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**ON THE CONSTRUCTION
OF GROUP INVARIANT
NON EXPANSIVE OPERATORS**

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*Alla mia famiglia,
e alla nonna Giuseppina.*

*”Per quanto sia in verità limitata la natura umana, essa porta con sé una
grandissima porzione di infinito.”*

Georg Cantor

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Chapter 1

Introduction

In the last years the problem of data analysis and shape comparison has assumed a relevant role in some aspects of the real life, and a lot of scientific fields have started to become interested in it. Some geometrical techniques have given their contribute, and from the beginning persistent homology has proven itself quite efficient both in qualitative and topological comparison of spaces.

The first important assumption is that we are not interested exactly in a space X , but only in some information we get on X by means of measurements. In practical situations measurements can be expressed by continuous \mathbb{R}^m -valued functions defined on a topological space; indeed for example an image can be seen as a function from a rectangle to \mathbb{R}^3 where each colour is indicated by a triple of numbers, and we can usually assume that the colour changes continuously, possibly omitting a set of null measure. However in this thesis we consider only real-valued functions, since the multidimensional case is still under study.

Therefore we want to compare these functions on a topological space X and investigate if they are similar, trying to reach a quantitative measure of similarity. The principal mathematical tool we refer to is the natural pseudo-distance, which is a metric on the space of data whose aim is to find the best correspondence between two functions on X with respect to a group of homeomorphisms of X . However this distance is generally difficult to compute, and hence a new approach is necessary. A possible solution is given by a combination of the technique of persistence homology and a new employment of the data by means of group-invariant non-expansive operators, in order to reach an approximation of the natural pseudo-distance with arbitrary precision.

The continuity of the data enables to apply persistent homology, which studies the birth and the death of the k -dimensional holes during the analysis of the space X ; precisely every map $\varphi : X \rightarrow \mathbb{R}^m$ defines a (multi)filtration on X , and persistent homology aims to establish which topological features mainly characterize this space when we move along the filtration. Moreover this procedure is invariant

with respect to all homeomorphisms of X , that is if $g \in \text{Homeo}(X)$, then φ and $\varphi \circ g$ induce on X two filtrations which have exactly the same topological properties, under the point of view of the persistence. For further and more detailed information about persistent homology, we refer the reader to [1, 5].

Unfortunately in the context of shape comparison this invariance is a limit, since we can possibly be interested in distinguishing φ and $\varphi \circ g$; indeed we can require an invariance with respect to only one specific class of homeomorphisms, in particular to a subgroup of $\text{Homeo}(X)$. This request immediately emerges in several simple cases, for example if we want to recognize a numerical symbol, the group of invariance cannot include the rotations, since the numbers 6 and 9 will result equal.

In order to formalize the method, a deeper study on the concept of shape is necessary. Indeed another fundamental assumption is that the observer has a strict relation with the space X , since he/she influences the procedure of comparison in a direct way, and we could say that he/she is himself/herself an element of this procedure. In this sense we study not directly the space X , but the pair (space, observer) [4].

In practice, the observer receives the data on X and has the ‘power’ to modify them in order to manipulate better the information; formally a set of specific operators acts on the collection of data, with the same aim. This kind of group-invariant non-expansive operators (GINOs), which transform functions on X into other more suitable functions still on X , is the object of study in this work.

These operators are the ‘glasses’ through which the observer studies the space X , and they reflect the invariance he/she is interested in. Indeed another advantage of this approach is that we can treat the group of invariance as a variable of the problem, since generally a change of the observer corresponds to a change of the invariance we want to analyse. Combining persistent homology and invariance with respect to a group, the use of these operators enables to manipulate information on a topological space, rather than a direct study of data. Unfortunately, from the best of our knowledge, this approach gives only theoretical results so far; the aim of this work is to study the structure of the space of these operators, trying to find constructive methods that allow to build them. As many operators we know, as many possibilities we have to reach a good approximation of this space.

From the beginning it is important to underline that even if the attention is focused on the algebraic properties of the variables of the problem, our aim always remains to build a sufficiently rich set of GINO which allows to achieve a suitable approximation of the natural pseudo-distance.

Outline of the thesis

In this thesis we start explaining the mathematical setting where our research will take place. Then Proposition 3.1 shows up a strong relation between operators and particular subsets of the group of invariance, and hereafter Proposition 3.2 gives a constructive method to find this kind of subsets. It follows a section in which we generalize the previous results, especially trying to obtain a larger set of GINOs.

The work ends with some ‘negative’ results, where we look for conditions that make ineffective the previous procedure.

Chapter 2

Mathematical setting

Let X be a (non-empty) topological space, and let $C_b^0(X, \mathbb{R})$ be the topological space of real-valued functions on X which are all bounded, with the topology induced by the sup-norm $\|\cdot\|_\infty$.

Let Φ be a topological subspace of $C_b^0(X, \mathbb{R})$, whose elements represent the data we have on X ; the functions in Φ will be called *admissible filtering functions* on the space X .

The kind of invariance we are interested in is given with respect to a subgroup G of the group $\text{Homeo}(X)$ of all homeomorphisms of X . We assume that G acts on Φ by composition on the right, that means that the action satisfies the property that for every $\varphi \in \Phi$ and every $g \in G$ the map $\varphi \circ g$ is still in Φ .

From now on in this thesis we always assume that X is a topological space, $\Phi \subseteq C_b^0(X, \mathbb{R})$ and G is a subgroup of $\text{Homeo}(X)$ that acts on Φ by composition on the right.

Definition 2.1. The pseudo-distance $d_G : \Phi \times \Phi \rightarrow \mathbb{R}$ is defined by setting

$$d_G(\varphi_1, \varphi_2) = \inf_{g \in G} \|\varphi_1 - \varphi_2 \circ g\|_\infty. \quad (2.0.1)$$

It is called *natural pseudo-distance* associated with the group G acting on Φ .

Remark 2.1. We recall that a pseudo-distance d on a set Φ is a distance without the request that

$$d(\varphi_1, \varphi_2) = 0 \implies \varphi_1 = \varphi_2, \quad \forall \varphi_1, \varphi_2 \in \Phi.$$

Remark 2.2. It is clear from the definition that if G is the trivial subgroup $\{id_X\}$, the natural pseudo-distance corresponds to the distance

$$d_\infty(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_\infty.$$

Moreover if G_1 and G_2 are two subgroups of $\text{Homeo}(X)$ that act on Φ and $G_1 \subseteq G_2$, the inequality $d_{G_2}(\varphi_1, \varphi_2) \leq d_{G_1}(\varphi_1, \varphi_2)$ holds.

Therefore if G is a subgroup of $\text{Homeo}(X)$, preserving Φ , we obtain that

$$d_{\text{Homeo}(X)}(\varphi_1, \varphi_2) \leq d_G(\varphi_1, \varphi_2) \leq d_\infty(\varphi_1, \varphi_2), \quad \forall \varphi_1, \varphi_2 \in \Phi.$$

Unfortunately the pseudo-distance d_G is generally difficult to compute, even in the case that the group G has good properties. A method based on persistent homology and group invariant non-expansive operators gives a way to approximate this distance. For more details, in a slightly different setting, see [2].

Now we can define the operators we will use to manipulate these data on X , respecting the invariance and exploiting some geometrical and algebraic properties of the problem.

Definition 2.2. A G -invariant non-expansive operator (GINO) for the pair (Φ, G) is an operator

$$F : \Phi \rightarrow \Phi$$

that satisfies the following properties:

1. $F(\varphi \circ g) = F(\varphi) \circ g, \quad \forall \varphi \in \Phi, \quad \forall g \in G;$
2. $\|F(\varphi_1) - F(\varphi_2)\|_\infty \leq \|\varphi_1 - \varphi_2\|_\infty, \quad \forall \varphi_1, \varphi_2 \in \Phi.$

We will say that F is a GINO for (Φ, G) .

The first property shows up the request of invariance, since we want that the operation of combining an admissible filtering function first with an homeomorphism and then with the operator, is exactly equal to the inverse procedure; instead the second one highlights the non expansivity of this kind of operators, since we require a control on the norm.

Definition 2.3. We define $\mathcal{F}(\Phi, G)$ to be the set of all G -invariant non-expansive operators for (Φ, G) .

Obviously $\mathcal{F}(\Phi, G)$ is not empty because it contains at least the identity operator.

Remark 2.3. The non expansivity property even adds that the operators in $\mathcal{F}(\Phi, G)$ are 1-Lipschitz and hence continuous. Instead we do not require that these operators have to be linear, even if all the cases we will study in this work have this feature.

Example 2.1. Let $X = [a, b] \subset \mathbb{R}$ be a compact interval, with $a, b \in \mathbb{R}, a < b$. Let Φ be a subset of $C_b^0(X, \mathbb{R})$ and G the group $\{id, g\}$, where g is the homeomorphism

$$\begin{aligned} g : X &\rightarrow X \\ x &\mapsto b + a - x. \end{aligned}$$

We define the operator F_1 by setting

$$F_1(\varphi)(x) := \frac{1}{2}(\varphi(x) + (\varphi \circ g)(x))$$

for every $\varphi \in \Phi$ and every $x \in X$.

If F_1 preserves Φ , it is not difficult to show that F_1 is a GINO for (Φ, G) .

Moreover we can add a simple example where the operator is not linear: defining $F_2(\varphi)(x) := \varphi(x) + \alpha$ for every $\varphi \in \Phi$ and every $x \in X$, where α is a real constant, we can check that F_2 is a non-linear GINO for (Φ, G) , provided that $F_2(\Phi) \subseteq \Phi$.

The Example 2.1 introduces to the problem we take care of in this work, that is: how can we find G -invariant non-expansive operators, once a set of admissible data and a group of invariance are fixed? Are there constructive methods to do it?

Furthermore the non-linear example we reported above is obtained only adding a constant to the function φ ; therefore a natural question arises: are there operators that are more interesting, or important, than the others?

We give a general method to construct GINOs under suitable hypotheses, in order to get informations about the structure of $\mathcal{F}(\Phi, G)$.

Chapter 3

Results on GINOs

In the following lines some new results about GINOs are illustrated; especially the attempt is focused on methods which allow to exploit the algebraic features of the variables of the problem in order to obtain operators.

3.1 A general method for the construction of GINOs

Let X be a topological space and Φ a subset of $C_b^0(X, \mathbb{R})$. Let G be a subgroup of $\text{Homeo}(X)$, and suppose that a *finite* subset $H = \{h_1, \dots, h_n\}$ of $\text{Homeo}(X)$ exists such that the map

$$\begin{aligned}\alpha_g : H &\rightarrow H \\ h &\mapsto g \circ h \circ g^{-1}\end{aligned}$$

is a permutation of H for every $g \in G$. In other words, we require that α_g takes H into H .

Definition 3.1. A finite set $H \subseteq \text{Homeo}(X)$ such that $\alpha_g(H) \subseteq H$ for every $g \in G$ will be called a *permutant* for G .

We observe that a permutant is not required to be a subgroup of $\text{Homeo}(X)$. Now we consider the operator

$$F_H : C_b^0 \rightarrow C_b^0$$

defined by setting

$$F_H(\varphi) := \frac{1}{n}(\varphi \circ h_1 + \dots + \varphi \circ h_n)$$

Proposition 3.1. *If $F_H(\Phi) \subseteq \Phi$ then F_H is a GINO for (Φ, G) .*

Proof. At first we prove that F_H is G -invariant. Let $\tilde{\alpha}_g : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be an index permutation such that $\tilde{\alpha}_g(i) = \text{index of the image of } h_i \text{ through the conjugacy action of } g$, that is

$$\alpha_g(h_i) = g \circ h_i \circ g^{-1} = h_{\tilde{\alpha}_g(i)}, \quad \forall i \in \{1, \dots, n\}.$$

We obtain that

$$g \circ h_i = h_{\tilde{\alpha}_g(i)} \circ g.$$

Exploiting this relation we can conclude:

$$\begin{aligned} F_H(\varphi \circ g) &= \frac{1}{n}(\varphi \circ g \circ h_1 + \dots + \varphi \circ g \circ h_n) \\ &= \frac{1}{n}(\varphi \circ h_{\tilde{\alpha}_g(1)} \circ g + \dots + \varphi \circ h_{\tilde{\alpha}_g(n)} \circ g). \end{aligned}$$

Now it is enough to put in order the indexes, and we get

$$F_H(\varphi \circ g) = \frac{1}{n}(\varphi \circ h_1 \circ g + \dots + \varphi \circ h_n \circ g) = F_H(\varphi) \circ g, \quad \forall \varphi \in \Phi, \quad \forall g \in G.$$

It remains to show the non expansivity

$$\begin{aligned} \|F_H(\varphi_1) - F_H(\varphi_2)\|_\infty &= \left\| \frac{1}{n}(\varphi_1 \circ h_1 + \dots + \varphi_1 \circ h_n) - \frac{1}{n}(\varphi_2 \circ h_1 + \dots + \varphi_2 \circ h_n) \right\|_\infty \\ &= \frac{1}{n} \|(\varphi_1 \circ h_1 - \varphi_2 \circ h_1) + \dots + (\varphi_1 \circ h_n - \varphi_2 \circ h_n)\|_\infty \\ &\leq \frac{1}{n} (\|\varphi_1 \circ h_1 - \varphi_2 \circ h_1\|_\infty + \dots + \|\varphi_1 \circ h_n - \varphi_2 \circ h_n\|_\infty) \\ &= \|\varphi_1 - \varphi_2\|_\infty \end{aligned}$$

and this fact holds for every $\varphi_1, \varphi_2 \in \Phi$. □

Remark 3.1. Obviously $H = \{id\}$ fulfills all the hypotheses of Proposition 3.1 for any subgroup G of $\text{Homeo}(X)$, but this subset corresponds to the identity operator on Φ , that is not so interesting in our research.

Remark 3.2. If the group G is Abelian, every finite subset of G is a permutant for G , since the conjugacy action is just the identity.

Hence in this setting, chosen g_1, \dots, g_n elements of G , we can consider the set $H = \{g_1, \dots, g_n\}$ and we get that

$$F_H(\varphi) = \frac{1}{n}(\varphi \circ g_1 + \dots + \varphi \circ g_n)$$

is a G -invariant non-expansive operator for (Φ, G) , if F_H preserves Φ . In particular it is important to underline that when G is Abelian we can choose even only one element of G in order to obtain an operator; indeed we can define

$$F_g(\varphi) = \varphi \circ g$$

that is still a GINO, provided that $F_g(\Phi) \subseteq \Phi$.

Example 3.1. A non-commutative example.

Let $X = \mathbb{R}$ and $\Phi \subseteq C_b^0(X, \mathbb{R})$. As an invariant group G we consider the group of all isometries of the real line, i.e. homeomorphisms of \mathbb{R} of the form

$$g(x) = ax + b, \quad a, b \in \mathbb{R}, \quad a = \pm 1.$$

We also consider the translation $h(x) = x + t$ and its inverse transformation $h^{-1}(x) = x - t$, with $t \in \mathbb{R}$.

We claim that $H = \{h, h^{-1}\}$ fulfills the hypotheses of Proposition 3.1. Let $g(x) = ax + b$ be a generic element of G , then $g^{-1}(x) = \frac{x - b}{a}$, and we want to show that the conjugacy action α_g is a permutation of H .

We divide the proof into two cases, according to the value of the coefficient a :

- $a = 1$

In this case also g is a translation, and so the conjugacy action α_g is the identity on H , since translations commute.

- $a = -1$

When g does not preserve the orientation of X , its effect is to exchange the elements of H , indeed in this setting

$$\begin{aligned} \alpha_g(h)(x) &= (g^{-1} \circ h \circ g)(x) = (g^{-1} \circ h)(-x + b) = \\ &= g^{-1}(-x + b + t) = x - b - t + b = \\ &= x - t = \\ &= h^{-1}(x), \quad \forall x \in X \end{aligned}$$

and

$$\begin{aligned} \alpha_g(h^{-1})(x) &= (g^{-1} \circ h^{-1} \circ g)(x) = (g^{-1} \circ h^{-1})(-x + b) = \\ &= g^{-1}(-x + b - t) = x - b + t + b = \\ &= x + t = \\ &= h(x), \quad \forall x \in X. \end{aligned}$$

All things considered, if g preserves the orientation, i.e. $a = 1$, the conjugation acts on H as the identity, while for $a = -1$ the elements of H are exchanged. Therefore we can conclude that $H = \{h, h^{-1}\}$ is a permutant for G . As final result, Proposition 3.1 allows us to construct the operator

$$F_H(\varphi) = \frac{1}{2}(\varphi \circ h + \varphi \circ h^{-1})$$

which is a GINO for (Φ, G) , provided that $F_H(\Phi) \subseteq \Phi$.

Remark 3.3. The operators studied in Proposition 3.1 are linear. Indeed, fixed a permutant $H = \{h_1, \dots, h_n\}$ for G and the associated operator $F_H(\varphi) = \frac{1}{n}(\varphi \circ h_1 + \dots + \varphi \circ h_n)$, for every $a, b \in \mathbb{R}$ and every $\varphi, \psi \in \Phi$ we have

$$\begin{aligned} F_H(a\varphi + b\psi) &= \frac{1}{n} \left((a\varphi + b\psi) \circ h_1 + \dots + (a\varphi + b\psi) \circ h_n \right) \\ &= \frac{1}{n} \left(a(\varphi \circ h_1) + b(\psi \circ h_1) + \dots + a(\varphi \circ h_n) + b(\psi \circ h_n) \right) \\ &= \frac{1}{n} \left(a(\varphi \circ h_1) + \dots + a(\varphi \circ h_n) \right) + \frac{1}{n} \left(b(\psi \circ h_1) + \dots + b(\psi \circ h_n) \right) \\ &= a \left[\frac{1}{n}(\varphi \circ h_1 + \dots + \varphi \circ h_n) \right] + b \left[\frac{1}{n}(\psi \circ h_1 + \dots + \psi \circ h_n) \right] \\ &= aF_H(\varphi) + bF_H(\psi) \end{aligned}$$

provided that the sum $aF_H(\varphi) + bF_H(\psi)$ is still in $\mathcal{F}(\Phi, G)$.

3.1.1 A general condition for the existence of non-trivial permutants

The previous proposition gives us a general method for the construction of GINOs and leads us to the study of which conditions the group G has to satisfy so that non-trivial permutants for G exist. Below a first result is shown.

Definition 3.2. Let G_1 and G_2 be two subgroups of $\text{Homeo}(X)$, we define $\langle G_1, G_2 \rangle$ to be the subgroup of $\text{Homeo}(X)$ generated by G_1 and G_2 , that is the smallest subgroup of $\text{Homeo}(X)$ containing all the finite compositions of elements of G_1 and G_2 .

Remark 3.4. G_1 and G_2 are trivially two subgroups of $\langle G_1, G_2 \rangle$.

Proposition 3.2. Let X be a topological space. Let G_1 and G_2 be two subgroups of $\text{Homeo}(X)$. Let us assume that G_1 and G_2 fulfill the following conditions:

- G_1 is Abelian and normal in $G = \langle G_1, G_2 \rangle$;
- G_2 is cyclic with finite order n , i.e. an element $p \in G_2$ exists such that it has order n and $G_2 = \langle p \rangle$.

Then, fixed any element $\bar{g} \in G_1$, the set

$$H = \{\bar{g}, p \circ \bar{g} \circ p^{-1}, \dots, p^{n-1} \circ \bar{g} \circ p^{-(n-1)}\}$$

is a permutant for G .

Proof. We denote the set H in the following way

$$H = \{\bar{g}^{(0)}, \bar{g}^{(1)}, \dots, \bar{g}^{(n-1)}\}$$

where the power indicates the element of G_2 that acts, that is

$$\bar{g}^{(i)} = p^i \circ \bar{g} \circ p^{-i}, \quad i \in \{0, \dots, n-1\}.$$

In order to prove the proposition we show that the map

$$\begin{aligned} \alpha_f : H &\rightarrow H \\ \bar{g}^{(i)} &\mapsto f \circ \bar{g}^{(i)} \circ f^{-1} \end{aligned}$$

is a permutation of H for every $f \in G$.

First of all we observe that α_f is injective and $H \subseteq G_1$, since G_1 is normal in G . Therefore we have the following possibilities:

1. $f = f_1 \in G_1$:

As we have already remarked, H is a subset of G_1 which is commutative, hence the conjugacy action of f_1 has the same effect of the identity, indeed

$$\alpha_{f_1}(\bar{g}^{(i)}) = f_1 \circ \bar{g}^{(i)} \circ f_1^{-1} = f_1 \circ f_1^{-1} \circ \bar{g}^{(i)} = \bar{g}^{(i)}, \quad \forall i = 0, \dots, n-1.$$

2. $f \in G_2$:

f can be written in the form $f = p^j$, with $j \in \{0, \dots, n-1\}$.

It follows that

$$\alpha_{p^j}(\bar{g}^{(i)}) = p^j \circ \bar{g}^{(i)} \circ p^{-j}$$

and expanding $\bar{g}^{(i)}$

$$= p^j \circ p^i \circ \bar{g} \circ p^{-i} \circ p^{-j} = p^{j+i} \circ \bar{g} \circ p^{-i-j}.$$

Now, since $i, j \in \{0, \dots, n-1\}$ and since G_2 is cyclic, this action permutes the elements of H , ‘translating’ of j places and restarting from the initial element \bar{g} when $i + j = n$.

3. $f = g_{j_1} \circ \cdots \circ g_{j_m}$ a generic element of G , with g_{j_k} in G_1 or G_2 .

It is sufficient to note that as functions on H the following equalities hold

$$\alpha_f \equiv \alpha_{g_{j_1} \circ \cdots \circ g_{j_m}} \equiv \alpha_{g_{j_1}} \circ \cdots \circ \alpha_{g_{j_m}}$$

and so, being a composition of permutations as follows from the previous points, also α_f is a permutation.

To summarize we have shown that fixed an $f \in G$, the conjugacy action α_f is a permutation on H , and this fact holds for every $f \in G$. Hence the set H is a permutant for G . \square

Remark 3.5. The set H is generally indicated with $G_2\bar{g}$, that is the G_2 -orbit of \bar{g} where the action of the group in our case is the conjugation. However we keep denoting this set H , or $H_{\bar{g}}$ with reference to the element \bar{g} , in order to maintain the notation of Proposition 3.1.

Although the hypotheses of Proposition 3.2 could appear really restrictive, in the following notes we propose some simple cases where our method can be applied.

Example 3.2. We can look at Example 3.1 as a particular application of Proposition 3.2. If $X = \mathbb{R}$ and G is the group of all isometries on X , we can write G as $\langle G_1, G_2 \rangle$ with G_1 the group of the translations and G_2 the group $\{id, p\}$, where p is the reflection with respect to the origin.

We know that G_1 is Abelian and normal in G , G_2 is cyclic with order 2, generated by p , and so we can apply the proposition. After fixing a translation $\bar{g}(x) = x + b$, we can construct the set

$$H = \{\bar{g}, p \circ \bar{g} \circ p\}$$

which is a permutant for the group of all isometries of the real line.

It is interesting to notice how this result exactly corresponds to the previous one, indeed

$$(p \circ \bar{g} \circ p)(x) = (p \circ \bar{g})(-x) = p(-x + b) = x - b = \bar{g}^{-1}(x), \quad \forall x \in \mathbb{R}$$

and so the set H is precisely $\{\bar{g}, \bar{g}^{-1}\}$.

Example 3.3. Let $X = \mathbb{R}^2$. If we consider the group G of the isometries of X , it is not possible to see G as a group generated by other two groups verifying the properties required by Proposition 3.2. Hence our method fails and as a matter of fact we will show in Section 4 that no non-trivial permutant exists for G .

However, if we restrict G to a smaller group G' of rigid motions of the plane, we can find permutants for G' .

In particular, we suppose to be interested in the invariance with respect to all the translations and a particular reflection p of the plane, so that we can write

$G_1 = \{\text{translations}\}$ and $G_2 = \{id, p\}$, where p is an arbitrarily chosen element of G of order 2.

Therefore $G' = \langle G_1, G_2 \rangle$ is a subgroup of G , with G_1 normal in G' and Abelian, G_2 cyclic of order 2. Now Proposition 3.2 applies and we can get a permutant H for G' .

Similarly if we are interested in the invariance with respect to the translations and to the multiplies of a rotation ρ_θ of a fixed angle θ , then we can set $G_1 = \{\text{translations}\}$, G_2 is the group generated by the rotation ρ_θ . With a suitable choice of the angle θ (see Appendix 1), G_2 is cyclic of finite order, and again we can apply Proposition 3.2 since all the hypotheses are fulfilled by these groups.

Our method proposed in Proposition 3.1 takes place in a bigger and more complicated problem whose aim is to study the structure of $\mathcal{F}(\Phi, G)$. In this context it is important to know as many operators as possible, so we add this simple remark which helps us to increase the set of available GINOs.

Remark 3.6. If H and K are two permutants for G , then also the union $H \cup K$ and the intersection $H \cap K$ are two permutants for G (provided that $H \cap K \neq \emptyset$).

Example 3.4. If X is equal to \mathbb{R} and G is the group of the isometries, we have seen how we can construct a GINO simply fixing a translation g and considering the set $H = \{g, g^{-1}\}$; if we take another translation $f \neq g$, we get the same result with $K = \{f, f^{-1}\}$.

Therefore we can consider the union $H \cup K = \{g, f, g^{-1}, f^{-1}\}$ which keep satisfying the properties we need, and given a set of admissible data $\Phi \subseteq C_b^0(X, \mathbb{R})$ it defines the operator

$$F_{H \cup K}(\varphi) = \frac{1}{4}(\varphi \circ g + \varphi \circ f + \varphi \circ g^{-1} + \varphi \circ f^{-1})$$

always assuming that $F_{H \cup K} \in \mathcal{F}(\Phi, G)$.

3.2 Generalization of the previous results

We have already underlined the importance of knowing as many operators as possible. To this end, it is not difficult to generalize the previous results.

Precisely, two aspects interestingly arise from the previous part of the thesis.

3.2.1 Finding permutants not contained in the invariance group

All the examples we have illustrated in the previous section have a common feature, i.e. the permutant H is always a subset of the group G .

On the other hand Proposition 3.1 claims that H could be generally a subset of $\text{Homeo}(X)$ without links with the group of invariance. In order not to lose the powerful generality of the statement, in the next lines we want to report two examples where it is clear how the previous work can help us to find operators built on subsets of $\text{Homeo}(X)$, which are not subsets of G .

Example 3.5. Let $X = S^1$ and $G = \langle \rho_\theta \rangle$ the cyclic group generated by the rotation of angle θ , with $\theta = \frac{p}{q}\pi$, $p, q \in \mathbb{N}^+$. Let R be the group of all rotations of X , so G is a subgroup of R .

Let h be another element of R not belonging to G . After fixing a set $\Phi \subseteq C_b^0(X, \mathbb{R})$ we have that

$$F_h(\varphi) := \varphi \circ h$$

is a GINO for (Φ, G) if $F(\Phi) \subseteq \Phi$. Indeed $H = \{h\}$ is a permutant for G , not contained in G , since h commutes with all elements of G .

Similarly we can build a permutant $H = \{h_1, \dots, h_n\} \subseteq R \setminus G$ by using again the commutativity of R .

Example 3.6. Proposition 3.2 gives us a method to find permutants for $\langle G_1, G_2 \rangle$, if G_1 and G_2 fulfill certain conditions. We remark that H is not required to be a subset of G_2 , and this fact can be useful for our aim. Because of our proposition, H is a permutant for G , and hence even for G_2 .

Now it is convenient to think G_2 as the invariant group in place of G . Therefore we get another example where H is outside of the invariant group, assuming that $G_1 \cap G_2 = \{id\}$.

3.2.2 Generalization on coefficients of GINOs

In Proposition 3.1 the operator F is built by means of a sum where all the addenda have the same coefficient, and the amount of these numbers is exactly equal to 1.

We can extend this result a little bit.

Proposition 3.3. *Under the same hypotheses of Proposition 3.1, we define the operator*

$$F(\varphi) := \bar{a}(\varphi \circ h_1 + \cdots + \varphi \circ h_n).$$

If F preserves Φ and $n|\bar{a}| \leq 1$, then F is a GINO for (Φ, G) .

Proof. The fact that all the coefficients keep being equal allows us to prove the G-invariance, instead the inequality is necessary for the non expansivity.

We proceed showing that F is G-invariant. By taking $\varphi \in \Phi$ and $g \in G$ and denoting with $\tilde{\alpha}_g$ the index permutation induced by the conjugation α_g ,

$$\begin{aligned} F(\varphi \circ g) &= \bar{a}(\varphi \circ g \circ h_1 + \cdots + \varphi \circ g \circ h_n) \\ &= \bar{a}(\varphi \circ h_{\tilde{\alpha}_g(1)} \circ g + \cdots + \varphi \circ h_{\tilde{\alpha}_g(n)} \circ g) \\ &= F(\varphi) \circ g \end{aligned}$$

where the last equality is given simply rearranging the indexes in the sum.

On the other side the non expansivity:

$$\begin{aligned} \|F(\varphi_1) - F(\varphi_2)\|_\infty &= \|\bar{a}(\varphi_1 \circ h_1 + \cdots + \varphi_1 \circ h_n) - \bar{a}(\varphi_2 \circ h_1 + \cdots + \varphi_2 \circ h_n)\|_\infty \\ &= |\bar{a}| \|(\varphi_1 \circ h_1 - \varphi_2 \circ h_1) + \cdots + (\varphi_1 \circ h_n - \varphi_2 \circ h_n)\|_\infty \\ &\leq |\bar{a}| (\|\varphi_1 \circ h_1 - \varphi_2 \circ h_1\|_\infty + \cdots + \|\varphi_1 \circ h_n - \varphi_2 \circ h_n\|_\infty) \\ &= n|\bar{a}| \|\varphi_1 - \varphi_2\|_\infty \\ &\leq \|\varphi_1 - \varphi_2\|_\infty \end{aligned}$$

and this statement holds for every $\varphi_1, \varphi_2 \in \Phi$. □

We include in this section other two results presented in [3], which explain how we can combine operators in order to get new GINOs.

Proposition 3.4. *Let X be a topological space, $\Phi \subseteq C_b^0(X, \mathbb{R})$ and $G \subseteq \text{Homeo}(X)$. Let F_1, \dots, F_n be GINOs for (Φ, G) , and consider a linear combination*

$$F_\Sigma(\varphi) = \sum_{i=1}^n a_i F_i(\varphi), \quad (a_1, \dots, a_n) \in \mathbb{R}^n.$$

If $\sum_{i=1}^n |a_i| \leq 1$ and $F_\Sigma(\Phi) \subseteq \Phi$, then F_Σ is a GINO for (Φ, G) .

Proof. See Appendix 2. □

Moreover it is possible to extend the result to the family of all the functions that combine single operators respecting the properties requested by the definition of GINO.

Precisely, let L be a 1-Lipschitzian map from \mathbb{R}^n to \mathbb{R} , where \mathbb{R}^n is endowed with the sup-norm $\| (x_1, \dots, x_n) \|_\infty = \max_{1 \leq i \leq n} |x_i|$.

Now, assuming as before that F_1, \dots, F_n are GINOs for (Φ, G) , we consider the function

$$L^*(F_1, \dots, F_n)(\varphi) = [L(F_1(\varphi), \dots, F_n(\varphi))]$$

from Φ to $C^0(X, \mathbb{R})$, where $[L(F_1(\varphi), \dots, F_n(\varphi))]$ is defined by setting

$$[L(F_1(\varphi), \dots, F_n(\varphi))](x) = L(F_1(\varphi)(x), \dots, F_n(\varphi)(x)).$$

Proposition 3.5. *If $L^*(F_1, \dots, F_n)(\Phi) \subseteq \Phi$, then $L^*(F_1, \dots, F_n)$ is a GINO for (Φ, G) .*

Proof. See Appendix 3. □

Remark 3.7. We finish the section studying again the case of G Abelian. Suppose that $H = \{h_1, \dots, h_n\}$ is a permutant for G . Proposition 3.1 builds the operator

$$F_H(\varphi) := \frac{1}{n}(\varphi \circ h_1 + \dots + \varphi \circ h_n)$$

and thanks to Proposition 3.3 we can vary the coefficient.

However, if G is Abelian, every single addend in the sum is individually an operator. Therefore from Proposition 3.4 it follows that, under the hypothesis of G Abelian, it is not necessary that all coefficients are equal, we only need that their sum is less or equal than 1.

3.3 Relation between G and the space of GINOs

We have seen how we can produce a GINO under the hypotheses of Proposition 3.2, where the first step consists in selecting a specific element \bar{g} of G_1 .

In the last section we generalized Proposition 3.1 by replacing the constant $\frac{1}{n}$ with

a value \bar{a} such that $|\bar{a}| \leq \frac{1}{n}$.

In other words we can define a map

$$\begin{aligned} op : G_1 \times \left[-\frac{1}{n}, \frac{1}{n} \right] &\rightarrow \mathcal{F}(\Phi, G) \\ (\bar{g}, \bar{a}) &\mapsto F_{\bar{g}, \bar{a}} \end{aligned}$$

where $F_{\bar{g}, \bar{a}}$ denotes the operator

$$\begin{aligned} F_{\bar{g}, \bar{a}} : \Phi &\rightarrow \Phi \\ \varphi &\mapsto \bar{a}(\varphi \circ \bar{g} + \varphi \circ p \circ \bar{g} \circ p^{-1} + \dots + \varphi \circ p^{n-1} \circ \bar{g} \circ p^{-(n-1)}). \end{aligned}$$

Now a question arises. After fixing the constant $\bar{a} \in \left[-\frac{1}{n}, \frac{1}{n} \right]$, does any element of G_1 generate a different operator? More formally, if f and g are two different homeomorphisms of G_1 , can we conclude that $F_{f, \bar{a}}$ and $F_{g, \bar{a}}$ are different?

This question is equivalent to require if the map

$$\begin{aligned} op_{\bar{a}} : G_1 &\rightarrow \mathcal{F}(\Phi, G) \\ g &\mapsto F_{g, \bar{a}} \end{aligned}$$

defined by setting

$$op_{\bar{a}}(g) := op(g, \bar{a}), \quad \forall \bar{a} \in \left[-\frac{1}{n}, \frac{1}{n} \right],$$

is injective.

It is important to note that the operators $op_{\bar{a}}(g)$ depends on the construction of the set

$$H_g = \{g, p \circ g \circ p^{-1}, \dots, p^{n-1} \circ g \circ p^{-(n-1)}\}.$$

So the question “is the map injective?” corresponds to “is it true that for every $f, g \in G_1$, $f \neq g$, the set H_f is different from H_g ?”.

The answer is negative because if we fix an element $g \in G_1$, then H_g is a subset of G_1 , and the choice of any other element of H_g leads to build the same set and therefore the same operator.

Indeed if $f \in G_1$, $f = p^i \circ g \circ p^{-i}$ with $i \in \{0, \dots, n-1\}$, we can write

$$\begin{aligned} H_f &= \{f, p \circ f \circ p^{-1}, \dots, p^{n-1} \circ f \circ p^{-(n-1)}\} \\ &= \{p^i \circ g \circ p^{-i}, p^{i+1} \circ g \circ p^{-i-1}, \dots, p^{n-1} \circ g \circ p^{-(n-1)}, g, \dots, p^{i-1} \circ g \circ p^{-i+1}\} \\ &= H_g \end{aligned}$$

To summarize, if $f = p^i \circ g \circ p^{-i}$ with $i \in \{0, \dots, n-1\}$

$$op_{\bar{a}}(f) = op_{\bar{a}}(g)$$

and obviously $f \neq g$, hence the map $op_{\bar{a}}$ is not injective.

Remark 3.8. Now it is immediate to prove that if $h \in G$ and no $i \in \{0, \dots, n-1\}$ exists such that h can be written in the form $h = p^i \circ g \circ p^{-i}$, then $H_h \neq H_g$. However in this case it is not possible to directly conclude that g and h generate two different operators.

Chapter 4

Negative results

In our research we are interested also in negative results, that is in understanding which properties the group G has to satisfy so that permutants for G do not exist.

Definition 4.1. Let G be a group that acts on a set X . We say that G is *versatile* if for every triple $(x, y, z) \in X^3$, with $x \neq z$, and for every finite subset S of X , at least one element $g \in G$ exists such that

1. $g(x) = y$;
2. $g(z) \notin S$.

Example 4.1. It is easy to check that the group of the isometries of the plane is versatile, while the group of isometries on \mathbb{R} is not.

Proposition 4.1. Let G be a subgroup of $\text{Homeo}(X)$ which acts on a set X , and let $H = \{h_1, \dots, h_n\}$ be a permutant for G .

If G is versatile, then $H = \{id\}$.

Proof. It is sufficient to prove that if H contains an element $h \neq id$, then G is not versatile. We can assume that $h \equiv h_1$. Since h_1 is different from the identity, a point $x \in X$ exists such that $h_1(x) \neq x$. Let us consider the triple $(h_1(x), x, x)$ and the set $S = \{h_1^{-1}(x), \dots, h_n^{-1}(x)\}$.

Suppose that a $g \in G$ exists which satisfies the first property, that is $g(h_1(x)) = x$. Now we want to show that g cannot satisfy even the second property.

Since the conjugacy action of g on H is a permutation, we can find an element of H which is equal to $g \circ h_1 \circ g^{-1}$, and let us indicate this with h_2 . This fact implies that $h_2(g(x)) = g(h_1(x)) = x$ and so $g(x) = h_2^{-1}(x) \in S$.

Hence we can conclude that no g exists such that it verifies both the two properties in the definition of versatile group, i.e. G is not versatile. \square

Remark 4.1. Let $H \subseteq G$ be two groups that act on the same set X . If H is versatile, then even G is.

Indeed, fixed a generic triple $(x, y, z) \in X^3$ and a finite subset S of X , we can find an element $h \in H$ which satisfies both the two properties since H is versatile. But now we can trivially think h as an element of G , and we get immediately the versatility of this second group.

Combining this simple remark with the previous proposition, we can conclude for example that any group G which acts on $X = \mathbb{R}^2$ and contains the isometries, is versatile and so we cannot exploit the argument proposed for the construction of GINOs, once fixed a set of admissible data.

Remark 4.2. Another simple note can be added about the relation between versatile groups and GINOs: if G is a finite group, then G is not versatile.

This result follows immediately from the definition of versatility. Indeed fixed a triple $(x, y, z) \in X^3$, with $x \neq z$, we construct the set

$$G_z = \{g(z), g \in G\} \subseteq X$$

that is finite. If we choose the finite subset S such that $G_z \subseteq S$, there are no possibilities that the second condition is satisfied by these points.

Chapter 5

Conclusions

This thesis focuses on methods for the construction of GINOs. Our final goal is the one of writing a sort of dictionary of operators to be used for a specific group; as we have already underlined the study of group invariant non-expansive operators allows to treat the group G as a variable, so we want to generalize the study only on the algebraic properties of the elements.

For sure this work is still not sufficient to reach a good approximation of the space $\mathcal{F}(\Phi, G)$, and no metric aspects are analysed here. However it could be a first step in this direction, and above all the methods we describe could be a good starting point. Indeed we give a way, actually a quite simple way, to manipulate Abelian groups, and rotations and translations are two examples of commutative transformations we can be easily interested in shape comparison. At the same time Proposition 3.2 indicates a technique to treat ‘larger’ groups, trying to see them as a groups generated by smaller structures.

On the other side we can possibly get some indications even from the algebraic form of the operators studied in this thesis; precisely they look like a finite ‘average’ of a filtering function modified with specific homeomorphisms. This fact suggests a possible generalization for infinite sum, and hence integrals, so that operators preserve the structure of GINO, even when there are no permutants for G . (In this work we do not solve this problem, since some stronger mathematical tools seem to be necessary.)

Another aspect that needs a more detailed study is the relation between the elements of G and $\mathcal{F}(\Phi, G)$; in Remark 3.8 we have shown how two different elements g and h of G can produce the same operators, but we cannot directly conclude that if H_g and H_h are different, even the associated operators are not equal. In other words which kind of informations about the structure of $\mathcal{F}(\Phi, G)$ can we get from the homeomorphisms in the invariance group? Is it possible to find subsets of $\mathcal{F}(\Phi, G)$ with algebraic features related with the group G ?

The answers of these questions could possibly light up the properties of $\mathcal{F}(\Phi, G)$,

in particular if there exist some GINOs that mainly characterize and approximate this space.

Appendix 1: About the choice of the angle θ in Example 3.3

In order to get GINOs, we need that the group $G_2 = \langle \rho_\theta \rangle$ generated by the rotation of angle θ is cyclic of finite order. This can be obtained by suitable assumption on θ .

Precisely, it is sufficient to set $\theta = \frac{p}{q}\pi$, with $p, q \in \mathbb{N}^+$. In this way G_2 has finite order equal to $n = \frac{2kq}{p}$ where k is the smallest natural number such that $\frac{2kq}{p}$ is in \mathbb{N} .

Indeed we want to find the smallest natural number n such that

$$(\rho_\theta)^n = id = \rho_0 = \rho_{2k\pi}, \quad k \in \mathbb{N}.$$

Since $(\rho_\theta)^n$ corresponds to the rotation of angle $n\theta = n\frac{p}{q}\pi$, then

$$n\frac{p}{q}\pi = 2k\pi \quad \iff \quad n = \frac{2kq}{p}.$$

Hence the research of the smallest n such that $(\rho_\theta)^n = id$ is equivalent to look for the smallest k such that $\frac{2kq}{p}$ is a natural number, and of course this k exists since in the worse case $k = p$ suits our requests.

Appendix 2: Proof of Proposition 3.4

Proof. F_Σ is G-invariant because F_1, \dots, F_n are individually G-invariant:

$$F_\Sigma(\varphi \circ g) = \sum_{i=1}^n a_i F_i(\varphi \circ g) = \left(\sum_{i=1}^n a_i F_i(\varphi) \right) \circ g = F_\Sigma(\varphi) \circ g$$

for every $\varphi \in \Phi$ and for every $g \in G$.

Similarly the non expansivity of F_Σ derives directly from the non expansivity of the single operators and from the condition on the coefficients:

$$\begin{aligned}
\| F_{\Sigma}(\varphi_1) - F_{\Sigma}(\varphi_2) \|_{\infty} &= \left\| \sum_{i=1}^n a_i F_i(\varphi_1) - \sum_{i=1}^n a_i F_i(\varphi_2) \right\|_{\infty} \\
&= \left\| \sum_{i=1}^n a_i (F_i(\varphi_1) - F_i(\varphi_2)) \right\|_{\infty} \\
&\leq \sum_{i=1}^n |a_i| \| F_i(\varphi_1) - F_i(\varphi_2) \|_{\infty} \\
&\leq \sum_{i=1}^n |a_i| \| \varphi_1 - \varphi_2 \|_{\infty} \leq \| \varphi_1 - \varphi_2 \|_{\infty}
\end{aligned}$$

for every $\varphi_1, \varphi_2 \in \Phi$. □

Appendix 3: Proof of Proposition 3.5

Proof. $L^*(F_1, \dots, F_n)$ is G -invariant since the operators F_1, \dots, F_n are G -invariant:

$$\begin{aligned}
L^*(F_1, \dots, F_n)(\varphi \circ g) &= [L(F_1(\varphi \circ g), \dots, F_n(\varphi \circ g))] \\
&= [L(F_1(\varphi) \circ g, \dots, F_n(\varphi) \circ g)] \\
&= [L(F_1(\varphi), \dots, F_n(\varphi))] \circ g \\
&= L^*(F_1, \dots, F_n)(\varphi) \circ g
\end{aligned}$$

for every $\varphi \in \Phi$ and every $g \in G$.

The non expansivity of F_1, \dots, F_n and the hypothesis of L 1-Lipschitz imply that L^* is non expansive; in order to prove it, previously we show that for every $x \in X$ and every $\varphi_1, \varphi_2 \in \Phi$

$$\begin{aligned}
&|L(F_1(\varphi_1)(x), \dots, F_n(\varphi_1)(x)) - L(F_1(\varphi_2)(x), \dots, F_n(\varphi_2)(x))| \\
&\leq \| (F_1(\varphi_1(x)) - F_1(\varphi_2(x)), \dots, F_n(\varphi_1(x)) - F_n(\varphi_2(x))) \|_{\infty} \\
&= \max_{1 \leq i \leq n} |F_i(\varphi_1(x)) - F_i(\varphi_2(x))| \\
&\leq \max_{1 \leq i \leq n} \| F_i(\varphi_1) - F_i(\varphi_2) \|_{\infty} \\
&\leq \| \varphi_1 - \varphi_2 \|_{\infty} .
\end{aligned}$$

All things considered, we prove that

$$\| L^*(F_1, \dots, F_n)(\varphi_1) - L^*(F_1, \dots, F_n)(\varphi_2) \|_{\infty} \leq \| \varphi_1 - \varphi_2 \|_{\infty}$$

and hence $L^*(F_1, \dots, F_n)$ is non expansive. □

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