The Generalized Mathieu equation at the *self dual point* of the Liouville coupling $b = 1$ reduces to the the *modified-Mathieu equation*.[26]

$$-\frac{d^2}{dy^2}u(y) + (P^2 + 2e^{\alpha}\cosh y)\psi(y) \qquad b = 1 \tag{11.0.1}$$

If we change variable so as to consider this equation on the imaginary axis

$$y = -iz$$

we get the well known *Mathieu equation*

$$\boxed{\frac{d^2}{dz^2}u(z) + (P^2 + 2e^{\alpha}\cos z)\psi(z) = 0} \tag{11.0.2}$$

It is well known that, as a consequence of Floquet theorem,[24] there exist a particular solution $f(z)$ of the Mathieu equation for which the following *monodromy relation*[26] holds

$$\boxed{f(z + 2\pi\nu) = e^{2\pi\nu}f(z)} \tag{11.0.3}$$

where ν is called *Floquet index*.
In his draft,[2] Al. Zamolodchikov conjectured that the Baxter's T function for the self dual Liouville model ($b = 1$) is exactly equal to the cosine of the Floquet index. In particular,[2][1]

$$T(\alpha) = 2\cos 2\pi\nu \tag{11.0.4}$$

In his seminar[1] told that Zamolodhikov proved this interesting relation. However, there is no analytical proof in Al. Zamolodchikov's draft,[2] but only numerical computations. In this section, we are going to give two different proofs of this relation. The first proof we give is directly through Floquet theorem and is *exact*; the second proof we give is through the calculation of the complete asymptotic expansion of T and ν and thus is just an *asymptotic* proof.

11.1 Proof by Floquet theory

11.1.1 Floquet theorem and Hill determinant

Note that the points at infinity transform as

$$\Re y \to \pm\infty \qquad \longleftrightarrow \qquad \Im z \to \pm\infty \tag{11.1.1}$$

while the Stokes sectors of $\mathbb{C}$ for the Mathieu equation are[26]

$$\begin{aligned}
\bar{D}_k^+ &= \{z \in \mathbb{C} | (2k-1)\pi < \Im(-iz) < (2k+1)\pi\}, \quad k \in \mathbb{Z} \\
\bar{D}_k^- &= \{z \in \mathbb{C} | (2k-1)\pi < \Im(-iz) < (2k+1)\pi\}, \quad k \in \mathbb{Z} \\
\bar{D}_k^+ &= \{z \in \mathbb{C} | (2k-1)\pi < \Re z < (2k+1)\pi\}, \quad k \in \mathbb{Z} \qquad (11.1.2) \\
\bar{D}_k^- &= \{z \in \mathbb{C} | (2k-1)\pi < \Re z < (2k+1)\pi\}, \quad k \in \mathbb{Z} \qquad (11.1.3)
\end{aligned}$$

In order to match exactly the *standard* Mathieu equation we simply divide by two the coordinate

$$z = 2z' \tag{11.1.4}$$

so that the periodicity of the Mathieu equation becomes π, rather than 2π.

$$\frac{d^2}{dz'^2}u(z') + (4P^2 + 8e^{\alpha}\cos 2z')u(z') = 0 \tag{11.1.5}$$

The Mathieu equation is a particular case of the Hill equation[24]

$$\frac{d^2u}{dz'^2}u(z') + \left(\theta_0 + 2\sum_{n=1}^{\infty}\theta_n \cos 2nz'\right)u(z') = 0 \tag{11.1.6}$$

with

$$\theta_0 = 4P^2 \quad \theta_{\pm 1} = 4e^{\alpha} \tag{11.1.7}$$

the other parameters being zero. In the following, we shall drop the prime from z.
According to *Floquet theorem*,[24] the Mathieu equation has two linearly independent solutions which factorize into an exponential part and a periodic part. They can be written as

$$\boxed{f_1(z) = e^{i\nu z}p(z) \qquad f_2(z) = e^{-i\nu z}p(-z)} \tag{11.1.8}$$

where the function $p(x)$ is even[37] and periodic: $p(x+2\pi) = p(x)$. The (parameter dependent) number ν is called *Floquet index*.
Following the standard[24] and,[1] we expand in modes the Bloch eigenfunctions of the Floquet theorem

$$f_{\pm}(z) = e^{\pm 2i\nu z}p(\pm z) \tag{11.1.9}$$

$$p_{\pm}(z) = \sum_{k=-\infty}^{k=+\infty}\hat{p}_{\pm}(k)e^{2ikz} \quad p(\pm(z+\pi)) = p(\pm z) \tag{11.1.10}$$

Then, by substituting these expressions in the Mathieu equation, we find

$$\sum_{k=-\infty}^{\infty}(2i\nu + 2ik)^2 b_k e^{(2\nu+2ik)z} + \left(\sum_{n=-1}^{1}\theta_n e^{2inz}\right)\left(\sum_{k=-\infty}^{\infty}b_k e^{(2\nu+2ik)z}\right) = 0 \tag{11.1.11}$$

and equating the coefficients of the same exponentials $e^{(2\nu+2ik)z}$

$$(2i\nu + 2ik)^2 p_k + \theta_{-1}p_{k+1} + \theta_0 p_k + \theta_1 p_{k-1} = 0 \tag{11.1.12}$$

It is clear that, in order for the modes p_k to be all non-null the determinant of the following matrix must be zero

$$\begin{pmatrix} (-2\nu+4)^2-\theta_0 & -\theta_1 & 0 & 0 & 0 \\ -\theta_1 & (-2\nu+2)^2-\theta_0 & -\theta_1 & 0 & 0 \\ 0 & -\theta_1 & (-2\nu)^2-\theta_0 & -\theta_1 & 0 \\ 0 & 0 & -\theta_1 & (-2\nu-2)^2-\theta_0 & -\theta_1 \\ 0 & 0 & 0 & -\theta_1 & (-2\nu-4)^2-\theta_0 \end{pmatrix} \tag{11.1.13}$$

To be precise, in the standard definition of Hill's determinant each line is multiplied by a suitable factor

$$\Delta(-\nu) = \det\begin{bmatrix} \frac{(-2\nu+4)^2-\theta_0}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & 0 & 0 & 0 \\ \frac{-\theta_1}{2^2-\theta_0} & \frac{(-2\nu+2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & 0 & 0 \\ 0 & \frac{-\theta_1}{0^2-\theta_0} & \frac{(-2\nu)^2-\theta_0}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & 0 \\ 0 & 0 & \frac{-\theta_1}{2^2-\theta_0} & \frac{(-2\nu-2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} \\ 0 & 0 & 0 & \frac{-\theta_1}{4^2-\theta_0} & \frac{(-2\nu-4)^2-\theta_0}{4^2-\theta_0} \end{bmatrix} = 0 \tag{11.1.14}$$

[37] That the function $p(z)$ is even follows from the fact that the Mathieu is such.

By considering well-known infinite products, it can be proven the expression[24]

$$\Delta(-\nu) = \Delta(0) - \frac{\sin^2 \pi\nu}{\sin^2 \pi P} \tag{11.1.15}$$

The roots ν of the Hill determinant are thus shown to satisfy the simplified equation[1]

$$\Delta(0) \sin^2 \pi P = \sin^2 \pi\nu \tag{11.1.16}$$

which can be written also as

$$2\cos 2\pi\nu = 2(\Delta(0) - 1) + 2\Delta(0)\cos 2\pi P \tag{11.1.17}$$

11.1.2 Proof of Zamolodchikov's conjecture through Floquet theorem

Considering numerical calculations, in his draft,[2] Al. Zamolodchikov conjectured that

$$\boxed{T(\alpha) = 2\cos 2\pi\nu} \tag{11.1.18}$$

We now prove it, directly by the use of Floquet theorem. In fact, consider the TQ relation at $b = 1$

$$\boxed{T(\alpha) = \frac{\bar{X}(\alpha + i\pi)}{\bar{X}(\alpha)} + \frac{\bar{X}(\alpha - i\pi)}{\bar{X}(\alpha)}} \tag{11.1.19}$$

where the terms on the right side ("Q side") can be constructed through the wronskians of the fundamental solutions of the *modified* Mathieu equation.

$$\bar{X}(\alpha) = W[V_0, U_0] \qquad \bar{X}(\alpha \pm i\pi) = W[V_0, U_{\pm 1}] \tag{11.1.20}$$

$$U_0(y) \simeq \frac{1}{\sqrt{2}} e^{-(\alpha+y)/4} e^{-2e^{(\alpha+y)/2}} \qquad \Re y \to \infty \tag{11.1.21}$$

$$U_{\pm 1}(y) \simeq \frac{\mp i}{\sqrt{2}} e^{-(\alpha+y)/4} e^{2e^{(\alpha+y)/2}} \qquad \Re y \to \infty \tag{11.1.22}$$

On the imaginary axis of y these are solutions of the Mathieu equation (11.0.2). We report only the correspondence between the periodic shifts, on the imaginary and real axis of y, respectively

$$z \to z + 2\pi \quad \leftrightarrow \quad y \to y - 2\pi i \tag{11.1.23}$$

We can expand the fundamental solutions in the Floquet theorem's basis (11.1.8)

$$V_0(y) = a_1 e^{-\nu y} p_1(y) + a_2 e^{\nu y} p_2(y)$$
$$U_0(y) = c_1 e^{-\nu y} p_1(y) + c_2 e^{\nu y} p_2(y)$$

the coefficients being understood as Stokes coefficients as in section 2. Note that the functions p_i $i = 1, 2$ are periodic only on the imaginary y-axis; therefore, for $\Re y \to +\infty$ the varying dominant/subdominant behavior of the solutions $U_{\pm 1}, U_0$ is admitted.
Bazhanov[1] defined the $U_{\pm 1}$ functions by application of the Ω_b ($b = 1$) symmetry to the U_0 function, as in (8.2.11)

$$U_{\pm 1}(y; \alpha) \coloneqq U_0(y \pm \pi i; \alpha \pm \pi i) = \Omega_1 U_0(y; \alpha) \tag{11.1.24}$$

We note that this definition permit us to obtain, from the asymptotic representation of the subdominant solution U_0, the asymptotic representation of the dominant solutions $U_{\pm 1}$, *staying within the same Stokes sector of* y. We can define two, in general, *different functions* $\tilde{U}_{\pm 1}$ as

$$\boxed{\tilde{U}_{\pm 1}(y; \alpha) \coloneqq U_0(y \pm 2\pi i; \alpha)} \tag{11.1.25}$$

which, however, for the leading order, have the same asymptotic *representation*

$$\tilde{U}_{\pm 1}(y;\alpha) \simeq U_{\pm 1}(y;\alpha) \simeq e^{2e^{(\alpha+y)/2}} \qquad \Re y \to +\infty \qquad \text{(LEADING ORDER)} \tag{11.1.26}$$

In fact, at the leading order in y, it is legitimate to use the approximate equation

$$\boxed{\left\{-\frac{d^2}{dy^2} + e^{\alpha+y} + P^2\right\}\psi(y) \simeq 0 \qquad \Re y \to +\infty} \tag{11.1.27}$$

Since, by Abel theorem,[27] the wronskians are independent from y, we can write their asymptotic calculation for $\Re y \to +\infty$ as

$$\begin{aligned} W[V_0(y;\alpha), U_{\pm 1}(y;\alpha)] &= W[\lim_{y\to+\infty} V_0(y;\alpha), \lim_{y\to+\infty} U_{\pm 1}(y;\alpha)] \\ &= W[\lim_{y\to+\infty} V_0(y;\alpha), \lim_{y\to+\infty} \tilde{U}_{\pm 1}(y;\alpha)] = W[V_0(y;\alpha), \tilde{U}_{\pm 1}(y;\alpha)] \end{aligned}$$

Thus we see that, *exactly at all orders*, it is true that we can use $\tilde{U}_{\pm 1}$ in the place of $U_{\pm 1}$ for the calculations of the wronskians

$$\boxed{W[V_0, U_{\pm 1}](\alpha) = W[V_0, \tilde{U}_{\pm 1}](\alpha)} \tag{11.1.28}$$

Note that α has been left untouched.
Thus, the proper "dominant" functions to be expanded in the Floquet basis are $\tilde{U}_{\pm 1}$

$$\tilde{U}_{-1}(y) = b_1 e^{-\nu y} p_1(y) + b_2 e^{\nu y} p_2(y) \tag{11.1.29}$$

$$\tilde{U}_1(y) = d_1 e^{-\nu y} p_1(y) + d_2 e^{\nu y} p_2(y) \tag{11.1.30}$$

in terms of some Stokes coefficients b_i and d_i. By considering the definition of $\tilde{U}_{\pm 1}$, we can write

$$\tilde{U}_{\pm 1}(y) \sim U_0(y \mp 2\pi i) \qquad \Re y \to +\infty \tag{11.1.31}$$

$$= c_1 e^{\pm 2\pi i\nu} e^{-\nu y} p_1(y) + c_2 e^{\mp 2\pi i\nu} e^{\nu y} p_2(y) \tag{11.1.32}$$

The key observation of our proof is the ensuing relation between the Stokes coefficients

$$b_1 = e^{-2\pi i\nu} c_1 \qquad b_2 = e^{2\pi i\nu} c_2 \tag{11.1.33}$$

$$d_1 = e^{2\pi i\nu} c_1 \qquad d_2 = e^{-2\pi i\nu} c_2 \tag{11.1.34}$$

we treat the Stokes coefficients as they were ordinary linear coefficients, because we just reach the border of a single Stokes sector (in which the Stokes coefficients are uniquely determined).
So, the wronskians of interest are expressed as

$$W[V_0, \tilde{U}_{-1}] = (a_1 b_2 - a_2 b_1) W[e^{-\nu y} p_1(y), e^{\nu y} p_2(y)] \tag{11.1.35}$$

$$W[V_0, \tilde{U}_1] = (a_1 d_2 - a_2 d_1) W[e^{-\nu y} p_1(y), e^{\nu y} p_2(y)] \tag{11.1.36}$$

$$W[V_0, U_0] = (a_1 c_2 - a_2 c_1) W[e^{-\nu y} p_1(y), e^{\nu y} p_2(y)] \tag{11.1.37}$$

Finally, the TQ relation can be *exactly written* as

$$\begin{aligned} T(\alpha) &= \frac{\bar{X}(\alpha + i\pi)}{\bar{X}(\alpha)} + \frac{\bar{X}(\alpha - i\pi)}{\bar{X}(\alpha)} = \frac{W[V_0, U_1]}{W[V_0, U_0]} + \frac{W[V_0, U_{-1}]}{W[V_0, U_0]} \\ &= \frac{W[V_0, \tilde{U}_1]}{W[V_0, U_0]} + \frac{W[V_0, \tilde{U}_{-1}]}{W[V_0, U_0]} = \frac{(a_1 d_2 - a_2 d_1) + (a_1 b_2 - a_2 b_1)}{a_1 c_2 - a_2 c_1} \\ &= \frac{(e^{-2\pi i\nu} a_1 c_2 - e^{2\pi i\nu} a_2 c_1) + (e^{2\pi i\nu} a_1 c_2 - e^{-2\pi i\nu} a_2 c_1)}{a_1 c_2 - a_2 c_1} \end{aligned} \tag{11.1.38}$$

$$= 2\cos 2\pi\nu \tag{11.1.39}$$

we thus conclude our proof of Zamolodchikov's conjecture (11.0.4).

$$T(\alpha) = 2\cos 2\pi\nu \tag{11.1.40}$$

11.1.3 Observations

We end this paragraph with an observations concerning the possibility of generalization of Zamolodchikov relation (11.0.4) for some $b \neq 1$.
For example, we know that at all rational b the Floquet theorem holds. Now, $e^{y/b}$ has period $2\pi i b$, e^{-yb} has period $2\pi i/b$. Let us suppose that b is a rational number, namely, suppose $b = \frac{m}{n}$, with m and n integers. Then the two periods are commensurable and there exist a common period.

$$A 2\pi i b = B 2\pi i/b \tag{11.1.41}$$

$$A\frac{m}{n} = B\frac{n}{m} \qquad n, m \in \mathbb{Z} \tag{11.1.42}$$

$$A = n^2, \qquad B = m^2 \tag{11.1.43}$$

With this choice, $e^{y/b} + e^{-yb}$ is invariant under the shift

$$y \to y \pm 2\pi i n^2 b = y \pm 2\pi i m^2/b \tag{11.1.44}$$

Then, by Floquet theorem, the generalized Mathieu equation (8.5.4), for rational b, has two linearly independent solutions

$$\psi_1(y) = e^{-\nu y} f(y) \qquad \psi_2(y) = e^{+\nu y} f(y) \tag{11.1.45}$$

$$f(y \pm 2\pi i n^2 b) = f(y \pm 2\pi i m^2/b) = f(y) \tag{11.1.46}$$

Nevertheless, the problem is that the TQ relation doesn't generalize to a relation between the wronskians of U_0 and $U_{n^2 b}$.
Hence, it is not trivial to obtain a generalization of Zamolodchikov's relation (11.0.4) between T and ν for $b \neq 1$, perhaps it is impossible.

11.2 Proof by integrability theory

In this section, we are going to give an *asymptotic proof*, for large rapidity θ, of Zamolodchikov's relation (11.0.4). This shall be done by direct calculation, of both the Floquet index ν asymptotic expansion and the usual $\log Q$ asymptotic expansion. The asymptotic proof amounts to a check that, for each n-th asymptotic coefficient, the only difference between $2\pi i\nu$ and $\log Q$ is just the ratio (1.6.21) of $\tilde{C}_n$ and $\tilde{B}_n$ between the constant of normalization of the Baxter's $\log T$ function expansion and $\log Q$ expansion, which is consistent with the TQ relation (1.6.13)

$$2\cos\left(2\pi\nu(\theta)\right) = \frac{Q(\theta + i\pi/2)}{Q(\theta)} + \frac{Q(\theta - i\pi/2)}{Q(\theta)} \tag{11.2.1}$$

which asymptotically, for large rapidity, reads (1.6.13)[38]

$$2\pi i\nu(\theta) \simeq \log Q(\theta + i\pi/2) - \log Q(\theta) \qquad \theta \to +\infty \tag{11.2.2}$$

[38]The asymptotic approximation $2\cos 2\pi\nu \sim e^{2\pi i\nu}$ is true only in particular Stokes sectors of θ which we don't specify, for simplicity, because the only other possibility amounts to a trivial minus sign. In fact, the leading order is proportional to e^{θ}; if the real part of the proportionality constant is positive, the contribution is dominant, instead, if the real part is negative, the contribution is subdominant.

11.2.1 Floquet exponent and T function

Consider the self-dual case $b = 1$. The generalized Mathieu equation becomes the standard modified Mathieu equation[26] and, by a rotation of the axis $z = iy$, becomes the well-known Mathieu equation.

$$\boxed{\frac{d^2}{dz^2}\psi(z) + (2e^{2\theta}\cos z + P^2)\psi(z) = 0} \tag{11.2.3}$$

The Mathieu equation is of the Liouville form (4.1.1) with $p_-(z) = -2\cos z$ and $u_- = -P^2$. The Bäcklund potential (4.1.8) is

$$\begin{aligned} U_-(z) &= \frac{P^2}{2\cos z} + \frac{1}{4}\frac{1}{2\cos z} + \frac{5}{16}\frac{\tan^2 z}{2\cos^3 z} \\ &= \left(P^2 - \frac{1}{16}\right)\frac{1}{2\cos z} + \frac{5}{4}\frac{1}{(2\cos z)^3} \end{aligned} \tag{11.2.4}$$

We put a subscript $-$ at p and u, to keep in mind that the rotation $z = iy$ introduces a minus sign, with respect with the usual $p = e^{y/b} + e^{-yb}$. Moreover, also the functions χ, S and R (nut not ψ!) need such reminder. The Bäcklund-Schrödinger equation is

$$\left[\frac{d^2}{dw_-^2} - U(w_-)\right]\chi_-(w_-) = e^{2\theta}\chi_-(w_-) \tag{11.2.5}$$

with $dw_- = \sqrt{p_-}dz$. Our aim is to asymptotically calculate the *Floquet index* ν, that is, the monodromy for the wave function around the cycle (period) $-\pi < z < \pi$. for the wave function solution ψ.

$$\psi(z + 2\pi) = e^{2\pi i\nu}\psi(z) \tag{11.2.6}$$

or, in the Zamolodchikov's rotated varible y

$$\boxed{\psi(y - 2\pi i) = e^{2\pi i\nu}\psi(y)} \tag{11.2.7}$$

We note that we can equivalently consider the monodromy of the Bäcklund-transformed wave function χ_-, since the factor $\sqrt[4]{p(z)}$, by which these two functions differ, is 2π-periodic. Hence, making the correctly "rotated" Bäcklund change of variable in (4.2.1), we can define the Floquet exponent through the integrand of the eikonal as

$$\boxed{\begin{aligned} \nu(\theta, P) &= \frac{1}{2\pi i}\left[\log\chi_-(\pi) - \log\chi_-(-\pi)\right] && (11.2.8) \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} dz\, \sqrt{p_-(z)}S_-(z;\theta,P) && (11.2.9) \end{aligned}}$$

which can asymptotically written as

$$2\pi i\nu \sim \sum_{n=0}^{\infty} e^{\theta(1-2n)}\left(-\frac{1}{2n-1}\right)\int_{-\pi}^{\pi} dz\, \sqrt{p_-(z)}R_n(z) \qquad \theta \to +\infty \tag{11.2.10}$$

Note that the factor i is correct, from (11.2.7).
We can begin our calculations. It is convenient to introduce the variable

$$\boxed{t(z) = \frac{1}{2\cos z}} \tag{11.2.11}$$

Note that $t(z) = -1/p_-(z)$ and that is has a simple pole for $z = \pi/2 + k\pi$, with $k \in \mathbb{Z}$. The inverse transformation is

$$z = \cos^{-1}\frac{1}{2t} \tag{11.2.12}$$

$$\tag{11.2.13}$$

By this transformation, the measure of integration get transformed as

$$dz = \frac{dt}{t\sqrt{-1+4t^2}} \qquad dw = \sqrt{-\frac{1}{t(z)}}dz = \frac{dt}{\sqrt{t^3}\sqrt{1-4t^2}} \tag{11.2.14}$$

while the Bäcklund potential assumes a (functional) polynomial form

$$U[t(z)] = \Big(P^2 - \frac{1}{16}\Big)t + \frac{5}{4}t^3 \tag{11.2.15}$$

By direct inspection of the first polynomials, we conjecture the following ansatz

$$\boxed{R_n[t(z)] = \sum_{m=0}^{n} a_{n,m}t^{n+2m}} \tag{11.2.16}$$

We begin with the calculation of the coefficients $a_{n,m}$ through the Gelfand Dikii recursion relation (4.3.13)

$$\frac{dR_{n+1}}{dw} = -\frac{1}{4}\frac{d^3R_n}{dw^3} + U\frac{dR_n}{dw} + \frac{1}{2}\frac{dU}{dw}R_n \tag{11.2.17}$$

We transform the derivatives with respect to w into derivatives with respect to t

$$\begin{aligned}
\frac{d}{dw} &= \sqrt{1-4t^2}\sqrt{t^3}\frac{d}{dt} \\
\frac{d^2}{dw^2} &= (-4t^5+t^3)\frac{d^2}{dt^2} + (-10t^4 + \frac{3}{2}t^2)\frac{d}{dt} \\
\frac{d^3}{dw^3} &= \sqrt{1-4t^2}\sqrt{t^3}\Big[(-4t^5+t^3)\frac{d^3}{dt^3} + (-30t^4 + \frac{9}{2}t^2)\frac{d^2}{dt^2} + (-40t^3+3t)\frac{d}{dt}\Big]
\end{aligned}$$

therefore the Gelfand Dikii recursion (4.3.13) becomes

$$\frac{dR_{n+1}}{dt} = (t^5 - \frac{1}{4}t^3)\frac{d^3}{dt^3}R_n + (\frac{15}{2}t^4 - \frac{9}{8}t^2)\frac{d^2}{dt^2}R_n + \Big(\frac{45}{4}t^3 + (P^2 - \frac{13}{16})t\Big)\frac{d}{dt}R_n \tag{11.2.18}$$

$$+ \Big(\frac{P^2}{2} - \frac{1}{32} + \frac{15}{8}t^2\Big)R_n \tag{11.2.19}$$

where we used also the fact that

$$\frac{dU}{dt} = P^2 - \frac{1}{16} + \frac{15}{4}t^2 \tag{11.2.20}$$

Expanding this equation in powers of $t(z)$ (by using the ansatz (11.2.16)) and equating the respective coefficients, we obtain a recursion relation for the Gelfand-Dikii coefficients

$$\begin{aligned}
(n+2m+1)a_{n+1,m} = \Big[&(n+2m+\frac{1}{2})P^2 - \frac{1}{4}(n+2m)(n+2m-1)(n+2m-2) - \frac{9}{8}(n+2m)(n+2m-1) \\
&- \frac{13}{16}(n+2m) - \frac{1}{32}\Big]a_{n,m} + \Big[(n+2m-2)(n+2m-3)(n+2m-4) \\
&+ \frac{15}{2}(n+2m-2)(n+2m-3) + \frac{45}{4}(n+2m-2) + \frac{15}{8}\Big]a_{n,m-1}
\end{aligned} \tag{11.2.21}$$

which, continuing the calculations, becomes

$$\begin{aligned}(n+2m+1)a_{n+1,m} &= \Big[(n+2m+\frac{1}{2})P^2 - \frac{1}{4}(n+2m)(n+2m-1)(n+2m-2) - \frac{9}{8}(n+2m)(n+2m-1) \\ &- \frac{13}{16}(n+2m) - \frac{1}{32}\Big]a_{n,m} + \Big[(n+2m-2)(n+2m-3)(n+2m-4) \\ &+ \frac{15}{2}(n+2m-2)(n+2m-3) + \frac{45}{4}(n+2m-2) + \frac{15}{8}\Big]a_{n,m-1} \\ &= \Big\{P^2(n+2m+\frac{1}{2}) - \frac{1}{4}\Big[n+2m+\frac{1}{2}\Big]^3\Big\}a_{n,m} \\ &+ \Big\{(n+2m-\frac{1}{2})^3 - (n+2m-\frac{1}{2})\Big\}a_{n,m-1}\end{aligned}$$

Simplifying again and a making also an index rearrangement $(n+1 \to n)$, we finally get

$$\boxed{a_{n,m} = \Big[P^2\frac{n+2m-\frac{1}{2}}{n+2m} - \frac{1}{4}\frac{(n+2m-\frac{1}{2})^3}{n+2m}\Big]a_{n-1,m} + \Big[\frac{(n+2m-\frac{3}{2})^3}{n+2m} - \frac{n+2m-\frac{3}{2}}{n+2m}\Big]a_{n-1,m-1}} \tag{11.2.22}$$

It is perhaps already evident that this is a very simple recursion. However, its simplicity can be emphasized by introducing the variable $\eta(n,m)$, which is function of the indexes n, m[39]

$$\boxed{\eta(n,m) = n+2m+\frac{1}{2}} \tag{11.2.23}$$

and functions of it $f(\eta)$ and $g(\eta)$

$$f(\eta) = P^2 - \frac{1}{4}\eta^2 \tag{11.2.24}$$

$$g(\eta) = \eta^2 - 1 \tag{11.2.25}$$

the recursion can then be written as

$$a_{n,m} = \frac{1}{\eta-\frac{1}{2}}\Big[f(\eta-1)(\eta-1)a_{n-1,m} + g(\eta-2)(\eta-2)a_{n-1,m-1}\Big] \tag{11.2.26}$$

Such simplicity makes us hope to solve it completely, with initial condition

$$a_{00} = 1 \tag{11.2.27}$$

We now want to calculate the general integral of the functional coefficients. The functional basis is made from positive power of the meromorphic function $t(z)$[40], the singularities must be avoided in the integration, by using the symmetries of the simple trigonometric function $\cos z$.

$$\begin{aligned}\int_{-\pi}^{\pi} dz\,\sqrt{-2\cos z}\,R_n(z) &= \sum_{m=0}^{n} a_{n,m}\int_{-\pi}^{\pi} dz\,\sqrt{-2\cos z}\left(\frac{1}{2\cos z}\right)^{n+2m} \\ &= \sum_{m=0}^{n} a_{n,m}\Big\{\int_{-\pi}^{\pi} dz\,\sqrt{-2\cos z}\Big(\frac{1}{-2\cos z}\Big)^{n+2m}(-1)^n\Big\} \\ &= \sum_{m=0}^{n} a_{n,m}\Big\{(-1)^n\int_{-\pi}^{\pi} dz\left(-\frac{1}{2\cos z}\right)^{n+2m-1/2}\Big\}\end{aligned}$$

[39] *Beware that in the following we shall drop the n, m dependence from $\eta(n,m)$.*

[40] Since $t(z)$ has just a simple pole, from the general form (11.2.16) we deduce that $R_n(z)$ has at most a $3n$ order pole for $z = \pi/2 + k\pi$, with $k \in \mathbb{Z}$

In particular, we split in two the interval of integration, according to whether $\cos z$ is positive or negative. It is clear, then, that the area enclosed between the curve of the integrand and the z axis is the same on every $\pi/2$-wide interval of z.

$$\begin{aligned}\int_{-\pi}^{\pi} dz\,\sqrt{-2\cos z}\,R_n(z) &= \sum_{m=0}^{n} a_{n,m}\left\{((-1)^{-1/2}+(-1)^n)\int_{-\pi/2}^{\pi/2} dz\left(\frac{1}{2\cos z}\right)^{n+2m-1/2}\right\}\\ &= \sum_{m=0}^{n} a_{n,m}\left\{2(\mp i+(-1)^n)\int_{0}^{\pi/2} dz\left(\frac{1}{2\cos z}\right)^{n+2m-1/2}\right\}\end{aligned}$$

It is perhaps useful to define a "constant" δ_n[41], in order to better manage this clumsy aspect of the calculation

$$\boxed{2\delta_n = 2(\mp i+(-1)^n)} \tag{11.2.28}$$

The basis $\mathcal{I}_{n,m}$ of the integrals is defined as

$$\mathcal{I}_{n,m} = \int_{0}^{\pi/2} dz\left(\frac{1}{2\cos z}\right)^{n+2m-1/2} \tag{11.2.29}$$

so that we can write the integral we are interested in as

$$\boxed{\int_{-\pi}^{\pi} dz\,\sqrt{-2\cos z}\,R_n(z) = 2\delta(n)\sum_{m=0}^{n} a_{n,m}\mathcal{I}_{n,m}} \tag{11.2.30}$$

We proceed to calculate the generic basis integral $\mathcal{I}_{n,m}$

$$\begin{aligned}\mathcal{I}_{n,m} &= \int_{0}^{\pi/2} dz\left(\frac{1}{2\cos z}\right)^{n+2m-1/2}\\ &= \int_{\frac{1}{2}}^{\infty} \frac{dt}{\sqrt{1-4t^2}} t^{2m+n-3/2}\end{aligned}$$

Change variable $s = 1/4t^2$, $t = \frac{1}{2}s^{-1/2}$, $dt = -\frac{1}{4}s^{-3/2}$.

$$\begin{aligned}\mathcal{I}_{n,m} &= \frac{1}{4}2^{-n-2m+3/2}\int_{0}^{1}\frac{ds}{\sqrt{s-1}}s^{-3/2-m-n/2+3/4+1/2}\\ &= 2^{-n-2m-1/2}B(\frac{1}{2},-m-\frac{n}{2}+\frac{3}{4})\end{aligned}$$

in terms of the well-known Euler Beta function. Therefore, the generic basis integral $\mathcal{I}_{n,m}$ can be expressed as

$$\boxed{\mathcal{I}_{n,m} = 2^{-n-2m-1/2}\sqrt{\pi}\frac{\Gamma(-m-\frac{n}{2}+\frac{3}{4})}{\Gamma(-m-\frac{n}{2}+\frac{5}{4})}} \tag{11.2.31}$$

In conclusion, the Floquet exponent (11.2.10) can be expanded, for large Bäcklund energy $e^{2\theta}$[42] in terms of the Gelfand Dikii coefficients as

$$\boxed{2\pi i\nu(\theta,P) \sim \sum_{n=0}^{\infty} e^{\theta(1-2n)}2\delta_n\left\{\sqrt{\pi}\frac{\sqrt{2}^{-1-2n}}{1-2n}\sum_{m=0}^{n} a_{n,m}(P)\frac{\Gamma(-m-\frac{n}{2}+\frac{3}{4})}{\Gamma(-m-\frac{n}{2}+\frac{5}{4})}\right\} \qquad \theta\to+\infty} \tag{11.2.32}$$

[41] More properly, $\delta(n)$ is a function of the (now) fixed index n.

[42] This limit can be more properly interpreted as a large rapidity θ limit

We can define the expansion coefficients of ν by

$$\nu \sim \sum_{n=0}^{\infty} e^{\theta(1-2n)} \nu_n \qquad \theta \to +\infty \tag{11.2.33}$$

or

$$2\pi i \nu_n = 2\delta_n \left\{ \sqrt{\pi} \frac{\sqrt{2}^{-1-2n}}{1-2n} \sum_{m=0}^{n} a_{n,m}(P) \frac{\Gamma(-m-\frac{n}{2}+\frac{3}{4})}{\Gamma(-m-\frac{n}{2}+\frac{5}{4})} \right\} \tag{11.2.34}$$

11.2.2 Baxter's Q function

Thanks to the standard[14] identification (**??**), we can calculate the Baxter's Q function for the self-dual case, using the wave function solution of the *modified Mathieu equation*[43], calculated in the transformed (by (8.1.21)) of $x = 0$.

$$\boxed{\frac{d^2}{dy^2}\psi(y) - (2e^{2\theta}\cosh y + P^2)\psi(y) = 0} \tag{11.2.35}$$

In particular, whether the integration extremes were $+\infty$ and 0 in the non-transformed equation (8.1.1), now, by the transformation (8.1.21), they get transformed respectively into $-\infty$ and $+\infty$. Thus, by (**??**) and (4.2.1)

$$\log Q \simeq \lim_{y\to+\infty} \log\psi(y) \tag{11.2.36}$$

$$= \int_{-\infty}^{\infty} dy\, \sqrt{p(y)} S(y) \tag{11.2.37}$$

The Bäcklund transformation involves now the functions $p(y) = 2\cosh y$ and $u = +P^2$; while the Bäcklund potential (4.1.8) is

$$\begin{aligned} U(y) &= \frac{P^2}{2\cosh y} + \frac{1}{4}\frac{1}{2\cosh y} - \frac{5}{16}\frac{\tanh^2 y}{2\cosh y} \\ &= \left(P^2 - \frac{1}{16}\right)\frac{1}{2\cosh y} + \frac{5}{4}\frac{1}{(2\cosh y)^3} \end{aligned} \tag{11.2.38}$$

Our aim in this paragraph is to calculate the asymptotic expansion of $\log Q$ by the following formula

$$\log Q \sim \sum_{n=0}^{\infty} e^{\theta(1-2n)} \left(-\frac{1}{2n-1}\right) \int_{-\infty}^{\infty} dy\, \sqrt{p(y)} R_n(y) \qquad \theta \to +\infty \tag{11.2.39}$$

We can now begin this calculation. It is convenient to introduce the variable v as

$$\boxed{v(y) = \frac{1}{2\cosh y} > 0 \quad \forall y \in \mathbb{R}} \tag{11.2.40}$$

We observe that, thanks to the properties $\cosh y$, this new variable $v(y)$ is always positive and continuous on the real axis of y. Note also that $v = 1/p$. The inverse transformation is

$$y = \cosh^{-1}\frac{1}{2v} \tag{11.2.41}$$

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[43] We recall that, the generalized Mathieu equation (8.1.19), for the self dual Liouville model at $b = 1$, reduces to the standard[26] modified Mathieu equation. With respect to the calculation of the Floquet exponent, we don't need to make any rotation of y.

while the differentials get transformed as

$$dy = -\frac{dv}{v\sqrt{1-4v^2}} \qquad dw = -\frac{dv}{\sqrt{v^3}\sqrt{1-4v^2}} \tag{11.2.42}$$

In terms of the function $v(y)$, we can express the Bäcklund potential simply as a functional polynomial

$$U[v(y)] = \Big(P^2 - \frac{1}{16}\Big)v + \frac{5}{4}v^3 \tag{11.2.43}$$

We note that, *thanks to the definition (11.2.40) of the variable v, the Bäcklund potential for the modified Mathieu equation, as well as the expressions for the derivatives, assume the very same form which they had with respect to variable t above, for the Mathieu equation.* If we choose also the *same ansatz form*[44]

$$\boxed{R_n[v(y)] = \sum_{m=0}^{n} a_{nm} v^{n+2m}} \tag{11.2.45}$$

we find also *the very same recursion relation* (using the initial condition $a_{00} = 1$).

$$a_{n,m} = \left[P^2 \frac{n+2m-\frac{1}{2}}{n+2m} - \frac{1}{4}\frac{(n+2m-\frac{1}{2})^3}{n+2m}\right]a_{n-1,m} + \left[\frac{(n+2m-\frac{3}{2})^3}{n+2m} - \frac{n+2m-\frac{3}{2}}{n+2m}\right]a_{n-1,m-1} \tag{11.2.46}$$

However, the basic integrals seem different. In fact, to calculate $\log Q$, the integral of the eikonal is taken over the whole real axis of y, rather than on a $2\pi i$-long segment on the imaginary axis of y. *If we were able to show that the two different basic integrals are actually equal, we would show that, apart perhaps the correction of constant, the two different integrals and hence T and ν would indeed be equal.* More precisely, the basic integrals read

$$\int_{-\infty}^{\infty} dy\, \sqrt{2\cosh y}[v(y)]^{n+2m} = 2\int_{0}^{\infty} dy\, \sqrt{2\cosh y}[v(y)]^{n+2m} := 2\mathcal{J}_{n,m}$$

where we defined, for convenience of comparison with the Floquet index ν expansion, the generic basis integral as

$$\mathcal{J}_{n,m} = \int_0^{\frac{1}{2}} \frac{dv}{\sqrt{1-4v^2}} v^{2m+n-3/2} \tag{11.2.47}$$

Therefore, the general Gelfand-Dikii integral we want to calculate is

$$\boxed{\int_{-\infty}^{\infty} dy\, \sqrt{2\cosh y} R_n(y) = 2\sum_{m=0}^{n} a_{n,m}(P)\mathcal{J}_{n,m}} \tag{11.2.48}$$

We reduce it to the usual Euler Beta function, by the change of variable $r = 4v^2$, $v = \frac{1}{2}\sqrt{r}$, with $dv = \frac{1}{4}r^{-1/2}$.

$$\begin{aligned}\mathcal{J}_{n,m} &= \frac{1}{2}2^{-n-2m+1/2}\int_0^1 \frac{dr}{\sqrt{1-r}} r^{m+n/2-3/4-1/2} \\ &= 2^{-n-2m-1/2}B(\frac{1}{2}, m+\frac{n}{2}-\frac{1}{4})\end{aligned}$$

[44] which is equivalent to

$$R_n(t) = \sum_{k=n}^{3n} (-1)^k a_{n,(k-n)/2} \frac{1}{p^k} \tag{11.2.44}$$

where if $k-n$ is not an even number the relative contribution is null. Thus, the ansatz for $b=1$, We observe that, while the $b=1$ Liouville model is of course a particular case of the general Liouville model; the present ansatz is only similar to the particular case, for $b=1$, of the general Liouville ansatz (10.1.5). Namely, the factor $e^{(n-k)y}$ is lacking.

The generic basic integral $\mathcal{J}_{n,m}$ can be finally expressed as

$$\boxed{\mathcal{J}_{n,m} = 2^{-n-2m-1/2}\sqrt{\pi}\frac{\Gamma(m+\frac{n}{2}-\frac{1}{4})}{\Gamma(m+\frac{n}{2}+\frac{1}{4})}} \tag{11.2.49}$$

Now, in order to compare with the analogue basic integral (11.2.29) for the Floquet index, we manipulate a bit the Gamma functions which compose the Beta function. We use the reflection formula[28]

$$\Gamma(z) = \frac{\pi}{\sin\pi z}\frac{1}{\Gamma(1-z)} \tag{11.2.50}$$

so, the Beta function of the $\log Q$ basic integral (11.2.49)

$$\begin{aligned}
B(\frac{1}{2}, m+\frac{n}{2}-\frac{1}{4}) &= \Gamma(\frac{1}{2})\frac{\Gamma(m+\frac{n}{2}-\frac{1}{4})}{\Gamma(m+\frac{n}{2}+\frac{1}{4})} \\
&= \Gamma(\frac{1}{2})\frac{\Gamma(-m-\frac{n}{2}+\frac{3}{4})}{\Gamma(-m-\frac{n}{2}+\frac{5}{4})}\frac{\sin\pi(m+\frac{n}{2}+\frac{1}{4})}{\sin\pi(m+\frac{n}{2}-\frac{1}{4})} \\
&= -\frac{\Gamma(\frac{1}{2})}{\tan\pi(m+\frac{n}{2}-\frac{1}{4})}\frac{\Gamma(-m-\frac{n}{2}+\frac{3}{4})}{\Gamma(-m-\frac{n}{2}+\frac{5}{4})} \\
&= (-1)^{n+1}B(\frac{1}{2}, -m-\frac{n}{2}+\frac{3}{4})
\end{aligned}$$

apart the alternating sign, becomes exactly the the Beta function of the $2\pi i\nu$ basic integral (11.2.31). *The two generic basic integrals are related as simply as*

$$\boxed{\mathcal{J}_{n,m} = (-1)^{n+1}\mathcal{I}_{n,m}} \tag{11.2.51}$$

The expansion of $\log Q$ (or $2\pi i\nu$) can therefore be written in two equivalent ways

$$\log Q(\theta,P) \sim \sum_{n=0}^{\infty} e^{\theta(1-2n)}2\left\{\sqrt{\pi}\,\frac{\sqrt{2}^{-1-2n}}{1-2n}\sum_{m=0}^{n} a_{n,m}(P)\frac{\Gamma(m+\frac{n}{2}-\frac{1}{4})}{\Gamma(m+\frac{n}{2}+\frac{1}{4})}\right\} \qquad \theta\to+\infty \tag{11.2.52}$$

$$= \sum_{n=0}^{\infty} e^{\theta(1-2n)}2(-1)^{n+1}\left\{\sqrt{\pi}\,\frac{\sqrt{2}^{-1-2n}}{1-2n}\sum_{m=0}^{n} a_{n,m}(P)\frac{\Gamma(-m-\frac{n}{2}+\frac{3}{4})}{\Gamma(-m-\frac{n}{2}+\frac{5}{4})}\right\} \tag{11.2.53}$$

Comparing with the expansion (11.2.32) of $2\pi i\nu$ we deduce that

$$2\pi i\nu_n = (-1)^{n+1}\delta_n\tilde{B}_n I_{2n-1} = [-1+i(-1)^n]\tilde{B}_n I_{2n-1} \tag{11.2.54}$$

The "global" factor $(-1)^{n+1}\delta_n$ permits perfect agreement with the TQ expansion (1.6.21). In fact, the normalizations constants of the I_{2n-1} asymptotic coefficients of Baxter's T and Q functions are related as

$$\tilde{C}_n = [-1+i(-1)^n]\tilde{B}_n \tag{11.2.55}$$

multiplying this relation by the local integrals of motion we conclude our proof, because, since the n-th term of the ν asymptotic expansion now is

$$\boxed{2\pi i\nu_n = \tilde{C}_n I_{2n-1}} \tag{11.2.56}$$

by a general theorem[23] of asymptotic expansions, starting from $n=0$, we can calculate all the coefficients through a recursive relation

$$\nu_n = \lim_{\theta\to+\infty}\frac{\nu(\theta)-\sum_{k=0}^{n-1}\nu_k e^{\theta(1-2k)}}{e^{\theta(1-2n)}} = \frac{\tilde{C}_n I_{2n-1}}{2\pi i} \qquad n=0,1,2,... \tag{11.2.57}$$

$$\tilde{C}_n I_{2n-1} = \lim_{\theta\to+\infty}\frac{\log T(\theta)-\sum_{k=0}^{n-1}\tilde{C}_k I_{2k-1}e^{\theta(1-2k)}}{e^{\theta(1-2n)}} = 2\pi i\nu_n \qquad n=0,1,2,... \tag{11.2.58}$$

which, inverted, establishes that the asymptotic expansion of the Floquet index $2\pi i\nu$ is equal to the asymptotic expansion of the Baxter's $\log T$ function. *In other words, we have given an asymptotic proof of Zamolodhikov conjecture* (11.0.4).

11.2.3 Test

We tested directly[45] the Gelfand Dikii recursion for $b = 1$, for $R_0 \to R_1$, and $R_1 \to R_2$. Starting from $a_{00} = 1$, we obtain recursively from (11.2.22)

$$a_{10} = \frac{P^2}{2} - \frac{1}{32} \qquad a_{11} = \frac{5}{8} \tag{11.2.59}$$

$$a_{20} = \frac{3}{8}P^4 - \frac{15}{64}P^2 + \frac{27}{2048} \qquad a_{21} = \frac{35}{16}P^2 - \frac{455}{256} \qquad a_{22} = \frac{1155}{128} \tag{11.2.60}$$

Using these Gelfand-Dikii coefficients and the relative basic integrals (11.2.49), we calculate the first local integrals of motion (or charges). The first charge I_1 is

$$\begin{aligned} I_1 &= -2\Big[a_{10} + 2^{-2}\frac{1}{3}a_{11}\Big] && (11.2.61)\\ &= -2\Big[\frac{P^2}{2} + \frac{1}{48}\Big] && (11.2.62)\\ &= \Delta - \frac{25}{24} && (11.2.63) \end{aligned}$$

The second charge I_3 is

$$\begin{aligned} I_3 &= +\frac{8}{3}\Big[a_{20} + 2^{-2}\frac{3}{5}a_{21} + 2^{-4}\frac{3\times 7}{5\times 9}a_{22}\Big] && (11.2.64)\\ &= P^4 + \frac{1}{4}P^2 + \frac{5}{192} && (11.2.65)\\ &= \Delta^2 - \frac{9}{4}\Delta + \frac{245}{192} && (11.2.66) \end{aligned}$$

Thus, we checked that our results match those of,[29] for $b = 1$, and of,[10] for $c = 25$.[46]

[45] That is, from the general expression of Gelfand Dikii polynomials (4.3.16) (4.3.17)

[46] We note also that, for $b = 1$, $\Delta = 1 - P^2$

11.3 WKB expansion of Zamolodchikov's relation

In this section, we expand the TQ relation for $b = 1$ in WKB series, with expansion parameter

$$\boxed{\epsilon(\theta) = \pm\frac{1}{2i}e^{-\theta}} \tag{11.3.1}$$

which, of course, has the role of the Planck constant $\hbar$. We use the following ansatz for the $Q(\theta)$ function[18]

$$Q(\theta) = e^{\frac{1}{\epsilon(\theta)}\phi(\theta)} \tag{11.3.2}$$

$$\phi(\theta) = \sum_{n=0}^{\infty}\phi_n\epsilon^n \tag{11.3.3}$$

$$\tag{11.3.4}$$

with the coefficients ϕ_n (independent from ϵ and therefore also from θ). The WKB expansion of the Floquet index has been treated in section 7. In this subsection we are going to show that the coefficients of the two expansion are related as

$$\boxed{\begin{aligned} 2\pi i\nu_n &= (-1+(-1)^n i)\phi_{2n-1} && (11.3.5)\\ &= \frac{\tilde{C}_n}{\tilde{B}_n}\phi_{2n-1} && (11.3.6)\end{aligned}}$$

where we used the usual (1.6.21). However, this is only true *assuming* Zamolodchikov's relation (11.0.4), since, in this subsection, we just expand the TQ relation.

11.3.1 WKB expansion of the TQ relation

The $T(\theta)$ function satisfies the TQ relation (because $p = \frac{1}{2}$ in (9.1.4))

$$\boxed{T(\theta) = \frac{Q(\theta + i\frac{\pi}{2})}{Q(\theta)} + \frac{Q(\theta - i\frac{\pi}{2})}{Q(\theta)}} \tag{11.3.7}$$

In this paragraph, we are going to examine the WKB expansion of this TQ relation, using for Q the ansatz (11.3.3) .
Because θ is divergent, it is convenient to change variable, defining an infinitesimal η as

$$\eta = e^{-\theta} \tag{11.3.8}$$

$$= \pm 2i\epsilon \to 0 \tag{11.3.9}$$

$$\theta = -\ln\eta \to \infty \tag{11.3.10}$$

For the sake of generality, set $\delta = i\frac{\pi}{2}$. We make a Taylor expansion at infinity

$$\phi(\theta+\delta) = \phi(-\ln\eta + \delta) \tag{11.3.11}$$

$$= \sum_{n=0}^{\infty}\frac{\delta^n}{n!}\frac{d^n\phi}{d(-\ln\eta)^n}\bigg|_{-\ln\eta} \tag{11.3.12}$$

$$= \phi(-\ln\eta) + \delta\frac{d\phi}{d(-\ln\eta)}\bigg|_{-\ln\eta} + \frac{\delta^2}{2!}\frac{d^2\phi}{d(-\ln\eta)^2}\bigg|_{-\ln\eta} + ...$$

and transform the derivatives in $\theta = -\ln\eta$ in derivatives in η, using the *Stirling numbers of the second kind* $\mathfrak{s}_n^{(m)}$

$$\frac{d^n\phi}{d(\ln\eta)^n} = \sum_{m=0}^{n} \mathfrak{s}_n^{(m)} \eta^m \frac{d^m\phi}{d\eta^m} \tag{11.3.13}$$

$$\frac{d^n\phi}{d(-\ln\eta)^n} = \sum_{m=0}^{n} (-1)^n \mathfrak{s}_n^{(m)} \eta^m \frac{d^m\phi}{d\eta^m} \tag{11.3.14}$$

$$\tag{11.3.15}$$

This relations can be proven showing that some generic coefficients of expansion, actually satisfy the characteristic recursion relation of the Stirling numbers of the second kind,[28] that is $\mathfrak{s}_n^{(m)}$.

$$\mathfrak{s}_{n+1}^{(m)} = m\mathfrak{s}_n^{(m)} + \mathfrak{s}_n^{(m-1)} \tag{11.3.16}$$

A brief collection of properties of the $\mathfrak{s}_n^{(m)}$ numbers can be found in appendix C.2. Thus Taylor expansion becomes

$$\begin{aligned}
\phi(\theta+\delta) &= \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \sum_{m=0}^{n} (-1)^n \mathfrak{s}_n^{(m)} \eta^m \frac{d^m\phi}{d\eta^m}\bigg|_{-\ln\eta} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[\frac{\delta^n}{n!}(-1)^n \mathfrak{s}_n^{(m)}\right] \eta^m \frac{d^m\phi}{d\eta^m}\bigg|_{-\ln\eta} \\
&= \sum_{m=0}^{\infty} \left[\sum_{n=m}^{\infty} \frac{\delta^n}{n!}(-1)^n \mathfrak{s}_n^{(m)}\right] \epsilon^m \frac{d^m\phi}{d\epsilon^m}\bigg|_{\theta(\epsilon)} \\
&= \sum_{n=0}^{\infty} \left[\frac{\delta^n}{n!}(-1)^n \mathfrak{s}_n^{(0)}\right] \phi\bigg|_{\theta(\epsilon)} + \sum_{m=1}^{\infty} \left[\sum_{n=m}^{\infty} \frac{\delta^n}{n!}(-1)^n \mathfrak{s}_n^{(m)}\right] \epsilon^m \frac{d^m\phi}{d\epsilon^m}\bigg|_{\theta(\epsilon)}
\end{aligned} \tag{11.3.17}$$

As explained in appendix C.2, it is a property of the Stirling numbers that $\mathfrak{s}_n^{(0)} = \delta_{0n}$, hence our expansion becomes

$$\phi(\theta+\delta) = \phi(\theta) + \sum_{m=1}^{\infty} \left[\sum_{n=m}^{\infty} \frac{\delta^n}{n!}(-1)^n \mathfrak{s}_n^{(m)}\right] \epsilon^m \frac{d^m\phi}{d\epsilon^m}\bigg|_{\theta(\epsilon)} \tag{11.3.18}$$

We must be careful in writing the TQ relation, because the parameter of expansion depends on θ

$$\epsilon(\theta + i\frac{\pi}{2}) = \pm\frac{1}{2i} e^{-\theta - i\pi/2} \tag{11.3.19}$$

$$= \pm\frac{1}{2i}\frac{1}{i} e^{-\theta} = -i\epsilon(\theta) \tag{11.3.20}$$

so that one part of the TQ relation can be written as

$$\begin{aligned}
\frac{Q(\theta+\delta)}{Q(\theta)} &= \exp\left\{\frac{1}{(-i\epsilon)}\phi(\theta+\delta) - \frac{1}{\epsilon}\phi(\theta)]\right\} \\
&= \exp\left\{\frac{i}{\epsilon}[\phi(\theta+\delta) - \phi(\theta)] + \frac{i-1}{\epsilon}\phi(\theta)]\right\} \\
&= \exp\left\{\frac{i}{\epsilon}\left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} \frac{\delta^m}{m!}(-1)^m \mathfrak{s}_m^{(n)}\right) \epsilon^n \frac{d^n\phi}{d\epsilon^n}\bigg|_{\theta(\epsilon)}\right] + \frac{i-1}{\epsilon}\phi(\theta)\right.
\end{aligned} \tag{11.3.21}$$

We now take into account the expansion (11.3.3) of $\phi(\theta)$ itself in the parameter ϵ. Since ϕ_l is independent from θ, the general n-th derivative is

$$\frac{d^n\phi}{d\epsilon^n} = \sum_{l=n}^{\infty} \phi_l \epsilon^{l-n} \frac{l!}{(l-n)!} \tag{11.3.22}$$

Substituting this expression in (11.3.18), we obtain

$$
\begin{aligned}
\phi(\theta+\delta)-\phi(\theta) &= \sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty}\frac{\delta^m}{m!}(-1)^m\mathfrak{s}_m^{(n)}\right)\epsilon^n\frac{d^n\phi}{d\epsilon^n}\bigg|_{\theta(\epsilon)} \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty}\frac{\delta^m}{m!}(-1)^m\mathfrak{s}_m^{(n)}\right)\sum_{N=n}^{\infty}\phi_N\epsilon^N\frac{N!}{(N-n)!} \\
&= \sum_{n=1}^{\infty}\sum_{N=n}^{\infty}\phi_N\epsilon^N\frac{N!}{(N-n)!}\left(\sum_{m=n}^{\infty}\frac{\delta^m}{m!}(-1)^m\mathfrak{s}_m^{(n)}\right) \\
&= \sum_{N=1}^{\infty}\epsilon^N\left\{\phi_N\sum_{n=1}^{N}\frac{N!}{(N-n)!}\left(\sum_{m=n}^{\infty}\frac{\delta^m}{m!}(-1)^m\mathfrak{s}_m^{(n)}\right)\right\}
\end{aligned}
\tag{11.3.23}
$$

so that the part of the TQ relation we are considering (11.3.21) becomes

$$
\begin{aligned}
\frac{Q(\theta+\delta)}{Q(\theta)} &= \exp\left\{\frac{i}{\epsilon}\left[\sum_{n=0}^{\infty}\left(\sum_{m=n}^{\infty}\frac{\delta^m}{m!}(-1)^m\mathfrak{s}_m^{(n)}\right)\epsilon^n\frac{d^n\phi}{d\epsilon^n}\bigg|_{\theta(\epsilon)}\right]-\frac{1}{\epsilon}\phi(\theta)\right. \\
&= \exp\left\{\frac{1}{\epsilon}\sum_{N=0}^{\infty}\epsilon^N\left[-\phi_N+i\phi_N\sum_{n=0}^{N}\frac{N!}{(N-n)!}\left(\sum_{m=n}^{\infty}\frac{\delta^m}{m!}(-1)^m\mathfrak{s}_m^{(n)}\right)\right]\right\}
\end{aligned}
\tag{11.3.24}
$$

The Stirling numbers of the second kind satisfy the following property[28]

$$
(e^x-1)^n = n!\sum_{m=n}^{\infty}\mathfrak{s}_m^{(n)}\frac{x^m}{m!}
\tag{11.3.25}
$$

which, substituted in our expression, entails

$$
\begin{aligned}
\frac{Q(\theta+\delta)}{Q(\theta)} &= \exp\left\{\frac{1}{\epsilon}\sum_{N=0}^{\infty}\epsilon^N\left[-\phi_N+i\phi_N\sum_{n=0}^{N}\frac{N!}{(N-n)!}\left(\frac{(e^{-\delta}-1)^n}{n!}\right)\right]\right\} \\
&= \exp\left\{\frac{1}{\epsilon}\sum_{N=0}^{\infty}\epsilon^N\left[-\phi_N+i\phi_N e^{-N\delta}\right]\right\}
\end{aligned}
\tag{11.3.26}
$$

The other term of the TQ relation is obtained simply by sending δ into $-\delta$. In conclusion, *for a generic shift* δ we can write

$$
\boxed{
\begin{aligned}
\frac{Q(\theta+\delta)}{Q(\theta)} &= \exp\left\{\frac{1}{\epsilon}\sum_{N=0}^{\infty}\epsilon^N\left[-\phi_N+i\phi_N e^{-N\delta}\right]\right\} && (11.3.27)\\
\frac{Q(\theta-\delta)}{Q(\theta)} &= \exp\left\{\frac{1}{\epsilon}\sum_{N=0}^{\infty}\epsilon^N\left[-\phi_N+i\phi_N e^{N\delta}\right]\right\} && (11.3.28)\\
& && (11.3.29)
\end{aligned}
}
$$

If we define, trivially, a new shift parameter $\tilde{\delta}$ by $\delta = i\tilde{\delta}$, the TQ relation is written, for general shift

$$
\boxed{
\begin{aligned}
T(\theta) &= \frac{Q(\theta+\delta)}{Q(\theta)}+\frac{Q(\theta-\delta)}{Q(\theta)} && (11.3.30)\\
&= \exp\left\{\frac{1}{\epsilon}\sum_{N=0}^{\infty}\epsilon^N\left[-\phi_N+i\phi_N\cos(N\tilde{\delta})\right]\right\}2\cosh\left\{\frac{1}{\epsilon}\sum_{n=0}^{N}\epsilon^N\left[\phi_N\sin(N\tilde{\delta})\right]\right\} && (11.3.31)\\
& && (11.3.32)
\end{aligned}
}
$$

Restoring the particular value of δ appears now essential in order to separate the odd and even terms in the expansion of $\phi(\theta)$

$$2\cos 2\pi\nu = T(\theta) = \frac{Q(\theta + i\frac{\pi}{2})}{Q(\theta)} + \frac{Q(\theta - i\frac{\pi}{2})}{Q(\theta)} \tag{11.3.33}$$

$$= \exp\left\{\frac{1}{\epsilon}\sum_{N=0}^{\infty}\epsilon^{2N}\phi_{2N}[-1+(-1)^N i]\right\} 2\cosh\left\{\frac{1}{\epsilon}\sum_{N=1}^{\infty}\epsilon^{2N-1}\phi_{2N-1}(-1)^N\right\} \tag{11.3.34}$$

Asymptotically, in some Stokes sector of θ

$$e^{2\pi i\nu} \simeq 2\cos 2\pi\nu = \frac{Q(\theta + i\frac{\pi}{2})}{Q(\theta)} + \frac{Q(\theta - i\frac{\pi}{2})}{Q(\theta)} \qquad \Re\theta \to +\infty \qquad \Im\theta \neq 0 \tag{11.3.35}$$

$$= \exp\left\{\frac{1}{\epsilon}\sum_{N=0}^{\infty}\epsilon^{2N}\nu_{2N}\right\} \tag{11.3.36}$$

It must be true that

$$\boxed{\begin{aligned} \nu_0 &= (-1+i)\phi_0 && (11.3.37)\\ 0 &= \phi_{2N}(w) && (11.3.38)\\ \nu_{2N}(w) &= \phi_{2N-1}(w)[-1+(-1)^N i] && (11.3.39) \end{aligned}}$$

where we set the even modes ϕ_{2N} to zero, because we assume that the ϕ_{2N} are the integrals over the whole space of the even modes of the eikonal integrand, as usual (6.2.5).
It might be worthy to note that the correcting constant for the WKB expansion is again the $\delta_n(-1)^{n+1}$ found for the large energy expansion (11.2.54), (11.2.56) and (1.6.21).

11.3.2 From large energy expansion to WKB expansion

We know that, in the WKB approximation, $P \sim e^{\theta} \to +\infty$; more precisely,

$$\boxed{P = \sqrt{-2u}e^{\theta}} \tag{11.3.40}$$

with u constant. Thus, we see that the *divergence of* P *is necessary* because $e^{-\theta}$ is infinitesimal ($\hbar \to 0$). We recall the $\mathcal{N} = 2$ form for the Mathieu equation

$$\frac{\epsilon^2}{2}\frac{d^2}{dz^2}\psi(z) + [u - \cos 2z]\psi(z) = 0 \tag{11.3.41}$$

We already know that the local integrals of motion can be expressed as polynomials in P^2, with some coefficients $\mathfrak{p}_{nl}$

$$I_{2n-1} = \sum_{l=0}^{n}\mathfrak{p}_{nl}P^{2l} \tag{11.3.42}$$

B in the WKB expansion, Considering the WKB expansion of Baxter's Q function, because $P \sim e^{\theta}$ as $\theta \to +\infty$, there is a rearrangement of the coefficients. More precisely,

$$\begin{aligned} \log Q &= \sum_{n=0}^{\infty} e^{(1-2n)\theta}\tilde{B}_n\Big(\lim_{P^2\to+\infty} I_{2n-1}(P)\Big) = e^{\theta}\sum_{n=0}^{\infty}e^{-2n\theta}\sum_{l=0}^{n}\tilde{B}_n\mathfrak{p}_{nl}P^{2l} \\ &= e^{\theta}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\tilde{B}_n\mathfrak{p}_{nl}(-2u)^l e^{2\theta(l-n)} = e^{\theta}\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\tilde{B}_n\mathfrak{p}_{nl}(-2u)^l e^{2\theta(l-n)} \\ (N = n-l) \qquad &= e^{\theta}\sum_{l=0}^{\infty}\sum_{N=0}^{\infty}\tilde{B}_{N+l}\mathfrak{p}_{N+l,l}(-2u)^l e^{-2N\theta} = \sum_{N=0}^{\infty}e^{(1-2N)\theta}\left[\sum_{l=0}^{\infty}\tilde{B}_{N+l}\mathfrak{p}_{N+l,l}(-2u)^l\right] \end{aligned} \tag{11.3.43}$$

Thus we see that, while for the large energy (rapidity) expansion the coefficients of the Q function were the local integrals of motion I_{2n-1}, which are polynomials in P^2 with coefficients $\mathfrak{p}_{nl}$

$$\boxed{\phi_{2n-1}\Big|_{\text{large } E} = \tilde{B}_n I_{2n-1} = \tilde{B}_n \sum_{l=0}^{n} \mathfrak{p}_l P^{2l}} \tag{11.3.44}$$

for the WKB expansion, the coefficients $\mathfrak{p}_{nl}$ *of undergoes a rearrangement over all but a finite number of charges* I_{2k-1}*, starting from the* n*-th.* $(k = n, n+1, ...)$ The formula for ϕ_n in the WKB case is thus an infinite series.

$$\boxed{\phi_{2n-1}\Big|_{\text{small } \hbar} = \sum_{l=0}^{\infty} \tilde{B}_{N+l} \mathfrak{p}_{N+l,l} (-2u)^l} \tag{11.3.45}$$

On the other hand, the Floquet exponent for the Mathieu equation, as calculated in section 7 can be expressed as a differential polynomial in $\ln u$

$$2\pi i \nu_n = u^n \sum_{l=0}^{n} \hat{C}_{n,n+l} u^l \frac{d^l}{du^l} \left\{ \frac{d^n}{du^n} \int_{-\pi/2}^{\pi/2} dz \sqrt{2(\cos 2z - u)} \right\} \tag{11.3.46}$$

Taking into account the rescaling of the expansion parameter ϵ, one can easily deduce the precise relation between the ϕ_{2n-1} modes of Baxter's Q function and the $2\pi i \nu_n$ coefficients of the Floquet index and then find an expression for the n-th ϵ_1 mode of the *deformed Seiberg-Witten prepotential in terms of the* P^2*-coefficients* [47] $\mathfrak{p}_{n,l}$ *of the local integrals of motions* I_{2n-1}.

$$\boxed{\nu_n = [-1 + i(-1)^n] 2^{2n+1} i (-1)^n \sum_{l=0}^{\infty} \tilde{B}_{N+l} \mathfrak{p}_{N+l,l} (-2u)^l} \tag{11.3.47}$$

thanks to Zamolodchikov's fundamental relation (11.0.4). In appendix E we began to outline a method for exactly solving the Gelfand Dikii coefficients recursion for the self-dual Liouville model. If, in a next work, we were to obtain a general formula for the Gelfand-Dikii coefficient

$$a_{n,m}(P) \qquad \forall n, m \in \mathbb{N} \qquad \text{such that } 0 \le m \le n \tag{11.3.48}$$

then it should not be difficult to find also the P^2 subcoefficient and thereby applying the general formula (11.3.47) to calculate the ν_n, hence the n-th mode of expansion of the *deformed Seiberg-Witten prepotential for* $\mathcal{N} = 2$ *gauge theory.*

[47] Which we recall that, by $\Delta = 1 - P^2$, correspond to coefficients with respect to the conformal weight Δ.

A Expansion modes for the Riccati equation

We report the first few modes for the eikonal integrand $S(w)$ of the Riccati equation (4.2.9)

$$\begin{aligned}
S_2^{\pm} &= \mp\frac{1}{2}S_1'^{\pm} = -\frac{1}{4}U' \\
S_3^{\pm} &= \mp\frac{1}{2}\left(S_2'^{\,\pm} + S_1^{\pm 2}\right) = \pm\frac{1}{8}(U'' - U^2) \\
S_4^{\pm} &= \mp\frac{1}{2}\left(S_3'^{\,\pm} + 2S_1 \pm S_2^{\pm}\right) = -\frac{1}{16}U''' + \frac{1}{4}UU'
\end{aligned} \tag{A.0.1}$$

$$\begin{aligned}
S_5^{\pm} &= \mp\frac{1}{2}\left(S_4'^{\,\pm} + 2S_1^{\pm}S_3^{\pm} + S_2^{\pm 2}\right) \\
&= \pm\frac{1}{32}U'''' \mp \frac{1}{8}(U'^2 + UU'') \mp \frac{1}{16}(UU'' - U^3) \mp \frac{1}{16}U'^2 \\
&= \pm\frac{1}{32}U'''' \mp \frac{3}{16}(UU'' + U'^2) \mp \frac{1}{16}U^3
\end{aligned} \tag{A.0.2}$$

here the ' stays always for the derivative with respect to w

We prove also a very simple theorem by induction

Theorem 2. *In general, we have*

$$S_{2n}^{+} = S_{2n}^{-} \qquad n \in \mathbb{N}$$
$$S_{2n+1}^{+} = -S_{2n+1}^{-}$$

In fact, let's assume that by direct calculation we have verified this statement for $k \in \mathbb{N}$, that is

$$S_{2k}^{+} = S_{2k}^{-}$$
$$S_{2k+1}^{+} = -S_{2k+1}^{-}$$

Taking $n = 2k+1$ we can write (4.2.12) as

$$\begin{aligned}
S_{2k+2}^{\pm} &= \mp\frac{1}{2}\Big(\pm S_{2k+1}'^{\,+} + (\pm S_1^{+})S_{2k}^{+} + S_2^{+}(\pm S_2k - 1)^{+} + ... \\
&= +... + S_{2k}^{+}(\pm S_1^{+})\Big) \\
&= -\frac{1}{2}\left(S_{2k+1}'^{\,+} + \sum_{m=1}^{2k} S_m^{+}S_{2k+1-m}^{+}\right) \\
S_{2k+2}^{+} &= S_{2k+2}^{-}
\end{aligned}$$

Now take $n = 2k+2$

$$\begin{aligned}
S_{2k+3}^{\pm} &= \mp\frac{1}{2}\Big(S_{2k+2}'^{\,+} + (\pm S_1^{+})(\pm S_{2k+1}^{+}) + S_2^{+}S_2k^{+} + (\pm S_3^{+})(\pm S_{2k-1}^{+}) + ... \\
&= +... + (\pm S_{2k+1}^{+})(\pm S_1^{+})\Big) \\
&= \mp\frac{1}{2}\left(S_{2k+1}'^{\,+} + \sum_{m=1}^{2k+1} S_m^{+}S_{2k+1-m}^{+}\right) \\
S_{2k+3}^{+} &= -S_{2k+3}^{-}
\end{aligned}$$

We have proved that

$$S_{2k+2}^{+} = S_{2k+2}^{-}$$
$$S_{2k+3}^{+} = -S_{2k+3}^{-}$$

by induction this equalities hold for all $k \in \mathbb{N}$, as was to be proved.
We now prove also that the leading term of the Gelfand Dikii polynomials is[16]

$$R_n[U] = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}n!}U^n + O(U^{n-1}) \qquad |U| \to +\infty \tag{A.0.3}$$

$$\tag{A.0.4}$$

In fact:

$$\begin{aligned}
\frac{p_{n-1}}{q_{n-1}}U^{n-1} \to \frac{p_{n-1}}{q_{n-1}}U^n - \frac{1}{2}\int dw \frac{dU}{dw}\frac{p_{n-1}}{q_{n-1}}U^{n-1} \\
&= \frac{p_{n-1}}{q_{n-1}}\left[1-\frac{1}{2n}\right]U^n \\
&= \frac{p_{n-1}}{q_{n-1}}\frac{2n-1}{2n}U^n \\
p_n &= (2n-1)p_{n-1} \\
q_n &= 2nq_n \\
p_1 &= 1 \\
q_1 &= 2 \\
p_n &= (2n-1)!! \qquad (n-1 \text{ factors, the last being } 1) \\
q_n &= (2n)!! \qquad (n-1 \text{ factors, the last being } 2) \\
\frac{p_n}{q_n} &= \frac{(n-\frac{1}{2})(n-\frac{3}{2})\cdots 2^{(n-1)}}{n!2^{(n-1)}} \\
&= \frac{(n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2})2^{(n-1)}}{n!2^{(n-1)}\Gamma(\frac{1}{2})} \\
&= \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}n!}
\end{aligned}$$

B Gauss Hypergeometric function

The Gauss hypergeometric function is defined by the series

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n \tag{B.0.1}$$

where c is not zero nor a negative integer. $(a)_n$ stands for the Pochhammer symbol

$$(a)_n \coloneqq \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1) \quad n \geq 1 \tag{B.0.2}$$

$$(a)_0 \coloneqq 0 \tag{B.0.3}$$

The series (B.0.1) converges absolutely for $|z| < 1$. On the circle $|z| = 1$ is absolutely convergent provided $\Re(c-a-b) > 0$; it converges but not absolutely and not at $z = 1$ if $-1 < \Re(c-a-b) \leq 0$ and diverges if $\Re(c-a-b) \leq -1$.[26] The Gauss-hypergeometric function (as well as the binomial expansion) convenes only in the circle of radius 1.
It can be proven that for $\Re b > 0, \Re c > 0$ the hypergeometric function has the integral representation

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\, \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \tag{B.0.4}$$

for $z \in \mathbb{C} \setminus (1, \infty)$. In our case $z = -e^{yQ} \to -\infty$ as $y \to \infty$, so

$$
\begin{aligned}
{}_2F_1(\alpha, \beta; \gamma; -z) &\sim \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} z^{-\alpha} \int_0^1 dt\, t^{\beta-\alpha-1}(1-t)^{\gamma-\beta-1} \quad z \to \infty \\
&= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} \frac{1}{z^a}
\end{aligned} \tag{B.0.5}
$$

The following formula[28] permit to expand the Gauss hypergeometric function around infinity

$$
\begin{aligned}
{}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\ {}_2F_1(a, 1-c+a; 1-b+a; \frac{1}{z}) \\
&+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\ {}_2F_1(b, 1-c+b; 1-a+b; \frac{1}{z})
\end{aligned} \tag{B.0.6}
$$

B.1 Indefinite integrals for ODE/IM

Using directly the defintion of the hypergeometric function (B.0.1) we calculated the*Bäcklund independent variable* (4.1.2) and also the *first nontrivial eikonal density*. They are both *indefinite integrals* and, in general, are far more difficult with respect to the Euler Beta function.
The Bäcklund independent variable for the Liouville models is

$$
w(y) = \int dy' \sqrt{e^{y'/b} + e^{-y'b}} \tag{B.1.1}
$$

$$
= 2b\sqrt{e^{\frac{y}{b}} + e^{-yb}} - 2Qe^{-yb/2}\ {}_2F_1\Big(\frac{1}{2}, -\frac{p}{2}, -\frac{p}{2}+1; -e^{yQ}\Big) + const. \tag{B.1.2}
$$

while for the minimal models we calculated it only for $s = 1$

$$
w(z) = \int dz \sqrt{z^{2M} - 1} \tag{B.1.3}
$$

$$
= -\frac{M}{M+1} iz\ {}_2F_1(\frac{1}{2}, \frac{1}{2M}; \frac{2M+1}{2M}; z^{2M}) + \frac{1}{M+1} z\sqrt{z^{2M}-1} + const. \tag{B.1.4}
$$

The indefinite integral for the first Gelfand-Dikii polynomial $R_1 = \frac{1}{2}U$ is given through the integral of the *Bäcklund potential*. In the Liouville case:

$$
\begin{aligned}
\int dy U(y) &= \int dy \sqrt{p(y)} \frac{1}{p(y)} \Big(+P^2 + \frac{1}{4} \frac{p''(y)}{p(y)} - \frac{5}{16} \frac{p'^{\,2}(y)}{p^2(y)} \Big) \\
&= +P^2 \int dy \frac{1}{\sqrt{e^{y/b} + e^{-yb}}} + \frac{1}{4} \int dy \frac{\frac{1}{b^2} e^{y/b} + b^2 e^{-yb}}{(e^{y/b} + e^{-yb})^{3/2}} \\
&- \frac{5}{16} \int dy \frac{\frac{1}{b^2} e^{2y/b} - 2e^{y/b - yb} + b^2 e^{-2yb}}{(e^{y/b} + e^{-yb})^{5/2}} \\
&= \Big(+P^2 \frac{2}{b} + \frac{1}{2b} - \frac{5}{12b} \Big) e^{yb/2} F_{21}(\frac{1}{2}, \frac{b^2}{2b^2+2}, \frac{3b^2+2}{2b^2+2}, -e^{yQ}) \\
&+ \frac{e^{yb/2}}{(1+e^{yQ})^{3/2}} \Big[\frac{1}{2b}(b^2-1)(1+e^{yQ}) - \frac{5}{48b}[6b^2 - 4 + (4b^2-6)e^{yQ}] \Big] \\
&= \Big(+P^2 \frac{2}{b} + \frac{1}{12b} \Big) e^{yb/2} F_{21}(\frac{1}{2}, \frac{b^2}{2b^2+2}, \frac{3b^2+2}{2b^2+2}, -e^{yQ}) \\
&+ \frac{1}{24b} \Big[-2 - 3b^2 + (3+2b^2)e^{yQ} \Big] \frac{e^{yb/2}}{(1+e^{yQ})^{3/2}}
\end{aligned}
$$

In the minimal models case:

$$
\begin{aligned}
\int dz\, U(z) &= \Big(\frac{2}{3} - \frac{11}{6} M \pm 4P^2(M+1)^2 \Big) \frac{i}{z}\ {}_2F_1(\frac{1}{2}, -\frac{1}{2M}; -\frac{1}{2M}+1; z^{2M}) \\
&+ \Big(-\frac{11}{12} + \frac{9}{4} M \Big) \frac{1}{z\sqrt{z^{2M}-1}} + \frac{5}{12} M \frac{1}{z(z^{2M}-1)^{3/2}}
\end{aligned}
$$

C Stirling numbers

C.1 Stirling numbers of the first kind

The *Stirling numbers of the first kind* $S_n^{(m)}$ can be given the following combinatorial calculus definition: $(-1)^{m-n}S_n^{(m)}$ is the number of permutations of n symbols which have exactly m cycles.[28]
Their generating function is

$$x(x-1)...(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} = \sum_{m=0}^{n} S_n^{(m)} x^m \tag{C.1.1}$$

The Stirling numbers of the first kind satisfies the recurrence relation

$$S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)} \qquad n \geq m \geq 1 \tag{C.1.2}$$

with initial condition $S_0^{(0)} = 1$ Some examples are

$$S_n^{(0)} = \delta_{0n} \tag{C.1.3}$$

$$\tag{C.1.4}$$

$$S_n^{(1)} = (-1)^{n-1}(n-1)! \tag{C.1.5}$$

$$S_n^{(n-1)} = -\binom{n}{2} \tag{C.1.6}$$

$$S_n^{(n)} = 1 \tag{C.1.7}$$

C.1.1 "Correcting polynomials" to Gelfand-Dikii coefficients

Using the signed Stirling numbers of the first kind[28] $S_N^{(l)}$, we can express the "descending factorial" of x

$$\frac{\Gamma(x+1)}{\Gamma(x-N+1)} = x(x-1)\cdots(x-N+1) = \sum_{l=0}^{N} S_N^{(l)} x^l \tag{C.1.8}$$

The descending factorial can be expressed in sum of powers of its argument, with as coefficients (up to a sign) the Stirling numbers of the first kind.
We give an example of application of this formula to the calculation of minimal models basis integrals of subsection (5.3)

$$\Gamma(\frac{2n-1}{2M} + \frac{2m-1}{2}) = \frac{\Gamma(\frac{2n-1}{2M} + \frac{2m-1}{2})}{\Gamma(\frac{2n-1}{2M} + \frac{2n-1}{2})}\Gamma(\frac{2n-1}{2M} + \frac{2n-1}{2})$$

$$= \Gamma(\frac{2n-1}{2M} + \frac{2n-1}{2})\frac{\Gamma\Big((n-\frac{1}{2})(\xi+1) + m - n\Big)}{\Gamma((n-\frac{1}{2})(\xi+1))}$$

$$\Gamma(m+\frac{1}{2}) = \frac{(2m-1)!!}{2^m\sqrt{\pi}}$$

Thus we can write a polynomial in $\xi = 1/M$ in terms of the Stirling numbers

$$\begin{aligned}
\frac{\Gamma\Big((n-\frac{1}{2})(\xi+1)+m-n\Big)}{\Gamma((n-\frac{1}{2})(\xi+1))} &= \sum_{l=0}^{m-n} S_{m-n}^{(l)}\Big(\frac{n-1/2}{M}+m-\frac{3}{2}\Big)^{l} \\
&= \sum_{l=0}^{m-n} S_{m-n}^{(l)} \sum_{r=0}^{l}\binom{l}{r}(n-\frac{1}{2})^{r}(m-\frac{3}{2})^{l-r}M^{-r} \\
&= \sum_{r=0}^{m-n}\alpha_{m-n}^{(r)}M^{-r} \\
\alpha_{m-n}^{(r)} &:= \sum_{l=r}^{m-n}\binom{l}{r}(n-\frac{1}{2})^{r}(m-\frac{3}{2})^{l-r}S_{m-n}^{(l)}
\end{aligned}$$

which will be the correcting factor to our expression for the local integrals of motions (5.3.11)

C.2 Stirling numbers of the second kind

We report the recursion relation for the Stirling numbers of the second kind $\mathfrak{s}_n^{(m)}$[28]

$$\boxed{\mathfrak{s}_{n+1}^{(m)} = m\mathfrak{s}_n^{(m)} + \mathfrak{s}_n^{(m-1)}} \tag{C.2.1}$$

with initial condition $\mathfrak{s}_0^{(0)} = 1$

The Stirling numbers of the second kind can be expressed in a closed form as

$$\mathfrak{s}_n^{(m)} = \frac{1}{m!}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{l}l^n \tag{C.2.2}$$

Some special cases are

$$\mathfrak{s}_n^{(0)} = \delta_{0n} \tag{C.2.3}$$

$$\mathfrak{s}_n^{(1)} = \mathfrak{s}_n^{(n)} = 1 \tag{C.2.4}$$

$$\mathfrak{s}_n^{(n-1)} = \binom{n}{2} \tag{C.2.5}$$

A possible generating function for the Stirling numbers of the second kind is $(e^x-1)^n$. More precisely[28]

$$(e^x-1)^n = n!\sum_{m=n}^{\infty}\mathfrak{s}_m^{(n)}\frac{x^m}{m!} \tag{C.2.6}$$

The Stirling numbers of the first kind can be expressed in terms of those of the second kind by the relation

$$S_n^{(m)} = \sum_{k=0}^{n-m}(-1)^k\binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k}\mathfrak{s}_{n-m+k}^{(k)} \tag{C.2.7}$$

Hence, also the Stirling numbers of the first kind have a closed form, though quite cumbersome.

D Further He-Miao operators examples

In this appendix, we give further examples of our algorithmic proof for He-Miao conjecture. In section 7 we reported the calculations for $n = 1, 2$, here we continue to $n = 3, 4$. We did also $n = 5$, but we don't write it. From $n = 1, 2, 3, 4$ we tested our results with those of He and Miao[34] (obtained through a far more laborious procedure). For $n = 5$ we tested our results with those of Basar and Dunne[35]

D.1 Less simple example: T_3

Let's begin with transformation of b_{39}, that is, the highest degree coefficient of R_3

$$\frac{b_{39}}{q^{11/2}} = -\frac{425425}{32}\left(u^2-1\right)^2 \frac{(u^2-1)}{q^{17/2}} \tag{D.1.1}$$

$$\equiv -\frac{425425}{32}\left(u^2-1\right)^2 \left[\frac{ud_1(9)}{q^{15/2}} + \frac{d_2(9)}{q^{13/2}}\right] \tag{D.1.2}$$

which implies that

$$b_{39}^{(1)} = 0 \tag{D.1.3}$$

$$b_{38}^{(1)} = b_{38} - \frac{425425}{32} d_1(9)(u^2-1)^2 u = -\frac{1106105}{96} u\left(u^2-1\right)^2 \tag{D.1.4}$$

$$b_{37}^{(1)} = b_{37} - \frac{425425}{32} d_2(9)(u^2-1)^2 = -\frac{143}{96}\left(9188u^4 - 10033u^2 + 845\right) \tag{D.1.5}$$

We operate now on (the transformed of) b_{38}

$$\frac{b_{38}^{(1)}}{q^{15/2}} = -\frac{1106105}{96} u\left(u^2-1\right) \frac{(u^2-1)}{q^{15/2}} \tag{D.1.6}$$

$$\equiv -\frac{1106105}{96} u(u^2-1)\left[\frac{ud_1(8)}{q^{13/2}} + \frac{d_2(8)}{q^{11/2}}\right] \tag{D.1.7}$$

which implies that

$$b_{38}^{(2)} = 0 \tag{D.1.8}$$

$$b_{37}^{(2)} = b_{37}^{(1)} - \frac{1106105}{96} u^2(u^2-1)d_1(8) = -\frac{143}{96}(u^2-1)\left(2048u^2 - 845\right) \tag{D.1.9}$$

$$b_{36}^{(1)} = b_{36} - \frac{1106105}{96} u(u^2-1)d_2(8) = -\frac{77}{384} u\left(15079u^2 - 6979\right) \tag{D.1.10}$$

We operate now on (the double transformed of) b_{37}

$$\frac{b_{37}^{(2)}}{q^{13/2}} = -\frac{143}{96}\left(2048u^2 - 845\right) \frac{(u^2-1)}{q^{13/2}} \tag{D.1.11}$$

$$\equiv -\frac{143}{96}\left(2048u^2 - 845\right)\left[\frac{ud_1(7)}{q^{11/2}} + \frac{d_2(7)}{q^{9/2}}\right] \tag{D.1.12}$$

which implies that

$$b_{37}^{(3)} = 0 \tag{D.1.13}$$

$$b_{36}^{(2)} = b_{36}^{1} - \frac{143}{96}\left(2048u^2 - 845\right) ud_1(7) = \frac{1}{128}\left(32661u - 32041u^3\right) \tag{D.1.14}$$

$$b_{35}^{(1)} = b_{35} - \frac{143}{96}\left(2048u^2 - 845\right) d_2(7) = -\frac{3}{512}\left(38509u^2 - 853\right) \tag{D.1.15}$$

We end thus the elimination of the "extra coefficients" (with $m > 2n = 6$). To proceed further we operate with the fundamental operation on (the second transformed of) b_{36}, but before it is necessary to expand it in the (7.3.25). To this end, we observe that in general

$$c_1 u + c_3 u^3 = d_1 u(u^2-1) + d_3 u^3 \tag{D.1.16}$$

$$d_1 = -c_1 \tag{D.1.17}$$

$$d_3 = c_1 + c_3 \tag{D.1.18}$$

so that

$$b_{36}^{(2)} = \frac{1}{128}\left(620u^3 - 32661u\left(u^2-1\right)\right) = \frac{155}{32}u^3 - \frac{32661}{128}u(u^2-1) \tag{D.1.19}$$

$$\frac{b_{36}^{(2)}}{q^{11/2}} \equiv \frac{155u^3}{32}\frac{1}{q^{11/2}} - \frac{32661}{128}u^2 d_1(6)\frac{1}{q^{9/2}} - \frac{32661}{128}ud_2(6)\frac{1}{q^{7/2}} \tag{D.1.20}$$

so that

$$b_{36}^{(3)} = \frac{155u^3}{32} \tag{D.1.21}$$

$$b_{35}^{(2)} = b_{35}^{(1)} - \frac{32661}{128}u^2 d_1(6) = \frac{1}{512}\left(601u^2+2559\right) \tag{D.1.22}$$

$$b_{34}^{(1)} = b_{34} - \frac{32661}{128}ud_2(6) = -\frac{441}{256}u \tag{D.1.23}$$

We proceed operating on the fifth coefficients, after having correctly expanded it.

$$b_{35}^{(2)} = \frac{395}{64}u^2 - \frac{2559}{512}\left(u^2-1\right) \tag{D.1.24}$$

$$\frac{b_{35}^{(2)}}{q^{9/2}} = \frac{395}{64}u^2\frac{1}{q^{(9/2)}} - \frac{2559}{512}ud_1(5)\frac{1}{q^{7/2}} - \frac{2559}{512}d_2(5)\frac{1}{q^{5/2}} \tag{D.1.25}$$

so that

$$b_{35}^{(3)} = \frac{395}{64}u^2 \tag{D.1.26}$$

$$b_{34}^{(2)} = b_{34}^{(1)} - \frac{2559}{512}ud_1(5) = \frac{2295u}{896} \tag{D.1.27}$$

$$b_{33}^{(1)} = b_{34} - \frac{2559}{512}d_2(5) = \frac{615}{1792} \tag{D.1.28}$$

Formula (7.3.41) now gives the coefficients of the differential operator in u

$$C_{36} = \frac{1}{2^6}\frac{124}{945}u^3 \tag{D.1.29}$$

$$C_{35} = \frac{1}{2^6}\frac{158}{105}u^2 \tag{D.1.30}$$

$$C_{34} = \frac{1}{2^6}\frac{153}{35}u \tag{D.1.31}$$

$$C_{33} = \frac{1}{2^6}\frac{41}{14} \tag{D.1.32}$$

which are exactly those of He and Miao in their article[34]
We note that

$$b_{37}^{(2)} = b_{37} + \frac{u}{u^2-1}d_1(8)b_{38} + \frac{1}{u^2-1}\left[d_2(9) + \frac{u^2}{u^2-1}d_1(9)d_1(8)\right]b_{39} \tag{D.1.33}$$

$$\begin{aligned} b_{36}^{(2)} &= b_{36} + \frac{u}{u^2-1}d_1(7)b_{37} + \frac{1}{u^2-1}\left[d_2(8) + \frac{u^2}{(u^2-1)}d_1(8)d_1(7)\right]b_{38} \\ &+ \frac{u}{(u^2-1)^2}\left[d_1(9)d_2(8) + d_2(9)d_1(7) + \frac{u^2}{(u^2-1)}d_1(9)d_1(8)d_1(7)\right]b_{39} \end{aligned} \tag{D.1.34}$$

We now write general formulae. Let's begin with c_{36} and $\Delta_{36}^{(2)}$

$$c_{36} = \Big[\beta_{363} + \beta_{361} + \beta_{374}d_1(7) - \beta_{370}d_1(7) + \beta_{385}d_1(8)d_1(7) + \beta_{396}d_1(9)d_1(8)d_1(7)\Big]u^3 \tag{D.1.35}$$

$$\Delta_{36}^{(2)} = \Big[-\beta_{361} + \beta_{370}d_1(7) + \beta_{385}d_2(8) + \beta_{369}[d_1(9)d_1(8) + d_2(9)d_1(7)]\Big]u(u^2-1) \tag{D.1.36}$$

Now we pass to c_{35} and $\Delta_{35}^{(1)}$

$$
\begin{aligned}
b_{35}^{(1)} &= b_{35} + \frac{b_{37}^{(2)}}{u^2-1} d_2(7) \\
&= b_{35} + \frac{b_{37}}{u^2-1} d_2(7) + \frac{b_{38}}{(u^2-1)^2} u d_1(8) d_2(7) + \frac{b_{39}}{(u^2-1)^3} \Big[(u^2-1) d_2(9) d_2(7) + u^2 d_1(9) d_1(8) d_2(7)\Big] \\
&= \beta_{352} u^2 + \beta_{350} + \beta_{374} d_2(7) u^2 - \beta_{370} d_2(7) + \beta_{385} u^2 d_1(8) d_2(7) \\
&+ \beta_{396}[(u^2-1) d_2(9) d_2(7) + u^2 d_1(9) d_1(8) d_2(7)] \\
&= \Big[\beta_{350} - \beta_{370} d_2(7) - \beta_{396} d_2(9) d_2(7)\Big] \\
&+ \Big[\beta_{352} + \beta_{374} d_2(7) + \beta_{385} d_1(8) d_2(7) \\
&+ \beta_{396}[d_2(9) d_2(7) + d_1(9) d_1(8) d_2(7)]\Big] u^2 \qquad \text{(D.1.37)}
\end{aligned}
$$

$$
\begin{aligned}
b_{35}^{(2)} &= b_{35}^{(1)} + \frac{\Delta_{36}^{(2)}}{u^2-1} u d_1(6) \qquad \text{(D.1.38)} \\
&= \Big[\beta_{350} - \beta_{370} d_2(7) - \beta_{396} d_2(9) d_2(7)\Big] + \Big\{ \Big[\beta_{352} + \beta_{374} d_2(7) + \beta_{385} d_1(8) d_2(7) \\
&+ \beta_{396}[d_2(9) d_2(7) + d_1(9) d_1(8) d_2(7)]\Big] + \Big[-\beta_{361} d_1(6) + \beta_{370} d_1(7) d_1(6) + \beta_{385} d_2(8) d_1(6) + \beta_{396}[d_1(9) d_1(8) d_1(6) + d
\end{aligned}
$$

$$\text{(D.1.39)}$$

So

$$
\begin{aligned}
c_{35} = \Big\{ & \beta_{350} + \beta_{352} - \beta_{361} d_1(6) + \beta_{370}[d_1(7) d_1(6) - d_2(7)] + \beta_{385}[d_1(8) d_2(7) + d_2(8) d_1(6)] \\
&+ \beta_{396}[d_1(9) d_1(8) d_1(6) + d_1(9) d_1(8) d_2(7) + d_1(9) d_1(8) + d_2(9) d_1(7) d_1(6)] \Big\} u^2 \qquad \text{(D.1.40)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{35}^{(1)} &= b_{35}^{(1)} - c_{35} = b_{35}^{(1)}(u) - \Big[b_{35}^{(1)}(u) + \frac{\Delta_{36}}{u^2-1} u d_1(6)\Big]\Big|_{u^2=1} u^2 \qquad \text{(D.1.41)} \\
&= \Big\{ -\Big[\beta_{350} - \beta_{370} d_2(7) - \beta_{396} d_2(9) d_2(7)\Big]\Big\} (u^2-1) \\
&- \Big[-\beta_{361} + \beta_{370} d_1(7) + \beta_{385} d_2(8) + \beta_{369}[d_1(9) d_1(8) + d_2(9) d_1(7)]\Big] d_1(6) u^2 \qquad \text{(D.1.42)}
\end{aligned}
$$

Now we pass to c_{34}

$$
\begin{aligned}
b_{34}^{(2)} &= b_{34} + \frac{\Delta_{35}^{(1)}}{u^2-1} u d_1(5) + \frac{\Delta_{36}^{(2)}}{(u^2-1)^2} [u^2 d_1(6) d_1(5) + (u^2-1) d_2(6)] \qquad \text{(D.1.43)} \\
&= \Big\{ \beta_{341} + \beta_{350} d_1(5) - \beta_{370} d_2(7) d_1(5) - \beta_{396} d_2(9) d_2(7) d_1(5) \\
&+ \Big[-\beta_{361} + \beta_{370} d_1(7) + \beta_{385} d_2(8) + \beta_{369}[d_1(9) d_1(8) + d_2(9) d_1(7)]\Big] d_2(6) \Big\} u \qquad \text{(D.1.44)}
\end{aligned}
$$

Note that $\Delta_{36}^{(2)}$ is not divisible by $(u^2-1)^2$ and $\Delta_{35}^{(1)}$ is not divisible by u^2-1. However, the rational (not polynomial) contributions cancel each other.
Finally, c_{33} is corrected not with $\Delta_{35}^{(1)}$ but with $\Delta_{35}^{(2)}$

$$
\begin{aligned}
b_{33}^{(1)} &= c_{33} = b_{33} + \frac{b_{35}^{(2)} - c_{35}}{u^2-1} d_2(5) = b_{33} + \frac{\Delta_{35}^{(1)}}{u^2-1} d_2(5) + \frac{\Delta_{36}^{(2)}}{(u^2-1)^2} u d_1(6) d_2(5) \qquad \text{(D.1.45)} \\
&= \beta_{330} + \beta_{350} d_2(5) - \beta_{370} d_2(7) d_2(5) - \beta_{396} d_2(9) d_2(7) d_2(5) \qquad \text{(D.1.46)}
\end{aligned}
$$

D.2 Less simple example: T_4

Let's begin with transformation of $b_{4,12}$, that is, the highest degree coefficient of R_4

$$\frac{b_{4,12}}{q^{23/2}} = \frac{1301375075}{256}\left(u^2-1\right)^3\frac{\left(u^2-1\right)}{q^{23/2}} \tag{D.2.1}$$

$$\equiv \frac{1301375075}{256}\left(u^2-1\right)^3\left[\frac{ud_1(12)}{q^{21/2}}+\frac{d_2(12)}{q^{19/2}}\right] \tag{D.2.2}$$

which implies that

$$b^{(1)}_{4,12} = 0 \tag{D.2.3}$$

$$b^{(1)}_{4,11} = b_{4,11} + \frac{1301375075}{256}\left(u^2-1\right)^3 ud_1(12) = \frac{706460755}{96}u\left(u^2-1\right)^3 \tag{D.2.4}$$

$$b^{(1)}_{4,10} = b_{4,10} + \frac{1301375075}{256}\left(u^2-1\right)^3 d_2(12) = \frac{46189\left(u^2-1\right)^2\left(728929u^2-66025\right)}{3072} \tag{D.2.5}$$

We operate now on (the transformed of) $b_{4,11}$

$$\frac{b^{(1)}_{4,11}}{q^{21/2}} = \frac{706460755}{96}u\left(u^2-1\right)^2\frac{\left(u^2-1\right)}{q^{21/2}} \tag{D.2.6}$$

$$\equiv \frac{706460755}{96}u\left(u^2-1\right)^2\left[\frac{ud_1(11)}{q^{19/2}}+\frac{d_2(11)}{q^{17/2}}\right] \tag{D.2.7}$$

which implies that

$$b^{(2)}_{4,11} = 0 \tag{D.2.8}$$

$$b^{(2)}_{4,10} = b^{(1)}_{4,10} + \frac{706460755}{96}u^2\left(u^2-1\right)^2 d_1(11) = \frac{46189\left(u^2-1\right)^2\left(265249u^2-66025\right)}{3072} \tag{D.2.9}$$

$$b^{(1)}_{4,9} = b_{4,9} + \frac{706460755}{96}u\left(u^2-1\right)^2 d_2(11) = \frac{12155}{768}u\left(299977u^4-403070u^2+103093\right) \tag{D.2.10}$$

We operate now on (the double transformed of) $b_{4,10}$

$$\frac{b^{(2)}_{4,10}}{q^{19/2}} = \frac{46189\left(u^2-1\right)\left(265249u^2-66025\right)}{3072}\frac{\left(u^2-1\right)}{q^{19/2}} \tag{D.2.11}$$

$$\equiv \frac{46189\left(u^2-1\right)\left(265249u^2-66025\right)}{3072}\left[\frac{ud_1(10)}{q^{17/2}}+\frac{d_2(10)}{q^{15/2}}\right] \tag{D.2.12}$$

which implies that

$$b^{(3)}_{4,10} = 0 \tag{D.2.13}$$

$$\begin{aligned} b^{(2)}_{4,9} &= b^1_{4,9} + \frac{46189u\left(u^2-1\right)\left(265249u^2-66025\right)}{3072}d_1(10) \\ &= \frac{143}{256}u\left(1779707\left(u^2-1\right)^2+531372\left(u^2-1\right)\right) \end{aligned} \tag{D.2.14}$$

$$\begin{aligned} b^{(1)}_{4,8} &= b_{4,8} + \frac{46189\left(u^2-1\right)\left(265249u^2-66025\right)}{3072}d_2(10) \\ &= \frac{715\left(5955443u^4-4565374u^2+197531\right)}{4096} \end{aligned} \tag{D.2.15}$$

We operate now on (the double transformed of) $b_{4,9}$

$$\frac{b^{(2)}_{4,9}}{q^{17/2}} = \frac{143}{256}u\left(1779707\left(u^2-1\right)+531372\right)\frac{\left(u^2-1\right)}{q^{19/2}} \tag{D.2.16}$$

$$\equiv \frac{143}{256}u\left(1779707\left(u^2-1\right)+531372\right)\left[\frac{ud_1(9)}{q^{15/2}}+\frac{d_2(9)}{q^{13/2}}\right] \tag{D.2.17}$$

which implies that

$$b_{4,9}^{(3)} = 0 \tag{D.2.18}$$

$$\begin{aligned} b_{4,8}^{(2)} &= b_{4,8}^{1} + \frac{143}{256}u^2\left(1779707\left(u^2-1\right)+531372\right)d_1(9) \\ &= \frac{143\left(48003857u^4-62776010u^2+14814825\right)}{61440} \end{aligned} \tag{D.2.19}$$

$$\begin{aligned} b_{4,7}^{(2)} &= b_{4,7} + \frac{143}{256}u\left(1779707\left(u^2-1\right)+531372\right)d_2(9) \\ &= \frac{143u\left(2892391u^2-975565\right)}{3840} \end{aligned} \tag{D.2.20}$$

We end thus the elimination of the "extra coefficients" (with $m > 2n = 8$). To proceed further we operate with the fundamental operation on (the second transformed of) $b_{4,8}$, but before it is necessary to expand it in the (7.3.25). To this end, we observe that in general

$$c_0 + c_2u^2 + c_4u^4 = d_0(u^2-1) + d_2(u^2-1)^2 + d_4u^4 \tag{D.2.21}$$

$$d_0 = -2c_0 - c_2 \tag{D.2.22}$$

$$d_2 = -c_0 - c_2 \tag{D.2.23}$$

$$d_4 = c_0 + c_2 + c_4 \tag{D.2.24}$$

so that

$$b_{4,8}^{(2)} = \frac{143}{61440}\left(42672u^4 + 47961185\left(u^2-1\right)^2 + 33146360\left(u^2-1\right)\right) \tag{D.2.25}$$

$$\frac{b_{4,8}^{(2)}}{q^{15/2}} \equiv \frac{127127u^4}{1280}\frac{1}{q^{15/2}} + \frac{143}{61440}\left(47961185\left(u^2-1\right)+33146360\right)\frac{u^2-1}{q^{15/2}} \tag{D.2.26}$$

$$\begin{aligned} &= \frac{127127u^4}{1280}\frac{1}{q^{15/2}} + \frac{143}{61440}\left(47961185\left(u^2-1\right)+33146360\right)ud_1(8)\frac{1}{q^{13/2}} \\ &+ \frac{143}{61440}\left(47961185\left(u^2-1\right)+33146360\right)d_2(8)\frac{1}{q^{11/2}} \end{aligned} \tag{D.2.27}$$

so that

$$b_{4,8}^{(3)} = \frac{127127u^4}{1280} \tag{D.2.28}$$

$$\begin{aligned} b_{4,7}^{(2)} &= b_{4,7}^{(1)} + \frac{143}{61440}\left(47961185\left(u^2-1\right)+33146360\right)ud_1(8) \\ &= \frac{71728547u^3}{15360} - \frac{13826791u}{3072} \end{aligned} \tag{D.2.29}$$

$$\begin{aligned} b_{4,6}^{(1)} &= b_{4,6} + \frac{143}{61440}\left(47961185\left(u^2-1\right)+33146360\right)d_2(8) \\ &= \frac{1989581u^2}{16384} - \frac{252461}{16384} \end{aligned} \tag{D.2.30}$$

We proceed operating on the seventh coefficient, after having correctly expanded it.

$$b_{4,7}^{(2)} = \frac{27027u^3}{160} + \frac{13826791\left(u^2-1\right)u}{3072} \tag{D.2.31}$$

$$\frac{b_{4,7}^{(2)}}{q^{13/2}} \equiv \frac{27027u^3}{160}\frac{1}{q^{13/2}} + \frac{13826791u}{3072}ud_1(7)\frac{1}{q^{11/2}} + \frac{13826791u}{3072}d_2(7)\frac{1}{q^{9/2}} \tag{D.2.32}$$

so that

$$b_{4,7}^{(3)} = \frac{27027u^3}{160} \tag{D.2.33}$$

$$b_{4,6}^{(2)} = b_{4,6}^{(1)} + \frac{13826791}{3072}u^2 d_1(7) = \frac{207085703u^2}{49152} - \frac{252461}{16384} \tag{D.2.34}$$

$$b_{4,5}^{(1)} = b_{4,5} + \frac{13826791u}{3072}d_2(7) = \frac{87405u}{2048} \tag{D.2.35}$$

We proceed operating on the sixth coefficient, after having correctly expanded it.

$$b_{4,6}^{(2)} = \frac{54285u^2}{512} + \frac{252461\left(u^2-1\right)}{16384} \tag{D.2.36}$$

$$\frac{b_{4,6}^{(2)}}{q^{11/2}} \equiv \frac{54285u^2}{512}\frac{1}{q^{11/2}} + \frac{252461}{16384}ud_1(6)\frac{1}{q^{9/2}} + \frac{252461}{16384}d_2(6)\frac{1}{q^{7/2}} \tag{D.2.37}$$

so that

$$b_{4,6}^{(3)} = \frac{54285u^2}{512} \tag{D.2.38}$$

$$b_{4,5}^{(2)} = b_{4,5}^{(1)} + \frac{252461}{16384}ud_1(6) = \frac{66773u}{2304} \tag{D.2.39}$$

$$b_{4,4}^{(1)} = b_{4,4} + \frac{13826791u}{3072}d_2(7) = \frac{106603}{36864} \tag{D.2.40}$$

Formula (7.3.41) now gives the coefficients of the differential operator in u

$$C_{4,8} = \frac{1}{2^7}\frac{127}{4725}u^4 \tag{D.2.41}$$

$$C_{4,7} = \frac{1}{2^4}\frac{13}{175}u^3 \tag{D.2.42}$$

$$C_{4,6} = \frac{1}{2^8}\frac{517}{63}u^2 \tag{D.2.43}$$

$$C_{4,5} = \frac{1}{2^7}\frac{9539}{945}u \tag{D.2.44}$$

$$C_{4,4} = \frac{1}{2^{11}}\frac{15229}{135} \tag{D.2.45}$$

which are exactly those of He and Miao in their article[34]
We now write general formulae. $b_{48}^{(2)}$ is directly expressed in terms of the Gelfand-Dikii coefficients as

$$\begin{aligned} b_{48}^{(2)} = {} & b_{48} + \frac{b_{49}}{u^2-1}ud_1(9) + \frac{b_{4,10}}{(u^2-1)^2}[u^2d_1(10)d_1(9) + (u^2-1)d_2(10)] + \frac{b_{4,11}}{(u^2-1)^3}\Big[u^3d_1(11)d_1(10)d_1(9) \\ & + u(u^2-1)d_1(11)d_2(10) + u(u^2-1)d_2(11)d_1(9)\Big] + \frac{b_{4,12}}{(u^2-1)^4}\Big[u^4d_1(12)d_1(11)d_1(10)d_1(9) \\ & + u^2(u^2-1)d_1(12)d_1(11)d_2(10) + u^2(u^2-1)d_1(12)d_2(11)d_1(9) + u^2(u^2-1)d_2(12)d_1(10)d_1(9) \\ & + u^2(u^2-1)d_2(12)d_2(10)\Big] \end{aligned} \tag{D.2.46}$$

E Recursion solving of Gelfand-Dikii coefficients for self-dual Liouville

The Gelfand-Dikii recursion relation for the coefficients $a_{n,m}$ is far simpler in the self-dual Liouville case (11.2.22), with respect to the general Liouville case (10.1.38). We report here such simple recursion, found incidentally in subsection 11.2 in the context of the asymptotic proof of Zamolodchikov's fundamental relation (11.0.4).

$$a_{n,m} = \left[P^2 \frac{n+2m-\frac{1}{2}}{n+2m} - \frac{1}{4}\frac{(n+2m-\frac{1}{2})^3}{n+2m}\right] a_{n-1,m} + \left[\frac{(n+2m-\frac{3}{2})^3}{n+2m} - \frac{n+2m-\frac{3}{2}}{n+2m}\right] a_{n-1,m-1} \tag{E.0.1}$$

The initial condition is

$$a_{00} = 1 \tag{E.0.2}$$

Its simplicity was already emphasized by a *first* change of variables. We defined the index function $\eta(n,m)$[48] by

$$\boxed{\eta(n,m) = n + 2m + \frac{1}{2}} \tag{E.0.3}$$

and then defined the defined the functions $f(\eta)$ and $g(\eta)$

$$f(\eta) = P^2 - \frac{1}{4}\eta^2 = -\frac{1}{4}(\eta - 2P)(\eta + 2P) \tag{E.0.4}$$

$$g(\eta) = \eta^2 - 1 = (\eta - 1)(\eta + 1) \tag{E.0.5}$$

Thus, a first simplified form for the recursion is

$$a_{n,m} = \frac{1}{\eta - \frac{1}{2}}\left[f(\eta - 1)(\eta - 1)a_{n-1,m} + g(\eta - 2)(\eta - 2)a_{n-1,m-1}\right] \tag{E.0.6}$$

Since the functions $f(\eta)$ and $g(\eta)$ are very similar, we can express one in terms of the other, for example $f(\eta)$ in terms of $g(\eta)$ as in

$$f(\eta) = -\frac{1}{4}\left[g(\eta) + 1 - 4P^2\right] \tag{E.0.7}$$

Hence, probably any finite arbitrary product of these functions can be computed. Noting the simplicity of this self-dual situation, we express *hope to solve the recursion for the Gelfand-Dikii coefficients $a_{n,m}$ completly.*

To get an even more simplified situation, we can make a *second* change of variable function, defining the functions

$$F(\eta) = \eta f(\eta) \tag{E.0.8}$$

$$= -\frac{1}{4}(\eta - 2P)\eta(\eta + 2P) \tag{E.0.9}$$

$$G(\eta) = (\eta + 1)g(\eta + 1) \tag{E.0.10}$$

$$= \eta(\eta + 1)(\eta + 2) \tag{E.0.11}$$

[48] *Beware that in the following we shall drop the n, m dependence from $\eta(n,m)$.*

where $\eta = n + 2m + 1/2$ decreases by 1 when n decreases by 1, but decreases by 2 when m decreases by 1. Equation (E.0.7) corresponds to

$$\begin{aligned} F(\eta) &= -\frac{1}{4}\frac{\eta}{\eta - 1}\left\{G(\eta-1) + \left[1 - 4P^2\right]\eta(\eta-1)\right\} && \text{(E.0.12)} \\ &= -\frac{1}{4}\frac{\eta}{\eta - 1}\left\{(\eta-1)\eta(\eta+1) + \left[1 - 4P^2\right]\eta(\eta-1)\right\} \\ &= -\frac{1}{4}\eta^2\left[\eta + 2 - 4P^2\right] && \text{(E.0.13)} \end{aligned}$$

The recursion now is written simply as

$$a_{n,m} = \frac{1}{\eta - 1/2}\left[F(\eta-1)a_{n-1,m} + G(\eta-3)a_{n-1,m-1}\right] \tag{E.0.14}$$

It appears now evident that the shift in the definition on G (with respect to g) can facilitate the calculation because in this way the total shift of both $F(\eta)$ and $G(\eta)$ equals the shift in η in the Gelfand Dikii coefficient this functions multiply.

Now, also the $\eta - 1/2$ divisor, already simple, can be included in the definition of a *third* couple of functions. In fact, define the *final solving functions* $\mathfrak{f}(\eta)$ *and* $\mathfrak{g}(\eta)$ by

$$\begin{aligned} \mathfrak{f}(\eta) &= \frac{F(\eta)}{\eta + 1/2)} && \text{(E.0.15)} \\ &= -\frac{1}{4}\frac{(\eta - 2P)\eta(\eta + 2P)}{\eta + \frac{1}{2}} && \text{(E.0.16)} \\ \mathfrak{g}(\eta) &= \frac{G(\eta)}{\eta + 5/2} && \text{(E.0.17)} \\ &= \frac{\eta(\eta+1)(\eta+2)}{\eta + \frac{5}{2}} = \frac{\Gamma(\eta+3)}{\Gamma(\eta)}\frac{1}{\eta + \frac{5}{2}} && \text{(E.0.18)} \end{aligned}$$

so that *the recursion finally becomes*

$$\boxed{a_{n,m} = \mathfrak{f}(\eta-1)a_{n-1,m} + \mathfrak{g}(\eta-3)a_{n-1,m-1}} \tag{E.0.19}$$

We easily show all its simplicity by iterating the first four steps: for the second

$$\begin{aligned} a_{n,m} = \mathfrak{f}(\eta-1)\mathfrak{f}(\eta-2)a_{n-2,m} + \Big[&\mathfrak{f}(\eta-1)\mathfrak{g}(\eta-4) \\ &+ \mathfrak{g}(\eta-3)\mathfrak{f}(\eta-4)\Big]a_{n-2,m-1} + \mathfrak{g}(\eta-3)\mathfrak{g}(\eta-6)a_{n-2,m-2} \end{aligned}$$

for the third

$$\begin{aligned} a_{n,m} = \mathfrak{f}(\eta-1)\mathfrak{f}(\eta-2)\mathfrak{f}(\eta-3)a_{n-3,m} + \Big[&\mathfrak{f}(\eta-1)\mathfrak{g}(\eta-4)\mathfrak{f}(\eta-5) + \mathfrak{g}(\eta-3)\mathfrak{f}(\eta-4)\mathfrak{f}(\eta-5) \\ + \mathfrak{f}(\eta-1)\mathfrak{f}(\eta-2)\mathfrak{g}(\eta-5)\Big]a_{n-3,m-1} + \Big[&\mathfrak{f}(\eta-1)\mathfrak{g}(\eta-4)\mathfrak{g}(\eta-7) + \mathfrak{g}(\eta-3)\mathfrak{f}(\eta-4)\mathfrak{g}(\eta-7) \\ + \mathfrak{g}(\eta-3)\mathfrak{g}(\eta-6)\mathfrak{f}(\eta-7)\Big]a_{n-3,m-2} + &\mathfrak{g}(\eta-3)\mathfrak{g}(\eta-6)\mathfrak{g}(\eta-9)a_{n-3,m-3} \end{aligned}$$

and for the fourth

$$
\begin{aligned}
a_{n,m} &= \mathfrak{f}(\eta-1)\mathfrak{f}(\eta-2)\mathfrak{f}(\eta-3)\mathfrak{f}(\eta-4)a_{n-4,m} + \mathfrak{g}(\eta-3)\mathfrak{g}(\eta-6)\mathfrak{g}(\eta-9)\mathfrak{g}(\eta-12)a_{n-4,m-4} \\
&+ \Big[\mathfrak{f}(\eta-1)\mathfrak{f}(\eta-2)\mathfrak{f}(\eta-3)\mathfrak{g}(\eta-6) + \mathfrak{f}(\eta-1)\mathfrak{f}(\eta-2)\mathfrak{g}(\eta-5)\mathfrak{f}(\eta-6) \\
&+ \mathfrak{f}(\eta-1)\mathfrak{g}(\eta-4)\mathfrak{f}(\eta-5)\mathfrak{f}(\eta-6) + \mathfrak{g}(\eta-3)\mathfrak{f}(\eta-4)\mathfrak{f}(\eta-5)\mathfrak{f}(\eta-6)\Big]a_{n-4,m-1} \\
&+ \Big[\mathfrak{g}(\eta-3)\mathfrak{g}(\eta-6)\mathfrak{g}(\eta-9)\mathfrak{f}(\eta-10) + \mathfrak{g}(\eta-3)\mathfrak{g}(\eta-6)\mathfrak{f}(\eta-7)\mathfrak{g}(\eta-10) \\
&+ \mathfrak{g}(\eta-3)\mathfrak{f}(\eta-4)\mathfrak{g}(\eta-7)\mathfrak{g}(\eta-10) + \mathfrak{f}(\eta-1)\mathfrak{g}(\eta-4)\mathfrak{g}(\eta-7)\mathfrak{g}(\eta-10)\Big]a_{n-4,m-3} \\
&+ \Big[\mathfrak{f}(\eta-1)\mathfrak{f}(\eta-2)\mathfrak{g}(\eta-5)\mathfrak{g}(\eta-8) + \mathfrak{f}(\eta-1)\mathfrak{g}(\eta-4)\mathfrak{f}(\eta-5)\mathfrak{g}(\eta-8) \\
&+ \mathfrak{g}(\eta-3)\mathfrak{f}(\eta-4)\mathfrak{f}(\eta-5)\mathfrak{g}(\eta-8) + \mathfrak{f}(\eta-1)\mathfrak{g}(\eta-4)\mathfrak{g}(\eta-7)\mathfrak{f}(\eta-8) \\
&+ \mathfrak{g}(\eta-3)\mathfrak{f}(\eta-4)\mathfrak{g}(\eta-7)\mathfrak{f}(\eta-8) + \mathfrak{g}(\eta-3)\mathfrak{g}(\eta-6)\mathfrak{f}(\eta-7)\mathfrak{f}(\eta-8)\Big]a_{n-4,m-2}
\end{aligned}
$$

In general, we can write for the h-th step

$$
a^{(h)}_{n,m} = \sum_{l=0}^{h} a_{n-h,m-l}\Big[\sum_{q=0}^{\binom{h}{l}} C^{(h)}_{l,q}[\mathfrak{f},\mathfrak{g}]\Big] \tag{E.0.20}
$$

and for the n-th *last step*

$$
\boxed{a^{(n)}_{n,m} = \sum_{l=0}^{n} \delta_{m,l}\Big[\sum_{q=0}^{\binom{n}{l}} C^{(n)}_{l,q}[\mathfrak{f},\mathfrak{g}]\Big]} \tag{E.0.21}
$$

where the coefficients $C^{(h)}_{l,q}[\mathfrak{f},\mathfrak{g}]$ are *obtained by the following rules.* A certain $C^{(h)}_{l,q}[\mathfrak{f},\mathfrak{g}]$ is product of h functions $\mathfrak{f}$, $\mathfrak{g}$ such that

1. *Number of* $\mathfrak{f}$ *and* $\mathfrak{g}$
   Given h, in each product, the total number of function (of whichever type) is h. Given l, distribute in all possible $C^{(h)}_{l,q}$ ways l functions of type $\mathfrak{f}$ and $h-l$ functions of type $\mathfrak{g}$

2. *Lowest argument*
   Given l, the lowest argument is
   $$\eta - (h+2l) \tag{E.0.22}$$
   or
   $$\Delta_{\max}\eta = \Delta n + 2\Delta m \tag{E.0.23}$$

3. *Growth of argument*
   Staring from the lowest argument: if one has a function $\mathfrak{f}(\eta_X)$, the nearby-argument function, whichever it is has augmented by $+1$; if one has a function $\mathfrak{g}(\eta_X)$, the nearby-argument function, whichever it is has augmented by $+3$

The highest and lowest coefficients are easily determined

$$a_{n,n} = \prod_{k=1}^{n} \mathfrak{g}(\eta - 3k) \tag{E.0.24}$$

$$= 3^n \frac{\Gamma(\eta)}{\Gamma(\eta - 3n)} \frac{\Gamma(\frac{2}{3}\eta - \frac{1}{3} - n)}{\Gamma(\frac{2}{3}\eta - \frac{1}{3})} \tag{E.0.25}$$

$$a_{n,0} = \prod_{k=1}^{n} \mathfrak{f}(\eta - k) \tag{E.0.26}$$

$$= \left(-\frac{1}{4}\right)^n \frac{\Gamma(\eta + \frac{1}{2} - n)}{\Gamma(\eta + \frac{1}{2})} \frac{\Gamma(\eta)}{\Gamma(\eta - n)} \left[\frac{\Gamma(\eta - 2P)}{\Gamma(\eta - 2P - n)} \frac{\Gamma(\eta + 2P)}{\Gamma(\eta + 2P - n)}\right] \tag{E.0.27}$$

Observe that the ratio of the Gamma functions in the explicit formula, for all possible orders n, for the lowest Gelfand Dikii coefficient a_{n0} can be expressed, through the Stirling numbers of the first kind, as a polynomial in the Liouville momentum P^2 with coefficients $\mathfrak{p}_{n,l}$ and to substitute in formula (11.3.47).

The next-to highest and lowest begin to be complex

$$a_{n,n-1} = \sum_{l=1}^{n} \left\{ \mathfrak{f}(\eta - 3(n-l) - 1) \prod_{k=1}^{n-l} \mathfrak{g}(\eta - 3k) \prod_{j=n-l}^{n-1} \mathfrak{g}(\eta - 3j - 4) \right\} \tag{E.0.28}$$

$$= \sum_{l=1}^{n} \mathfrak{f}(\eta - 3(n-l) - 1) \frac{\Gamma(\eta)}{\Gamma(\eta - 3(n-l))} \frac{\Gamma(\eta - 3(n-l) - 1)}{\Gamma(\eta - 3n + 2)} \times$$
$$\times \frac{(\eta + 1/2 - 3(n-l))!!!}{(\eta + 1/2)!!!} \frac{(\eta - 3n + 9/2)!!!}{(\eta - 3/2 - 3(n-l))!!!} \tag{E.0.29}$$

$$a_{n,1} = \sum_{l=1}^{n} \left\{ \mathfrak{g}(\eta - (n-l) - 3) \prod_{k=1}^{n-l} \mathfrak{f}(\eta - k) \prod_{j=n-l+3}^{n+1} \mathfrak{f}(\eta - j - 1) \right\} \tag{E.0.30}$$

In general, we conjecture that a *definite, tough very involved, formula, for the generic Gelfand-Dikii coefficient* $a_{n,m}$

$$a_{n,m}(P) \qquad \forall n, m \in \mathbb{N} \qquad \text{such that } 0 \le m \le n \tag{E.0.31}$$

might be found, following the rules we just outlined. In particular, the generic $a_{n,m}$ might be expressed in terms $[m/2]$ *finite sums*. If we were correct, the final outcome of the research project we outlined in this appendix, might be the determination of an *explicit formula for the generic Liouville charge* I_{2n-1} *at the self dual point* $b = 1$. In fact, we remind that the Gelfand Dikii polynomial, for the self dual Liouville case, is defined as

$$R_n(z) = \sum_{m=0}^{n} a_{n,m} \mathcal{J}_{n,m} \tag{E.0.32}$$

where for integral $\mathcal{J}_{n,m}$ of the Gelfand-Dikii functional part we already, far more easily, found a general formula (11.2.49).

$$\boxed{\mathcal{J}_{n,m} = 2^{-n-2m-1/2} \sqrt{\pi} \frac{\Gamma(m + \frac{n}{2} - \frac{1}{4})}{\Gamma(m + \frac{n}{2} + \frac{1}{4})}} \tag{E.0.33}$$

Then, following the procedure outlined in the second paragraph of subsection 11.3 (cf. (11.3.47)), it might be possible to obtain also a general formula (now involving the *infinite series* (11.3.47), however) also for generic ϵ_1-deformed Seiberg-Witten cycle $2\pi i \nu_n$ of $\mathcal{N} = 2$ pure gauge theory, thanks to Zamolodchikov's fundamental relation (11.0.4).

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