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Spin coupling to curved space-times: Melvin and double Kasner cosmologies.

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Abstract

Quantum field theory represents the modern approach to describe three fundamental interactions: strong, weak and electromagnetic. It is not possible to describe gravity as a quantum field theory because of its non-renormalizability. Quantum field theory on curved space-times represents a modern approach to describe interactions between quantum particles and gravitational fields. The target of this paper is to illustrate a completely general mathematical method to describe spinning particles in arbitrary gravitational fields, focusing on physical implications concerning the interaction between spin and gravitational fields. This apparatus is then applied to particular cosmologies, Melvin cosmology and double Kasner cosmology. The results obtained are analysed from a physical point of view showing effects emerging from spin-gravity coupling.

Sommario

La teoria dei campi quantistici rappresenta l'approccio moderno per descrivere tre interazioni fondamentali: la forte, la debole e l'elettromagnetica. Non è possibile descrivere la gravità come teoria di campo quantistico per via della sua non-rinormalizzabilità. La teoria dei campi quantistici su spazitempi curvi rappresenta un approccio moderno per descrivere le interazioni tra particelle quantistiche e campi gravitazionali. L'obiettivo di questo elaborato è di illustrare un metodo matematico completamente generale per descrivere particelle con spin in campi gravitazionali arbitrari, concentrando l'attenzione sulle implicazioni fisiche che riguardano l'interazione tra spin e campi gravitazionali. Questo apparato viene poi applicato a specifiche cosmologie, la cosmologia di Melvin e la cosmologia double Kasner. I risultati ottenuti vengono analizzati da un punto di vista fisico mostrando gli effetti che emergono dall'accoppiamento tra spin e gravità.

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Introduction.

Quantum field theory is the modern approach to describe strong, weak and electromagnetic interactions. Gravity is excluded from this theory because of the well known non-renormalizability of the theory[1]. Quantum field theory on curved space-time represents a semiclassical approach to study the interaction of quantum particles with gravitational fields.

After the notion of spin was introduced in physics, the study of spin dynamics was initiated. First efforts were concerned with the study of fermions in weak gravitational fields, i.e. for the case when the geometry does not deviate significantly from Minkowki manifold. Subsequently the attention was devoted to the description of fermions in geometries risen as exact solutions of Einstein field equations. It was only recently that the study of the dynamics of spinning particles in general manifolds has been developed [2] and this is the starting point of this paper.

In most applications of mathematical cosmology Friedmann cosmological models are considered. However, in the early universe, the effects of anisotropies could be essential. Some studies have already been devoted mainly on simple homogeneous Bianchi universes [3], in particular it has been studied the motion of Dirac particles in gravitational fields whose metric is represented by Bianchi universes. The goal of this paper is to study a couple of different anisotropic models, one emerging as a solution of Einstein-Maxwell field equation, the other is a generalisation of Kasner universe.

In the first chapter I explain the mathematical method of tetrads[4] used to describe spinors on arbitrary manifolds, until I define the "*spin connection*", which is analogous to the affine connection, necessary to describe parallel transport for vectors. This is done by generalizing Clifford algebra from the Minkowski metric to an arbitrary metric. Subsequently, I generalize the Dirac equation in arbitrary space-times using tetrad formalism.

In the second chapter I study the Foldy-Wouthuysen transformation [5]. The F-W transformation is a unitary transformation of the Dirac Hamiltonian used to separate spinors into two components with positive energy eigenvalues and two components with negative energy eigenvalues. This separation is particularly useful when studying the classical limit of the Dirac equation. This transformation is then generalized in arbitrary space-times [6], [7], [2], obtaining the transformation of the Hamiltonian as a power expansion in terms of $\hbar$. Then I study the Dirac equation in curved spaces and the equations of motion of polarization operator of spinning particles.

In the third chapter I apply what I have proved in the previous ones to particular anisotropic cosmologies: the Melvin cosmology and the double Kasner cosmology.

Melvin metric [8] is obtained as a static solution of Einstein-Maxwell field equations, having assumed that the universe is filled with a cylindrically symmetric magnetic field. I demonstrate that starting from a general diagonal metric whose coefficients depend only on a radial variable, it is possible to obtain the Melvin metric by imposing the condition that the scalar curvature R is 0, coherently with the case of the electromagnetic energy-momentum tensor whose trace is 0[9]. Subsequently I use the results obtained in chapters 1 and 2 to work out the angular velocity operators for the spinning particle in Melvin space-time. I solve the geodesics equations to find an expression for the velocities, necessary to find the angular velocity operators. I demonstrate the coherence between the exact result, used to solve the equation of motion of spin operators, and the approximated semiclassical result, used to solve the equation of motion of average spin. By using these previous results, I solve the equation of motion of spin operators and I show that spinning particles precede in a regular way approaching the early universe $t \to \infty$. Finally I use the relations $t \to iR$ and $r \to i\tau$ [10] to analytically continue the Melvin metric to obtain the Melvin cosmology, where the coefficients of the metric depend on a temporal coordinate instead of depending on a radial coordinate. The results obtained for the Melvin metric are then transposed to results for the Melvin cosmology.

Double Kasner cosmology [11] is a generalisation of the standard Kasner metric [9]. Kasner metric describes an homogeneous but anisotropic space-time and the coefficients of the diagonal metric are time-dependent only. Double Kasner metric generalises the standard Kasner metric by introducing the dependence on a spacial coordinate for the coefficients of the metric. I demonstrate that, starting from a diagonal metric with coefficients depending only on a temporal and on a spacial coordinate, the double Kasner metric emerges as a solution of vacuum field equations. This is analogous to what I show in an appendix, that is that Kasner solution can be found, starting from the anisotropic metric Bianchi I, by solving vacuum field equations. I work out the angular velocity operators in the semiclassical approximation to describe the motion of average spin. I solve asymptotically, near $t \to 0$, the geodesics equations for completely general values of the parameters appearing in the metric. This is necessary to find explicit expressions for angular velocity operators in order to solve the equations of motion of average spin. I solve then these equations and for the temporal trend I find complete agreement with the results found for the standard Kasner metric near the singularity $t \to 0$ [3].

The results obtained show the effects of the anisotropy on spinning particles and can be used to give an interpretation of some phenomena of the early universe.

Chapter 1

Mathematical tools.

1.1 Introduction.

In this chapter we introduce some useful mathematical instruments to study spinor fields in curved space-time. It is not trivial indeed to deduce the correct generalisation of spinor fields on Minkowski space, because the general Riemann space has much less symmetries. One possible approach is the covariant generalisation of flat space-time expressions; so, we start from the locally flat frame, then we perform the "covariantization" [4].

1.2 Tetrad formalism.

First of all, let us recall the definition of the metric tensor [12]

Def. 1.1 (Metric tensor): we define the *Metric tensor*, or simply the *Metric*, as the tensor of rank $(0, 2)$ that satisfies:

i) $g(v, v) = 0 \Leftrightarrow v = 0$, that is, g is not degenerate;

ii) $g(u, v) = g(v, u)$, that is, g is symmetric;

where u and v are vectors of the tangent space T_P of a point P of the manifold.

Expanding the expression of the metric to a general coordinate basis $\{e_\mu\}_{\mu=1}^{n}$, where n is the dimension of the manifold, we obtain

$$g(u, v) = g\left(u^\mu e_\mu, v^\nu e_\nu\right) = g\left(e_\mu, e_\nu\right) u^\mu v^\nu = g_{\mu\nu} u^\mu v^\nu. \tag{1.1}$$

The metric is also often defined starting from the line element

$$ds^2 = g_{\mu\nu} e^\mu e^\nu \tag{1.2}$$

where e^μ is the 1-form dual of the base vector e_ν, that is

$$e^\mu \left(e_\nu\right) = e_\nu \left(e^\mu\right) = \delta^\mu_\nu$$

or analogously

$$e^\mu = g^{\mu\nu} e_\nu \text{ with } g^{\mu\nu} \text{ the matrix inverse of } g_{\mu\nu}.$$

From (1.1) it is obvious that, for a general choice of vector basis, the components of the metric tensor depend on the coordinates of the space-time. In some applications it is very useful to use a particular vector basis $\{e_a\}_{a=1}^n$ where the metric coefficients are constants. In particular, it is possible to choose such vectors so that the metric coefficients reproduce the Minkowskian metric in n dimensions $\eta_{ab} = diag(+, \underbrace{-, \cdots, -}_{n-1\ times})$, so we have

$$ds^2 = \eta_{ab} e^a e^b \tag{1.3}$$

where the indexes are raised and lowered with the metric η.

It is possible to expand the new basis' vectors to the old basis, so that we obtain

$$e_a = e^\mu_a e_\mu \quad , \quad e^a = e^a_\mu e^\mu \quad \text{and vice versa} \quad e_\mu = e^a_\mu e_a \quad , \quad e^\mu = e^\mu_a e^a . \tag{1.4}$$

The transition coefficients are called *tetrads* [4] or analogously *vielbein fields* (from German *viel* = many, *bein* = legs), so a tetrad is just a change of basis. Using (1.4) in (1.2) and confronting the result with (1.3) we obtain

$$ds^2 = g_{\mu\nu} e^\mu e^\nu = g_{\mu\nu} e^\mu_a e^a e^\nu_b e^b \quad \Rightarrow \quad g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab} . \tag{1.5}$$

Defining $g = det g_{\mu\nu}$ from (1.5) we easily obtain

$$det\ \eta_{ab} = g\ det\left(e^\mu_a\right)^2 \quad \Rightarrow \quad det\left(e^a_\mu\right) = \sqrt{|g|} . \tag{1.6}$$

It is obvious that the choice of vectors e_a is not unique, because you can perform local transformations of the Lorentz group $O(1, n-1)$ on such vectors and the metric will remain the same η_{ab}. This way, we found a new local symmetry:

$$e^a_\mu \to \tilde{e}^a_\mu = \Lambda^a_b e^b_\mu \quad \text{where} \quad \Lambda^a_b \Lambda^c_d \eta_{ac} = \eta_{bd}$$

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \tilde{e}^a_\mu \tilde{e}^b_\nu \eta_{ab} = e^c_\mu e^d_\nu \Lambda^a_c \Lambda^b_d \eta_{ab} = e^c_\mu e^d_\nu \eta_{cd} = g_{\mu\nu} .$$

We expanded the symmetry of the general relativity to general co-ordinate transformations and local Lorentz transformations, often called *local generalized rotations*. This is the reason why $g_{\mu\nu}$ has $\frac{d(d+1)}{2}$ free components while e^a_μ has d^2.

1.2.1 Covariant derivative.

The covariant derivative is introduced in Riemannian geometry to define the concept of "parallel transport" of vectors, differential forms and so on. Let's recall the basic properties characterising this operator [12]:

i) let U and V be vectors and $\tilde{U}$ be a different parametrisation of U, so that $\tilde{U} = gU$ where g is a scalar, the covariant derivative is invariant under different parametrisations, so:

$$\nabla_{\tilde{U}} V = \nabla_{gU} V = g \nabla_U V \quad ;$$

ii) the covariant derivative is a differential operator, therefore it obeys Leibnitz rule; so, let U, V be vectors and f be a scalar, we have:

$$\nabla_U f V = f \nabla_U V + (\nabla_U f) V = f \nabla_U V + (Uf) V \quad ;$$

iii) the covariant derivative is a linear operator; so, let U, V, W be vectors and f, g be scalars, we have:

$$\nabla_{fU+gV} W = f \nabla_U W + g \nabla_V W \quad .$$

The *affine connection* is the rule that defines the way tensors are parallely transported and that implements the definition of the covariant derivative; let e_i and e_j be basis vectors, we have:

$$\nabla_{e_\mu} e_\nu = \Gamma^\lambda_{\nu\mu} e_\lambda \ . \tag{1.7}$$

In the context of a torsion-free geometry, if $\{e_i\}_{i=1}^n$ is a coordinate base (so that the anholonomy coefficients are equal to 0), the affine connection is called *Christoffel symbols*, and starting from the condition of accordance of affine connection and metric it is easy to obtain the following relation:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\tau} \left(\partial_\mu g_{\tau\nu} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu} \right) \ . \tag{1.8}$$

For a general vector, from (1.7) we obtain:

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda \ . \tag{1.9}$$

The generalisation to (p,q) tensors is trivial because of Leibnitz's rule.

Now let's show how to derive an analogous expression for the covariant derivative of objects with local Lorentz indices. Starting from a point P of the manifold, any movement from P means the shift from a tangent space to another. So, it is easy to understand that the covariant derivative of a vector V^b can not be trivially equal to $\partial_a V^b$. If we assume that the covariant derivative is a linear differential operator, we come to the expression:

$$\nabla_a V^b = \partial_a V^b + \tilde{\omega}^b_{\cdot ca} V^c \ , \tag{1.10}$$

where $\tilde{\omega}^b_{\cdot ca}$ are unknown coefficients. From (1.4) and $\nabla_\nu V^\mu = e^a_\nu e^b_\mu \nabla_a V^b$ we come to the following chain of equalities:

$$\begin{aligned}\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\tau\nu} V^\tau = e^a_\nu e^\mu_b \nabla_a V^b = e^a_\nu e^\mu_b \left(\partial_a V^b + \tilde{\omega}^b_{\cdot ca} V^c\right) = \\ = \delta^\mu_\tau \partial_\nu V^\tau + V^\tau e^a_\nu e^\mu_b \partial_a e^b_\tau + e^a_\nu e^\mu_b e^c_\tau \tilde{\omega}^b_{\cdot ca} V^\tau = \\ = \partial_\nu V^\mu + V^\tau \left(e^\mu_b \partial_\nu e^b_\tau + e^\mu_b e^c_\tau \tilde{\omega}^b_{\cdot c\nu}\right) \ .\end{aligned} \tag{1.11}$$

So, we have obtained

$$\Gamma^\mu_{\tau\nu} = e^\mu_b \partial_\nu e^b_\tau + e^\mu_b e^c_\tau \tilde{\omega}^b_{\cdot c\nu} \ . \tag{1.12}$$

By multiplying (1.12) by $e^a_\mu e^{\tau d}$, we obtain

$$\tilde{\omega}^{ad}_{\cdot\cdot\nu} = e^a_\mu e^{\tau d} \Gamma^\mu_{\tau\nu} - e^{\tau d} \partial_\nu e^a_\tau \ . \tag{1.13}$$

(1.13) allows us to calculate the covariant derivative using only vielbein fields and the affine connection. It is easy to demonstrate the antisymmetry of $\tilde{\omega}^{ad}_{\cdot\cdot\nu}$ for $a \leftrightarrow d$ using (1.13), as shown below:

$$\begin{aligned}\tilde{\omega}^{ad}_{\cdot\cdot\nu} + \tilde{\omega}^{da}_{\cdot\cdot\nu} = e^a_\mu e^{\tau d}\Gamma^\mu_{\tau\nu} + e^d_\mu e^{\tau a}\Gamma^\mu_{\tau\nu} - e^{\tau d}\partial_\nu e^a_\tau - e^{\tau a}\partial_\nu e^d_\tau = \\ = \frac{1}{2} g^{\mu\lambda}\left(\partial_\tau g_{\lambda\nu} + \partial_\nu g_{\lambda\tau} - \partial_\lambda g_{\tau\nu}\right)\left(e^a_\mu e^{\tau d} + e^d_\mu e^{\tau a}\right) - e^{\tau d}\partial_\nu e^a_\tau - e^{\tau a}\partial_\nu e^d_\tau = \\ = \frac{1}{2}\left(\partial_\tau g_{\lambda\nu} + \partial_\nu g_{\lambda\tau} - \partial_\lambda g_{\tau\nu}\right)\left(e^{a\lambda} e^{\tau d} + e^{d\lambda} e^{\tau a}\right) - e^{\tau d}\partial_\nu e^a_\tau - e^{\tau a}\partial_\nu e^d_\tau = \\ = e^{a\lambda} e^{\tau d} e^c_\lambda \partial_\nu e_{c\tau} + e^{a\lambda} e^{\tau d} e_{c\tau}\partial_\nu e^c_\lambda - e^{\tau d}\partial_\nu e^a_\tau - e^{\tau a}\partial_\nu e^d_\tau = 0 \ ,\end{aligned}$$

where the third equality is allowed by the symmetry, under $\lambda \leftrightarrow \tau$, of the tetrad term in brackets, and by the antisymmetry of the first and third metric terms in brackets, and the last equality is consequential to the contraction of indices.

Another useful formula follows from (1.5):

$$0 = \nabla_\tau g_{\mu\nu} = \nabla_\tau \left(e^a_\mu e^b_\nu \eta_{ab}\right) = 2\eta_{ab} e^a_\mu \nabla_\tau e^b_\nu = 0 \ \Leftrightarrow \ \nabla_\tau e^b_\nu = 0 \ . \tag{1.14}$$

1.2.2 Covariant derivative of fermions and spin connection.

Now let's study the application of the previous section to define the covariant derivative for Dirac fermions and then to describe spinor fields on general manifolds. Spinor fields are described in a flat four-dimensional metric by the Dirac action

$$S = \int dx^4 \bar{\psi} \left(i\gamma^a \partial_a - m\right) \psi \ . \tag{1.15}$$

The generalisation to curved manifolds is trivially given by

$$S = \int dx^4 \sqrt{|g|} \bar{\psi} \left(i\gamma^\mu \nabla_\mu - m\right) \psi \ , \tag{1.16}$$

but the definition of covariant derivative is not trivial. The necessity of using tetrads to formulate a consistent spinor field theory emerges because there is no covering group for the group of general coordinate transformations. So, we start from a flat metric and then use tetrads to generalize to curved metrics.

On a flat metric, gamma matrices satisfy the $Clifford\ algebra$ [1], that is

$$\left\{\gamma^a, \gamma^b\right\} = \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} \ . \tag{1.17}$$

Moving to a general curved space-time, we can use tetrads to obtain the following result:

$$\gamma^a = e^a_\mu \gamma^\mu \quad \Rightarrow \quad \left\{e^a_\mu \gamma^\mu, e^b_\nu \gamma^\nu\right\} = 2\eta^{ab} \quad \Rightarrow \quad \left\{\gamma^\mu, \gamma^\nu\right\} = 2g^{\mu\nu} \ , \tag{1.18}$$

that is the curved-space version of the Clifford algebra.

Let's define

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{i}{2} \omega_\mu{}^{ab} \sigma_{ab} \psi \ , \quad \text{where} \quad \sigma_{ab} = \frac{i}{2} \left[\gamma_a, \gamma_b\right] \ . \tag{1.19}$$

The coefficients $\omega_\mu{}^{ab}$ are called *spinor* (or simply *spin*) *connection*. The conjugate of (1.19) is

$$\nabla_\mu \bar{\psi} = \partial_\mu \psi - \frac{i}{2} \omega_\mu{}^{ab} \bar{\psi} \sigma_{ab} \ . \tag{1.20}$$

By using (1.19), (1.20) and Leibniz's rule, we can get these simple results:

$$\nabla_\mu (\bar{\psi}\psi) = \left(\nabla_\mu \bar{\psi}\right) \psi + \bar{\psi} \left(\nabla_\mu \psi\right) =$$

$$= \left(\partial_\mu \bar{\psi} - \frac{i}{2}\omega_\mu{}^{ab}\bar{\psi}\sigma_{ab}\right)\psi + \bar{\psi}\left(\partial_\mu\psi + \frac{i}{2}\omega_\mu{}^{ab}\sigma_{ab}\psi\right) =$$

$$= \left(\partial_\mu\bar{\psi}\right)\psi + \bar{\psi}\left(\partial_\mu\psi\right) = \partial_\mu\left(\bar{\psi}\psi\right) \ , \tag{1.21}$$

which is the result we expected for the scalar $\bar{\psi}\psi$;

$$\nabla_\mu\gamma^\nu = \nabla_\mu\left(e_a^\nu\gamma^a\right) = \gamma^a\nabla_\mu e_a^\nu = 0 \ , \tag{1.22}$$

having used (1.14).

Now we can find a way to calculate the spin connection through a systematic use of Leibniz's rule, starting from the covariant derivative of the vector $\bar{\psi}\gamma^\alpha\psi$:

$$\nabla_\mu\left(\bar{\psi}\gamma^\alpha\psi\right) = \partial_\mu\left(\bar{\psi}\gamma^\alpha\psi\right) + \Gamma^\alpha_{\nu\mu}\bar{\psi}\gamma^\nu\psi =$$

$$= \left(\partial_\mu\bar{\psi}\right)\gamma^\alpha\psi + \bar{\psi}\left(\partial_\mu e_c^\alpha\right)\gamma^c\psi + \bar{\psi}\gamma^\alpha\left(\partial_\mu\psi\right) + \Gamma^\alpha_{\nu\mu}\bar{\psi}\gamma^\nu\psi \ . \tag{1.23}$$

This result is to be compared with the following result:

$$\nabla_\mu\left(\bar{\psi}\gamma^\alpha\psi\right) = \left(\nabla_\mu\bar{\psi}\right)\gamma^\alpha\psi + \bar{\psi}\left(\nabla_\mu\gamma^\alpha\right)\psi + \bar{\psi}\gamma^\alpha\left(\nabla_\mu\psi\right) =$$

$$= \left(\partial_\mu\bar{\psi}\right)\gamma^\alpha\psi - \frac{i}{2}\omega_\mu{}^{ab}\bar{\psi}\sigma_{ab}\gamma^\alpha\psi + \bar{\psi}\gamma^\alpha\left(\partial_\mu\psi\right) + \frac{i}{2}\bar{\psi}\gamma^\alpha\omega_\mu{}^{ab}\sigma_{ab}\psi \ , \tag{1.24}$$

having used (1.22). Since the equality must be valid for any field ψ, by comparing (1.23) and (1.24) we obtain:

$$-\frac{1}{4}\omega_\mu{}^{ab}e_c^\alpha\left(\gamma^c\left(\gamma_a\gamma_b - \gamma_b\gamma_a\right) - \left(\gamma_a\gamma_b - \gamma_b\gamma_a\right)\gamma^c\right) = \gamma^c\left(e_c^\nu\Gamma^\alpha_{\nu\mu} + \partial_\mu e_c^\alpha\right) \ . \tag{1.25}$$

By making use of the Clifford algebra relation

$$\gamma^c\gamma_a\gamma_b = 2\delta_a^c\gamma_b - \gamma_a\gamma^c\gamma_b$$

we get

$$\omega_\mu{}^{ab}\left(e_b^\alpha\gamma_a - e_a^\alpha\gamma_b\right) = \gamma^c\left(e_c^\nu\Gamma^\alpha_{\nu\mu} + \partial_\mu e_c^\alpha\right) \ . \tag{1.26}$$

We can solve this equation by taking advantage of the antisymmetry of ω, and we obtain

$$\omega_\mu{}^{ab} = \frac{1}{2}\tilde{\omega}^{ab}{}_\mu = \frac{1}{2}\left(e_\tau^b e^{\lambda a}\Gamma^\tau_{\lambda\mu} - e^{\lambda a}\partial_\mu e_\lambda^b\right) \ , \tag{1.27}$$

having defined

$$e^{\lambda a} = \eta^{ab}e_b^\lambda \ . \tag{1.28}$$

We have here established the relation between (1.13) and the spin connection, and this gives us an explicit way to calculate the covariant derivative of fermions.

1.3 Foldy-Wouthuysen transformation.

The Foldy-Wouthuysen transformation [5] is a unitary transformation of the Dirac equation, and has the advantage of separating the spinor into four components, of which two have positive energy eigenvalues, and two have negative energy eigenvalues. This is particularly useful for the interpretation of the non-relativistic limit, that is the Pauli equation. This type of transformation had already been studied previously by Newton and Wigner while they were studying properties of the position operator; they found that states localized in position cannot be formed solely from positive energy states or solely from negative energy states in Dirac-Pauli representation. Starting from reasonable invariance requirements, they discovered that states split into pure positive-energy and pure negative-energy states could be found, and that they are unique. The studies of Foldy and Wouthuysen generalized those of Newton and Wigner to the case of particles in electromagnetic fields [13].

Here we analyse their work, and consider the case of gravitational fields as well.

Starting from the Dirac equation in Hamiltonian form, we have

$$i\hbar\frac{\partial}{\partial t}\psi = \mathcal{H}\psi \quad , \quad \text{where} \quad \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} . \tag{1.29}$$

We can now perform a unitary transformation by setting $\psi' = U\psi = e^{iS}\psi$, where U is a unitary operator; this transformation implies:

$$i\hbar\frac{\partial}{\partial t}\left(U^{-1}\psi'\right) = U^{-1}i\hbar\frac{\partial\psi'}{\partial t} + i\hbar\left(\frac{\partial}{\partial t}U^{-1}\right)\psi' = \mathcal{H}U^{-1}\psi'$$

and, rearranging it into Hamiltonian form,

$$i\hbar\frac{\partial}{\partial t}\psi' = \left(U\mathcal{H}U^{-1} - i\hbar U\frac{\partial U^{-1}}{\partial t}\right)\psi' = \mathcal{H}'\psi' . \tag{1.30}$$

The Hamiltonian can be split into operators that commute ("even operators") and anticommute ("odd operators") with the operator β:

$$\mathcal{H} = \beta\mathcal{M} + \mathcal{E}\mathbb{I} + \mathcal{O} \ , \ [\beta,\mathcal{M}] = 0 \ , \ [\beta,\mathcal{E}\mathbb{I}] = 0 \ , \ \{\beta,\mathcal{O}\} = 0 \ , \tag{1.31}$$

where $\mathcal{H}$ is hermitian and we assume $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{O}$ to be hermitian as well. From now on, we will use the common notation for the β and α^i matrices [14], so we have

$$\beta = \gamma^0 \ , \ \alpha^i = \gamma^0\gamma^i \ , \ i = 1, 2, 3 \ ,$$

where $\{\gamma^a\}_{a=0}^3$ are the Dirac matrices. Therefore,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \quad , \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix} \ ,$$

where σ^α are the Pauli matrices.

1.3.1 Free Dirac particle.

In the case of free Dirac particles, we have

$$\mathcal{E} = 0 \ , \ \mathcal{O} = \boldsymbol{\alpha} \cdot \mathbf{p} \quad \Rightarrow \quad \mathcal{H} = \beta m + \boldsymbol{\alpha} \cdot \mathbf{p} \ , \tag{1.32}$$

where m is the particle mass; we can set

$$S = -\frac{i}{2m} \beta \boldsymbol{\alpha} \cdot \mathbf{p} \theta(\mathbf{p}) \tag{1.33}$$

so that θ becomes a function to be determined in order for $\mathcal{H}'$ to be free from odd operators. S commutes with the Hamiltonian because it is not explicitly time-dependent,therefore the operator e^{iS} does as well, because it is defined by its power expansion as

$$e^{iS} = \sum_{n=0}^{\infty} \frac{i^n}{n!} S^n \ ,$$

since each term of the sum commutes with the Hamiltonian because S does.

The transformation of the Hamiltonian is

$$\mathcal{H}' = e^{iS} \mathcal{H} e^{-iS} = e^{i2S} \mathcal{H} \tag{1.34}$$

and, by performing the power expansion of e^{i2S}, and using (1.33) and trivial identities of $\boldsymbol{\alpha}$ and β matrices, we can easily obtain

$$e^{i2S} = \cos\left(\frac{p\theta}{m}\right) + \left(\frac{\beta \boldsymbol{\alpha} \cdot \mathbf{p}}{p}\right) \sin\left(\frac{p\theta}{m}\right) . \tag{1.35}$$

By using this result in (1.34) we obtain

$$\mathcal{H}' = \left[\cos\left(\frac{p\theta}{m}\right) + \left(\frac{\beta \boldsymbol{\alpha} \cdot \mathbf{p}}{p}\right) \sin\left(\frac{p\theta}{m}\right)\right] (\beta m + \boldsymbol{\alpha} \cdot \mathbf{p}) =$$

$$= \beta \left[m \cos\left(\frac{p\theta}{m}\right) + p \sin\left(\frac{p\theta}{m}\right)\right] + \frac{\boldsymbol{\alpha} \cdot \mathbf{p}}{p} \left[p \cos\left(\frac{p\theta}{m}\right) - m \sin\left(\frac{p\theta}{m}\right)\right] . \tag{1.36}$$

From this equation, by putting the second parenthesis equal to 0, we easily obtain

$$\theta(p) = \frac{m}{p} \arctan\left(\frac{p}{m}\right) \ . \tag{1.37}$$

With this θ we obtained the odd operators-free Hamiltonian

$$\mathcal{H}' = \beta\sqrt{m^2 + p^2} = \beta E_p \ . \tag{1.38}$$

The solutions of (1.30) are such that upper components represent positive energies and lower components represent negative energies. We can split ψ' as follows:

$$\psi' = \phi' + \chi' \ , \ \text{where} \ , \ \phi' = \frac{1+\beta}{2}\psi' \ , \ \chi' = \frac{1-\beta}{2}\psi' \ . \tag{1.39}$$

Now (1.30) can be split explicitly into

$$\beta E_p \phi' = \frac{i\partial}{\partial t}\phi' \ , \tag{1.40}$$

$$-\beta E_p \chi' = \frac{i\partial}{\partial t}\chi' \ . \tag{1.41}$$

Thus, we separated the positive energy and the negative energy terms.

1.3.2 Exact Foldy-Wouthuysen transformation.

First of all, we can notice that the operator $\mathcal{O}^2$ is even because of (1.31), therefore we have

$$\beta\mathcal{O}^2 = -\mathcal{O}\beta\mathcal{O} = \mathcal{O}^2\beta \ . \tag{1.42}$$

We now define the operator S in a similar way to (1.31) as

$$S = -i\frac{\beta\mathcal{O}}{\mathcal{C}}\theta \ , \tag{1.43}$$

where C and θ are function of $\mathcal{O}^2$, and C is defined unambiguously by the relations

$$C^2 = \mathcal{O}^2 \ , \ [\beta, C] = 0 \ , \ \sqrt{\mathbb{I}} = \mathbb{I} \ . \tag{1.44}$$

(1.44) defines exactly the square root of the operator $\mathcal{O}^2$:

$$C = \sqrt{\mathcal{O}^2} \ . \tag{1.45}$$

We observe that the operator S, and therefore the operator U, are not explicitly time-dependent, so we can use the relation $\mathcal{H}' = U\mathcal{H}U^{-1}$.

For example, to understand what the square root of an operator is we consider the free particle case:

$$\mathcal{O} = \boldsymbol{\alpha} \cdot \mathbf{p} \ , \ \mathcal{O}^2 = \mathbb{I}|^2 \ , \ C = \mathbb{I}\sqrt{\mathbf{p}^2} = \mathbb{I}|\mathbf{p}| \ \ .$$

The operator U can be broken down via a series expansion into

$$U = \cos\theta + \frac{\beta\mathcal{O}}{\mathcal{C}} \sin\theta \ . \tag{1.46}$$

A sufficient, but not necessary, condition to have an exact transformation is [6]

$$[\mathcal{E}, \mathcal{O}] = 0 \tag{1.47}$$

(we have omitted the identity symbol multiplied by $\mathcal{E}$), and this implies

$$[\mathcal{E}, \beta\mathcal{O}] = [\mathcal{E}, \beta]\mathcal{O} + \beta[\mathcal{E}, \mathcal{O}] = 0 \ , \tag{1.48}$$

having used (1.31) and (1.47). Using (1.46) and (1.48) we can derive the transformed Hamiltonian:

$$\begin{aligned}
\mathcal{H}' = U\mathcal{H}U^{-1} &= \left(\cos\theta + \frac{\beta\mathcal{O}}{\mathcal{C}}\sin\theta\right)\mathcal{H}\left(\cos\theta - \frac{\beta\mathcal{O}}{\mathcal{C}}\sin\theta\right) = \\
&= (\beta m + \mathcal{O})\left(\cos\theta - \frac{\beta\mathcal{O}}{\mathcal{C}}\sin\theta\right)^2 + \mathcal{E} = \\
&= (\beta m + \mathcal{O})\left(\cos 2\theta - \frac{\beta\mathcal{O}}{\mathcal{C}}\sin 2\theta\right) + \mathcal{E} = \\
&= \beta\left(m\cos 2\theta + C\sin 2\theta\right) + \mathcal{O}\left(\cos 2\theta - \frac{m}{C}\sin 2\theta\right) + \mathcal{E} \ .
\end{aligned} \tag{1.49}$$

From this result we can put the parenthesis multiplied by $\mathcal{O}$ to be equal to 0 in order to obtain an even Hamiltonian:

$$\tan 2\theta = \frac{C}{m} \ , \tag{1.50}$$

and using trivial goniometric relations

$$\tan\theta_{\pm} = \pm\frac{C}{\epsilon \pm m} \quad \text{where} \quad \epsilon = \sqrt{m^2 + C^2} = \sqrt{m^2 + \mathcal{O}^2} \ . \tag{1.51}$$

Therefore, we found two values of θ that make the Hamiltonian even, θ_+ realises the F-W transformation, and by substituting in (1.49) we obtain

$$\mathcal{H}' = \beta\epsilon + \mathcal{E} \quad , \quad \epsilon = \sqrt{m^2 + \mathcal{O}^2} \ . \tag{1.52}$$

Through a substitution of the values of θ in (1.46) we find

$$U^{\pm} = \frac{\epsilon + m \pm \beta\mathcal{O}}{\sqrt{2\epsilon\left(\epsilon + m\right)}} \ , \tag{1.53}$$

that fully agrees with what can be found in the free particle case. We also notice (this will be useful a bit later) that $U^{-} = U^{-1}$.

1.3.3 General case.

In the general case, the operator U is time-dependent [6], so it is not possible to remove odd operators to all orders. So, we will perform a power expansion. To do this, first we need to calculate some commutators that appear during calculations:

$$U\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)U^{-1}=\mathcal{E}-i\frac{\partial}{\partial t}+\left[U,\mathcal{E}-i\frac{\partial}{\partial t}\right]U^{-1}\ . \tag{1.54}$$

To calculate the commutators we will use these formulas:

$$\text{i) }\left[A^{-1},B\right]=A^{-1}\left[B,A\right]A^{-1}\ , \tag{1.55}$$

where the proof is

$$A^{-1}\left[B,A\right]A^{-1}=A^{-1}\left(BA-AB\right)A^{-1}=A^{-1}B-BA^{-1}=\left[A^{-1},B\right]\ \ ;$$

$$\text{ii)}ABA=\frac{1}{2}\left(\left\{A^2,B\right\}-\left[A,\left[A,B\right]\right]\right)\ , \tag{1.56}$$

where the proof is

$$\begin{aligned}&\frac{1}{2}\left(\left\{A^2,B\right\}-\left[A,\left[A,B\right]\right]\right)=\frac{1}{2}\left(A^2B+BA^2-\left[A,AB-BA\right]\right)=\\&=\frac{1}{2}\left(A^2B+BA^2-A^2B+ABA+ABA-BA^2\right)=ABA\ \ ;\end{aligned}$$

$$\text{iii) }\left[A,B\right]=\frac{1}{4}\left\{A^{-1},\left[A^2,B\right]\right\}-\frac{1}{4}\left[\left[A,\left[A,B\right]\right],A^{-1}\right]\ , \tag{1.57}$$

where the proof is

$$\begin{aligned}&\frac{1}{4}\left\{A^{-1},\left[A^2,B\right]\right\}-\frac{1}{4}\left[\left[A,\left[A,B\right]\right],A^{-1}\right]=\\&=\frac{1}{4}\left\{A^{-1},A^2B-BA^2\right\}-\frac{1}{4}\left[\left[A,AB-BA\right],A^{-1}\right]=\\&=\frac{1}{4}\left(AB-A^{-1}BA^2+A^2BA^{-1}-BA\right)-\\&-\frac{1}{4}\left[A^2B-ABA-ABA+BA^2,A^{-1}\right]=\\&=\frac{1}{4}(AB-A^{-1}BA^2+A^2BA^{-1}-BA-A^2BA^{-1}+AB+AB-BA+\\&+AB-BA-BA+A^{-1}BA^2)=\left[A,B\right]\ .\end{aligned}$$

If the commutator of two operators is small compared to their product, i.e.

$$\left|\left[A,B\right]\right|\ll\left|AB\right|\ , \tag{1.58}$$

we can use (1.55) and (1.56) to evaluate $[A, B]$ with any accuracy by performing successive approximations.

Now we perform a transformation using (1.53), and through a substitution in (1.30) we obtain

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}' \ , \tag{1.59}$$

where the operators $\mathcal{E}'$ and $\mathcal{O}'$ are simply given by

$$\begin{aligned}\mathcal{E}' = i\frac{\partial}{\partial t} + \frac{\epsilon + m}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}} - \\ - \frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}\end{aligned} \tag{1.60}$$

and

$$\begin{aligned}\mathcal{O}' = \frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}} - \\ - \frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}} \ .\end{aligned} \tag{1.61}$$

Using (1.56) in (1.60), the two anticommutators and the time derivative cancel each other out, therefore we have

$$\begin{aligned}A = \frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}} \ , \ A^2 = \frac{\epsilon+m}{2\epsilon} \ , \ B = \mathcal{E} - i\frac{\partial}{\partial t} \quad \Rightarrow \\ \Rightarrow \quad \left\{A^2, B\right\} = \left(\frac{\epsilon+m}{2\epsilon}\right)\left(\mathcal{E} - i\frac{\partial}{\partial t}\right) + \left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\left(\frac{\epsilon+m}{2\epsilon}\right)\end{aligned}$$

for the first term and

$$\begin{aligned}A = \frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}} \ , \ A^2 = \frac{\beta\mathcal{O}\beta\mathcal{O}}{2\epsilon(\epsilon+m)} = \frac{-\mathcal{O}^2}{2\epsilon(\epsilon+m)} \ , \ B = \mathcal{E} - i\frac{\partial}{\partial t} \quad \Rightarrow \\ \Rightarrow \quad \left\{A^2, B\right\} = \left(\frac{-\mathcal{O}^2}{2\epsilon(\epsilon+m)}\right)\left(\mathcal{E} - \frac{\partial}{\partial t}\right) + \left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\left(\frac{-\mathcal{O}^2}{2\epsilon(\epsilon+m)}\right)\end{aligned}$$

for the second term. By combining together the terms with the temporal derivative we obtain

$$\frac{\epsilon+m}{2\epsilon} + \frac{\mathcal{O}^2}{2\epsilon(\epsilon+m)} = \frac{\epsilon^2 + m^2 + 2m\epsilon + \mathcal{O}^2}{2\epsilon(\epsilon+m)} = \mathbb{I} \ .$$

As explicitly shown in the previous manipulations, the terms with time derivatives are cancelled, and (1.60) becomes

$$\mathcal{E}' = \mathcal{E} - \frac{1}{4}\left[\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}, \left[\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}, \left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\right]\right] +$$

$$+\frac{1}{4}\left[\frac{\beta\mathcal{O}}{\sqrt{2\epsilon\left(\epsilon+m\right)}},\left[\frac{\beta\mathcal{O}}{\sqrt{2\epsilon\left(\epsilon+m\right)}},\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\right]\right] \ . \qquad (1.62)$$

At this point we perform a second transformation using the following operator:

$$U'=\exp\left(iS'\right)\ ,\ S'=-\frac{i}{4}\beta\left\{\mathcal{O}',\frac{1}{\epsilon}\right\}=-\frac{i}{4}\left[\frac{\beta}{\epsilon},\mathcal{O}'\right]\ , \qquad (1.63)$$

and we want to perform a power expansion from this. To this end, we use the relation [14]

$$e^{iS}He^{-iS}=H+i\left[S,H\right]+\frac{i^2}{2!}\left[S,\left[S,H\right]\right]+\ldots+$$
$$+\frac{i^n}{n!}\left[S,\left[S,\ldots\left[S,H\right]\right]\ldots\right]+\ldots \quad . \qquad (1.64)$$

This relation is verified as follows: by considering

$$F\left(\lambda\right)=e^{i\lambda S}He^{-i\lambda S} \qquad (1.65)$$

and expanding it to a Taylor series at $\lambda=0$, we obtain

$$F\left(\lambda\right)=\sum_{i=0}^{\infty}\frac{\lambda^n}{n!}\left(\frac{\partial^n F}{\partial\lambda^n}\right)_{\lambda=0}\ , \qquad (1.66)$$

and by direct calculation we have

$$\frac{\partial F}{\partial\lambda}=ie^{i\lambda S}SHe^{-i\lambda S}-ie^{i\lambda S}HSe^{-i\lambda S}=ie^{i\lambda S}\left[S,H\right]e^{-i\lambda S}$$

$$\frac{\partial^2 F}{\partial\lambda^2}=i^2e^{i\lambda S}S\left[S,H\right]e^{-i\lambda S}-i^2e^{i\lambda S}\left[S,H\right]Se^{-i\lambda S}=i^2e^{i\lambda S}\left[S,\left[S,H\right]\right]e^{-i\lambda S}$$

up until

$$\frac{\partial^n F}{\partial\lambda^n}=i^ne^{i\lambda S}\left[S,\left[S,\ldots\left[S,H\right]\right]\ldots\right]e^{-i\lambda S}$$

and so on. By putting these results in (1.66) and calculating $F(1)$, we obtain (1.64). If we use $H\equiv\mathcal{H}'$ and (1.63), and limiting ourselves to major corrections, we obtain

$$\mathcal{H}''\approx\beta\epsilon+\mathcal{E}'+\frac{1}{4}\beta\left\{\mathcal{O}'^2,\frac{1}{\epsilon}\right\}\ . \qquad (1.67)$$

If you want to achieve higher accuracy, you should repeat the transformation using (1.63) several times.

Thanks to this procedure, we obtained the Hamiltonian in F-W representation in the general case to any order of accuracy.

1.3.4 Strong external fields.

In the case of strong external fields, we need to generalize the method developed in the previous section [7]. The natural generalization of (1.53) is

$$U = \frac{\beta\epsilon + \beta\mathcal{M} - \mathcal{O}}{\sqrt{(\beta\epsilon + \beta\mathcal{M} - \mathcal{O})^2}}\beta \quad , \quad U^{-1} = \beta\frac{\beta\epsilon + \beta\mathcal{M} - \mathcal{O}}{\sqrt{(\beta\epsilon + \beta\mathcal{M} - \mathcal{O})^2}}$$

$$\epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2} \ . \tag{1.68}$$

This operator is used to perform the F-W transformation in the general case. The calculations are analogous to those in the previous section, so if we define

$$T = \sqrt{(\beta\epsilon + \beta\mathcal{M} - \mathcal{O})^2} \ \text{ and } \ \mathcal{F} = \mathcal{E} - i\hbar\frac{\partial}{\partial t} \ , \tag{1.69}$$

then the transformed Hamiltonian is

$$\begin{aligned} \mathcal{H}' = \beta\epsilon + \mathcal{E} + \frac{1}{2T}([T,[T,(\beta\epsilon + \mathcal{F})]] + \beta\,[\mathcal{O},[\mathcal{O},\mathcal{M}]] - \\ - [\mathcal{O},[\mathcal{O},\mathcal{F}]] - [(\epsilon + \mathcal{M}),[(\epsilon + \mathcal{M}),\mathcal{F}]] - [(\epsilon + \mathcal{M}),[\mathcal{M},\mathcal{O}]] - \\ -\beta\,\{\mathcal{O},[(\epsilon + \mathcal{M}),\mathcal{F}]\} + \beta\,\{(\epsilon + \mathcal{M}),[\mathcal{O},\mathcal{F}]\})\frac{1}{T} \end{aligned} \tag{1.70}$$

and it can be presented as usual in the form of

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}' \ , \ [\mathcal{E}',\beta] = 0 \ , \ \{\beta,\mathcal{O}'\} = 0 \quad , \tag{1.71}$$

having explicitly separated even and odd terms. The even and the odd terms are defined by

$$\mathcal{E}' = \frac{1}{2}(\mathcal{H}' + \beta\mathcal{H}'\beta) - \beta\epsilon \quad , \quad \mathcal{O}' = \frac{1}{2}(\mathcal{H}' - \beta\mathcal{H}'\beta) \quad ,$$

therefore we have

$$\begin{aligned} \frac{1}{2}(\mathcal{H}' + \beta\mathcal{H}'\beta) - \beta\epsilon = \frac{1}{2}(\beta\epsilon + \mathcal{E}' + \mathcal{O}' + \beta\,(\beta\epsilon + \mathcal{E}' + \mathcal{O}')\,\beta) - \beta\epsilon = \\ = \frac{1}{2}(\beta\epsilon + \mathcal{E}' + \mathcal{O}' + \epsilon\beta + \beta\mathcal{E}'\beta + \beta\mathcal{O}'\beta) - \beta\epsilon = \\ = \beta\epsilon + \mathcal{E}' + \frac{1}{2}(\mathcal{O}' - \mathcal{O}') - \beta\epsilon = \mathcal{E}' \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}(\mathcal{H}' - \beta\mathcal{H}'\beta) = \frac{1}{2}(\beta\epsilon + \mathcal{E}' + \mathcal{O}' - \beta\,(\beta\epsilon + \mathcal{E}' + \mathcal{O}')\,\beta) = \\ \frac{1}{2}(\beta\epsilon + \mathcal{E}' + \mathcal{O}' - \epsilon\beta - \beta\mathcal{E}'\beta - \beta\mathcal{O}'\beta) = \frac{1}{2}(\mathcal{O}' + \mathcal{O}') = \mathcal{O}' \quad , \end{aligned}$$

having used the relations in (1.71).

The second transformation is performed according to the previous section and the result, approximated to first terms, is

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}' + \frac{1}{4}\left\{\mathcal{O}'^2, \frac{1}{\epsilon}\right\} \ . \tag{1.72}$$

The formula obtained is very similar to (1.67) and, in an analogous way, further corrections can be calculated by repeating the transformation as explained in the previous section.

(1.72) describes the transformation in the general case. In special cases it can be strongly simplified [7]: for example, if $[\mathcal{M}, \mathcal{O}] = 0$, then (1.70) takes a simpler form and again we obtain results derived in previous sections.

Chapter 2

Spin and Gravitational Field.

In this chapter we apply the techniques developed in the previous one in order to describe particles with spin 1/2 in a general gravitational field. First, we use the tetrad formalism to describe a general metric, then we derive the Dirac Hamiltonian in a curved manifold [2] [15]. Then, we use the F-W transformation to obtain the precession operators equations of motion. We use the same notation as in chapter 1, where space-time indices are Greek letters from the middle of the alphabet $(\mu, \nu \ldots)$, tetrad indices (which are the same as Minkowski indices) are Latin letters from the first half of the alphabet $(a, b, c \ldots)$, spatial indices are Greek letters from the first half of the alphabet $(\alpha, \beta, \gamma \ldots)$, and finally we mark particular values of tetrad indices, being them spacial or zero, with top hats $(\hat{a}, \hat{b} \ldots)$.

2.1 Dirac equation.

Let $x^\mu(\tau)$ describe the position of the particle in space-time, where τ is the proper time and $S^{ab} = -S^{ba}$ the spin tensor. The 4-velocity of the particle is $U^\mu(\tau) = \dfrac{dx^\mu(\tau)}{d\tau}$ and it is normalized as usual, that is $g_{\mu\nu}U^\mu U^\nu = c^2$. Using the tetrad e^μ_a and its inverse e^a_μ we have $U^a(\tau) = e^a_\mu \dfrac{dx^\mu(\tau)}{d\tau}$ and $\eta_{ab}U^a U^b = c^2$, where $\eta_{ab} = diag(c^2, -1, -1, -1)$.

There are several different ways to describe a general metric; we use the following parametrisation for the line element [16]

$$ds^2 = V^2 c^2 dt^2 - \delta_{\hat{\alpha}\hat{\beta}} W^{\hat{\alpha}}_{\gamma} W^{\hat{\beta}}_{\delta} \left(dx^\gamma - K^\gamma c dt\right) \left(dx^\delta - K^\delta c dt\right) \ . \tag{2.1}$$

Parametrisation (2.1) involves functions V, K^α and $W^{\hat{\alpha}}_{\beta}$, so $1 + 3 + 9 = 13$ functions in total, more than the 10 free functions in a general metric in 4 dimensions. We notice that this choice of parametrisation is not the most general, because, as we stated in the previous chapter, a general choice of tetrads involves 16 parameters. The line element (2.1) is invariant for arbitrary local rotations, that is $W^{\hat{\alpha}}_{\beta} \to L^{\hat{\alpha}}_{\hat{\gamma}} W^{\hat{\gamma}}_{\beta}$, where $L^{\hat{\alpha}}_{\hat{\beta}} \in SO(3)$, indeed $L^{\hat{\alpha}}_{\hat{\beta}} L^{\hat{\gamma}}_{\hat{\delta}} \delta_{\hat{\alpha}\hat{\gamma}} = \delta_{\hat{\beta}\hat{\delta}}$. We can eliminate this freedom by choosing a gauge. A useful choice of gauge is the Schwinger gauge, that is $e^{\hat{0}}_{\alpha} = 0$, $\alpha = 1, 2, 3$. Confronting (2.1) with (1.5) and using the gauge fixing, we recognize the tetrad

$$e^{\hat{0}}_{\mu} = V \delta^{\hat{0}}_{\mu} \ , \ e^{\hat{\alpha}}_{\mu} = W^{\hat{\alpha}}_{\beta} \left(\delta^{\beta}_{\mu} - cK^{\beta} \delta^{0}_{\mu} \right) \ , \ \alpha = 1, 2, 3 \ , \tag{2.2}$$

and its inverse, which satisfies $e^{\mu}_{a} e^{a}_{\nu} = \delta^{\mu}_{\nu}$, is

$$e^{\mu}_{\hat{0}} = \frac{1}{V} \left(\delta^{\mu}_{\hat{0}} + \delta^{\mu}_{\alpha} cK^{\alpha} \right) \ , \ e^{\mu}_{\hat{\alpha}} = \delta^{\mu}_{\beta} W^{\beta}_{\hat{\alpha}} \ . \tag{2.3}$$

These inverses satisfy an analogous condition of gauge, that is $e^{0}_{\hat{\alpha}} = 0$. We also introduced the inverse of $W^{\hat{\alpha}}_{\beta}$, that is $W^{\beta}_{\hat{\alpha}}$ and satisfies $W^{\hat{\alpha}}_{\beta} W^{\gamma}_{\hat{\alpha}} = \delta^{\gamma}_{\beta}$.

The velocity of a particle in the orthonormal frame is $U^{a} = e^{a}_{\mu} U^{\mu}$, but it will be convenient to describe the 4-velocity using the spacial components $v^{\hat{\alpha}}$, therefore $U^{a} = \left(\gamma, \gamma v^{\hat{\alpha}} \right)$, where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$. By using (2.3), we obtain

$$U^{0} = \frac{dt}{d\tau} = e^{0}_{a} U^{a} = e^{0}_{\hat{0}} U^{\hat{0}} + e^{0}_{\hat{\alpha}} U^{\hat{\alpha}} = \frac{\gamma}{V} \ , \tag{2.4}$$

$$\begin{aligned} U^{\alpha} = \frac{dx^{\alpha}}{d\tau} &= e^{\alpha}_{a} U^{a} = e^{\alpha}_{\hat{0}} U^{\hat{0}} + e^{\alpha}_{\hat{\beta}} U^{\hat{\beta}} = \frac{1}{V} \delta^{\alpha}_{\gamma} cK^{\gamma} \gamma + \delta^{\alpha}_{\gamma} W^{\gamma}_{\hat{\beta}} \gamma v^{\hat{\beta}} = \\ &= \frac{\gamma}{V} \left(cK^{\alpha} + V W^{\alpha}_{\hat{\beta}} v^{\hat{\beta}} \right) \ . \end{aligned} \tag{2.5}$$

If we define

$$\mathcal{F}^{\alpha}_{\hat{\beta}} = V W^{\alpha}_{\hat{\beta}} \tag{2.6}$$

and combine (2.4) with (2.5), we obtain

$$\frac{dx^{\alpha}}{dt} = \mathcal{F}^{\alpha}_{\hat{\beta}} v^{\hat{\beta}} + cK^{\alpha} \ . \tag{2.7}$$

The Dirac equation in curved space-time that we obtained in chapter 1 reads

$$\left(i\hbar \gamma^{a} \nabla_{a} - mc \right) \Psi = 0 \ , \tag{2.8}$$

where

$$\nabla_{a} = e^{\mu}_{a} \nabla_{\mu} \quad , \quad \nabla_{\mu} = \partial_{\mu} + \frac{iq}{\hbar} A_{\mu} + \frac{i}{4} \sigma^{ab} \tilde{\omega}_{\mu ab} \ . \tag{2.9}$$

For completeness, we included the coupling of the electric charge q to the 4-potential A_μ of the electromagnetic field.

Now we have to calculate the components of the spin connection using (2.2) and (2.3), but first we have to extrapolate a formula that gives the coefficients of the spin connection as a function of tetrads. To do this, we begin calculating Christoffel symbols as functions of tetrads:

$$\Gamma^\nu_{\tau\mu} = \frac{1}{2} g^{\nu\sigma} \left(g_{\sigma\mu},_\tau + g_{\sigma\tau},_\mu - g_{\tau\mu},_\sigma \right) = \frac{1}{2} e^\nu_a e^\sigma_b \eta^{ab} \left(\left(e^c_\sigma e^d_\mu \eta_{cd} \right),_\tau + \right.$$

$$+ \left(e^c_\sigma e^d_\tau \eta_{cd} \right),_\mu - \left(e^c_\tau e^d_\mu \eta_{cd} \right),_\sigma \Big) = \frac{1}{2} \eta^{ab} \eta_{cd} e^\nu_a e^\sigma_b \left(e^c_\sigma,_\tau e^d_\mu + e^c_\sigma e^d_\mu,_\tau + \right.$$

$$+ e^c_\sigma,_\mu e^d_\tau + e^c_\sigma e^d_\tau,_\mu - e^c_\tau,_\sigma e^d_\mu - e^c_\tau e^d_\mu,_\sigma \Big) = \frac{1}{2} \eta^{ab} \eta_{cd} \left(e^\nu_a \delta^c_b e^d_\mu,_\tau + e^\nu_a \delta^c_b e^d_\tau,_\mu + \right.$$

$$+ e^\nu_a e^\sigma_b e^d_\mu e^c_\sigma,_\tau + e^\nu_a e^\sigma_b e^d_\tau e^c_\sigma,_\mu - e^\nu_a e^\sigma_b e^d_\mu e^c_\tau,_\sigma - e^\nu_a e^\sigma_b e^c_\tau e^d_\mu,_\sigma \Big) = \frac{1}{2} \left(e^\nu_a e^a_\mu,_\tau + \right.$$

$$+ e^\nu_a e^a_\tau,_\mu + e^\nu_a e^{\sigma a} e_{\mu c} e^c_\sigma,_\tau + e^\nu_a e^{\sigma a} e_{\tau c} e^c_\sigma,_\mu - e^\nu_a e^{\sigma a} e_{\mu c} e^c_\tau,_\sigma - e^\nu_a e^{\sigma a} e_{\tau c} e^c_\mu,_\sigma \Big) =$$

$$= \frac{1}{2} e^\nu_a \left(e^a_\mu,_\tau + e^a_\tau,_\mu \right) + \frac{1}{2} e^\nu_a e^{\sigma a} e_{\mu c} \left(e^c_\sigma,_\tau - e^c_\tau,_\sigma \right) + \frac{1}{2} e^\nu_a e^{\sigma a} e_{\tau c} \left(e^c_\sigma,_\mu - e^c_\mu,_\sigma \right) \; . \tag{2.10}$$

Now we use (2.10) in (1.13):

$$\tilde{\omega}_{\mu ab} = e_{\nu a} e^\tau_b \Gamma^\nu_{\tau\mu} - e^\tau_b e_{\tau a},_\mu = \frac{1}{2} e_{\nu a} e^\tau_b e^\nu_c \left(e^c_\mu,_\tau + e^c_\tau,_\mu \right) + \frac{1}{2} e_{\nu a} e^\tau_b e^\nu_c e^{\sigma c} e_{\mu d} \left(e^d_\sigma,_\tau - \right.$$

$$- e^d_\tau,_\sigma \Big) + \frac{1}{2} e_{\nu a} e^\tau_b e^\nu_c e^{\sigma c} e_{\tau d} \left(e^d_\sigma,_\mu - e^d_\mu,_\sigma \right) - e^\tau_b e_{\tau a},_\mu = \frac{1}{2} \eta_{ac} e^\tau_b \left(e^c_\mu,_\tau + e^c_\tau,_\mu \right) +$$

$$+ \frac{1}{2} \eta_{ac} e^\tau_b e^{\sigma c} e_{\mu d} \left(e^d_\sigma,_\tau - e^d_\tau,_\sigma \right) + \frac{1}{2} \eta_{ac} e^\tau_b e^{\sigma c} e_{\tau d} \left(e^d_\sigma,_\mu - e^d_\mu,_\sigma \right) - e^\tau_b e_{\tau a},_\mu =$$

$$= \frac{1}{2} e^\tau_b \left(e_{\mu a},_\tau - e_{\tau a},_\mu \right) + \frac{1}{2} e^\tau_b e^\sigma_a e_{\mu d} \left(e^d_\sigma,_\tau - e^d_\tau,_\sigma \right) + \frac{1}{2} e^\tau_a \left(e_{\tau b},_\mu - e_{\mu b},_\tau \right) \; . \tag{2.11}$$

(2.11) [17] gives us an explicit formula to calculate spin connection as functions of the vierbein field. We now use (2.2) and (2.3) and, after lengthy algebra, we obtain

$$\tilde{\omega}_{\mu\hat{\alpha}\hat{0}} = \frac{c^2}{V} W^\beta_{\hat{\alpha}} \partial_\beta V e^{\hat{0}}_\mu - \frac{c}{V} \mathcal{Q}_{(\hat{\alpha}\hat{\beta})} e^{\hat{\beta}}_\mu \tag{2.12}$$

and

$$\tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}} = \frac{c}{V} \mathcal{Q}_{[\hat{\alpha}\hat{\beta}]} e^{\hat{0}}_\mu + \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + \mathcal{C}_{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \mathcal{C}_{\hat{\gamma}\hat{\beta}\hat{\alpha}} \right) e^{\hat{\gamma}}_\mu \; , \tag{2.13}$$

where we introduced the following objects:

$$\mathcal{Q}_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\gamma}} W^{\delta}_{\hat{\beta}} \left(\frac{1}{c} \dot{W}^{\hat{\gamma}}_{\delta} + K^{\epsilon}\partial_{\epsilon} W^{\hat{\gamma}}_{\delta} + W^{\hat{\gamma}}_{\epsilon}\partial_{\delta} K^{\epsilon} \right) \ , \tag{2.14}$$

$$\mathcal{C}_{\hat{\alpha}\hat{\beta}}^{\ \ \hat{\gamma}} = W^{\delta}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \partial_{[\delta} W^{\hat{\gamma}}_{\epsilon]} \quad , \quad \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = \eta_{\hat{\gamma}\hat{\delta}} \mathcal{C}_{\hat{\alpha}\hat{\beta}}^{\ \ \hat{\delta}} \ . \tag{2.15}$$

In this context, the dot "$\cdot$" means, as usual, the derivative with respect to the coordinate time t. We also recognise in $\mathcal{C}_{\hat{\alpha}\hat{\beta}}^{\ \ \hat{\gamma}} = -\mathcal{C}_{\hat{\beta}\hat{\alpha}}^{\ \ \hat{\gamma}}$ the anholonomity coefficients for the spacial triad W.

2.1.1 Hermiticity of the Hamiltonian.

The Dirac equation can be obtained, as usual, from an action principle using the action integral

$$I = \int d^4x \sqrt{-g} L \quad , \quad L = \frac{i\hbar}{2} \left(\bar{\Psi}\gamma^a D_a \Psi - D_a \bar{\Psi}\gamma^a \Psi \right) - mc\bar{\Psi}\Psi \ , \tag{2.16}$$

but the Schrödinger form of the derived Dirac equation involves a non-hermitian Hamiltonian [18]. We can demonstrate this starting from the Dirac equation and its hermitian conjugate:

$$\left(i\hbar\gamma^{\mu}\nabla_{\mu} - mc \right) \Psi = 0 \quad , \quad \bar{\Psi} \left(i\hbar\gamma^{\mu} \overleftarrow{\nabla}_{\mu} + mc \right) = 0 \quad .$$

By multiplying the first equation by $\bar{\Psi}$ from the left, the second equation by Ψ from the right, and using (2.9) and the related hermitian conjugate, we obtain

$$\begin{aligned} 0 = \bar{\Psi} i\hbar\gamma^{\mu} \left(\partial_{\mu}\Psi \right) - qA_{\mu}\bar{\Psi}\gamma^{\mu}\Psi - \frac{\hbar}{4}\bar{\Psi}\gamma^{\mu}\sigma^{ab}\tilde{\omega}_{\mu ab}\Psi - mc\bar{\Psi}\Psi + \\ + \left(\partial_{\mu}\bar{\Psi} \right) i\hbar\gamma^{\mu}\Psi + qA_{\mu}\bar{\Psi}\gamma^{\mu}\Psi + \frac{\hbar}{4}\bar{\Psi}\gamma^{\mu}\sigma^{ab}\tilde{\omega}_{\mu ab}\Psi + mc\bar{\Psi}\Psi \Rightarrow \\ \Rightarrow \partial_{\mu} \left(e\bar{\Psi}\gamma^{\mu}\Psi \right) = 0 \ , \end{aligned}$$

where $e = det\, e^{\mu}_{a} = \sqrt{|g|}$ (the constants have been omitted).

Now, we can integrate this relation over space and, omitting boundary terms (emerging by an application of Gauss's theorem), we obtain

$$0 = \frac{d}{dt} \int \sqrt{|g|} dx^3 \bar{\Psi}\gamma^t\Psi = \frac{d}{dt} \int \sqrt{|g|} dx^3 \Psi^{\dagger}\gamma^0\gamma^t\Psi = 0 \ . \tag{2.17}$$

This relation defines an inner product for spinor fields:

$$\left(\Psi_1, \Psi_2 \right) = \int \sqrt{|g|} dx^3 \Psi^{\dagger}_1 \gamma^0 \gamma^t \Psi_2 \ . \tag{2.18}$$

We observe that, for spinor fields in flat spaces, (2.18) becomes the usual inner product, where $\gamma^t = \gamma^0$ [1]. To verify the hermiticity of the Hamiltonian $\mathcal{H}$ we have to know if $(\Psi, \mathcal{H}\Psi) = (\mathcal{H}\Psi, \Psi)$ or not.

By using (2.17), the Dirac equations $i\partial_t\Psi = \mathcal{H}\Psi$, $-i\partial_t\Psi^\dagger = \Psi^\dagger\mathcal{H}^\dagger$ and $\gamma^t_{,t} = e^t_{a,t}\gamma^a$, we obtain

$$0 = \int \left[\sqrt{|g|}\left(\left(\partial_t\Psi^\dagger\right)\gamma^0\gamma^t\Psi + \Psi^\dagger\gamma^0\gamma^t\left(\partial_t\Psi\right)\right) + \left(\left(\partial_t\sqrt{|g|}\right)\Psi^\dagger\gamma^0\gamma^t\Psi + \right.\right.$$
$$\left.\left. + \sqrt{|g|}\Psi^\dagger\gamma^0\left(\partial_t\gamma^t\right)\Psi\right)\right] dx^3 \quad \Rightarrow \quad 0 = i\left[(\mathcal{H}\Psi, \Psi) - (\Psi, \mathcal{H}\Psi)\right] +$$
$$+ \left(\Psi, \left[\partial_t\left(\ln\left(\sqrt{|g|g^{tt}}\right)\right)\Psi\right]\right) \ ,$$

and this explicitly proves that, in general, the Hamiltonian is not hermitian. We have used $\gamma^t\gamma^t = g^{tt}$, taken from (1.18).

To solve this problem, we introduce a rescaled wave function

$$\psi = \left(\sqrt{|g|}e^0_{\hat{0}}\right)^{\frac{1}{2}}\Psi \ , \tag{2.19}$$

and we observe that by substituting (2.19) in (2.18) we recover the Minkowskian inner product for spinor fields. From this non-unitary transformation, the Dirac equation that emerges in the Schrödinger form involves an hermitian Hamiltonian:

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi \quad , \quad \mathcal{H} = \beta mc^2 V + q\Phi + \frac{c}{2}\left(\pi_\beta\mathcal{F}^\beta_{\hat{\alpha}}\alpha^{\hat{\alpha}} + \alpha^{\hat{\alpha}}\mathcal{F}^\beta_{\hat{\alpha}}\pi_\beta\right) +$$
$$+\frac{c}{2}\left(\mathbf{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\mathbf{K}\right) + \frac{\hbar c}{4}\left(\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma} - \Upsilon\gamma_5\right) \ . \tag{2.20}$$

Here we used $\mathbf{K} = \{K^\alpha\}$ and the momentum operator $\boldsymbol{\pi} = \{\pi_\alpha\}$, where $\pi_\alpha = -i\hbar\partial_\alpha + qA_\alpha = p_\alpha + qA_\alpha$ takes account of the interaction with the electromagnetic field described by the 4-potential $A_\mu = (\Phi, A_\alpha)$. We also have, as usual, $\beta = \gamma^{\hat{0}}$, $\boldsymbol{\alpha} = \{\alpha^{\hat{\alpha}}\}$, $\boldsymbol{\Sigma} = \{\Sigma^{\hat{\alpha}}\}$ with $\alpha^{\hat{\alpha}} = \gamma^{\hat{0}}\gamma^{\hat{\alpha}}$, $\Sigma^{\hat{1}} = i\gamma^{\hat{2}}\gamma^{\hat{3}}$, $\Sigma^{\hat{2}} = i\gamma^{\hat{3}}\gamma^{\hat{1}}$ and $\Sigma^{\hat{3}} = i\gamma^{\hat{1}}\gamma^{\hat{2}}$. Finally, we introduced $\Upsilon = -V\epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}$ and $\boldsymbol{\Xi} = \{\Xi_{\hat{\alpha}}\}$, $\Xi_{\hat{\alpha}} = \epsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}}\mathcal{Q}^{\hat{\beta}\hat{\gamma}}$.

2.1.2 F-W transformation.

As we explained in the previous chapter, the physical content of the theory emerges in the Foldy-Wouthuysen representation. In the previous chapter we have reported this apparatus and derived the formulas to obtain the Dirac Hamiltonian in the F-W representation in an arbitrary gravitational field.

Now we apply these formulas to the current case [2] and, confronting (2.20) with (1.31), we deduce

$$\mathcal{M} = mc^2 V \ , \tag{2.21}$$

$$\mathcal{E} = q\Phi + \frac{c}{2}\left(\mathbf{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\mathbf{K}\right) + \frac{\hbar c}{4}\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma} \ , \tag{2.22}$$

$$\mathcal{O} = \frac{c}{2}\left(\pi_\beta \mathcal{F}^\beta_{\hat\alpha}\alpha^{\hat\alpha} + \alpha^{\hat\alpha}\mathcal{F}^\beta_{\hat\alpha}\pi_\beta\right) - \frac{\hbar c}{4}\Upsilon\gamma_5 \ . \tag{2.23}$$

We refer to (1.72) to find the F-W Hamiltonian, and to previous formulas to calculate the various terms. After lengthy algebra we obtain

$$\mathcal{H}_{FW} = \mathcal{H}^{(1)}_{FW} + \mathcal{H}^{(2)}_{FW} \ , \tag{2.24}$$

where the two terms are

$$\mathcal{H}^{(1)}_{FW} = \beta\epsilon' + \frac{\hbar c^2}{16}\left\{\frac{1}{\epsilon'}, \left(2\epsilon^{\hat\gamma\hat\alpha\hat\epsilon}\Pi_{\hat\epsilon}\left\{\pi_\beta, \mathcal{F}^\delta_{\hat\gamma}\partial_\delta\mathcal{F}^\beta_{\hat\alpha}\right\} + \Pi^{\hat\alpha}\left\{\pi_\beta, \mathcal{F}^\beta_{\hat\alpha}\Upsilon\right\}\right)\right\} +$$

$$+\frac{\hbar mc^4}{4}\epsilon^{\hat\gamma\hat\alpha\hat\epsilon}\Pi_{\hat\epsilon}\left\{\frac{1}{\mathcal{T}}, \left\{\pi_\delta, \mathcal{F}^\delta_{\hat\gamma}\mathcal{F}^\beta_{\hat\alpha}\partial_\beta V\right\}\right\} \ , \tag{2.25}$$

$$\mathcal{H}^{(2)}_{FW} = \frac{c}{2}\left(K^\alpha\pi_\alpha + \pi_\alpha K^\alpha\right) + \frac{\hbar c}{4}\Sigma_\alpha\Xi^\alpha + \frac{\hbar c^2}{16}\left\{\frac{1}{\mathcal{T}}, \left\{\Sigma_\alpha\left\{\pi_\epsilon, \mathcal{F}^\epsilon_\beta\right\}, \left\{\pi_\zeta, \left[\epsilon^{\alpha\beta\gamma}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left(\frac{1}{c}\dot{\mathcal{F}}^\zeta_\gamma - \mathcal{F}^\delta_\gamma\partial_\delta K^\zeta + K^\delta\partial_\delta\mathcal{F}^\zeta_\gamma\right) - \frac{1}{2}\mathcal{F}^\zeta_\delta\left(\delta^{\delta\beta}\Xi^\alpha - \delta^{\delta\alpha}\Xi^\beta\right)\right]\right\}\right\}\right\} \ , \tag{2.26}$$

where

$$\epsilon' = \sqrt{m^2c^4V^2 + \frac{c^2}{4}\delta^{\alpha\gamma}\left\{\pi_\beta, \mathcal{F}^\beta_\alpha\right\}\left\{\pi_\delta, \mathcal{F}^\delta_\gamma\right\}} \ , \ \mathcal{T} = 2\epsilon'^2 + \left\{\epsilon', mc^2V\right\} \ . \tag{2.27}$$

Now we can derive the equation of motion of spin. After introducing the polarization operator $\mathbf{\Pi} = \beta\mathbf{\Sigma}$, its equation of motion can be obtained by calculating the commutator with the Hamiltonian, that is

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}\left[\mathcal{H}_{FW}, \mathbf{\Pi}\right] = \mathbf{\Omega}_{(1)}\times\mathbf{\Sigma} + \mathbf{\Omega}_{(2)}\times\mathbf{\Pi} \ , \tag{2.28}$$

where $\mathbf{\Omega}_{(1)}$ and $\mathbf{\Omega}_{(2)}$ are the operators of the angular velocity of spin precessing in the exterior gravitational field, and they are calculated as follows:

$$\Omega^{\hat\alpha}_{(1)} = \frac{mc^4}{2}\left\{\frac{1}{\mathcal{T}}, \left\{\pi_\epsilon, \epsilon^{\hat\alpha\hat\beta\hat\gamma}\mathcal{F}^\epsilon_{\hat\beta}\mathcal{F}^\delta_{\hat\gamma}\partial_\delta V\right\}\right\} +$$

$$+\frac{c^2}{8}\left\{\frac{1}{\epsilon'}, \left\{\pi_\epsilon, \left(2\epsilon^{\hat\alpha\hat\beta\hat\gamma}\mathcal{F}^\delta_{\hat\beta}\partial_\delta\mathcal{F}^\epsilon_{\hat\gamma} + \delta^{\hat\alpha\hat\beta}\mathcal{F}^\epsilon_{\hat\beta}\Upsilon\right)\right\}\right\} \ , \tag{2.29}$$

$$\Omega^{\hat{\alpha}}_{(2)} = \frac{\hbar c^2}{8}\left\{\frac{1}{\mathcal{T}}, \left\{\left\{\pi_\epsilon, \mathcal{F}^\epsilon_{\hat{\beta}}\right\}, \left\{\pi_\zeta, \left(\epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\left(\frac{1}{c}\dot{\mathcal{F}}^\zeta_{\hat{\gamma}} - \mathcal{F}^\delta_{\hat{\gamma}}\partial_\delta K^\zeta + K^\delta\partial_\delta \mathcal{F}^\zeta_{\hat{\gamma}}\right) - \right.\right.\right.\right.$$
$$\left.\left.\left.\left. - \frac{1}{2}\mathcal{F}^\zeta_\delta\left(\delta^{\hat{\delta}\hat{\beta}}\Xi^{\hat{\alpha}} - \delta^{\hat{\delta}\hat{\alpha}}\Xi^{\hat{\beta}}\right)\right)\right\}\right\}\right\} + \frac{c}{2}\Xi^{\hat{\alpha}} \ . \tag{2.30}$$

We can now use these general results to derive the semiclassical expressions obtained by neglecting powers of $\hbar$ higher than 1; equations (2.28), (2.29) and (2.30) lead to the following equations, which describe the motion of the average spin $\mathbf{s}$:

$$\frac{d\mathbf{s}}{dt} = \left(\mathbf{\Omega}_{(1)} + \mathbf{\Omega}_{(2)}\right) \times \mathbf{s} \ , \tag{2.31}$$

$$\Omega^{\hat{\alpha}}_{(1)} = \frac{c^2}{\epsilon'}\mathcal{F}^\delta_{\hat{\gamma}}\pi_\delta\left(\frac{1}{2}\Upsilon\delta^{\hat{\alpha}\hat{\gamma}} - \epsilon^{\hat{\alpha}\hat{\epsilon}\hat{\chi}}V\mathcal{C}^{\hat{\gamma}}_{\hat{\epsilon}\hat{\chi}} + \frac{\epsilon'}{\epsilon' + mc^2V}\epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}}W^\epsilon_{\hat{\beta}}\partial_\epsilon V\right) \ , \tag{2.32}$$

$$\Omega^{\hat{\alpha}}_{(2)} = \frac{c}{2}\Xi^{\hat{\alpha}} - \frac{c^3}{\epsilon'\left(\epsilon' + mc^2V\right)}\epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}}Q_{\left(\hat{\beta}\hat{\delta}\right)}\delta^{\hat{\delta}\hat{\eta}}\mathcal{F}^\kappa_{\hat{\eta}}\pi_\kappa\mathcal{F}^\xi_{\hat{\gamma}}\pi_\xi \ . \tag{2.33}$$

In the semiclassical limit we have

$$\epsilon' = \sqrt{m^2c^4V^2 + c^2\delta^{\hat{\gamma}\hat{\delta}}\mathcal{F}^\alpha_{\hat{\gamma}}\mathcal{F}^\beta_{\hat{\delta}}\pi_\alpha\pi_\beta} \ . \tag{2.34}$$

We can now use these results in (2.24) to obtain the F-W Hamiltonian in terms of precession angular velocities:

$$\mathcal{H}_{FW} = \beta\epsilon' + \frac{c}{2}\left(\mathbf{K}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{K}\right) + \frac{\hbar}{2}\mathbf{\Pi}\cdot\mathbf{\Omega}_{(1)} + \frac{\hbar}{2}\mathbf{\Sigma}\cdot\mathbf{\Omega}_{(2)} \ . \tag{2.35}$$

By using this Hamiltonian, we can derive the velocity operators:

$$\frac{dx^\alpha}{dt} = \frac{i}{\hbar}\left[\mathcal{H}_{FW}, x^\alpha\right] = \beta\frac{\partial\epsilon'}{\partial\pi_\alpha} + cK^\alpha = \beta\frac{c^2}{\epsilon'}\mathcal{F}^\alpha_{\hat{\beta}}\delta^{\hat{\beta}\hat{\gamma}}\mathcal{F}^\delta_{\hat{\gamma}}\pi_\delta + cK^\alpha \ . \tag{2.36}$$

By comparing (2.36) with (2.7), we directly obtain:

$$\beta\frac{c^2}{\epsilon'}\mathcal{F}^\beta_{\hat{\alpha}}\pi_\beta = v_{\hat{\alpha}} \ . \tag{2.37}$$

From (2.37) we trivially obtain $\delta^{\hat{\gamma}\hat{\delta}}\mathcal{F}^\alpha_{\hat{\gamma}}\mathcal{F}^\beta_{\hat{\delta}}\pi_\alpha\pi_\beta = \left(\epsilon'\right)^2\frac{v^2}{c^2}$, and by substituting this identity in (2.34) we obtain

$$\left(\epsilon'\right)^2 = m^2c^4V^2 + \left(\epsilon'\right)^2\frac{v^2}{c^2} \ \Rightarrow \ \epsilon' = \gamma mc^2V \ , \tag{2.38}$$

where γ is the Lorentz factor $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. By using (2.38), we can do some manipulation to obtain useful results so as to simplify the expressions; in particular, we can use

$$\frac{\epsilon'}{\epsilon' + mc^2 V} = \frac{\gamma}{1+\gamma} \quad \text{and} \quad \frac{c^3}{\epsilon' \left(\epsilon' + mc^2 V\right)} \mathcal{F}^{\beta}_{\hat{\alpha}} \pi_{\beta} \mathcal{F}^{\delta}_{\hat{\gamma}} \pi_{\delta} = \frac{\gamma}{1+\gamma} \frac{v_{\hat{\alpha}} v_{\hat{\beta}}}{c} \; . \qquad (2.39)$$

Chapter 3

Cosmological applications.

In this chapter, our purpose is to consider specific cosmologies and to apply the methods developed in the previous chapters. We will then study the physical implications that emerge from the interaction between gravitational field and spin particles.

3.1 Melvin space-time.

The Melvin space-time is a solution of the Einstein-Maxwell equations that describes a universe filled with a cylindrical magnetic field [8] [19] [20]. Let's consider a static magnetic field the lines of which are perpendicular to a certain radial direction; this magnetic field falls as fast as $\frac{1}{r}$ far away from the symmetry axis. This magnetic field can be obtained by considering a four-potential A_μ, whose only non-vanishing component is $A_\phi = -\frac{B_0 r^2}{2f(r)}$, where $f(r)$ is a polynomial quadratic in r. In this section we describe this metric, we solve the geodesics equations and we solve the equation of motion of spin operators. Finally we describe the transition to the *Melvin Cosmology* [10], where the metric coefficients depend on a temporal coordinate, and describe the motion of polarisation operators.

Let's consider the Melvin space-time described by the line element

$$ds^2 = f^2(r)dt^2 - f^2(r)dr^2 - \frac{r^2}{f^2(r)}d\phi^2 - f^2(r)dz^2 \ , \tag{3.1}$$

where

$$f(r) = 1 + \frac{1}{4}B_0^2 r^2 \ , \tag{3.2}$$

and B_0 is a constant. Here we have set $c = 1$ for notation simplicity. For the sake of clarity, we will use $1, 2, 3, 4$ or t, r, ϕ, z interchangeably to represent indices. By using

$$g_{\mu\nu} = \begin{pmatrix} f^2(r) & 0 & 0 & 0 \\ 0 & -f^2(r) & 0 & 0 \\ 0 & 0 & -\frac{r^2}{f^2(r)} & 0 \\ 0 & 0 & 0 & -f^2(r) \end{pmatrix} \text{ and}$$

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.3}$$

in (1.5), it is easy to obtain $e_0^{\hat{0}} = f(r), e_0^{\hat{1}} = f(r), e_2^{\hat{2}} = \dfrac{r}{f(r)}, e_3^{\hat{3}} = f(r)$, therefore

$$e_\mu^a = \begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & f(r) & 0 & 0 \\ 0 & 0 & \frac{r}{f(r)} & 0 \\ 0 & 0 & 0 & f(r) \end{pmatrix} \tag{3.4}$$

and its inverse

$$e_a^\mu = \begin{pmatrix} \frac{1}{f(r)} & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & \frac{f(r)}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{f(r)} \end{pmatrix} . \tag{3.5}$$

Now we compare these expressions with the parametrisation (2.2) in the Schwinger gauge in order to obtain the metric coefficients expressed with the parametrisation used in the previous chapter:

$$e_0^{\hat{0}} = f(r) = V \quad ;$$
$$e_1^{\hat{1}} = f(r) = W_\beta^{\hat{1}} \left(\delta_1^\beta - K^\beta \delta_1^0\right) = W_1^{\hat{1}} \;\Rightarrow\; W_1^{\hat{1}} = f(r) = V \quad ;$$
$$e_2^{\hat{2}} = \frac{r}{f(r)} = W_\beta^{\hat{2}} \left(\delta_2^\beta - K^\beta \delta_2^0\right) = W_2^{\hat{2}} \;\Rightarrow W_2^{\hat{2}} = \frac{r}{f(r)} = \frac{r}{V} \quad ;$$
$$e_3^{\hat{3}} = f(r) = W_\beta^{\hat{3}} \left(\delta_3^\beta - K^\beta \delta_3^0\right) = W_3^{\hat{3}} \;\Rightarrow\; W_3^{\hat{3}} = f(r) = V \quad ;$$
$$e_2^{\hat{1}} = 0 = W_\beta^{\hat{1}} \left(\delta_2^\beta - K^\beta \delta_2^0\right) = W_2^{\hat{1}} \;\Rightarrow\; W_2^{\hat{1}} = 0 \quad ;$$
$$e_3^{\hat{1}} = 0 = W_\beta^{\hat{1}} \left(\delta_3^\beta - K^\beta \delta_3^0\right) = W_3^{\hat{1}} \;\Rightarrow\; W_3^{\hat{1}} = 0 \quad ;$$
$$e_0^{\hat{1}} = 0 = W_\beta^{\hat{1}} \left(\delta_0^\beta - K^\beta \delta_0^0\right) = -W_1^{\hat{1}} K^1 - W_2^{\hat{1}} K^2 - W_3^{\hat{1}} K^3 = 0 \;\Rightarrow\; K^1 = 0 \quad ;$$
$$e_1^{\hat{2}} = 0 = W_\beta^{\hat{2}} \left(\delta_1^\beta - K^\beta \delta_1^0\right) = W_1^{\hat{2}} \;\Rightarrow\; W_1^{\hat{2}} = 0 \quad ;$$

$$e^{\hat{2}}_{3} = 0 = W^{\hat{2}}_{\beta}\left(\delta^{\beta}_{3} - K^{\beta}\delta^{0}_{3}\right) = W^{\hat{2}}_{3} \;\Rightarrow\; W^{\hat{2}}_{3} = 0 \;\;;$$
$$e^{\hat{2}}_{0} = 0 = W^{\hat{2}}_{\beta}\left(\delta^{\beta}_{0} - K^{\beta}\delta^{0}_{0}\right) = -W^{\hat{2}}_{1}K^{1} - W^{\hat{2}}_{2}K^{2} - W^{\hat{2}}_{3}K^{3} = 0 \;\Rightarrow\; K^{2} = 0 \;\;;$$
$$e^{\hat{3}}_{1} = 0 = W^{\hat{3}}_{\beta}\left(\delta^{\beta}_{1} - K^{\beta}\delta^{0}_{1}\right) = W^{\hat{3}}_{1} \;\Rightarrow\; W^{\hat{3}}_{1} = 0 \;\;;$$
$$e^{\hat{3}}_{2} = 0 = W^{\hat{3}}_{\beta}\left(\delta^{\beta}_{2} - K^{\beta}\delta^{0}_{2}\right) = W^{\hat{3}}_{2} \;\Rightarrow\; W^{\hat{3}}_{2} = 0 \;\;;$$
$$e^{\hat{3}}_{0} = 0 = W^{\hat{3}}_{\beta}\left(\delta^{\beta}_{0} - K^{\beta}\delta^{0}_{0}\right) = -W^{\hat{3}}_{1}K^{1} - W^{\hat{3}}_{2}K^{2} - W^{\hat{3}}_{3}K^{3} = 0 \;\Rightarrow\; K^{3} = 0 \;\;.$$

So we have obtained

$$W^{\hat{\alpha}}_{\beta} = \begin{pmatrix} f(r) & 0 & 0 \\ 0 & \frac{r}{f(r)} & 0 \\ 0 & 0 & f(r) \end{pmatrix} = \begin{pmatrix} V & 0 & 0 \\ 0 & \frac{r}{V} & 0 \\ 0 & 0 & V \end{pmatrix} \tag{3.6}$$

and

$$K^{\alpha} = 0 \;, \alpha = 1, 2, 3 \;\;. \tag{3.7}$$

3.1.1 Calculation of angular velocity operators.

In order to continue our calculations let's refer to formulas (2.12) $\div$ (2.15). We first calculate $\mathcal{Q}_{\hat{\alpha}\hat{\beta}}$:

$$\mathcal{Q}_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\gamma}} W^{\delta}_{\hat{\beta}}\left(\dot{W}^{\hat{\gamma}}_{\delta} + K^{\epsilon}\partial_{\epsilon}W^{\hat{\gamma}}_{\delta} + W^{\hat{\gamma}}_{\epsilon}\partial_{\delta}K^{\epsilon}\right) = 0 \;. \tag{3.8}$$

Really, the first term between brackets vanishes because $W^{\hat{\gamma}}_{\delta}$ is not dependant on t, and the second and the third terms vanish because $K^{\epsilon} = 0, \forall\epsilon = 1, 2, 3$. Now it is easy to calculate the components of the spin connection. From (3.8) we can directly obtain:

$$\tilde{\omega}_{\mu\hat{\alpha}\hat{0}} = \frac{1}{V}W^{\beta}_{\hat{\alpha}}\partial_{\beta}V e^{\hat{0}}_{\mu} - \frac{1}{V}\mathcal{Q}_{(\hat{\alpha}\hat{\beta})}e^{\hat{\beta}}_{\mu} = \frac{1}{V}\left(W^{r}_{\hat{\alpha}}\partial_{r}V e^{\hat{0}}_{\mu} + W^{\phi}_{\hat{\alpha}}\partial_{\phi}V e^{\hat{0}}_{\mu} + \right.$$
$$\left. + W^{z}_{\hat{\alpha}}\partial_{z}V e^{\hat{0}}_{\mu}\right) = \frac{1}{V}W^{r}_{\hat{\alpha}}\partial_{r}V e^{\hat{0}}_{\mu} \;, \tag{3.9}$$

which, explicitly, is

$$\text{for } \mu = 0 \;\Rightarrow\; \tilde{\omega}_{0\hat{\alpha}\hat{0}} = \frac{1}{V}W^{r}_{\hat{\alpha}}\left(\partial_{r}V\right)V = W^{r}_{\hat{\alpha}}\left(\partial_{r}V\right) \;\Rightarrow$$

$$\Rightarrow\; \tilde{\omega}_{0\hat{\alpha}\hat{0}} = \begin{cases} W^{r}_{r}\left(\partial_{r}V\right) = \frac{1}{V}\partial_{r}V = \partial_{r}\ln\left(V\right) \;, & \text{if } \hat{\alpha} = r \;, \\ 0 \;, & \text{if } \hat{\alpha} = \phi, z \;; \end{cases} \tag{3.10}$$

$$\text{for } \mu = \beta \;\Rightarrow\; \tilde{\omega}_{\beta\hat{\alpha}\hat{0}} = 0 \;,\; \beta = 1, 2, 3. \tag{3.11}$$

In order to proceed, we calculate the anholonimity coefficients:

$$\mathcal{C}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}} = W^{\delta}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \partial_{[\delta} W^{\hat{\gamma}}_{\epsilon]} = \frac{1}{2} W^{\delta}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \left(\partial_{\delta} W^{\hat{\gamma}}_{\epsilon} - \partial_{\epsilon} W^{\hat{\gamma}}_{\delta} \right) =$$

$$= \frac{1}{2} \left[W^{r}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \left(\partial_{r} W^{\hat{\gamma}}_{\epsilon} - \partial_{\epsilon} W^{\hat{\gamma}}_{r} \right) - W^{\phi}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \left(\partial_{\epsilon} W^{\hat{\gamma}}_{\phi} \right) - W^{z}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \left(\partial_{\epsilon} W^{\hat{\gamma}}_{z} \right) \right] =$$

$$= \frac{1}{2} \left[W^{r}_{\hat{\alpha}} W^{r}_{\hat{\beta}} \left(\partial_{r} W^{\hat{\gamma}}_{r} - \partial_{r} W^{\hat{\gamma}}_{r} \right) + W^{r}_{\hat{\alpha}} W^{\phi}_{\hat{\beta}} \left(\partial_{r} W^{\hat{\gamma}}_{\phi} \right) + W^{r}_{\hat{\alpha}} W^{z}_{\hat{\beta}} \left(\partial_{r} W^{\hat{\gamma}}_{z} \right) - \right.$$

$$\left. - W^{\phi}_{\hat{\alpha}} W^{r}_{\hat{\beta}} \left(\partial_{r} W^{\hat{\gamma}}_{\phi} \right) - W^{z}_{\hat{\alpha}} W^{r}_{\hat{\beta}} \left(\partial_{r} W^{\hat{\gamma}}_{z} \right) \right] =$$

$$= W^{r}_{[\hat{\alpha}} W^{\phi}_{\hat{\beta}]} \left(\partial_{r} W^{\hat{\gamma}}_{\phi} \right) + W^{r}_{[\hat{\alpha}} W^{z}_{\hat{\beta}]} \left(\partial_{r} W^{\hat{\gamma}}_{z} \right) \ , \tag{3.12}$$

which, explicitly, is:

$$\text{for } \hat{\gamma} = \hat{r} \ \Rightarrow \ \mathcal{C}^{\hat{r}}_{\hat{\alpha}\hat{\beta}} = 0 \tag{3.13}$$

because $W^{\hat{\alpha}}_{\beta}$ is diagonal;

$$\text{for } \hat{\gamma} = \hat{\phi} \ \Rightarrow \ \mathcal{C}^{\hat{\phi}}_{\hat{\alpha}\hat{\beta}} = W^{r}_{[\hat{\alpha}} W^{\phi}_{\hat{\beta}]} \left(\partial_{r} W^{\hat{\phi}}_{\phi} \right) = W^{r}_{[\hat{\alpha}} W^{\phi}_{\hat{\beta}]} \frac{V - rV'}{V^2} =$$

$$= \begin{pmatrix} 0 & \frac{V-rV'}{2rV^2} & 0 \\ -\frac{V-rV'}{2rV^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ; \tag{3.14}$$

$$\text{for } \hat{\gamma} = \hat{z} \ \Rightarrow \ \mathcal{C}^{\hat{z}}_{\hat{\alpha}\hat{\beta}} = W^{r}_{[\hat{\alpha}} W^{z}_{\hat{\beta}]} \left(\partial_{r} W^{\hat{z}}_{z} \right) = W^{r}_{[\hat{\alpha}} W^{z}_{\hat{\beta}]} V' =$$

$$= \begin{pmatrix} 0 & 0 & \frac{V'}{2V^2} \\ 0 & 0 & 0 \\ -\frac{V'}{2V^2} & 0 & 0 \end{pmatrix} \ . \tag{3.15}$$

Here we used $'$ to indicate the derivative with respect to r. By using $\eta_{\hat{\gamma}\hat{\delta}} = -\delta_{\hat{\gamma}\hat{\delta}}$, we lower the $\hat{\gamma}$ index and obtain

$$\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{r}} = 0 \ , \ \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\phi}} = -\mathcal{C}^{\hat{\phi}}_{\hat{\alpha}\hat{\beta}} \ , \ \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{z}} = -\mathcal{C}^{\hat{z}}_{\hat{\alpha}\hat{\beta}} \ . \tag{3.16}$$

Now we use (3.15) and (3.16) to calculate (2.13):

$$\tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}} = \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + \mathcal{C}_{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \mathcal{C}_{\hat{\gamma}\hat{\beta}\hat{\alpha}} \right) e^{\hat{\gamma}}_{\mu} \tag{3.17}$$

which, explicitly, is

$$\text{for } \mu = 0 \ \Rightarrow \ \tilde{\omega}_{0\hat{\alpha}\hat{\beta}} = 0 \ ; \tag{3.18}$$

$$\text{for } \mu = r \;\Rightarrow\; \tilde{\omega}_{r\hat{\alpha}\hat{\beta}} = \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{r}} + \mathcal{C}_{\hat{\alpha}\hat{r}\hat{\beta}} + \mathcal{C}_{\hat{r}\hat{\beta}\hat{\alpha}}\right) V =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \tag{3.19}$$

$$\text{for } \mu = \phi \;\Rightarrow\; \tilde{\omega}_{\phi\hat{\alpha}\hat{\beta}} = \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\phi}} + \mathcal{C}_{\hat{\alpha}\hat{\phi}\hat{\beta}} + \mathcal{C}_{\hat{\phi}\hat{\beta}\hat{\alpha}}\right) \frac{r}{V} =$$

$$= \begin{pmatrix} 0 & -\frac{V-rV'}{V^3} & 0 \\ \frac{V-rV'}{V^3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \tag{3.20}$$

$$\text{for } \mu = z \;\Rightarrow\; \tilde{\omega}_{z\hat{\alpha}\hat{\beta}} = \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{z}} + \mathcal{C}_{\hat{\alpha}\hat{z}\hat{\beta}} + \mathcal{C}_{\hat{z}\hat{\beta}\hat{\alpha}}\right) V =$$

$$= \begin{pmatrix} 0 & 0 & -\frac{V'}{V} \\ 0 & 0 & 0 \\ \frac{V'}{V} & 0 & 0 \end{pmatrix} \; . \tag{3.21}$$

Starting from what we obtained, we want to calculate all the terms of the Hamiltonian (2.20), therefore we have:

$$\mathcal{F}^{\beta}_{\hat{\alpha}} = V W^{\beta}_{\hat{\alpha}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{V^2}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \; , \tag{3.22}$$

$$\Xi_{\hat{\alpha}} = 0 \; , \tag{3.23}$$

$$\Upsilon = -V \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = -V \left(\epsilon^{\hat{r}\hat{\phi}\hat{z}} \mathcal{C}_{\hat{r}\hat{\phi}\hat{z}} + \epsilon^{\hat{r}\hat{z}\hat{\phi}} \mathcal{C}_{\hat{r}\hat{z}\hat{\phi}} + \epsilon^{\hat{\phi}\hat{r}\hat{z}} \mathcal{C}_{\hat{\phi}\hat{r}\hat{z}} + \right.$$

$$\left. + \epsilon^{\hat{\phi}\hat{z}\hat{r}} \mathcal{C}_{\hat{\phi}\hat{z}\hat{r}} + \epsilon^{\hat{z}\hat{r}\hat{\phi}} \mathcal{C}_{\hat{z}\hat{r}\hat{\phi}} + \epsilon^{\hat{z}\hat{\phi}\hat{r}} \mathcal{C}_{\hat{z}\hat{\phi}\hat{r}} \right) = 0 \; . \tag{3.24}$$

By using these results as a replacement in (2.21) $\div$ (2.23), we obtain:

$$\mathcal{M} = mV \; , \tag{3.25}$$

$$\mathcal{E} = q\phi \; , \tag{3.26}$$

$$\mathcal{O} = \frac{1}{2} \left(\pi_r \alpha^r + \frac{V^2}{r} \pi_\phi \alpha^\phi + \pi_z \alpha^z + \alpha^r \pi_r + \frac{V^2}{r} \alpha^\phi \pi_\phi + \alpha^z \pi_z \right) =$$

$$= \alpha^r \pi_r + \frac{V^2}{r} \alpha^\phi \pi_\phi + \alpha^z \pi_z \; . \tag{3.27}$$

Now we want to calculate the equation of motion of spin, so we have to calculate the operators of the angular velocity of spin precession (2.29) and (2.30); we have:

$$\Omega^{\hat{\alpha}}_{(1)} = \frac{m}{2}\left\{\frac{1}{\mathcal{T}}, \left\{\pi_\epsilon, \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\mathcal{F}^{\epsilon}_{\hat{\beta}}\mathcal{F}^{\delta}_{\hat{\gamma}}\partial_\delta V\right\}\right\} +$$

$$+\frac{1}{8}\left\{\frac{1}{\epsilon'}, \left\{\pi_\epsilon, \left(2\epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\mathcal{F}^{\delta}_{\hat{\beta}}\partial_\delta\mathcal{F}^{\epsilon}_{\hat{\gamma}} + \delta^{\hat{\alpha}\hat{\beta}}\mathcal{F}^{\epsilon}_{\hat{\beta}}\Upsilon\right)\right\}\right\} =$$

$$= \frac{m}{2}\left\{\frac{1}{\mathcal{T}}, \left\{\pi_\epsilon, \epsilon^{\hat{\alpha}\hat{\beta}\hat{r}}\mathcal{F}^{\epsilon}_{\hat{\beta}}\partial_r V\right\}\right\} + \frac{1}{8}\left\{\frac{1}{\epsilon'}, \left\{\pi_\epsilon, 2\epsilon^{\hat{\alpha}\hat{r}\hat{\gamma}}\partial_r\mathcal{F}^{\epsilon}_{\hat{\gamma}}\right\}\right\} , \tag{3.28}$$

$$\Omega^{\hat{\alpha}}_{(2)} = 0 , \tag{3.29}$$

having relied on the fact that the only derivatives different from 0 are those with respect to r, and on the equations (3.23) and (3.24). To proceed with the calculations for (3.28) we separately work out the two internal anticommutators:

$$\left\{\pi_\epsilon, \epsilon^{\hat{\alpha}\hat{\beta}\hat{r}}\mathcal{F}^{\epsilon}_{\hat{\beta}}\partial_r V\right\} = \left\{\pi_r, \epsilon^{\hat{\alpha}\hat{\beta}\hat{r}}\mathcal{F}^{r}_{\hat{\beta}}\partial_r V\right\} + \left\{\pi_\phi, \epsilon^{\hat{\alpha}\hat{\beta}\hat{r}}\mathcal{F}^{\phi}_{\hat{\beta}}\partial_r V\right\} +$$

$$+\left\{\pi_z, \epsilon^{\hat{\alpha}\hat{\beta}\hat{r}}\mathcal{F}^{z}_{\hat{\beta}}\partial_r V\right\} = \left\{\pi_\phi, \epsilon^{\hat{\alpha}\hat{\phi}\hat{r}}\frac{V^2}{r}\partial_r V\right\} + \left\{\pi_z, \epsilon^{\hat{\alpha}\hat{z}\hat{r}}\partial_r V\right\} , \tag{3.30}$$

$$\left\{\pi_\epsilon, 2\epsilon^{\hat{\alpha}\hat{r}\hat{\gamma}}\partial_r\mathcal{F}^{\epsilon}_{\hat{\gamma}}\right\} = \left\{\pi_\phi, 2\epsilon^{\hat{\alpha}\hat{r}\hat{\phi}}\partial_r\frac{V^2}{r}\right\} . \tag{3.31}$$

Now we can calculate the three components of the operator $\Omega_{(1)}$:

$$\Omega^{\hat{r}}_{(1)} = 0 , \tag{3.32}$$

$$\Omega^{\hat{\phi}}_{(1)} = \frac{m}{2}\left\{\frac{1}{\mathcal{T}}, \{\pi_z, V'\}\right\} , \tag{3.33}$$

$$\Omega^{\hat{z}}_{(1)} = \frac{m}{2}\left\{\frac{1}{\mathcal{T}}, \left\{\pi_\phi, -\frac{V^2}{r}V'\right\}\right\} + \frac{1}{8}\left\{\frac{1}{\epsilon'}, \left\{\pi_\phi, 2\frac{2VV'r - V^2}{r^2}\right\}\right\} . \tag{3.34}$$

We observe that the energy ϵ' assumes the same form as in (2.34), since we have

$$\left\{\pi_\beta, \mathcal{F}^{\beta}_{\hat{\alpha}}\right\} = 2\mathcal{F}^{\beta}_{\hat{\alpha}}\pi_\beta ,$$

because $\mathcal{F}^{\phi}_{\hat{\phi}}$ is the only non constant element, but in this case it is associated with π_ϕ and they do commute. So we have

$$\epsilon' = \sqrt{m^2V^2 + \delta^{\hat{\gamma}\hat{\delta}}\mathcal{F}^{\alpha}_{\hat{\gamma}}\mathcal{F}^{\beta}_{\hat{\delta}}\pi_\alpha\pi_\beta}$$

and in particular (2.37) holds. Now we work out each term to determine the angular velocity operators:

$$\mathcal{T} = 2\epsilon'^2 + \{\epsilon', mV\} = 2\epsilon'^2 + 2\epsilon' mV = 2\epsilon'^2\left(1+\frac{1}{\gamma}\right) = 2\epsilon'^2\frac{1+\gamma}{\gamma} \ ,$$

$$\left\{\frac{1}{\mathcal{T}}, \{\pi_z, V'\}\right\} = \left\{\frac{1}{\mathcal{T}}, 2V'\pi_z\right\} = 4\frac{V'\pi_z}{\mathcal{T}} = 2V'\frac{\gamma}{1+\gamma}\frac{\pi_z}{\epsilon'^2} \quad \Rightarrow$$

$$\Rightarrow \quad \Omega^{\hat{\phi}}_{(1)} = mV'\frac{\gamma}{1+\gamma}\frac{\pi_z}{\epsilon'^2} = \frac{V'}{V}\frac{1}{1+\gamma}\frac{\pi_z}{\epsilon'} \ ; \tag{3.35}$$

$$\left\{\frac{1}{\mathcal{T}}, \left\{\pi_\phi, -\frac{V'V^2}{r}\right\}\right\} = \left\{\frac{1}{\mathcal{T}}, -2\frac{V'V^2}{r}\pi_\phi\right\} = -4\frac{V'V^2}{r}\frac{\pi_\phi}{\mathcal{T}} =$$

$$= -2\frac{V'V^2}{r}\frac{\gamma}{1+\gamma}\frac{\pi_\phi}{\epsilon'^2} \ ,$$

$$\left\{\frac{1}{\epsilon'}, \left\{\pi_\phi, 2\frac{2VV'r - V^2}{r^2}\right\}\right\} = \left\{\frac{1}{\epsilon'}, 4\frac{2VV'r - V^2}{r^2}\pi_\phi\right\} = 8\frac{2VV'r - V^2}{r^2}\frac{\pi_\phi}{\epsilon'} \ ,$$

$$\Rightarrow \quad \Omega^{\hat{z}}_{(1)} = -m\frac{V'V^2}{r}\frac{\gamma}{1+\gamma}\frac{\pi_\phi}{\epsilon'^2} + \frac{2VV'r - V^2}{r^2}\frac{\pi_\phi}{\epsilon'} =$$

$$= -\frac{V'V}{r}\frac{1}{1+\gamma}\frac{\pi_\phi}{\epsilon'} + \frac{2VV'r - V^2}{r^2}\frac{\pi_\phi}{\epsilon'} =$$

$$= \frac{VV'r - V^2 + 2\gamma VV'r - \gamma V^2}{r^2}\frac{1}{1+\gamma}\frac{\pi_\phi}{\epsilon'} \ . \tag{3.36}$$

Now we can use (2.37) to replace the momenta π_α with the velocities v_α, and we obtain:

$$v_{\hat{r}} = \beta\mathcal{F}^{\beta}_{\hat{r}}\frac{\pi_\beta}{\epsilon'} = \beta\frac{\pi_{\hat{r}}}{\epsilon'} \ ,$$

$$v_{\hat{\phi}} = \beta\mathcal{F}^{\beta}_{\hat{\phi}}\frac{\pi_\beta}{\epsilon'} = \beta\frac{V^2}{r}\frac{\pi_{\hat{\phi}}}{\epsilon'} \ ,$$

$$v_{\hat{z}} = \beta\mathcal{F}^{\beta}_{\hat{z}}\frac{\pi_\beta}{\epsilon'} = \beta\frac{\pi_{\hat{z}}}{\epsilon'} \ ,$$

and by substituting in (3.35) and (3.36), we obtain:

$$\Omega^{\hat{\phi}}_{(1)} = \beta\frac{V'}{V}\frac{1}{1+\gamma}v_{\hat{z}} \ , \tag{3.37}$$

$$\Omega^{\hat{z}}_{(1)} = \beta\frac{V'r - V + 2\gamma V'r - \gamma V}{rV}\frac{1}{1+\gamma}v_{\hat{\phi}} \ . \tag{3.38}$$

So (3.37) and (3.38) are the only angular velocity operators different from 0, and we will use this to solve the equation of motion of spin operators.

3.1.2 Semiclassical approximation.

It is interesting to observe that if we were to use the semiclassical formulas introduced in the previous chapter, we would obtain the same result. In this section we will prove this correspondence to be true: starting from (2.32) and (2.33) we can work out the angular velocity operators, and from (2.34) we obtain the energy, that can be expressed using the γ factor as shown in (2.38). In this semiclassical limit we have:

$$\Omega^{\hat{\alpha}}_{(2)} = 0 \ , \tag{3.39}$$

for the same reason as in the exact case, and

$$\begin{aligned}\Omega^{\hat{\alpha}}_{(1)} &= \frac{1}{\epsilon'}\mathcal{F}^{\delta}_{\hat{\gamma}}\pi_\delta\left(-\epsilon^{\hat{\alpha}\hat{\epsilon}\hat{\chi}}V\mathcal{C}^{\hat{\gamma}}_{\hat{\epsilon}\chi} + \frac{\gamma}{1+\gamma}\epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}}W^{\epsilon}_{\hat{\beta}}\partial_\epsilon V\right) = \\ &= -\frac{1}{\epsilon'}\left(\mathcal{F}^{r}_{\hat{\gamma}}\pi_r + \mathcal{F}^{\phi}_{\hat{\gamma}}\pi_\phi + \mathcal{F}^{z}_{\hat{\gamma}}\pi_z\right)\epsilon^{\hat{\alpha}\hat{\epsilon}\hat{\chi}}V\mathcal{C}^{\hat{\gamma}}_{\hat{\epsilon}\chi} + \frac{1}{\epsilon'}\left(\mathcal{F}^{r}_{\hat{\gamma}}\pi_r + \right. \\ &\quad \left. + \mathcal{F}^{\phi}_{\hat{\gamma}}\pi_\phi + \mathcal{F}^{z}_{\hat{\gamma}}\pi_z\right)\frac{\gamma}{1+\gamma}\epsilon^{\hat{\alpha}\hat{r}\hat{\gamma}}\frac{V'}{V} \ .\end{aligned} \tag{3.40}$$

Now we separately work out the two terms appearing in (3.40):

$$\begin{aligned}1)\quad & -\frac{1}{\epsilon'}\pi_r\epsilon^{\hat{\alpha}\hat{\epsilon}\hat{\chi}}V\mathcal{C}^{\hat{r}}_{\hat{\epsilon}\hat{\chi}} - \frac{1}{\epsilon'}\frac{V^2}{r}\pi_\phi\epsilon^{\hat{\alpha}\hat{\epsilon}\hat{\chi}}V\mathcal{C}^{\hat{\phi}}_{\hat{\epsilon}\hat{\chi}} - \frac{1}{\epsilon'}\pi_z\epsilon^{\hat{\alpha}\hat{\epsilon}\hat{\chi}}V\mathcal{C}^{\hat{z}}_{\hat{\epsilon}\hat{\chi}} = \\ &= -\frac{1}{\epsilon}\frac{V^2}{r}\pi_\phi\left(\epsilon^{\hat{\alpha}\hat{r}\hat{\phi}}V\frac{V-rV'}{2rV^2} - \epsilon^{\hat{\alpha}\hat{\phi}\hat{r}}V\frac{V-rV'}{2rV^2}\right) - \\ &\quad -\frac{1}{\epsilon'}\pi_z\left(\epsilon^{\hat{\alpha}\hat{r}\hat{z}}V\frac{V'}{2V} - \epsilon^{\hat{\alpha}\hat{z}\hat{r}}V\frac{V'}{2V}\right) \ ;\end{aligned} \tag{3.41}$$

$$\begin{aligned}2)\quad & \frac{1}{\epsilon'}\pi_r\frac{\gamma}{1+\gamma}\epsilon^{\hat{\alpha}\hat{r}\hat{r}}\frac{V'}{V} + \frac{1}{\epsilon'}\frac{V^2}{r}\pi_\phi\frac{\gamma}{1+\gamma}\epsilon^{\hat{\alpha}\hat{r}\hat{\phi}}\frac{V'}{V} + \frac{1}{\epsilon'}\pi_z\frac{\gamma}{1+\gamma}\epsilon^{\hat{\alpha}\hat{r}\hat{z}}\frac{V'}{V} = \\ &= \frac{1}{\epsilon'}\frac{V^2}{r}\pi_\phi\frac{\gamma}{1+\gamma}\epsilon^{\hat{\alpha}\hat{r}\hat{\phi}}\frac{V'}{V} + \frac{1}{\epsilon'}\pi_z\frac{\gamma}{1+\gamma}\epsilon^{\hat{\alpha}\hat{r}\hat{z}}\frac{V'}{V} \ .\end{aligned} \tag{3.42}$$

By putting these results together, and keeping in mind the definition of the Levi-Civita symbols, we obtain:

$$\Omega^{\hat{r}}_{(1)} = 0 \ , \tag{3.43}$$

$$\Omega^{\hat{\phi}}_{(1)} = -\frac{1}{\epsilon'}\pi_z\left(-\frac{V'}{2V} - \frac{V'}{2V}\right) - \frac{1}{\epsilon'}\pi_z\frac{\gamma}{1+\gamma}\frac{V'}{V} =$$

$$= \frac{1}{\epsilon'}\pi_z \left(\frac{V'}{V} - \frac{\gamma V'}{(1+\gamma)\,V}\right) = \frac{1}{\epsilon'}\pi_z \frac{V'}{(1+\gamma)\,V} \ , \tag{3.44}$$

$$\Omega^{\hat{z}}_{(1)} = -\frac{1}{\epsilon'}\frac{V^2}{r}\pi_\phi \left(\frac{V-rV'}{2rV} + \frac{V-rV'}{2rV}\right) + \frac{1}{\epsilon'}\frac{V^2}{r}\pi_\phi \frac{\gamma}{1+\gamma}\frac{V'}{V} =$$

$$= -\frac{1}{\epsilon'}\frac{V^2}{r}\pi_\phi \frac{V - rV' + \gamma V - 2\gamma r V'}{(1+\gamma)\,rV} \ . \tag{3.45}$$

(3.43) is equal to the corresponding exact term (3.32), and if we rewrite (3.44) and (3.45) in terms of the velocities we obtain

$$\Omega^{\hat{\phi}}_{(1)} = v_{\hat{z}}\frac{V'}{V}\frac{1}{1+\gamma} \ , \tag{3.46}$$

$$\Omega^{\hat{z}}_{(1)} = -v_{\hat{\phi}}\frac{V - rV' + \gamma V - 2\gamma r V'}{(1+\gamma)\,rV} \ . \tag{3.47}$$

So, the angular velocities calculated in the semiclassical approximation reproduce the exact operators as anticipated, the only difference being the β matrix that appears only in the operators (3.37) and (3.38).

3.1.3 Geodesics equations: direct approach.

The equations of motion of the polarization operator (2.28) are

$$\frac{d\mathbf{\Pi}}{dt} = \mathbf{\Omega}_{(1)} \times \mathbf{\Sigma} + \mathbf{\Omega}_{(2)} \times \mathbf{\Pi} = \mathbf{\Omega}_{(1)} \times \mathbf{\Sigma} \ , \tag{3.48}$$

because $\mathbf{\Omega}_{(2)} = 0$. By rewriting (3.39) using the component notation, we have

$$\frac{d\Pi^{\hat{\alpha}}}{dt} = \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\Omega_{(1)\hat{\beta}}\Sigma_{\hat{\gamma}} = \epsilon^{\hat{\alpha}\hat{\beta}\hat{r}}\Omega_{(1)\hat{\beta}}\Sigma_{\hat{r}} + \epsilon^{\hat{\alpha}\hat{\beta}\hat{\phi}}\Omega_{(1)\hat{\beta}}\Sigma_{\hat{\phi}} + \epsilon^{\hat{\alpha}\hat{\beta}\hat{z}}\Omega_{(1)\hat{\beta}}\Sigma_{\hat{z}} =$$

$$= \epsilon^{\hat{\alpha}\hat{\phi}\hat{r}}\Omega_{(1)\hat{\phi}}\Sigma_{\hat{r}} + \epsilon^{\hat{\alpha}\hat{z}\hat{r}}\Omega_{(1)\hat{z}}\Sigma_{\hat{r}} + \epsilon^{\hat{\alpha}\hat{r}\hat{\phi}}\Omega_{(1)\hat{r}}\Sigma_{\hat{\phi}} + \epsilon^{\hat{\alpha}\hat{z}\hat{\phi}}\Omega_{(1)\hat{z}}\Sigma_{\hat{\phi}} + \epsilon^{\hat{\alpha}\hat{r}\hat{z}}\Omega_{(1)\hat{r}}\Sigma_{\hat{z}} +$$

$$+\epsilon^{\hat{\alpha}\hat{\phi}\hat{z}}\Omega_{(1)\hat{\phi}}\Sigma_{\hat{z}} = \epsilon^{\hat{\alpha}\hat{\phi}\hat{r}}\Omega_{(1)\hat{\phi}}\Sigma_{\hat{r}} + \epsilon^{\hat{\alpha}\hat{z}\hat{r}}\Omega_{(1)\hat{z}}\Sigma_{\hat{r}} + \epsilon^{\hat{\alpha}\hat{z}\hat{\phi}}\Omega_{(1)\hat{z}}\Sigma_{\hat{\phi}} + \epsilon^{\hat{\alpha}\hat{\phi}\hat{z}}\Omega_{(1)\hat{\phi}}\Sigma_{\hat{z}} \ , \tag{3.49}$$

having used (3.32). By making (4.39) explicit:

$$\frac{d\Pi^{\hat{r}}}{dt} = \Omega_{(1)\hat{\phi}}\Sigma_{\hat{z}} - \Omega_{(1)\hat{z}}\Sigma_{\hat{\phi}} \ , \tag{3.50}$$

$$\frac{d\Pi^{\hat{\phi}}}{dt} = \Omega_{(1)\hat{z}}\Sigma_{\hat{r}} \ , \tag{3.51}$$

$$\frac{d\Pi^{\hat{z}}}{dt} = -\Omega_{(1)\hat{\phi}}\Sigma_{\hat{r}} \ , \tag{3.52}$$

having used the antisymmetry property of ϵ and (3.32) again. In operators (3.37) and (3.38) the velocities of a particle appear explicitly, so we need to find an expression for the velocities of a particle moving in a Melvin universe by solving the geodesics equations [9]:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \ . \tag{3.53}$$

First of all, we work out the Christoffel symbols:

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu},_\sigma + g_{\rho\sigma},_\nu - g_{\nu\sigma},_\rho\right) \ ,$$

which, explicitly, are

$$\Gamma^0_{\nu\sigma} = \frac{1}{2}g^{0\rho}\left(g_{\rho\nu},_\sigma + g_{\rho\sigma},_\nu - g_{\nu\sigma},_\rho\right) = \frac{1}{2}g^{00}\left(g_{0\nu},_\sigma + g_{0\sigma},_\nu - g_{\nu\sigma},_0\right) =$$

$$= \frac{1}{2V^2}\left(g_{0\nu},_\sigma + g_{0\sigma},_\nu\right) = \frac{1}{2V^2}\begin{pmatrix} 0 & 2VV' & 0 & 0 \\ 2VV' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{3.54}$$

$$\Gamma^1_{\nu\sigma} = \frac{1}{2}g^{1\rho}\left(g_{\rho\nu},_\sigma + g_{\rho\sigma},_\nu - g_{\nu\sigma},_\rho\right) = \frac{1}{2}g^{11}\left(g_{1\nu},_\sigma + g_{1\sigma},_\nu - g_{\nu\sigma},_1\right) =$$

$$= -\frac{1}{2V^2}\left(g_{1\nu},_\sigma + g_{1\sigma},_\nu - g_{\nu\sigma},_1\right) =$$

$$= -\frac{1}{2V^2}\begin{pmatrix} -2VV' & 0 & 0 & 0 \\ 0 & -2VV' & 0 & 0 \\ 0 & 0 & \frac{2rV^2-2r^2VV'}{V^4} & 0 \\ 0 & 0 & 0 & 2VV' \end{pmatrix} \ , \tag{3.55}$$

$$\Gamma^2_{\nu\sigma} = \frac{1}{2}g^{2\rho}\left(g_{\rho\nu},_\sigma + g_{\rho\sigma},_\nu - g_{\nu\sigma},_\rho\right) = \frac{1}{2}g^{22}\left(g_{2\nu},_\sigma + g_{2\sigma},_\nu - g_{\nu\sigma},_2\right) =$$

$$= -\frac{V^2}{2r^2}\left(g_{2\nu},_\sigma + g_{2\sigma},_\nu\right) = -\frac{V^2}{2r^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2rV^2-2r^2VV'}{V^4} & 0 \\ 0 & -\frac{2rV^2-2r^2VV'}{V^4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{3.56}$$

$$\Gamma^3_{\nu\sigma} = \frac{1}{2}g^{3\rho}\left(g_{\rho\nu},_\sigma + g_{\rho\sigma},_\nu - g_{\nu\sigma},_\rho\right) = \frac{1}{2}g^{33}\left(g_{3\nu},_\sigma + g_{3\sigma},_\nu - g_{\nu\sigma},_3\right) =$$

$$= -\frac{1}{2V^2}\left(g_{3\nu},_\sigma + g_{3\sigma},_\nu\right) = -\frac{1}{2V^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2VV' \\ 0 & 0 & 0 & 0 \\ 0 & -2VV' & 0 & 0 \end{pmatrix} . \tag{3.57}$$

Thus, the geodesics equations are:

$$\text{for } \mu = 0 \;\Rightarrow\; 0 = \frac{d^2t}{d\tau^2} + \Gamma^0_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2t}{d\tau^2} + \Gamma^0_{0\sigma}\frac{dx^0}{d\tau}\frac{dx^\sigma}{d\tau} +$$

$$+\Gamma^0_{1\sigma}\frac{dx^1}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^0_{2\sigma}\frac{dx^2}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^0_{3\sigma}\frac{dx^3}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2t}{d\tau^2} + 2\frac{V'}{V}\frac{dt}{d\tau}\frac{dr}{d\tau} \;\Rightarrow$$

$$\Rightarrow\; \frac{d^2t}{d\tau^2} + 2\frac{V'}{V}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0 \;, \tag{3.58}$$

$$\text{for } \mu = 1 \;\Rightarrow\; 0 = \frac{d^2r}{d\tau^2} + \Gamma^1_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2r}{d\tau^2} + \Gamma^1_{0\sigma}\frac{dx^0}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^1_{1\sigma}\frac{dx^1}{d\tau}\frac{dx^\sigma}{d\tau} +$$

$$+\Gamma^1_{2\sigma}\frac{dx^2}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^1_{3\sigma}\frac{dx^3}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2r}{d\tau^2} + \frac{V'}{V}\left(\frac{dt}{d\tau}\right)^2 + \frac{V'}{V}\left(\frac{dr}{d\tau}\right)^2 +$$

$$+\frac{r^2V' - rV}{V^5}\left(\frac{d\phi}{d\tau}\right)^2 - \frac{V'}{V}\left(\frac{dz}{d\tau}\right)^2 \;\Rightarrow$$

$$\Rightarrow\; \frac{d^2r}{d\tau^2} + \frac{V'}{V}\left(\frac{dt}{d\tau}\right)^2 + \frac{V'}{V}\left(\frac{dr}{d\tau}\right)^2 + \frac{r^2V' - rV}{V^5}\left(\frac{d\phi}{d\tau}\right)^2 - \frac{V'}{V}\left(\frac{dz}{d\tau}\right)^2 = 0 \;, \tag{3.59}$$

$$\text{for } \mu = 2 \;\Rightarrow\; 0 = \frac{d^2\phi}{d\tau^2} + \Gamma^2_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2\phi}{d\tau^2} + \Gamma^2_{0\sigma}\frac{dx^0}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^2_{1\sigma}\frac{dx^1}{d\tau}\frac{dx^\sigma}{d\tau} +$$

$$+\Gamma^2_{2\sigma}\frac{dx^2}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^2_{3\sigma}\frac{dx^3}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2\phi}{d\tau^2} + 2\frac{V - rV'}{rV}\frac{dr}{d\tau}\frac{d\phi}{d\tau} \;\Rightarrow$$

$$\Rightarrow\; \frac{d^2\phi}{d\tau^2} + 2\frac{V - rV'}{rV}\frac{dr}{d\tau}\frac{d\phi}{d\tau} = 0 \;, \tag{3.60}$$

$$\text{for } \mu = 3 \;\Rightarrow\; 0 = \frac{d^2z}{d\tau^2} + \Gamma^3_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2z}{d\tau^2} + \Gamma^3_{0\sigma}\frac{dx^0}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^3_{1\sigma}\frac{dx^1}{d\tau}\frac{dx^\sigma}{d\tau} +$$

$$+\Gamma^3_{2\sigma}\frac{dx^2}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^3_{3\sigma}\frac{dx^3}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2z}{d\tau^2} + 2\frac{V'}{V}\frac{dz}{d\tau}\frac{dr}{d\tau} \;\Rightarrow$$

$$\Rightarrow\; \frac{d^2z}{d\tau^2} + 2\frac{V'}{V}\frac{dz}{d\tau}\frac{dr}{d\tau} = 0 \;. \tag{3.61}$$

In order to solve these equations, we first solve (3.58), (3.60) and (3.61), then we use the results obtained to solve (3.59). From (3.58) we have:

$$\frac{V'}{V} = \frac{d(\ln V)}{dr} \;\Rightarrow\; \frac{d(\ln V)}{dr}\frac{dr}{d\tau} = \frac{d(\ln V)}{d\tau} \;\Rightarrow$$

$$\Rightarrow \frac{d^2t}{d\tau^2} = -2\frac{d\left(\ln V\right)}{d\tau}\frac{dt}{d\tau} \Rightarrow \frac{dt}{d\tau} = C_0 \exp\left[-2\ln V\right] = \frac{C_0}{V^2} \ , \tag{3.62}$$

so

$$U^0 = \frac{dt}{d\tau} = \frac{C_0}{V^2} \ . \tag{3.63}$$

Analogously, for (3.61) we have

$$U^3 = \frac{dz}{d\tau} = \frac{C_3}{V^2} \ . \tag{3.64}$$

From (3.60) we have

$$2\frac{V - rV'}{rV} = 2\left(\frac{r}{V}\right)'\frac{V}{r} = 2\frac{d\left(\ln\frac{r}{V}\right)}{dr} = \frac{d\left(\ln\frac{r^2}{V^2}\right)}{dr} \Rightarrow$$

$$\Rightarrow \frac{d\left(\ln\frac{r^2}{V^2}\right)}{dr}\frac{dr}{d\tau} = \frac{d\left(\ln\frac{r^2}{V^2}\right)}{d\tau} \Rightarrow$$

$$\Rightarrow \frac{d^2\phi}{d\tau^2} = -\frac{d\left(\ln\frac{r^2}{V^2}\right)}{d\tau}\frac{d\phi}{d\tau} \Rightarrow \frac{d\phi}{d\tau} = C_2\exp\left[-\ln\frac{r^2}{V^2}\right] = \frac{C_2V^2}{r^2} \ , \tag{3.65}$$

so

$$U^2 = \frac{d\phi}{d\tau} = \frac{C_2V^2}{r^2} \ . \tag{3.66}$$

Substituting (3.63), (3.64) and (3.66) in (3.59) we obtain:

$$\frac{d^2r}{d\tau^2} + \frac{C_0^2V'}{V^5} + \frac{V'}{V}\left(\frac{dr}{d\tau}\right)^2 + C_2^2\frac{rV' - V}{r^3V} - \frac{C_3^2V'}{V^5} = 0 \Rightarrow$$

$$\Rightarrow \frac{d^2r}{d\tau^2} + \frac{d\ln V}{dr}\left(\frac{dr}{d\tau}\right)^2 + \frac{d}{dr}\left[-\frac{C_0^2}{4}\frac{1}{V^4} + C_2^2\int_0^r \frac{V'}{\tilde{r}^2V}d\tilde{r} + \right.$$

$$\left. + \frac{C_2^2}{2}\frac{1}{r^2} + \frac{C_3^2}{4}\frac{1}{V^4}\right] = 0 \Rightarrow \frac{d^2r}{d\tau^2} + \frac{d\ln V}{dr}\left(\frac{dr}{d\tau}\right)^2 + \frac{dA}{dr} = 0 \ , \tag{3.67}$$

where $A = -\frac{C_0^2}{4}\frac{1}{V^4} + C_2^2\int_0^r \frac{V'}{\tilde{r}^2V}d\tilde{r} + \frac{C_2^2}{2}\frac{1}{r^2} + \frac{C_3^2}{4}\frac{1}{V^4}$.
By using

$$\frac{d\ln V}{dr}\left(\frac{dr}{d\tau}\right)^2 = \frac{d\ln V}{d\tau}\frac{dr}{d\tau} \quad \text{and} \quad \frac{dA}{dr} = \frac{dA}{d\tau}\frac{d\tau}{dr} \ ,$$

in (3.67), we obtain

$$\frac{d^2r}{d\tau^2} + \frac{d\ln V}{d\tau}\frac{dr}{d\tau} + \frac{dA}{d\tau}\left(\frac{dr}{d\tau}\right)^{-1} = 0 \;\Rightarrow$$

$$\Rightarrow\; \frac{d}{d\tau}\left[\left(\frac{dr}{d\tau}\right)^2\right] + 2\frac{d\ln V}{d\tau}\left(\frac{dr}{d\tau}\right)^2 + 2\frac{dA}{d\tau} = 0 \;\Rightarrow$$

$$\Rightarrow\; \left(\frac{dr}{d\tau}\right)^2 = C_1 e^{-2\ln V} - 2A = \frac{C_1}{V^2} - 2A \;, \tag{3.68}$$

therefore

$$U^1 = \frac{dr}{d\tau} = \sqrt{\frac{C_1}{V^2} + \frac{C_0^2}{2}\frac{1}{V^4} - \frac{C_2^2}{2}\int_0^r \frac{V'}{\tilde{r}^2 V}d\tilde{r} - C_2^2\frac{1}{r^2} - \frac{C_3^2}{2}\frac{1}{V^4}} \;. \tag{3.69}$$

The velocities can be obtained using the chain rule, and therefore are

$$v^1 = \frac{dx}{dt} = \frac{V^2}{C_0}U^1 \;,\; v^2 = \frac{dy}{dt} = \frac{V^2}{C_0}U^2 \;,\; v^3 = \frac{dz}{dt} = \frac{V^2}{C_0}U^3 \;. \tag{3.70}$$

By using the tetrads (3.4), we can obtain the tetrad velocities:

$$v^{\hat{1}} = v^{\hat{r}} = Vv^1 \;,\; v^{\hat{2}} = v^{\hat{\phi}} = \frac{r}{V}v^2 \;,\; v^{\hat{3}} = v^{\hat{z}} = Vv^3 \;. \tag{3.71}$$

These are the velocities to be substituted in (3.46) and (3.47).

3.1.4 Geodesics equations: Killing approach.

In this section we will solve the geodesics equations in a more elegant way. We make use of *Killing vector fields*. As we demonstrated in appendix B, if the metric is independent of one coordinate, then a trivial Killing vector field is the generator of translations along that coordinate. In the case of the line element (3.2), the metric is independent of the coordinates t, ϕ and z, so three Killing vector fields are

$$\xi_t^\mu = (1,0,0,0) \;,\; \xi_\phi^\mu = (0,0,1,0) \;,\; \xi_z^\mu = (0,0,0,1) \;. \tag{3.72}$$

As we have shown in appendix B, we can build constants of motion from these Killing vector fields; in particular we have

$$\xi_t^\mu U_\mu = D_0 \;,\; \xi_\phi^\mu U_\mu = D_2 \;,\; \xi_z^\mu U_\mu = D_3 \;,$$

and from these relations we obtain

$$U_0 = D_0 \;\Rightarrow\; U^0 = \frac{D_0}{V^2} \;, \tag{3.73}$$

$$U_2 = D_2 \;\Rightarrow\; U^2 = -\frac{V^2 D_2}{r^2} \;, \tag{3.74}$$

$$U_3 = D_3 \;\Rightarrow\; U^3 = -\frac{D_3}{V^2} \;. \tag{3.75}$$

We observe that these three components have exactly the same structure as in (3.63), (3.64) and (3.66). The component U^1 can be found by imposing a normalization condition, that is

$$U_\mu U^\mu = 1 \;\Rightarrow\; \frac{D_0^2}{V^2} - V^2\left(U^1\right)^2 - \frac{V^2 D_2^2}{r^2} - \frac{D_3^2}{V^2} = 1 \;. \tag{3.76}$$

By solving (3.76), we find that U^1 has the same structure as in (3.69), the only difference being that in (3.69) four integration constants appear but only three are really independent, because no condition of normalization was imposed.

3.1.5 Ricci tensor in Melvin space-time.

Now that we have calculated the components of the affine connection, it is interesting to verify the coherence of the emerging geometry with the assumption (3.2) of the metric. Here we will calculate the Ricci tensor and show that, by imposing the scalar curvature to be 0, we will rediscover the structure of the Melvin solution. To work out the Ricci tensor we have:

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} \;, \tag{3.77}$$

and by using (3.54) $\div$ (3.57) we can work out each component of the tensor:

$$\begin{aligned} R_{00} &= \Gamma^\rho_{00,\rho} - \Gamma^\rho_{0\rho,0} + \Gamma^\rho_{00}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{0\rho}\Gamma^\rho_{0\lambda} = \\ &= \partial_r\left(\frac{V'}{V}\right) + \frac{V'}{V}\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right) - \left(\Gamma^0_{0\rho}\Gamma^\rho_{00} + \Gamma^1_{0\rho}\Gamma^\rho_{01} + \right. \\ &\left. + \Gamma^2_{0\rho}\Gamma^\rho_{02} + \Gamma^3_{0\rho}\Gamma^\rho_{03}\right) = \partial_r\left(\frac{V'}{V}\right) + \left(\frac{V'}{V}\right)^2 + \left(\frac{V'}{V}\right)^2 + \left(\frac{V'}{V}\right)\frac{V - rV'}{rV} + \\ &+ \left(\frac{V'}{V}\right)^2 - \left[\left(\frac{V'}{V}\right)^2 + \left(\frac{V'}{V}\right)^2\right] = \partial_r\left(\frac{V'}{V}\right) + \frac{V'}{rV} \;; \end{aligned} \tag{3.78}$$

$$R_{11} = \Gamma^{\rho}_{11,\rho} - \Gamma^{\rho}_{1\rho,1} + \Gamma^{\rho}_{11}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{1\rho}\Gamma^{\rho}_{1\lambda} =$$

$$= \partial_r\left(\frac{V'}{V}\right) - \left[\Gamma^0_{10,1} + \Gamma^1_{11,1} + \Gamma^2_{12,1} + \Gamma^3_{13,1}\right] + \Gamma^1_{11}\left[\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right] -$$

$$- \left[\Gamma^0_{1\rho}\Gamma^{\rho}_{10} + \Gamma^1_{1\rho}\Gamma^{\rho}_{11} + \Gamma^2_{1\rho}\Gamma^{\rho}_{12} + \Gamma^3_{1\rho}\Gamma^{\rho}_{13}\right] = \partial_r\left(\frac{V'}{V}\right) - \left[\partial_r\left(\frac{V'}{V}\right) +\right.$$

$$\left.\partial_r\left(\frac{V'}{V}\right) + \partial_r\left(\frac{V - rV'}{rV}\right) + \partial_r\left(\frac{V'}{V}\right)\right] + \frac{V'}{V}\left[\frac{V'}{V} + \frac{V'}{V} + \frac{V - rV'}{rV} + \frac{V'}{V}\right] -$$

$$- \left[\left(\frac{V'}{V}\right)^2 + \left(\frac{V'}{V}\right)^2 + \left(\frac{V - rV'}{rV}\right)^2 + \left(\frac{V'}{V}\right)^2\right] = -\partial_r\left(\frac{1}{r}\right) - \partial_r\left(\frac{V'}{V}\right) +$$

$$+3\left(\frac{V'}{V}\right)^2 + \frac{V'}{rV} - \left(\frac{V'}{V}\right)^2 - 3\left(\frac{V'}{V}\right)^2 - \frac{1}{r^2} - \left(\frac{V'}{V}\right)^2 + \frac{2V'}{rV} =$$

$$= -\partial_r\left(\frac{V'}{V}\right) + \frac{3V'}{rV} - 2\left(\frac{V'}{V}\right)^2 \ ; \tag{3.79}$$

$$R_{22} = \Gamma^{\rho}_{22,\rho} - \Gamma^{\rho}_{2\rho,2} + \Gamma^{\rho}_{22}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{2\rho}\Gamma^{\rho}_{2\lambda} =$$

$$= \partial_r\left(\frac{-rV + r^2V'}{V^5}\right) + \Gamma^1_{22}\left[\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right] -$$

$$- \left[\Gamma^{\lambda}_{20}\Gamma^0_{2\lambda} + \Gamma^{\lambda}_{21}\Gamma^1_{2\lambda} + \Gamma^{\lambda}_{22}\Gamma^2_{2\lambda} + \Gamma^{\lambda}_{23}\Gamma^3_{2\lambda}\right] =$$

$$= \partial_r\left(\frac{-rV + r^2V'}{V^5}\right) + \frac{-rV + r^2V'}{V^5}\left[\frac{V'}{V} + \frac{V'}{V} + \frac{V - rV'}{rV} + \frac{V'}{V}\right] -$$

$$- \left[\frac{V - rV'}{rV}\frac{-rV + r^2V'}{V^5} + \frac{-rV + r^2V'}{V^5}\frac{V - rV'}{rV}\right] =$$

$$= \partial_r\left(\frac{-rV + r^2V'}{V^5}\right) + 3\frac{V'}{V}\frac{-rV + r^2V'}{V^5} + \frac{-rV + r^2V'}{V^5}\frac{V - rV'}{rV} -$$

$$-2\frac{-rV + r^2V'}{V^5} - 2\frac{-rV + r^2V'}{V^5}\frac{V - rV'}{rV} =$$

$$= \partial_r\left(\frac{-rV + r^2V'}{V^5}\right) + 4\frac{V'}{V}\frac{-rV + r^2V'}{V^5} - \frac{-V + rV'}{V^5} \ ; \tag{3.80}$$

$$R_{33} = \Gamma^{\rho}_{33,\rho} - \Gamma^{\rho}_{3\rho,3} + \Gamma^{\rho}_{33}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{3\rho}\Gamma^{\rho}_{3\lambda} =$$

$$= -\partial_r\left(\frac{V'}{V}\right) + \Gamma^1_{33}\left[\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right] -$$

$$- \left[\Gamma^0_{3\rho}\Gamma^{\rho}_{30} + \Gamma^1_{3\rho}\Gamma^{\rho}_{31} + \Gamma^2_{3\rho}\Gamma^{\rho}_{32} + \Gamma^3_{3\rho}\Gamma^{\rho}_{33}\right] =$$

$$= -\partial_r\left(\frac{V'}{V}\right) - \frac{V'}{V}\left[\frac{V'}{V} + \frac{V'}{V} + \frac{V - rV'}{rV} + \frac{V'}{V}\right] - \left[-\left(\frac{V'}{V}\right)^2 - \left(\frac{V'}{V}\right)^2\right] =$$

$$= -\partial_r\left(\frac{V'}{V}\right) - 2\left(\frac{V'}{V}\right)^2 - \frac{V'}{rV} + 2\left(\frac{V'}{V}\right)^2 = -\partial_r\left(\frac{V'}{V}\right) - \frac{V'}{rV} = -R_{00}\ ; \tag{3.81}$$

$$R_{01} = \Gamma^{\rho}_{10,\rho} - \Gamma^{\rho}_{0\rho,1} + \Gamma^{\rho}_{01}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{0\rho}\Gamma^{\rho}_{1\lambda} =$$

$$= -\left[\Gamma^{0}_{1\lambda}\Gamma^{\lambda}_{00} + \Gamma^{1}_{1\lambda}\Gamma^{\lambda}_{01} + \Gamma^{2}_{1\lambda}\Gamma^{\lambda}_{02} + \Gamma^{3}_{1\lambda}\Gamma^{\lambda}_{03}\right] = 0\ ; \tag{3.82}$$

$$R_{02} = \Gamma^{\rho}_{20,\rho} - \Gamma^{\rho}_{0\rho,2} + \Gamma^{\rho}_{02}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{0\rho}\Gamma^{\rho}_{2\lambda} =$$

$$= -\left[\Gamma^{0}_{2\lambda}\Gamma^{\lambda}_{00} + \Gamma^{1}_{2\lambda}\Gamma^{\lambda}_{01} + \Gamma^{2}_{2\lambda}\Gamma^{\lambda}_{02} + \Gamma^{3}_{2\lambda}\Gamma^{\lambda}_{03}\right] = 0\ ; \tag{3.83}$$

$$R_{03} = \Gamma^{\rho}_{30,\rho} - \Gamma^{\rho}_{0\rho,3} + \Gamma^{\rho}_{03}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{0\rho}\Gamma^{\rho}_{3\lambda} =$$

$$= -\left[\Gamma^{0}_{3\lambda}\Gamma^{\lambda}_{00} + \Gamma^{1}_{3\lambda}\Gamma^{\lambda}_{01} + \Gamma^{2}_{3\lambda}\Gamma^{\lambda}_{02} + \Gamma^{3}_{3\lambda}\Gamma^{\lambda}_{03}\right] = 0\ ; \tag{3.84}$$

$$R_{12} = \Gamma^{\rho}_{12,\rho} - \Gamma^{\rho}_{1\rho,2} + \Gamma^{\rho}_{12}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{1\rho}\Gamma^{\rho}_{2\lambda} =$$

$$= \Gamma^{2}_{12}\left[\Gamma^{0}_{02} + \Gamma^{1}_{12} + \Gamma^{2}_{22} + \Gamma^{3}_{32}\right] -$$

$$- \left[\Gamma^{\lambda}_{10}\Gamma^{0}_{2\lambda} + \Gamma^{\lambda}_{11}\Gamma^{1}_{2\lambda} + \Gamma^{\lambda}_{12}\Gamma^{2}_{2\lambda} + \Gamma^{\lambda}_{13}\Gamma^{3}_{2\lambda}\right] = 0\ ; \tag{3.85}$$

$$R_{13} = \Gamma^{\rho}_{13,\rho} - \Gamma^{\rho}_{1\rho,3} + \Gamma^{\rho}_{13}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{1\rho}\Gamma^{\rho}_{3\lambda} =$$

$$= \Gamma^{3}_{13}\left[\Gamma^{0}_{03} + \Gamma^{1}_{13} + \Gamma^{2}_{23} + \Gamma^{3}_{33}\right] -$$

$$- \left[\Gamma^{\lambda}_{10}\Gamma^{0}_{3\lambda} + \Gamma^{\lambda}_{11}\Gamma^{1}_{3\lambda} + \Gamma^{\lambda}_{12}\Gamma^{2}_{3\lambda} + \Gamma^{\lambda}_{13}\Gamma^{3}_{3\lambda}\right] = 0\ ; \tag{3.86}$$

$$R_{23} = \Gamma^{\rho}_{23,\rho} - \Gamma^{\rho}_{2\rho,3} + \Gamma^{\rho}_{23}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{2\rho}\Gamma^{\rho}_{3\lambda} =$$

$$= -\left[\Gamma^{\lambda}_{20}\Gamma^{0}_{3\lambda} + \Gamma^{\lambda}_{21}\Gamma^{1}_{3\lambda} + \Gamma^{\lambda}_{22}\Gamma^{2}_{3\lambda} + \Gamma^{\lambda}_{23}\Gamma^{3}_{3\lambda}\right] = 0\ . \tag{3.87}$$

We observe that the Ricci tensor is diagonal, as expected. In order to work out the scalar curvature, we need to raise one of the indices:

$$R^{\mu}_{\nu} = g^{\mu\rho}R_{\rho\nu}\ ,$$

so each component different from 0 becomes:

$$R^{0}_{0} = g^{0\rho}R_{\rho 0} = g^{00}R_{00} = \frac{1}{V^2}\left(\frac{V''}{V} - \frac{V'^2}{V^2} + \frac{V'}{rV}\right) =$$

$$= \frac{V''}{V^3} - \frac{V'^2}{V^4} + \frac{V'}{rV^3}\ , \tag{3.88}$$

$$R^{1}_{1} = g^{1\rho}R_{\rho 1} = g^{11}R_{11} = -\frac{1}{V^2}\left(-\frac{V''}{V} + \frac{3V'}{rV} - \frac{V'^2}{V^2}\right) =$$

$$= \frac{V''}{V^3} - \frac{3V'}{rV^3} + \frac{V'^2}{V^4} \ , \tag{3.89}$$

$$R^2_2 = g^{2\rho}R_{\rho 2} = g^{22}R_{22} = -\frac{V^2}{r^2}\left(\frac{r^2V''}{V^5} + \frac{rV'}{V^5} - \frac{r^2V'^2}{V^6}\right) =$$

$$= -\frac{V''}{V^3} - \frac{V'}{rV^3} + \frac{V'^2}{V^4} \ , \tag{3.90}$$

$$R^3_3 = g^{3\rho}R_{\rho 3} = g^{33}R_{33} = -\frac{1}{V^2}\left(-\frac{V''}{V} + \frac{V'^2}{V^2} - \frac{V'}{rV}\right) =$$

$$= \frac{V''}{V^3} - \frac{V'^2}{V^4} + \frac{V'}{rV^3} \ . \tag{3.91}$$

The scalar curvature is then

$$R = R^\mu_\mu = \frac{V''}{V^3} - \frac{V'^2}{V^4} + \frac{V'}{rV^3} + \frac{V''}{V^3} - \frac{3V'}{rV^3} + \frac{V'^2}{V^4} - \frac{V''}{V^3} -$$

$$-\frac{V'}{rV^3} + \frac{V'^2}{V^4} + \frac{V''}{V^3} - \frac{V'^2}{V^4} + \frac{V'}{rV^3} = \frac{2V''}{V^3} - \frac{2V'}{rV^3} \ . \tag{3.92}$$

The trace of the energy-momentum tensor of the electromagnetic field is 0 [9]. Coherently with this constraint we impose $R = 0$, so we have

$$V'' - \frac{V'}{r} = 0 \ , \tag{3.93}$$

and by solving this equation we obtain:

$$V' = Ar \quad \Rightarrow \quad V = \frac{A}{2}r^2 + B \ , \tag{3.94}$$

where A and B are integration constants. (3.94) is in perfect agreement with (3.2), as expected.

3.1.6 Equations of spin operators.

Now we have all the ingredients necessary to solve equations (3.50) $\div$ (3.52); first we observe that both $\Omega_{(1)\hat{\phi}}$ and $\Omega_{(1)\hat{z}}$ contain a β matrix, and that if we multiply this matrix by $\Sigma_{\hat{r}}, \Sigma_{\hat{\phi}}$ or $\Sigma_{\hat{z}}$ we obtain $\Pi_{\hat{r}}, \Pi_{\hat{\phi}}$ or $\Pi_{\hat{z}}$ respectively. By renaming the angular velocities with the same notations without β, the set of differential equations can be written as

$$\frac{d}{dt}\begin{pmatrix}\Pi^{\hat{r}}\\ \Pi^{\hat{\phi}}\\ \Pi^{\hat{z}}\end{pmatrix} = \begin{pmatrix} 0 & -\Omega_{(1)\hat{z}} & \Omega_{(1)\hat{\phi}} \\ \Omega_{(1)\hat{z}} & 0 & 0 \\ -\Omega_{(1)\hat{\phi}} & 0 & 0 \end{pmatrix}\begin{pmatrix}\Pi_{\hat{r}}\\ \Pi_{\hat{\phi}}\\ \Pi_{\hat{z}}\end{pmatrix} \ . \tag{3.95}$$

Using the spacial Minkowski metric $diag\,(-1,-1,-1)$ to raise the hatted indices, system (3.95) becomes

$$\frac{d}{dt}\begin{pmatrix}\Pi^{\hat{r}}\\ \Pi^{\hat{\phi}}\\ \Pi^{\hat{z}}\end{pmatrix}=\begin{pmatrix}0 & -\Omega^{\hat{z}}_{(1)} & \Omega^{\hat{\phi}}_{(1)}\\ \Omega^{\hat{z}}_{(1)} & 0 & 0\\ -\Omega^{\hat{\phi}}_{(1)} & 0 & 0\end{pmatrix}\begin{pmatrix}\Pi^{\hat{r}}\\ \Pi^{\hat{\phi}}\\ \Pi^{\hat{z}}\end{pmatrix}\ . \tag{3.96}$$

(3.96) is a linear system of differential equations with coefficients constant in time t, so we can solve it easily with common methods of mathematical analysis; we first evaluate the eigenvalues of the matrix of coefficients by solving the secular equation $det\,(A-\lambda\mathbb{I})=0$:

$$0=det\begin{pmatrix}-\lambda & -\Omega^{\hat{z}}_{(1)} & \Omega^{\hat{\phi}}_{(1)}\\ \Omega^{\hat{z}}_{(1)} & -\lambda & 0\\ -\Omega^{\hat{\phi}}_{(1)} & 0 & -\lambda\end{pmatrix}=-\lambda^3-\left(\lambda\Omega^{\hat{\phi}2}_{(1)}+\lambda\Omega^{\hat{z}2}_{(1)}\right)=0\ \Rightarrow$$

$$\Rightarrow\ \lambda=0\ \ ,\ \ \lambda_{\pm}=\pm i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}\ . \tag{3.97}$$

Now we work out the corresponding eigenvectors:

$$\text{for } \lambda=0\ \Rightarrow\ \begin{pmatrix}0 & -\Omega^{\hat{z}}_{(1)} & \Omega^{\hat{\phi}}_{(1)}\\ \Omega^{\hat{z}}_{(1)} & 0 & 0\\ -\Omega^{\hat{\phi}}_{(1)} & 0 & 0\end{pmatrix}\begin{pmatrix}\Pi^{\hat{r}}\\ \Pi^{\hat{\phi}}\\ \Pi^{\hat{z}}\end{pmatrix}=0\ \Rightarrow$$

$$\Rightarrow\begin{cases}-\Omega^{\hat{z}}_{(1)}\Pi^{\hat{\phi}}+\Omega^{\hat{\phi}}_{(1)}\Pi^{\hat{z}}=0\\ \Omega^{\hat{z}}_{(1)}\Pi^{\hat{r}}=0\\ -\Omega^{\hat{\phi}}_{(1)}\Pi^{\hat{r}}=0\end{cases}\quad\text{, an eigenvector is}\ \begin{pmatrix}0\\ \frac{\Omega^{\hat{\phi}}_{(1)}}{\Omega^{\hat{z}}_{(1)}}\\ 1\end{pmatrix}=\mathbf{u}\ . \tag{3.98}$$

The other two eigenvalues are complex conjugates, so we need to find an eigenvector corresponding to λ_+ and then split it into its real and imaginary parts:

$$\text{for } \lambda_+=i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}\ \Rightarrow$$

$$\Rightarrow\begin{pmatrix}i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}} & -\Omega^{\hat{z}}_{(1)} & \Omega^{\hat{\phi}}_{(1)}\\ \Omega^{\hat{z}}_{(1)} & i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}} & 0\\ -\Omega^{\hat{\phi}}_{(1)} & 0 & i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}\end{pmatrix}\begin{pmatrix}\Pi^{\hat{r}}\\ \Pi^{\hat{\phi}}\\ \Pi^{\hat{z}}\end{pmatrix}=0\ \Rightarrow$$

$$\Rightarrow\begin{cases}i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}\Pi^{\hat{r}}-\Omega^{\hat{z}}_{(1)}\Pi^{\hat{\phi}}+\Omega^{\hat{\phi}}_{(1)}\Pi^{\hat{z}}=0\\ \Omega^{\hat{z}}_{(1)}\Pi^{\hat{r}}+i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}\Pi^{\hat{\phi}}=0\\ -\Omega^{\hat{\phi}}_{(1)}\Pi^{\hat{r}}+i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}\Pi^{\hat{z}}=0\end{cases}\quad\text{, an eigenvector is}$$

$$\begin{pmatrix} 1 \\ -\frac{\Omega^{\hat{z}}_{(1)}}{i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}} \\ \frac{\Omega^{\hat{\phi}}_{(1)}}{i\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ \frac{\Omega^{\hat{z}}_{(1)}}{\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}} \\ -\frac{\Omega^{\hat{\phi}}_{(1)}}{\sqrt{\Omega^{\hat{\phi}2}_{(1)}+\Omega^{\hat{z}2}_{(1)}}} \end{pmatrix} = \mathbf{v} + i\mathbf{w} \ . \tag{3.99}$$

The solution to the system is therefore

$$\begin{pmatrix} \Pi^{\hat{r}} \\ \Pi^{\hat{\phi}} \\ \Pi^{\hat{z}} \end{pmatrix} = s_1 \exp\left[0t\right] \mathbf{u} + s_2 \left(\mathbf{v} \cos\left[\omega t\right] - \mathbf{w} \sin\left[\omega t\right]\right) + s_3 \left(\mathbf{v} \sin\left[\omega t\right] + \right.$$

$$\left. + \mathbf{w} \cos\left[\omega t\right]\right) = \begin{pmatrix} s_2 \cos\left[\omega t\right] + s_3 \sin\left[\omega t\right] \\ s_1 \frac{\Omega^{\hat{\phi}}_{(1)}}{\Omega^{\hat{z}}_{(1)}} - s_2 \frac{\Omega^{\hat{z}}_{(1)}}{\omega} \sin\left[\omega t\right] + s_3 \frac{\Omega^{\hat{z}}_{(1)}}{\omega} \cos\left[\omega t\right] \\ s_1 + s_2 \frac{\Omega^{\hat{\phi}}_{(1)}}{\omega} \sin\left[\omega t\right] - s_3 \frac{\Omega^{\hat{\phi}}_{(1)}}{\omega} \cos\left[\omega t\right] \end{pmatrix} , \tag{3.100}$$

where s_1, s_2 and s_3 are integration constants; by rewriting (3.100) using $s_2 \cos\left[\omega t\right] + s_3 \sin\left[\omega t\right] = s \cos\left[\omega t + \phi_0\right]$ and analogously $s_3 \cos\left[\omega t\right] - s_2 \sin\left[\omega t\right] = s \sin\left[\omega t + \phi_0\right]$, we obtain

$$\begin{pmatrix} \Pi^{\hat{r}} \\ \Pi^{\hat{\phi}} \\ \Pi^{\hat{z}} \end{pmatrix} = \begin{pmatrix} s \cos\left[\omega t + \phi_0\right] \\ s_1 \frac{\Omega^{\hat{\phi}}_{(1)}}{\Omega^{\hat{z}}_{(1)}} + s \frac{\Omega^{\hat{z}}_{(1)}}{\omega} \sin\left[\omega t + \phi_0\right] \\ s_1 - s \frac{\Omega^{\hat{\phi}}_{(1)}}{\omega} \sin\left[\omega t + \phi_0\right] \end{pmatrix} , \tag{3.101}$$

where $\omega = \sqrt{\Omega^{\hat{\phi}2}_{(1)} + \Omega^{\hat{z}2}_{(1)}}$, s and ϕ_0 are integration constants derived from s_2 and s_3. We should keep in mind that multiplication for the identity matrix $\mathbb{I}$ has been omitted.

3.1.7 Melvin cosmology.

(3.1) describes what is called the "*Melvin fluxtube*", which is a region of space with cylindrical symmetry where a magnetic field is parallel to the walls of the cylinder. Through analytic continuation, (3.1) describes the domain wall for the anisotropic Melvin cosmology [10]. Here we show how to perform this analytic continuation. Let's rescale the angular coordinate $\phi \in [0; 2\pi]$ using $\phi = B_0 y$, where we assume $y \in (-\infty; +\infty)$. The Melvin solution takes on the form of

$$ds^2 = f^2(r)dt^2 - f^2(r)dr^2 - f^2(r)dz^2 - \frac{B_0^2 r^2}{f^2(r)} dy^2 \ . \tag{3.102}$$

The wall is spatially homogeneous and time-translations invariant; we also notice that it is invariant for boosts in the tz planes. The anisotropy is due to the inequality $g_{yy} \neq g_{zz}$. Now, in order to obtain the Melvin cosmology we perform the substitutions $r = iT$, $t = ix$ and, in order to keep the 4-potential real, $B_0 = -iE_0$. So, (3.101) becomes

$$ds^2 = f^2(T)dT^2 - f^2(T)dx^2 - f^2(T)dz^2 - \frac{E_0^2T^2}{f^2(T)}dy^2 \ , \tag{3.103}$$

where

$$A_y = \frac{E_0^2T^2}{2f(T)} \ , f(T) = 1 + \frac{E^2T^2}{4} \ . \tag{3.104}$$

Melvin cosmology (3.103) has homogeneous but anisotropic flat spatial slices and a spatially uniform electric field pointing to the y direction. We also notice that the boost in tz planes has become a rotational symmetry on yz planes.

After this introduction we study the spin precession in the case of the Melvin cosmology, then we confront the results with those obtained for the standard Melvin metric.

By using metric (3.103) we can easily obtain

$$e^a_\mu = \begin{pmatrix} f(T) & 0 & 0 & 0 \\ 0 & f(T) & 0 & 0 \\ 0 & 0 & \frac{E_0T}{f(T)} & 0 \\ 0 & 0 & 0 & f(T) \end{pmatrix} \ , \tag{3.105}$$

and its inverse

$$e^a_\mu = \begin{pmatrix} \frac{1}{f(T)} & 0 & 0 & 0 \\ 0 & \frac{1}{f(T)} & 0 & 0 \\ 0 & 0 & \frac{f(T)}{E_0T} & 0 \\ 0 & 0 & 0 & \frac{1}{f(T)} \end{pmatrix} \ . \tag{3.106}$$

Through lengthy calculations, analogous to those performed for the Melvin metric, it is easy to obtain

$$W^{\hat{\alpha}}_{\beta} = \begin{pmatrix} f(T) & 0 & 0 \\ 0 & \frac{E_0T}{f(T)} & 0 \\ 0 & 0 & f(T) \end{pmatrix} \ , \text{its inverse } W^{\beta}_{\hat{\alpha}} = \begin{pmatrix} \frac{1}{f(T)} & 0 & 0 \\ 0 & \frac{f(T)}{E_0T} & 0 \\ 0 & 0 & \frac{1}{f(T)} \end{pmatrix} \ , \tag{3.107}$$

$$K^{\hat{\alpha}} = 0 \ , \hat{\alpha} = 1,2,3 \ \text{ and } \ f(T) = V \ . \tag{3.108}$$

Starting from definition (2.14) we obtain

$$\mathcal{Q}_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\gamma}} W^{\delta}_{\hat{\beta}} \left(\dot{W}^{\hat{\gamma}}_{\delta} + K^{\epsilon}\partial_{\epsilon} W^{\hat{\gamma}}_{\delta} + W^{\hat{\gamma}}_{\epsilon}\partial_{\delta} K^{\epsilon} \right) = \eta_{\hat{\alpha}\hat{\gamma}} W^{\delta}_{\hat{\beta}} \dot{W}^{\hat{\gamma}}_{\delta} \ ,$$

knowing that the $K^{\hat{\alpha}}$'s all vanish as shown in (3.108). So we explicitly evaluate

$$\mathcal{Q}_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{1}} W^{\delta}_{\hat{\beta}} \dot{W}^{\hat{1}}_{\delta} + \eta_{\hat{\alpha}\hat{2}} W^{\delta}_{\hat{\beta}} \dot{W}^{\hat{2}}_{\delta} + \eta_{\hat{\alpha}\hat{3}} W^{\delta}_{\hat{\beta}} \dot{W}^{\hat{3}}_{\delta} = \eta_{\hat{\alpha}\hat{1}} W^{1}_{\hat{\beta}} \dot{W}^{\hat{1}}_{1} + \eta_{\hat{\alpha}\hat{2}} W^{2}_{\hat{\beta}} \dot{W}^{\hat{2}}_{2} +$$

$$+\eta_{\hat{\alpha}\hat{3}} W^{2}_{\hat{\beta}} \dot{W}^{\hat{3}}_{2} = \begin{pmatrix} -\frac{\dot{V}}{V} & 0 & 0 \\ 0 & -\frac{V}{T}\left(\frac{\dot{T}}{V}\right) & 0 \\ 0 & 0 & -\frac{\dot{V}}{V} \end{pmatrix} \ . \tag{3.109}$$

Now we can evaluate each term appearing in the expressions of angular velocity operators separately. In particular, we have

$$\mathcal{F}^{\alpha}_{\hat{\beta}} = V W^{\alpha}_{\hat{\beta}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{V^2}{E_0 T} & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ; \tag{3.110}$$

$$\Xi_{\hat{\alpha}} = \epsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}} \mathcal{Q}^{\hat{\beta}\hat{\gamma}} = 0 \ , \tag{3.111}$$

because $\mathcal{Q}$ is symmetric and the Levi Civita symbol is antisymmetric;

$$\mathcal{C}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}} = W^{\delta}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \partial_{[\delta} W^{\hat{\gamma}}_{\epsilon]} = 0 \ \Rightarrow \ \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = \eta_{\hat{\gamma}\hat{\delta}} \mathcal{C}^{\hat{\delta}}_{\hat{\alpha}\hat{\beta}} = 0 \ , \tag{3.112}$$

because W depends only on the temporal variable T, while in the definition of the anholonimity coefficients only derivatives with respect to spacial coordinates appear;

$$\Upsilon = -V \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = 0 \ , \tag{3.113}$$

as a consequence of (3.112).

Now we can use these results to explicitly evaluate the angular velocity operators. By substituting the previous results into (2.29) we directly obtain

$$\Omega^{\hat{\alpha}}_{(1)} = 0 \ , \tag{3.114}$$

because of (3.113) and because the functions involved in these calculations only depend on T, while in (2.29) only spacial derivatives are considered.

Now we want to work out (2.30) for the Melvin cosmology:

$$\Omega^{\hat{\alpha}}_{(2)} = \frac{1}{8} \left\{ \frac{1}{\mathcal{T}}, \left\{ \left\{ \pi_{\epsilon}, \mathcal{F}^{\epsilon}_{\hat{\beta}} \right\}, \left\{ \pi_{\zeta}, \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \dot{\mathcal{F}}^{\zeta}_{\hat{\gamma}} \right\} \right\} \right\} \ , \tag{3.115}$$

because of (3.108) and (3.111). The anticommutators are trivial to evaluate because π_{ϵ} is a differential operator involving only spacial derivatives, while all the functions in the expression for $\Omega_{(2)}$ are time-dependent only.

It is useful to observe that because of the trivial anticommutators, analogously to what happens in the Melvin space-time, the energy takes on the same form as in (2.34). In particular, (2.37) holds. After these considerations, (3.115) becomes

$$\Omega^{\hat{\alpha}}_{(2)} = \frac{2}{\mathcal{T}} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \mathcal{F}^{\epsilon}_{\hat{\beta}} \pi_{\epsilon} \dot{\mathcal{F}}^{\zeta}_{\hat{\gamma}} \pi_{\zeta} \ , \tag{3.116}$$

where

$$\mathcal{T} = 2\epsilon'^2 + \{\epsilon', mV\} = 2\epsilon'\left(\epsilon' + mV\right) = 2\epsilon'^2 \frac{1+\gamma}{\gamma} \ ,$$

because of (2.38). From (2.37) we obtain an expression for the momenta giving them in terms of the velocities. So, using the velocities, (3.116) becomes

$$\begin{aligned}
\Omega^{\hat{\alpha}}_{(2)} &= \frac{\gamma}{1+\gamma} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \mathcal{F}^{\epsilon}_{\hat{\beta}} v_{\hat{\delta}} \mathcal{F}^{\hat{\delta}}_{\epsilon} \dot{\mathcal{F}}^{\zeta}_{\hat{\gamma}} v_{\hat{\eta}} \mathcal{F}^{\hat{\eta}}_{\zeta} = \frac{\gamma}{1+\gamma} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} v_{\hat{\beta}} v_{\hat{\eta}} \mathcal{F}^{\hat{\eta}}_{\zeta} \dot{\mathcal{F}}^{\zeta}_{\hat{\gamma}} = \\
&= \frac{\gamma}{1+\gamma} \left(\epsilon^{\hat{\alpha}\hat{1}\hat{\gamma}} v_{\hat{1}} v_{\hat{\eta}} \mathcal{F}^{\hat{\eta}}_{\zeta} \dot{\mathcal{F}}^{\zeta}_{\hat{\gamma}} + \epsilon^{\hat{\alpha}\hat{2}\hat{\gamma}} v_{\hat{2}} v_{\hat{\eta}} \mathcal{F}^{\hat{\eta}}_{\zeta} \dot{\mathcal{F}}^{\zeta}_{\hat{\gamma}} + \epsilon^{\hat{\alpha}\hat{3}\hat{\gamma}} v_{\hat{3}} v_{\hat{\eta}} \mathcal{F}^{\hat{\eta}}_{\zeta} \dot{\mathcal{F}}^{\zeta}_{\hat{\gamma}} \right) = \\
&= \frac{\gamma}{1+\gamma} \Big(\epsilon^{\hat{\alpha}\hat{1}\hat{2}} v_{\hat{1}} v_{\hat{2}} \mathcal{F}^{\hat{2}}_{2} \dot{\mathcal{F}}^{2}_{\hat{2}} + \epsilon^{\hat{\alpha}\hat{1}\hat{3}} v_{\hat{1}} v_{\hat{3}} \mathcal{F}^{\hat{3}}_{3} \dot{\mathcal{F}}^{3}_{\hat{3}} + \epsilon^{\hat{\alpha}\hat{2}\hat{1}} v_{\hat{2}} v_{\hat{1}} \mathcal{F}^{\hat{1}}_{1} \dot{\mathcal{F}}^{1}_{\hat{1}} + \\
&\quad + \epsilon^{\hat{\alpha}\hat{2}\hat{3}} v_{\hat{2}} v_{\hat{3}} \mathcal{F}^{\hat{3}}_{3} \dot{\mathcal{F}}^{3}_{\hat{3}} + \epsilon^{\hat{\alpha}\hat{3}\hat{1}} v_{\hat{3}} v_{\hat{1}} \mathcal{F}^{\hat{1}}_{1} \dot{\mathcal{F}}^{1}_{\hat{1}} + \epsilon^{\hat{\alpha}\hat{3}\hat{2}} v_{\hat{3}} v_{\hat{2}} \mathcal{F}^{\hat{2}}_{2} \dot{\mathcal{F}}^{2}_{\hat{2}} \Big) = \\
&= \frac{\gamma}{1+\gamma} \left(\epsilon^{\hat{\alpha}\hat{1}\hat{2}} v_{\hat{1}} v_{\hat{2}} \mathcal{F}^{\hat{2}}_{2} \dot{\mathcal{F}}^{2}_{\hat{2}} + \epsilon^{\hat{\alpha}\hat{3}\hat{2}} v_{\hat{3}} v_{\hat{2}} \mathcal{F}^{\hat{2}}_{2} \dot{\mathcal{F}}^{2}_{\hat{2}} \right) \ .
\end{aligned} \tag{3.117}$$

If we write the three components explicitly we have:

$$\Omega^{\hat{1}}_{(2)} = -\frac{\gamma}{1+\gamma} v_{\hat{3}} v_{\hat{2}} \mathcal{F}^{\hat{2}}_{2} \dot{\mathcal{F}}^{2}_{\hat{2}} \ ; \tag{3.118}$$

$$\Omega^{\hat{2}}_{(2)} = 0 \ ; \tag{3.119}$$

$$\Omega^{\hat{3}}_{(2)} = \frac{\gamma}{1+\gamma} v_{\hat{1}} v_{\hat{2}} \mathcal{F}^{\hat{2}}_{2} \dot{\mathcal{F}}^{2}_{\hat{2}} \ . \tag{3.120}$$

In order to find an expression for the velocities, we need to solve the geodesics equations. We observe that metric (3.103) does not depend on the spacial variables x, y and z, so, by using the results demonstrated in appendix B, we immediately obtain

$$U_1 = C_1 \ , \ U_2 = C_2 \ , \ U_3 = C_3 \ , \tag{3.121}$$

where C_1, C_2 and C_3 are constants. Using metric (3.103) we immediately obtain

$$U^1 = -\frac{C_1}{V^2} \ , \ U^2 = -\frac{C_2 V^2}{E_0^2 T^2} \ , \ U^3 = -\frac{C_3}{V^2} \ . \tag{3.122}$$

The velocity corresponding to the temporal coordinate can be easily obtained by imposing a normalisation condition, that is

$$U^\mu U_\mu = 1 \;\Rightarrow\; V^2 \left(U^0\right)^2 = 1 + \frac{C_1^2}{V^2} + \frac{C_2^2 V^2}{E_0^2 T^2} + \frac{C_3^2}{V^2} \,. \tag{3.123}$$

From this condition we obtain that $U^0 = \dfrac{dt}{d\tau}$ is nothing but the γ factor divided by V. So, by using the chain rule we obtain

$$v^1 = \frac{dx}{dt} = -\frac{C_1}{\gamma V} \;,\; v^2 = \frac{dy}{dt} = -\frac{C_2 V^3}{\gamma E_0^2 T^2} \;,\; v^3 = \frac{dz}{dt} = -\frac{C_3}{\gamma V} \;, \tag{3.124}$$

using the tetrads (3.105) we obtain

$$v^{\hat{1}} = \frac{dx}{dt} = -\frac{C_1}{\gamma} \;,\; v^{\hat{2}} = \frac{dy}{dt} = -\frac{C_2 V}{\gamma E_0 T} \;,\; v^{\hat{3}} = \frac{dz}{dt} = -\frac{C_3}{\gamma} \;,$$

and lowering the indices we finally obtain

$$v_{\hat{1}} = \frac{dx}{dt} = \frac{C_1}{\gamma} \;,\; v_{\hat{2}} = \frac{dy}{dt} = \frac{C_2 V^2}{\gamma E_0 T} \;,\; v_{\hat{3}} = \frac{dz}{dt} = \frac{C_3}{\gamma} \,. \tag{3.125}$$

By using

$$\mathcal{F}_2^{\hat{2}} \dot{\mathcal{F}}_{\hat{2}}^2 = \frac{2\dot{V}T - V}{VT} \;,$$

we finally obtain

$$\Omega^{\hat{1}}_{(2)} = -\frac{C_2 C_3}{\gamma\,(1+\gamma)} \frac{2VV'T - V^2}{E_0 T^2} \tag{3.126}$$

and

$$\Omega^{\hat{3}}_{(2)} = \frac{C_2 C_1}{\gamma\,(1+\gamma)} \frac{2VV'T - V^2}{E_0 T^2} \,. \tag{3.127}$$

We are interested in solving the equations for spin operators near the early universe $T \to 0$, therefore we must find an asymptotic expression [21] for the γ factor:

$$\gamma \underset{T\to 0}{\sim} \frac{C_2}{E_0 T} \;\Rightarrow\; \gamma\,(1+\gamma) \underset{T\to 0}{\sim} \gamma^2 \underset{T\to 0}{\sim} \frac{C_2^2}{E_0^2 T^2} \,.$$

By using these asymptotic expressions in (3.126) and (3.127) we obtain

$$\Omega^{\hat{1}}_{(2)} \underset{T\to 0}{\sim} -\frac{C_3}{C_2} E_0 \left(2VV'T - V^2\right) \underset{T\to 0}{\sim} \frac{C_3}{C_2} E_0 \tag{3.128}$$

and

$$\Omega^{\hat{3}}_{(2)} \underset{T\to 0}{\sim} \frac{C_1}{C_2} E_0 \left(2VV'T - V^2\right) \underset{T\to 0}{\sim} -\frac{C_1}{C_2} E_0 \ , \tag{3.129}$$

as a consequence of $V \underset{T\to 0}{\sim} 1$. So we have obtained that the angular velocity operators are constant in the early universe. This is interesting because it is in complete agreement with what we found for the standard Melvin metric, where the angular velocity operators did depend on the radial coordinate only, so they were constant in time. So the solution

3.2 Double Kasner space-time.

The double Kasner space-time is a generalisation of the standard Kasner solution of Einstein vacuum field equations [22] [11]. The standard Kasner metric is described in some details in appendix A, and it is shown that it can be found as a vacuum solution starting from a Bianchi I-type metric. The double Kasner metric generalises the standard Kasner metric by introducing a dependence from a spacial coordinate for the metric coefficients. It can be shown [11] that if we perform a non-linear superposition of the space-like Kasner metric and the time-like Kasner metric (both described in appendix A) we obtain the double Kasner metric. In this section we study this metric in depth, solve the geodesics equations asymptotically near the origin of time $t \to 0$, and solve the equations of motion of average spin. We therefore obtain an exhaustive description of spin motion in the early universe in this space-time.

Let's consider the double Kasner space-time described by the line element

$$ds^2 = e^{2q_0 x} dt^2 - g_1^2 dx^2 - g_2^2 dy^2 - g_3^2 dz^2 \ , \tag{3.130}$$

where

$$g_i = e^{q_i x} t^{p_i} \ , \ i = 1, 2, 3 \ . \tag{3.131}$$

Here p_i are constants satisfying

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \ , \tag{3.132}$$

$$q_0 + q_2 + q_3 = q_1 \ , \ q_0^2 + q_2^2 + q_3^2 = q_1^2 \ , \tag{3.133}$$

$$q_0 \left(p_2 + p_3\right) = q_2 \left(p_2 - p_1\right) + q_3 \left(p_3 - p_1\right) \ . \tag{3.134}$$

Analogously to the study of the Melvin metric, here we will consider $c = 1$ and interchangeably use $1, 2, 3, 4$ or t, x, y, z respectively to represent indices.

Using the metric

$$g_{\mu\nu} = \begin{pmatrix} e^{2q_0x} & 0 & 0 & 0 \\ 0 & -e^{2q_1x}t^{2p_1} & 0 & 0 \\ 0 & 0 & -e^{2q_2x}t^{2p_2} & 0 \\ 0 & 0 & 0 & -e^{2q_3x}t^{2p_3} \end{pmatrix} \tag{3.135}$$

and the Schwinger gauge we obtain the tetrads

$$e^a_\mu = \begin{pmatrix} e^{q_0x} & 0 & 0 & 0 \\ 0 & e^{q_1x}t^{p_1} & 0 & 0 \\ 0 & 0 & e^{q_2x}t^{p_2} & 0 \\ 0 & 0 & 0 & e^{q_3x}t^{p_3} \end{pmatrix} \tag{3.136}$$

and their inverses

$$e^\mu_a = \begin{pmatrix} e^{-q_0x} & 0 & 0 & 0 \\ 0 & e^{-q_1x}t^{-p_1} & 0 & 0 \\ 0 & 0 & e^{-q_2x}t^{-p_2} & 0 \\ 0 & 0 & 0 & e^{-q_3x}t^{-p_3} \end{pmatrix} . \tag{3.137}$$

Now we compare these expressions with (2.2) to obtain the coefficients of parametrization (2.1) in the Schwinger gauge:

$$\begin{aligned}
e^{\hat{0}}_0 &= e^{q_0x} = V \ ; \\
e^{\hat{1}}_1 &= e^{q_1x}t^{p_1} = W^{\hat{1}}_\beta\left(\delta^\beta_1 - K^\beta\delta^0_1\right) = W^{\hat{1}}_1 \ \Rightarrow \ W^{\hat{1}}_1 = e^{q_1x}t^{p_1} \ ; \\
e^{\hat{2}}_2 &= e^{q_2x}t^{p_2} = W^{\hat{2}}_\beta\left(\delta^\beta_2 - K^\beta\delta^0_2\right) = W^{\hat{2}}_2 \ \Rightarrow \ W^{\hat{2}}_2 = e^{q_2x}t^{p_2} \ ; \\
e^{\hat{3}}_3 &= e^{q_3x}t^{p_3} = W^{\hat{3}}_\beta\left(\delta^\beta_3 - K^\beta\delta^0_3\right) = W^{\hat{3}}_3 \ \Rightarrow \ W^{\hat{3}}_3 = e^{q_3x}t^{p_3} \ ; \\
e^{\hat{1}}_2 &= 0 = W^{\hat{1}}_\beta\left(\delta^\beta_2 - K^\beta\delta^0_2\right) = W^{\hat{1}}_2 \ \Rightarrow \ W^{\hat{1}}_2 = 0 \ ; \\
e^{\hat{1}}_3 &= 0 = W^{\hat{1}}_\beta\left(\delta^\beta_3 - K^\beta\delta^0_3\right) = W^{\hat{1}}_3 \ \Rightarrow \ W^{\hat{1}}_3 = 0 \ ; \\
e^{\hat{1}}_0 &= 0 = W^{\hat{1}}_\beta\left(\delta^\beta_0 - K^\beta\delta^0_0\right) = -\left(K^1W^{\hat{1}}_1 + K^2W^{\hat{1}}_2 + K^3W^{\hat{1}}_3\right) \ \Rightarrow \ K^1 = 0 \ ; \\
e^{\hat{2}}_1 &= 0 = W^{\hat{2}}_\beta\left(\delta^\beta_1 - K^\beta\delta^0_1\right) = W^{\hat{2}}_1 \ \Rightarrow \ W^{\hat{2}}_1 = 0 \ ; \\
e^{\hat{2}}_3 &= 0 = W^{\hat{2}}_\beta\left(\delta^\beta_3 - K^\beta\delta^0_3\right) = W^{\hat{2}}_3 \ \Rightarrow \ W^{\hat{2}}_3 = 0 \ ; \\
e^{\hat{2}}_0 &= 0 = W^{\hat{2}}_\beta\left(\delta^\beta_0 - K^\beta\delta^0_0\right) = -\left(K^1W^{\hat{2}}_1 + K^2W^{\hat{2}}_2 + K^3W^{\hat{2}}_3\right) \ \Rightarrow \ K^2 = 0 \ ; \\
e^{\hat{3}}_1 &= 0 = W^{\hat{3}}_\beta\left(\delta^\beta_1 - K^\beta\delta^0_1\right) = W^{\hat{3}}_1 \ \Rightarrow \ W^{\hat{3}}_1 = 0 \ ; \\
e^{\hat{3}}_2 &= 0 = W^{\hat{3}}_\beta\left(\delta^\beta_2 - K^\beta\delta^0_2\right) = W^{\hat{3}}_2 \ \Rightarrow \ W^{\hat{3}}_2 = 0 \ ; \\
e^{\hat{3}}_0 &= 0 = W^{\hat{3}}_\beta\left(\delta^\beta_0 - K^\beta\delta^0_0\right) = -\left(K^1W^{\hat{3}}_1 + K^2W^{\hat{3}}_2 + K^3W^{\hat{3}}_3\right) \ \Rightarrow \ K^3 = 0 \ .
\end{aligned}$$

So, we obtained

$$W^{\hat\alpha}_{\beta} = \begin{pmatrix} e^{q_1 x}t^{p_1} & 0 & 0 \\ 0 & e^{q_2 x}t^{p_2} & 0 \\ 0 & 0 & e^{q_3 x}t^{p_3} \end{pmatrix} \ , \tag{3.138}$$

its inverse

$$W^{\beta}_{\hat\alpha} = \begin{pmatrix} e^{-q_1 x}t^{-p_1} & 0 & 0 \\ 0 & e^{-q_2 x}t^{-p_2} & 0 \\ 0 & 0 & e^{-q_3 x}t^{-p_3} \end{pmatrix} \tag{3.139}$$

and

$$K^{\alpha} = 0 \ , \ \alpha = 1, 2, 3 \ . \tag{3.140}$$

3.2.1 Double Kasner solution as a vacuum solution.

Appendix A contains the derivation of the Kasner solution from the Einstein equations in vacuum, and it is analysed both in timelike and spacelike forms. We expect the double Kasner metric, as a superposition of these two forms, to satisfy the vacuum equations as well, and this is what we demonstrate in this section.

First, we work out the coefficients of the affine connection

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(g_{\mu\lambda},_{\nu} + g_{\nu\lambda},_{\mu} - g_{\mu\nu},_{\lambda} \right) \ ,$$

and by calculating each component explicitly we find:

$$\Gamma^{0}_{\mu\nu} = \frac{1}{2} g^{0\lambda} \left(g_{\mu\lambda},_{\nu} + g_{\nu\lambda},_{\mu} - g_{\mu\nu},_{\lambda} \right) = \frac{1}{2} e^{-2q_0 x} \left(g_{\mu 0},_{\nu} + g_{\nu 0},_{\mu} - g_{\mu\nu},_{0} \right) =$$

$$= \frac{e^{-2q_0 x}}{2} \begin{pmatrix} 0 & 2q_0 e^{2q_0 x} & 0 & 0 \\ 2q_0 e^{2q_o x} & 2p_1 e^{2q_1 x} t^{2p_1 - 1} & 0 & 0 \\ 0 & 0 & 2p_2 e^{2q_2 x} t^{2p_2 - 1} & 0 \\ 0 & 0 & 0 & 2p_3 e^{2q_3 x} t^{2p_3 - 1} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & q_0 & 0 & 0 \\ q_0 & p_1 e^{2(q_1 - q_0)x} t^{2p_1 - 1} & 0 & 0 \\ 0 & 0 & p_2 e^{2(q_2 - q_0)x} t^{2p_2 - 1} & 0 \\ 0 & 0 & 0 & p_3 e^{2(q_3 - q_0)x} t^{2p_3 - 1} \end{pmatrix} \ ; \tag{3.141}$$

$$\Gamma^{1}_{\mu\nu} = \frac{1}{2} g^{1\lambda} \left(g_{\mu\lambda},_{\nu} + g_{\nu\lambda},_{\mu} - g_{\mu\nu},_{\lambda} \right) = -\frac{e^{-2q_1 x} t^{-2p_1}}{2} \left(g_{\mu 1},_{\nu} + g_{\nu 1},_{\mu} - g_{\mu\nu},_{1} \right) =$$

$$= -\frac{e^{-2q_1 x} t^{-2p_1}}{2} \begin{pmatrix} -2q_0 e^{2q_0 x} & -2p_1 e^{2q_1 x} t^{2p_1 - 1} & 0 & 0 \\ -2p_1 e^{2q_1 x} t^{2p_1 - 1} & -2q_1 e^{2q_1 x} t^{2p_1 - 1} & 0 & 0 \\ 0 & 0 & 2q_2 e^{2q_2 x} t^{2p_2} & 0 \\ 0 & 0 & 0 & 2q_3 e^{2q_3 x} t^{2p_3} \end{pmatrix} =$$

$$= \begin{pmatrix} q_0 e^{2(q_0-q_1)x} t^{-2p_1} & \frac{p_1}{t} & 0 & 0 \\ \frac{p_1}{t} & q_1 & 0 & 0 \\ 0 & 0 & -q_2 e^{2(q_2-q_1)x} t^{2(p_2-p_1)} & 0 \\ 0 & 0 & 0 & -q_3 e^{2(q_3-q_1)x} t^{2(p_3-p_1)} \end{pmatrix} ; \tag{3.142}$$

$$\Gamma^2_{\mu\nu} = \frac{1}{2} g^{2\lambda} \left(g_{\mu\lambda},_\nu + g_{\nu\lambda},_\mu - g_{\mu\nu},_\lambda \right) = -\frac{e^{-2q_2x} t^{-2p_2}}{2} \left(g_{\mu 2},_\nu + g_{\nu 2},_\mu \right) =$$

$$= -\frac{e^{-2q_2x} t^{-2p_2}}{2} \begin{pmatrix} 0 & 0 & -2p_2 e^{2q_2x} t^{2p_2-1} & 0 \\ 0 & 0 & -2q_2 e^{2q_2x} t^{2p_2} & 0 \\ -2p_2 e^{2q_2x} t^{2p_2-1} & -2q_2 e^{2q_2x} t^{2p_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & \frac{p_2}{t} & 0 \\ 0 & 0 & q_2 & 0 \\ \frac{p_2}{t} & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \tag{3.143}$$

$$\Gamma^3_{\mu\nu} = \frac{1}{2} g^{3\lambda} \left(g_{\mu\lambda},_\nu + g_{\nu\lambda},_\mu - g_{\mu\nu},_\lambda \right) = -\frac{e^{-2q_3x} t^{-2p_3}}{2} \left(g_{\mu 3},_\nu + g_{\nu 3},_\mu \right) =$$

$$= -\frac{e^{-2q_3x} t^{-2p_3}}{2} \begin{pmatrix} 0 & 0 & 0 & -2p_3 e^{2q_3x} t^{2p_3-1} \\ 0 & 0 & 0 & -2q_3 e^{2q_3x} t^{2p_3} \\ 0 & 0 & 0 & 0 \\ -2p_3 e^{2q_3x} t^{2p_3-1} & -2q_3 e^{2q_3x} t^{2p_3} & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \frac{p_3}{t} \\ 0 & 0 & 0 & q_3 \\ 0 & 0 & 0 & 0 \\ \frac{p_3}{t} & q_3 & 0 & 0 \end{pmatrix} . \tag{3.144}$$

To calculate the components of the Ricci tensor we use (3.77), so, by making extensive use of (3.132) $\div$ (3.134), each component reads explicitly:

$$R_{00} = \Gamma^\rho_{00,\rho} - \Gamma^\rho_{0\rho,0} + \Gamma^\rho_{00}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{0\rho}\Gamma^\rho_{0\lambda} =$$

$$= \Gamma^0_{00,0} + \Gamma^1_{00,1} - \Gamma^0_{00,0} - \Gamma^1_{01,0} - \Gamma^2_{02,0} - \Gamma^3_{03,0} + \Gamma^1_{00}\Gamma^\lambda_{1\lambda} - \Gamma^0_{0\rho}\Gamma^\rho_{00} - \Gamma^1_{0\rho}\Gamma^\rho_{01} -$$

$$-\Gamma^2_{0\rho}\Gamma^\rho_{02} - \Gamma^3_{0\rho}\Gamma^\rho_{03} = 2q_0 \left(q_0 - q_1 \right) e^{2(q_0-q_1)x} t^{-2p_1} + \frac{p_1}{r} + \frac{p_2}{t} + \frac{p_3}{t} +$$

$$+ q_0 e^{2(q_0-q_1)x} t^{-2p_1} \left(q_0 + q_1 + q_2 + q_3 \right) - q_0^2 e^{2(q_0-q_1)x} t^{-2p_1} -$$

$$- \left(q_0^2 e^{2(q_0-q_1)x} t^{-2p_1} + \frac{p_1^2}{t^2} \right) - \frac{p_2^2}{t^2} - \frac{p_3^2}{t^2} = 0 ; \tag{3.145}$$

$$R_{11} = \Gamma^\rho_{11,\rho} - \Gamma^\rho_{1\rho,1} + \Gamma^\rho_{11}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{1\rho}\Gamma^\rho_{1\lambda} =$$

$$= \Gamma^0_{11,0} + \Gamma^1_{11,1} - \Gamma^0_{10,1} - \Gamma^1_{11,1} - \Gamma^2_{12,1} - \Gamma^3_{13,1} + \Gamma^0_{11}\Gamma^\lambda_{0\lambda} + \Gamma^1_{11}\Gamma^\lambda_{1\lambda} - \Gamma^0_{1\rho}\Gamma^\rho_{10} - \Gamma^2_{1\rho}\Gamma^\rho_{12} -$$

$$-\Gamma^2_{1\rho}\Gamma^\rho_{12} - \Gamma^3_{1\rho}\Gamma^\rho_{13} = (2p_1 - 1)\, p_1 e^{2(q_1-q_0)x} t^{2p_1-2} + p_1 e^{2(q_1-q_0)x} t^{2p_1-1} \left(\frac{p_1}{t} + \right.$$

$$\left. + \frac{p_2}{t} + \frac{p_3}{t}\right) + q_1 (q_0 + q_1 + q_2 + q_3) - \left(q_0^2 + p_1^2 e^{2(q_1-q_0)x} t^{2p_1-2}\right) -$$

$$- \left(p_1^2 e^{2(q_1-q_0)x} t^{2p_1-2} + q_1^2\right) - q_2^2 - q_3^2 = 0\ ; \qquad (3.146)$$

$$R_{22} = \Gamma^\rho_{22,\rho} - \Gamma^\rho_{2\rho,2} + \Gamma^\rho_{22}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{2\rho}\Gamma^\rho_{2\lambda} =$$

$$= \Gamma^0_{22,0} + \Gamma^1_{22,1} + \Gamma^0_{22}\Gamma^\lambda_{0\lambda} + \Gamma^1_{22}\Gamma^\lambda_{1\lambda} - \Gamma^0_{2\rho}\Gamma^\rho_{20} - \Gamma^1_{2\rho}\Gamma^\rho_{21} - \Gamma^2_{2\rho}\Gamma^\rho_{22} - \Gamma^3_{2\rho}\Gamma^\rho_{23} =$$

$$= (2p_2 - 1)\, p_2 e^{2(q_2-q_0)x} t^{2p_2-2} - 2q_2^2 e^{2(q_2-q_1)x} t^{2(p_2-p_1)} + p_2 e^{2(q_2-q_0)x} t^{2p_2-1} \left(\frac{p_1}{t} + \right.$$

$$\left. + \frac{p_2}{t} + \frac{p_3}{t}\right) - q_2 e^{2(q_2-q_1)x} t^{2(p_2-p_1)} (q_0 + q_1 + q_2 + q_3) - p_2^2 e^{2(q_2-q_0)x} t^{2p_2-2} +$$

$$+ q_2^2 e^{2(q_2-q_1)x} t^{2(p_2-p_1)} - \left(p_2^2 e^{2(q_2-q_0)x} t^{2p_2-2} - q_2^2 e^{2(q_2-q_1)x} t^{2(p_2-p_1)}\right) = 0\ ; \quad (3.147)$$

$$R_{33} = \Gamma^\rho_{33,\rho} - \Gamma^\rho_{3\rho,3} + \Gamma^\rho_{33}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{3\rho}\Gamma^\rho_{3\lambda} =$$

$$= \Gamma^0_{33,0} + \Gamma^1_{33,1} + \Gamma^0_{33}\Gamma^\lambda_{0\lambda} + \Gamma^1_{33}\Gamma^\lambda_{1\lambda} - \Gamma^0_{3\rho}\Gamma^\rho_{30} - \Gamma^1_{3\rho}\Gamma^\rho_{31} - \Gamma^2_{3\rho}\Gamma^\rho_{32} - \Gamma^3_{3\rho}\Gamma^\rho_{33} =$$

$$= (2p_3 - 1)\, p_3 e^{2(q_3-q_0)x} t^{2p_3-2} - 2q_3^2 e^{2(q_3-q_1)x} t^{2(p_3-p_1)} + p_3 e^{2(q_3-q_0)x} t^{2p_3-1} \left(\frac{p_1}{t} + \right.$$

$$\left. + \frac{p_2}{t} + \frac{p_3}{t}\right) - q_3 e^{2(q_3-q_1)x} t^{2(p_3-p_1)} (q_0 + q_1 + q_2 + q_3) - p_3^2 e^{2(q_3-q_0)x} t^{2p_3-2} +$$

$$+ q_3^2 e^{2(q_3-q_1)x} t^{2(p_3-p_1)} - \left(p_3^2 e^{2(q_3-q_0)x} t^{2p_3-2} - q_3^2 e^{2(q_3-q_1)x} t^{2(p_3-p_1)}\right) = 0\ ; \quad (3.148)$$

$$R_{01} = \Gamma^\rho_{01,\rho} - \Gamma^\rho_{0\rho,1} + \Gamma^\rho_{01}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{0\rho}\Gamma^\rho_{1\lambda} =$$

$$= \Gamma^0_{01}\Gamma^\lambda_{\lambda 0} + \Gamma^1_{01}\Gamma^\lambda_{\lambda 1} - \Gamma^0_{0\rho}\Gamma^\rho_{10} - \Gamma^1_{0\rho}\Gamma^\rho_{11} - \Gamma^2_{0\rho}\Gamma^\rho_{12} - \Gamma^3_{0\rho}\Gamma^\rho_{13} = q_0 \left(\frac{p_1}{t} + \frac{p_2}{t} + \frac{p_3}{t}\right) +$$

$$+ \frac{p_1}{t}(q_0 + q_1 + q_2 + q_3) - \frac{q_0 p_1}{t} - \left(\frac{q_0 p_1}{t} + \frac{p_1 q_1}{t}\right) - \frac{q_2 p_2}{t} - \frac{p_3 q_3}{t} = 0\ ; \quad (3.149)$$

$$R_{02} = \Gamma^\rho_{02,\rho} - \Gamma^\rho_{0\rho,2} + \Gamma^\rho_{02}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{0\rho}\Gamma^\rho_{2\lambda} =$$

$$= \Gamma^2_{02}\Gamma^\lambda_{\lambda 2} - \Gamma^0_{0\rho}\Gamma^\rho_{20} - \Gamma^1_{0\rho}\Gamma^\rho_{21} - \Gamma^2_{0\rho}\Gamma^\rho_{22} - \Gamma^3_{0\rho}\Gamma^\rho_{23} = 0\ ; \qquad (3.150)$$

$$R_{03} = \Gamma^\rho_{03,\rho} - \Gamma^\rho_{0\rho,3} + \Gamma^\rho_{03}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{0\rho}\Gamma^\rho_{3\lambda} =$$

$$= \Gamma^3_{03}\Gamma^\lambda_{\lambda 3} - \Gamma^0_{0\rho}\Gamma^\rho_{30} - \Gamma^1_{0\rho}\Gamma^\rho_{31} - \Gamma^2_{0\rho}\Gamma^\rho_{32} - \Gamma^3_{0\rho}\Gamma^\rho_{33} = 0\ ; \qquad (3.151)$$

$$R_{12} = \Gamma^\rho_{12,\rho} - \Gamma^\rho_{1\rho,2} + \Gamma^\rho_{12}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{1\rho}\Gamma^\rho_{2\lambda} =$$

$$= \Gamma^2_{12}\Gamma^\lambda_{\lambda 2} - \Gamma^0_{1\rho}\Gamma^\rho_{20} - \Gamma^1_{1\rho}\Gamma^\rho_{21} - \Gamma^2_{1\rho}\Gamma^\rho_{22} - \Gamma^3_{1\rho}\Gamma^\rho_{23} = 0\ ; \qquad (3.152)$$

$$R_{13} = \Gamma^\rho_{13,\rho} - \Gamma^\rho_{1\rho,3} + \Gamma^\rho_{13}\Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{1\rho}\Gamma^\rho_{3\lambda} =$$

$$= \Gamma^{3}_{13}\Gamma^{\lambda}_{\lambda 3} - \Gamma^{0}_{1\rho}\Gamma^{\rho}_{30} - \Gamma^{1}_{1\rho}\Gamma^{\rho}_{31} - \Gamma^{2}_{1\rho}\Gamma^{\rho}_{32} - \Gamma^{3}_{1\rho}\Gamma^{\rho}_{33} = 0 \ ; \tag{3.153}$$

$$R_{23} = \Gamma^{\rho}_{23,\rho} - \Gamma^{\rho}_{2\rho,3} + \Gamma^{\rho}_{23}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{2\rho}\Gamma^{\rho}_{3\lambda} =$$

$$= -\Gamma^{0}_{2\rho}\Gamma^{\rho}_{30} - \Gamma^{1}_{2\rho}\Gamma^{\rho}_{31} - \Gamma^{2}_{2\rho}\Gamma^{\rho}_{32} - \Gamma^{3}_{2\rho}\Gamma^{\rho}_{33} = 0 \ . \tag{3.154}$$

We have thus obtained $R_{\mu\nu} = 0$, as expected, so we have demonstrated explicitly that (3.130) is a solution of the vacuum Einstein field equations.

3.2.2 Calculation of angular velocity operators.

Now, our goal is to find an expression for the angular velocity operators, so that we can study the equation of motion of spin operators in this particular space-time. As we did in section 3.1.1, we have to work out the terms appearing in the Hamiltonian (2.20). First of all, we calculate the components of spin connection

$$\tilde{\omega}_{\mu\hat{\alpha}\hat{0}} = \frac{1}{V} W^{\beta}_{\hat{\alpha}} \partial_\beta V e^{\hat{0}}_{\mu} - \frac{1}{V}\mathcal{Q}_{(\hat{\alpha}\hat{\beta})} e^{\hat{\beta}}_{\mu}$$

and

$$\tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}} = \frac{1}{V}\mathcal{Q}_{[\hat{\alpha}\hat{\beta}]} e^{\hat{0}}_{\mu} + \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + \mathcal{C}_{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \mathcal{C}_{\hat{\gamma}\hat{\beta}\hat{\alpha}}\right) e^{\hat{\gamma}}_{\mu} \ .$$

In order to perform these calculations we have to work out $\mathcal{Q}_{\hat{\alpha}\hat{\beta}}$:

$$\mathcal{Q}_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\gamma}} W^{\delta}_{\hat{\beta}} \left(\dot{W}^{\hat{\gamma}}_{\delta} + K^{\epsilon}\partial_\epsilon W^{\hat{\gamma}}_{\delta} + W^{\hat{\gamma}}_{\epsilon}\partial_\delta K^{\epsilon}\right) = \eta_{\hat{\alpha}\hat{\gamma}} W^{\delta}_{\hat{\beta}} \dot{W}^{\hat{\gamma}}_{\delta} =$$

$$= \eta_{\hat{\alpha}\hat{\gamma}} W^{1}_{\hat{\beta}} \dot{W}^{\hat{\gamma}}_{1} + \eta_{\hat{\alpha}\hat{\gamma}} W^{2}_{\hat{\beta}} \dot{W}^{\hat{\gamma}}_{2} + \eta_{\hat{\alpha}\hat{\gamma}} W^{3}_{\hat{\beta}} \dot{W}^{\hat{\gamma}}_{3} = \eta_{\hat{\alpha}\hat{1}} W^{1}_{\hat{\beta}} \dot{W}^{\hat{1}}_{1} + \eta_{\hat{\alpha}\hat{2}} W^{2}_{\hat{\beta}} \dot{W}^{\hat{2}}_{2} +$$

$$+ \eta_{\hat{\alpha}\hat{3}} W^{3}_{\hat{\beta}} \dot{W}^{\hat{3}}_{3} = \begin{pmatrix} -\frac{p_1}{t} & 0 & 0 \\ 0 & -\frac{p_2}{t} & 0 \\ 0 & 0 & -\frac{p_3}{t} \end{pmatrix} \ . \tag{3.155}$$

We observe that the matrix $\mathcal{Q}_{\hat{\alpha}\hat{\beta}}$ is diagonal and therefore symmetric; this will be useful later. The coefficients of the spin connection read explicitly:

$$\text{for } \mu = 0 \ \Rightarrow \ \tilde{\omega}_{0\hat{\alpha}\hat{0}} = e^{-q_0 x} W^{\beta}_{\hat{\alpha}} \partial_\beta \left(e^{q_0 x}\right) e^{q_0 x} = W^{x}_{\hat{\alpha}} \partial_x e^{q_0 x} = q_0 e^{q_0 x} W^{x}_{\hat{\alpha}} =$$

$$= \begin{cases} q_0 e^{(q_0 - q_1)x} t^{-p_1} \ , & \text{if } \hat{\alpha} = x \ , \\ 0 \ , & \text{if } \hat{\alpha} = y, z \ ; \end{cases} \tag{3.156}$$

$$\text{for } \mu = 1 \ \Rightarrow \ \tilde{\omega}_{1\hat{\alpha}\hat{0}} = -e^{-q_0 x}\mathcal{Q}_{(\hat{\alpha}\hat{\beta})} e^{\hat{\beta}}_{1} = -e^{-q_0 x}\mathcal{Q}_{(\hat{\alpha}\hat{1})} e^{\hat{1}}_{1} = -e^{(q_1 - q_0)x} t^{p_1} \mathcal{Q}_{(\hat{\alpha}\hat{1})} =$$

$$= \begin{cases} p_1 e^{(q_1-q_0)x} t^{p_1-1} \ , & \text{if } \hat{\alpha} = x \ , \\ 0 \ , & \text{if } \hat{\alpha} = y, z \ ; \end{cases} \tag{3.157}$$

for $\mu = 2 \ \Rightarrow \ \tilde{\omega}_{2\hat{\alpha}\hat{0}} = -e^{-q_0 x} \mathcal{Q}_{\left(\hat{\alpha}\hat{\beta}\right)} e_2^{\hat{\beta}} = -e^{-q_0 x} \mathcal{Q}_{\left(\hat{\alpha}\hat{2}\right)} e_2^{\hat{2}} = -e^{(q_2-q_0)x} t^{p_2} \mathcal{Q}_{\left(\hat{\alpha}\hat{2}\right)} =$

$$= \begin{cases} p_2 e^{(q_2-q_0)x} t^{p_2-1} \ , & \text{if } \hat{\alpha} = y \ , \\ 0 \ , & \text{if } \hat{\alpha} = x, z \ ; \end{cases} \tag{3.158}$$

for $\mu = 3 \ \Rightarrow \ \tilde{\omega}_{3\hat{\alpha}\hat{0}} = -e^{-q_0 x} \mathcal{Q}_{\left(\hat{\alpha}\hat{\beta}\right)} e_3^{\hat{\beta}} = -e^{-q_0 x} \mathcal{Q}_{\left(\hat{\alpha}\hat{3}\right)} e_3^{\hat{3}} = -e^{(q_3-q_0)x} t^{p_3} \mathcal{Q}_{\left(\hat{\alpha}\hat{3}\right)} =$

$$= \begin{cases} p_3 e^{(q_3-q_0)x} t^{p_3-1} \ , & \text{if } \hat{\alpha} = z \ , \\ 0 \ , & \text{if } \hat{\alpha} = x, y \ . \end{cases} \tag{3.159}$$

To calculate the spacial set of spin connection coefficients we first calculate the anholonimity coefficients:

$$\mathcal{C}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}} = W^{\delta}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \partial_{[\delta} W^{\hat{\gamma}}_{\epsilon]} = \frac{1}{2} W^{\delta}_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \left(\partial_{\delta} W^{\hat{\gamma}}_{\epsilon} - \partial_{\epsilon} W^{\hat{\gamma}}_{\delta} \right) =$$

$$= \frac{1}{2} \left[W^x_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \left(\partial_x W^{\hat{\gamma}}_{\epsilon} - \partial_{\epsilon} W^{\hat{\gamma}}_{x} \right) - W^y_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \partial_{\epsilon} W^{\hat{\gamma}}_{y} - W^z_{\hat{\alpha}} W^{\epsilon}_{\hat{\beta}} \partial_{\epsilon} W^{\hat{\gamma}}_{z} \right] =$$

$$= \frac{1}{2} \left[W^x_{\hat{\alpha}} W^x_{\hat{\beta}} \partial_x W^{\hat{\gamma}}_{x} + W^x_{\hat{\alpha}} W^y_{\hat{\beta}} \partial_x W^{\hat{\gamma}}_{y} + W^x_{\hat{\alpha}} W^z_{\hat{\beta}} \partial_x W^{\hat{\gamma}}_{z} - \right.$$

$$\left. - W^x_{\hat{\alpha}} W^x_{\hat{\beta}} \partial_x W^{\hat{\gamma}}_{x} - W^y_{\hat{\alpha}} W^x_{\hat{\beta}} \partial_x W^{\hat{\gamma}}_{y} - W^z_{\hat{\alpha}} W^x_{\hat{\beta}} \partial_x W^{\hat{\gamma}}_{z} \right] =$$

$$= W^x_{[\hat{\alpha}} W^z_{\hat{\beta}]} \partial_x W^{\hat{\gamma}}_{z} + W^x_{[\hat{\alpha}} W^y_{\hat{\beta}]} \partial_x W^{\hat{\gamma}}_{y} \ , \tag{3.160}$$

so explicitly we obtain:

$$\text{if } \ \hat{\gamma} = \hat{x} \ \Rightarrow \ \mathcal{C}^{\hat{x}}_{\hat{\alpha}\hat{\beta}} = 0 \ , \text{ because } W \text{ is diagonal } \ ; \tag{3.161}$$

$$\text{if } \ \hat{\gamma} = \hat{y} \ \Rightarrow \ \mathcal{C}^{\hat{y}}_{\hat{\alpha}\hat{\beta}} = W^x_{[\hat{\alpha}} W^y_{\hat{\beta}]} \partial_x W^{\hat{y}}_{y} = q_2 e^{q_2 x} t^{p_2} \frac{1}{2} \left(W^x_{\hat{\alpha}} W^y_{\hat{\beta}} - W^x_{\hat{\beta}} W^y_{\hat{\alpha}} \right) =$$

$$= \frac{q_2}{2} e^{q_2 x} t^{p_2} \begin{pmatrix} 0 & e^{-(q_1+q_2)x} t^{-(p_1+p_2)} & 0 \\ -e^{-(q_1+q_2)x} t^{-(p_1+p_2)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \frac{q_2}{2} e^{-q_1 x} t^{-p_1} & 0 \\ -\frac{q_2}{2} e^{-q_1 x} t^{-p_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ; \tag{3.162}$$

$$\text{if } \ \hat{\gamma} = \hat{z} \ \Rightarrow \ \mathcal{C}^{\hat{z}}_{\hat{\alpha}\hat{\beta}} = W^x_{[\hat{\alpha}} W^z_{\hat{\beta}]} \partial_x W^{\hat{z}}_{z} = q_3 e^{q_3 x} t^{p_3} \frac{1}{2} \left(W^x_{\hat{\alpha}} W^z_{\hat{\beta}} - W^x_{\hat{\beta}} W^z_{\hat{\alpha}} \right) =$$

$$= \frac{q_3}{2} e^{q_3 x} t^{p_3} \begin{pmatrix} 0 & 0 & e^{-(q_1+q_3)x} t^{-(p_1+p_3)} \\ 0 & 0 & 0 \\ -e^{-(q_1+q_3)x} t^{-(p_1+p_3)} & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & \frac{q_3}{2} e^{-q_1 x} t^{-p_1} \\ 0 & 0 & 0 \\ -\frac{q_2}{2} e^{-q_1 x} t^{-p_1} & 0 & 0 \end{pmatrix} . \qquad (3.163)$$

As we did in (3.16), we lower the indices using the spacial Minkowski metric $diag\,(-1,-1,-1)$, so we have:

$$\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{x}} = 0 \ , \ \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{y}} = -\mathcal{C}^{\hat{y}}_{\hat{\alpha}\hat{\beta}} \ , \ \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{z}} = -\mathcal{C}^{\hat{z}}_{\hat{\alpha}\hat{\beta}} \ . \qquad (3.164)$$

Now we can calculate the spacial set of the spin connection; we use (2.13) and, remembering that $\mathcal{Q}$ is symmetric, the skew-symmetric part of $\mathcal{Q}$ is 0, so we have

$$\tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}} = \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + \mathcal{C}_{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \mathcal{C}_{\hat{\gamma}\hat{\beta}\hat{\alpha}} \right) e^{\hat{\gamma}}_{\mu} \ , \qquad (3.165)$$

therefore explicitly we have

$$\text{if} \ \ \mu = 0 \ \Rightarrow \ \tilde{\omega}_{0\hat{\alpha}\hat{\beta}} = 0 \ ; \qquad (3.166)$$

$$\text{if} \ \ \mu = 1 \ \Rightarrow \ \tilde{\omega}_{1\hat{\alpha}\hat{\beta}} = \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + \mathcal{C}_{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \mathcal{C}_{\hat{\gamma}\hat{\beta}\hat{\alpha}} \right) e^{\hat{\gamma}}_{1} = e^{q_1 x} t^{p_1} \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{1}} + \mathcal{C}_{\hat{\alpha}\hat{1}\hat{\beta}} + \right.$$

$$\left. + \mathcal{C}_{\hat{1}\hat{\beta}\hat{\alpha}} \right) = -e^{q_1 x} t^{p_1} \left(\mathcal{C}^{\hat{1}}_{\hat{\alpha}\hat{\beta}} + \mathcal{C}^{\hat{\beta}}_{\hat{\alpha}\hat{1}} + \mathcal{C}^{\hat{\alpha}}_{\hat{1}\hat{\beta}} \right) =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ; \qquad (3.167)$$

$$\text{if} \ \ \mu = 2 \ \Rightarrow \ \tilde{\omega}_{2\hat{\alpha}\hat{\beta}} = \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + \mathcal{C}_{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \mathcal{C}_{\hat{\gamma}\hat{\beta}\hat{\alpha}} \right) e^{\hat{\gamma}}_{2} = e^{q_2 x} t^{p_2} \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{2}} + \mathcal{C}_{\hat{\alpha}\hat{2}\hat{\beta}} + \right.$$

$$\left. + \mathcal{C}_{\hat{2}\hat{\beta}\hat{\alpha}} \right) = -e^{q_2 x} t^{p_2} \left(\mathcal{C}^{\hat{2}}_{\hat{\alpha}\hat{\beta}} + \mathcal{C}^{\hat{\beta}}_{\hat{\alpha}\hat{2}} + \mathcal{C}^{\hat{\alpha}}_{\hat{2}\hat{\beta}} \right) =$$

$$= \begin{pmatrix} 0 & -q_2 e^{(q_2-q_1)x} t^{p_2-p_1} & 0 \\ q_2 e^{(q_2-q_1)x} t^{p_2-p_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ; \qquad (3.168)$$

$$\text{if} \ \ \mu = 3 \ \Rightarrow \ \tilde{\omega}_{3\hat{\alpha}\hat{\beta}} = \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + \mathcal{C}_{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \mathcal{C}_{\hat{\gamma}\hat{\beta}\hat{\alpha}} \right) e^{\hat{\gamma}}_{3} = e^{q_3 x} t^{p_3} \left(\mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{3}} + \mathcal{C}_{\hat{\alpha}\hat{3}\hat{\beta}} + \right.$$

$$\left. + \mathcal{C}_{\hat{3}\hat{\beta}\hat{\alpha}} \right) = -e^{q_3 x} t^{p_3} \left(\mathcal{C}^{\hat{3}}_{\hat{\alpha}\hat{\beta}} + \mathcal{C}^{\hat{\beta}}_{\hat{\alpha}\hat{3}} + \mathcal{C}^{\hat{\alpha}}_{\hat{3}\hat{\beta}} \right) =$$

$$= \begin{pmatrix} 0 & 0 & -q_3 e^{(q_3-q_1)x} t^{p_3-p_1} \\ 0 & 0 & 0 \\ q_3 e^{(q_3-q_1)x} t^{p_3-p_1} & 0 & 0 \end{pmatrix} . \qquad (3.169)$$

So we have worked out the coefficients of the spin connection.

Now, we want to calculate the terms appearing in the Hamiltonian in order to work out the angular velocity operators in the F-W representation. Starting from (2.20), we consider each term separately:

$$\Xi_{\hat{\alpha}} = \epsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}} \mathcal{Q}^{\hat{\beta}\hat{\gamma}} = 0 \ , \tag{3.170}$$

because $\mathcal{Q}$ is symmetric and ϵ is skew-symmetric.

$$\mathcal{F}^{\alpha}_{\hat{\beta}} = VW^{\alpha}_{\hat{\beta}} = \begin{pmatrix} e^{(q_0-q_1)x} t^{-p_1} & 0 & 0 \\ 0 & e^{(q_0-q_2)x} t^{-p_2} & 0 \\ 0 & 0 & e^{(q_0-q_3)x} t^{-p_3} \end{pmatrix} \ ; \tag{3.171}$$

$$\Upsilon = -V\epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = -V\epsilon^{\hat{\alpha}\hat{\beta}\hat{y}} \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{y}} - V\epsilon^{\hat{\alpha}\hat{\beta}\hat{z}} \mathcal{C}_{\hat{\alpha}\hat{\beta}\hat{z}} = V\left(\epsilon^{\hat{\alpha}\hat{\beta}\hat{y}} \mathcal{C}^{\hat{y}}_{\hat{\alpha}\hat{\beta}} + \epsilon^{\hat{\alpha}\hat{\beta}\hat{z}} \mathcal{C}^{\hat{z}}_{\hat{\alpha}\hat{\beta}}\right) =$$

$$= V\left(\epsilon^{\hat{z}\hat{x}\hat{y}} \mathcal{C}^{\hat{y}}_{\hat{z}\hat{x}} + \epsilon^{\hat{x}\hat{z}\hat{y}} \mathcal{C}^{\hat{y}}_{\hat{x}\hat{z}} + \epsilon^{\hat{y}\hat{x}\hat{z}} \mathcal{C}^{\hat{z}}_{\hat{y}\hat{x}} + \epsilon^{\hat{x}\hat{y}\hat{z}} \mathcal{C}^{\hat{z}}_{\hat{x}\hat{y}}\right) = 0 \ . \tag{3.172}$$

Then the even and odd terms of the Hamiltonian read

$$\mathcal{M} = mV \ ; \tag{3.173}$$

$$\mathcal{E} = 0 \ ; \tag{3.174}$$

$$\mathcal{O} = \frac{1}{2}\left(p_\beta \mathcal{F}^{\beta}_{\hat{\alpha}} \alpha^{\hat{\alpha}} + \alpha^{\hat{\alpha}} \mathcal{F}^{\beta}_{\hat{\alpha}} p_\beta\right) = \frac{1}{2}\left(p_x \mathcal{F}^{x}_{\hat{x}} \alpha^{\hat{x}} + \alpha^{\hat{x}} \mathcal{F}^{x}_{\hat{x}} p_x + p_y \mathcal{F}^{y}_{\hat{y}} \alpha^{\hat{y}} + \alpha^{\hat{y}} \mathcal{F}^{y}_{\hat{y}} p_y + \right.$$

$$\left. + p_z \mathcal{F}^{z}_{\hat{z}} \alpha^{\hat{z}} + \alpha^{\hat{z}} \mathcal{F}^{z}_{\hat{z}} p_z\right) = e^{(q_0-q_1)x} t^{-p_1} \alpha^{\hat{x}} p_x + e^{(q_0-q_2)x} t^{-p_2} \alpha^{\hat{y}} p_y +$$

$$+ e^{(q_0-q_3)x} t^{-p_3} \alpha^{\hat{z}} p_z + \left(p_x e^{(q_0-q_1)x} t^{-p_1}\right) \alpha^{\hat{x}} \ . \tag{3.175}$$

We observe that p_α is a differential operator depending on ∂_α, so in (3.175) we move it from the left to the right of $\mathcal{F}$ using the well known Leibniz rule.

3.2.3 Semiclassical approach.

The dependence of the terms necessary to work out the angular velocity operators on the coordinates t and x is rather involved, and this makes the calculations quite complicated. To better understand the physical content of this model, it is preferable to study the semiclassical approximation first. The exact formulation will potentially be studied at a later date.

To proceed we refer to (2.32) and (2.33) together with (2.39), that in the case of the double Kasner space-time are reduced to

$$\Omega^{\hat{\alpha}}_{(1)} = \frac{1}{\epsilon'} \mathcal{F}^{\delta}_{\hat{\gamma}} p_\delta \left(-\epsilon^{\hat{\alpha}\hat{\epsilon}\hat{\chi}} V \mathcal{C}^{\hat{\gamma}}_{\hat{\epsilon}\hat{\chi}} + \frac{\gamma}{1+\gamma} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} W^{\epsilon}_{\hat{\beta}} \partial_\epsilon V\right) \ , \tag{3.176}$$

$$\Omega^{\hat{\alpha}}_{(2)} = -\frac{\gamma}{1+\gamma} v_{\hat{\eta}} v_{\hat{\gamma}} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \mathcal{Q}_{(\hat{\beta}\hat{\delta})} \delta^{\hat{\delta}\hat{\eta}} \ . \tag{3.177}$$

We calculate $\Omega^{\hat{\alpha}}_{(1)}$ first; we evaluate the two terms between brackets separately:

$$-\epsilon^{\hat{\alpha}\hat{\epsilon}\hat{\chi}} V \mathcal{C}^{\hat{\gamma}}_{\hat{\epsilon}\hat{\chi}} = -\epsilon^{\hat{\alpha}\hat{\epsilon}\hat{x}} V \mathcal{C}^{\hat{\gamma}}_{\hat{\epsilon}\hat{x}} - \epsilon^{\hat{\alpha}\hat{\epsilon}\hat{y}} V \mathcal{C}^{\hat{\gamma}}_{\hat{\epsilon}\hat{y}} - \epsilon^{\hat{\alpha}\hat{\epsilon}\hat{z}} V \mathcal{C}^{\hat{\gamma}}_{\hat{\epsilon}\hat{z}} =$$

$$= -\epsilon^{\hat{\alpha}\hat{y}\hat{x}} V \mathcal{C}^{\hat{\gamma}}_{\hat{y}\hat{x}} - \epsilon^{\hat{\alpha}\hat{z}\hat{x}} V \mathcal{C}^{\hat{\gamma}}_{\hat{z}\hat{x}} - \epsilon^{\hat{\alpha}\hat{x}\hat{y}} V \mathcal{C}^{\hat{\gamma}}_{\hat{x}\hat{y}} - \epsilon^{\hat{\alpha}\hat{z}\hat{y}} V \mathcal{C}^{\hat{\gamma}}_{\hat{z}\hat{y}} - \epsilon^{\hat{\alpha}\hat{x}\hat{z}} V \mathcal{C}^{\hat{\gamma}}_{\hat{x}\hat{z}} - \epsilon^{\hat{\alpha}\hat{y}\hat{z}} V \mathcal{C}^{\hat{\gamma}}_{\hat{y}\hat{z}} =$$

$$= -2\epsilon^{\hat{\alpha}\hat{x}\hat{y}} V \mathcal{C}^{\hat{\gamma}}_{\hat{x}\hat{y}} - 2\epsilon^{\hat{\alpha}\hat{z}\hat{x}} V \mathcal{C}^{\hat{\gamma}}_{\hat{z}\hat{x}} \ ;$$

$$\frac{\gamma}{1+\gamma} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} W^{\epsilon}_{\hat{\beta}} \partial_{\epsilon} V = \frac{\gamma}{1+\gamma} \epsilon^{\hat{\alpha}\hat{x}\hat{\gamma}} e^{-q_1 x} t^{-p_1} q_0 e^{q_0 x} = q_0 \frac{\gamma}{1+\gamma} \epsilon^{\hat{\alpha}\hat{x}\hat{\gamma}} e^{(q_0-q_1)x} t^{-p_1} \ .$$

By combining these two terms we obtain

$$\Omega^{\hat{\alpha}}_{(1)} = \frac{1}{\epsilon'} \mathcal{F}^{\delta}_{\hat{\gamma}} p_{\delta} \left(-2\epsilon^{\hat{\alpha}\hat{x}\hat{y}} e^{q_0 x} \mathcal{C}^{\hat{\gamma}}_{\hat{x}\hat{y}} - 2\epsilon^{\hat{\alpha}\hat{z}\hat{x}} e^{q_0 x} \mathcal{C}^{\hat{\gamma}}_{\hat{z}\hat{x}} + q_0 \frac{\gamma}{1+\gamma} \epsilon^{\hat{\alpha}\hat{x}\hat{\gamma}} e^{(q_0-q_1)x} t^{-p_1} \right) \ .$$

In particular, each component is

$$\Omega^{\hat{x}}_{(1)} = 0 \ ; \tag{3.178}$$

$$\begin{aligned}
\Omega^{\hat{y}}_{(1)} &= \frac{1}{\epsilon'} \mathcal{F}^{\delta}_{\hat{\gamma}} p_{\delta} \left(-2 e^{q_0 x} \mathcal{C}^{\hat{\gamma}}_{\hat{z}\hat{x}} + q_0 \frac{\gamma}{1+\gamma} \epsilon^{\hat{y}\hat{x}\hat{\gamma}} e^{(q_0-q_1)x} t^{-p_1} \right) = \\
&= \frac{1}{\epsilon'} \mathcal{F}^{z}_{\hat{z}} p_z q_2 e^{q_0 x} e^{-q_1 x} t^{-p_1} - \frac{1}{\epsilon'} \mathcal{F}^{z}_{\hat{z}} p_z \frac{\gamma}{1+\gamma} q_0 e^{(q_0-q_1)x} t^{-p_1} = \\
&= \frac{1}{\epsilon'} e^{(2q_0-q_1-q_3)x} t^{-(p_1+p_3)} p_z \left(q_2 - q_0 \frac{\gamma}{1+\gamma} \right) \ ;
\end{aligned} \tag{3.179}$$

$$\begin{aligned}
\Omega^{\hat{z}}_{(1)} &= \frac{1}{\epsilon'} \mathcal{F}^{\delta}_{\hat{\gamma}} p_{\delta} \left(-2 e^{q_0 x} \mathcal{C}^{\hat{\gamma}}_{\hat{x}\hat{y}} + q_0 \frac{\gamma}{1+\gamma} \epsilon^{\hat{z}\hat{x}\hat{\gamma}} e^{(q_0-q_1)x} t^{-p_1} \right) = \\
&= -\frac{1}{\epsilon'} \mathcal{F}^{y}_{\hat{y}} p_y q_2 e^{q_0 x} e^{-q_1 x} t^{-p_1} + \frac{1}{\epsilon'} \mathcal{F}^{y}_{\hat{y}} p_y \frac{\gamma}{1+\gamma} q_0 e^{(q_0-q_1)x} t^{-p_1} = \\
&= \frac{1}{\epsilon'} e^{(2q_0-q_1-q_2)x} t^{-(p_1+p_2)} p_y \left(-q_2 + q_0 \frac{\gamma}{1+\gamma} \right) \ .
\end{aligned} \tag{3.180}$$

Starting from (2.37) we have

$$\frac{p_x}{\epsilon'} = \mathcal{F}^{\hat{x}}_{x} v_{\hat{x}} = e^{(q_1-q_0)x} t^{p_1} v_{\hat{x}} \ ,$$

$$\frac{p_y}{\epsilon'} = \mathcal{F}^{\hat{y}}_{y} v_{\hat{y}} = e^{(q_2-q_0)x} t^{p_2} v_{\hat{y}} \ ,$$

$$\frac{p_z}{\epsilon'} = \mathcal{F}^{\hat{z}}_{z} v_{\hat{z}} = e^{(q_3-q_0)x} t^{p_3} v_{\hat{z}} \ .$$

By substituting these expressions in (3.154) and (3.155) we obtain

$$\Omega^{\hat{x}}_{(1)} = 0 \ , \tag{3.181}$$

$$\Omega^{\hat{y}}_{(1)} = e^{(q_0-q_1)x} t^{-p_1} v_{\hat{z}} \left(q_2 - q_0 \frac{\gamma}{1+\gamma} \right) \ , \tag{3.182}$$

$$\Omega^{\hat{z}}_{(1)} = e^{(q_0-q_1)x} t^{-p_1} v_{\hat{y}} \left(-q_2 + q_0 \frac{\gamma}{1+\gamma} \right) \ . \tag{3.183}$$

Now we evaluate $\Omega^{\hat{\alpha}}_{(2)}$; starting from (3.152) we have

$$\Omega^{\hat{\alpha}}_{(2)} = -\frac{\gamma}{1+\gamma} v_{\hat{x}} v_{\hat{\gamma}} \epsilon^{\hat{\alpha}\hat{x}\hat{\gamma}} \mathcal{Q}_{\hat{x}\hat{x}} - \frac{\gamma}{1+\gamma} v_{\hat{y}} v_{\hat{\gamma}} \epsilon^{\hat{\alpha}\hat{y}\hat{\gamma}} \mathcal{Q}_{\hat{y}\hat{y}} - -\frac{\gamma}{1+\gamma} v_{\hat{z}} v_{\hat{\gamma}} \epsilon^{\hat{\alpha}\hat{z}\hat{\gamma}} \mathcal{Q}_{\hat{z}\hat{z}} \ .$$

In particular, each component is

$$\Omega^{\hat{x}}_{(2)} = -\frac{\gamma}{1+\gamma} v_{\hat{y}} v_{\hat{z}} \mathcal{Q}_{\hat{y}\hat{y}} + \frac{\gamma}{1+\gamma} v_{\hat{y}} v_{\hat{z}} \mathcal{Q}_{\hat{z}\hat{z}} = \frac{\gamma}{1+\gamma} v_{\hat{y}} v_{\hat{z}} \left(\frac{p_2}{t} - \frac{p_3}{t} \right) \ ; \tag{3.184}$$

$$\Omega^{\hat{y}}_{(2)} = +\frac{\gamma}{1+\gamma} v_{\hat{x}} v_{\hat{z}} \mathcal{Q}_{\hat{x}\hat{x}} - \frac{\gamma}{1+\gamma} v_{\hat{x}} v_{\hat{z}} \mathcal{Q}_{\hat{z}\hat{z}} = \frac{\gamma}{1+\gamma} v_{\hat{x}} v_{\hat{z}} \left(\frac{p_3}{t} - \frac{p_1}{t} \right) \ ; \tag{3.185}$$

$$\Omega^{\hat{z}}_{(2)} = -\frac{\gamma}{1+\gamma} v_{\hat{y}} v_{\hat{x}} \mathcal{Q}_{\hat{x}\hat{x}} + \frac{\gamma}{1+\gamma} v_{\hat{y}} v_{\hat{x}} \mathcal{Q}_{\hat{y}\hat{y}} = \frac{\gamma}{1+\gamma} v_{\hat{y}} v_{\hat{x}} \left(\frac{p_1}{t} - \frac{p_2}{t} \right) \ . \tag{3.186}$$

So system (2.31) becomes

$$\begin{cases} \dfrac{ds^{\hat{x}}}{dt} = \Omega^{\hat{y}} s^{\hat{z}} - \Omega^{\hat{z}} s^{\hat{y}} \\ \dfrac{ds^{\hat{y}}}{dt} = -\Omega^{\hat{x}} s^{\hat{z}} + \Omega^{\hat{z}} s^{\hat{x}} \\ \dfrac{ds^{\hat{z}}}{dt} = \Omega^{\hat{x}} s^{\hat{y}} - \Omega^{\hat{y}} s^{\hat{x}} \end{cases} \ , \tag{3.187}$$

where we have raised the spacial indices using the spacial metric $\eta_{\hat{\alpha}\hat{\beta}} = = diag\,(-1,-1,-1)$. (3.187) represents the equations of motion of the average spin $\mathbf{s}$. Now we have to solve the geodesics equations in order to find expressions for the velocities, so that we can approach the solution of system (3.162).

3.2.4 Geodesics equation.

Metric (3.135) is independent of the two coordinates y and z, so, as we demonstrate in appendix B, it is easy to find the two corresponding velocities:

$$v_y = C_2 \Rightarrow -e^{2q_2 x}t^{2p_2}v^y = C_2 \Rightarrow v^y = -C_2 e^{-2q_2 x}t^{-2p_2} \ , \tag{3.188}$$

$$v_z = C_3 \Rightarrow -e^{2q_3 x}t^{2p_3}v^z = C_3 \Rightarrow v^z = -C_3 e^{-2q_3 x}t^{-2p_3} \ . \tag{3.189}$$

Normalisation condition $v^\mu v_\mu = 1$ becomes

$$e^{2q_0 x}\left(\frac{dt}{d\tau}\right)^2 - e^{2q_1 x}t^{2p_1}\left(\frac{dx}{d\tau}\right)^2 = 1 + C_2^2 e^{-2q_2 x}t^{-2p_2} + C_3^2 e^{-2q_3 x}t^{-2p_3} \ . \tag{3.190}$$

By using (3.188) and (3.189), the geodesics equations (3.53) for t and x are reduced to

$$\begin{aligned} &\text{for } \mu = 0 \Rightarrow 0 = \frac{d^2t}{d\tau^2} + \Gamma^0_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2t}{d\tau^2} + \Gamma^0_{0\sigma}\frac{dt}{d\tau}\frac{dx^\sigma}{d\tau} + \\ &+\Gamma^0_{1\sigma}\frac{dx}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^0_{2\sigma}\frac{dy}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^0_{3\sigma}\frac{dz}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2t}{d\tau^2} + 2q_0\frac{dt}{d\tau}\frac{dx}{d\tau} + \\ &+p_1 e^{2(q_1-q_0)x}t^{2p_1-1}\left(\frac{dx}{d\tau}\right)^2 + p_2 C_2^2 e^{-2(q_2+q_0)x}t^{-2p_2-1} + \\ &+p_3 C_3^2 e^{-2(q_3+q_0)x}t^{-2p_3-1} = 0 \ , \end{aligned} \tag{3.191}$$

$$\begin{aligned} &\text{for } \mu = 1 \Rightarrow 0 = \frac{d^2x}{d\tau^2} + \Gamma^1_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2x}{d\tau^2} + \Gamma^1_{0\sigma}\frac{dxt}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^1_{1\sigma}\frac{dx}{d\tau}\frac{dx^\sigma}{d\tau} + \\ &+\Gamma^1_{2\sigma}\frac{dy}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^1_{3\sigma}\frac{dz}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{d^2x}{d\tau^2} + \frac{2p_1}{t}\frac{dt}{d\tau}\frac{dx}{d\tau} + q_0 e^{2(q_0-q_1)x}t^{-2p_1}\left(\frac{dt}{d\tau}\right)^2 - \\ &-q_2 C_2^2 e^{-2(q_2+q_0)x}t^{-2(p_2+p_1)} - q_3 C_3^2 e^{-2(q_3+q_0)x}t^{-2(p_3+p_1)} = 0 \ . \end{aligned} \tag{3.192}$$

Starting from (3.190) we can obtain an expression for $\dfrac{dt}{d\tau}$ that we substitute in (3.192), producing

$$\begin{aligned} &e^{2q_1 x}t^{2p_1}\left[\frac{d^2x}{d\tau^2} + \frac{2p_1}{t}\frac{dt}{d\tau}\frac{dx}{d\tau} + (q_1+q_0)\left(\frac{dx}{d\tau}\right)^2\right] = \\ &= \left[-q_0 + (q_2-q_0)\,C_2^2 e^{-2q_2 x}t^{-2p_2} + (q_3-q_0)\,C_3^2 e^{-2q_3 x}t^{-2p_3}\right] \ . \end{aligned} \tag{3.193}$$

The system of differential equations for the geodesics is not solvable analytically, so we follow [11] to rewrite the left term of (3.193) parametrically; the authors define function

$$F = t^{2p_1}e^{(q_1+q_0)x}\frac{dx}{d\tau} \ , \tag{3.194}$$

therefore the system of differential equations for the geodesics becomes

$$\frac{dt}{d\tau} = e^{-q_0 x}\left[1 + C_2^2 e^{-2q_2 x} t^{-2p_2} + C_3^2 e^{-2q_3 x} t^{-2p_3} + e^{-2q_0 x} t^{-2p_1} F^2\right]^{\frac{1}{2}} \ , \quad (3.195)$$

$$\frac{dx}{d\tau} = e^{-(q_1+q_0)x} t^{-2p_1} F \ , \quad (3.196)$$

$$\frac{dy}{d\tau} = C_2 e^{-2q_2 x} t^{-2p_2} \ , \ \frac{dz}{d\tau} = C_3 e^{-2q_3 x} t^{-2p_3} \ , \quad (3.197)$$

$$\frac{dF}{d\tau} = e^{(q_0-q_1)x}\left[-q_0 + (q_2 - q_0)\, C_2^2 e^{-2q_2 x} t^{-2p_2} + (q_3 - q_0)\, C_3^2 e^{-2q_3 x} t^{-2p_3}\right] \ . \quad (3.198)$$

The velocities are obtained using the chain rule:

$$v^x = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} \ , \ v^y = \frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} \ , \ v^z = \frac{dz}{dt} = \frac{dz}{d\tau}\frac{d\tau}{dt} \ . \quad (3.199)$$

It is quite interesting to observe that if the q_i's all vanish, therefore F is a constant and the system of differential equations is reduced to the system of the standard Kasner universe. This will be the pivotal consideration for the asymptotic analysis [21] of geodesics equations we will perform here: in the standard Kasner geometry, as we show in appendix A, the universe undergoes a contraction along one spacial direction and an expansion along the other two. This is a consequence of the fact that one of the p_i's is negative, for example p_1 in appendix A, while the other two are positive. The consequence of this consideration is that $\frac{dx}{d\tau}$ tends to 0 as $t \to 0$, while $\frac{dy}{d\tau}$ and $\frac{dz}{d\tau}$ have values different from 0 as $t \to 0$. So our ansatz is

$$\frac{dx}{d\tau} \xrightarrow[t\to 0]{} 0 \ .$$

This results in an hypothesis for the trend of F as $t \to 0$, namely

$$e^{-(q_1+q_0)x} t^{-2p_1} F \xrightarrow[t\to 0]{} 0 \ \Rightarrow \ F \underset{t\to 0}{\sim} e^{(q_1+q_0)x} t^{2p_1+\epsilon} \ , \ \epsilon \ \geq 0 \ . \quad (3.200)$$

So, with the asymptotic trend of F we can solve asymptotically the set of differential equations. By substituting the asymptotic representation of F in (3.195) we obtain

$$\frac{dt}{d\tau} = e^{-q_0 x}\left[1 + C_2^2 e^{-2q_2 x} t^{-2p_2} + C_3^2 e^{-2q_3 x} t^{-2p_3} + e^{2q_1 x} t^{2p_1+2\epsilon}\right]^{\frac{1}{2}} \ .$$

In appendix A we show, by using the Lifshitz-Khalatnikov parametrization [23], that $|p_1|, |p_2| < |p_3|$. Therefore, within the four terms inside the bracket the dominating one as $t \to 0$ is $C_3^2 e^{-2q_3 x} t^{-2p_3}$. So (3.195) asymptotically becomes

$$\frac{dt}{d\tau} \approx C_3 e^{-(q_0+q_3)x} t^{-p_3} \ . \tag{3.201}$$

Now we can solve (3.198), which is necessary to solve (3.196); by multiplying (3.198) by $\frac{d\tau}{dt}$ and using the chain rule, we obtain

$$\frac{dF}{dt} \approx C_3 \left(q_3 - q_0\right) e^{(2q_0-q_1-q_3)x} t^{-p_3} \ \Rightarrow$$
$$\Rightarrow \ F \approx \frac{C_3 + (q_3 - q_0)}{1 - p_3} e^{(2q_0-q_1-q_3)x} t^{1-p_3} + D \underset{t\to 0}{\sim} D \ , \tag{3.202}$$

so that F is approximately constant for $t \to 0$. We observe that this result verifies the ansatz (3.200), and in particular we found that $\epsilon = -2p_1 > 0$. Integration constant D may be fixed according to (3.200), therefore $D = e^{(q_1+q_0)x}$. By substituting (3.202) in (3.196) we find

$$\frac{dx}{d\tau} \approx t^{-2p_1} \ ,$$

and the corresponding velocity is

$$v^x \approx \frac{1}{C_3} e^{(q_3+q_0)x} t^{p_3-2p_1} \ . \tag{3.203}$$

The other two velocities read explicitly

$$v^y = \frac{dy}{dt} = \frac{C_2}{C_3} e^{(q_0+q_3-2q_2)x} t^{p_3-2p_2} \ , \ v^z = \frac{dz}{dt} = e^{(q_0-q_3)x} t^{-p_3} \ . \tag{3.204}$$

Using tetrads (3.136) we obtain the tetrad velocities

$$v^{\hat{x}} = \frac{1}{C_3} e^{(q_3+q_1+q_0)x} t^{p_3-p_1} \ , \ v^{\hat{y}} = \frac{C_2}{C_3} e^{(q_0+q_3-q_2)x} t^{p_3-p_2} \ , \ v^{\hat{z}} = e^{q_0 x} \ . \tag{3.205}$$

So far we have obtained an asymptotic expression for the tetrad velocities in the vicinity of the singularity $t \to 0$. Looking back at what we did in chapter two, we also obtain an explicit expression for the γ factor

$$\gamma = U^{\hat{0}} = e^{\hat{0}}_0 U^0 = e^{q_0 x} \frac{dt}{d\tau} = C_3 e^{-q_3 x} t^{-p_3} \ . \tag{3.206}$$

Finally, we observe that in obtaining (3.202) we have implicitly assumed $p_3 \neq 1$; this is reasonable, because in the standard Kasner space-time $p_3 = 1$ implies $p_1 = p_2 = 0$, and the Kasner metric would be a reformulation of the flat space-time, as shown in appendix A.

Now we use (3.205) and (3.206) to study the equations of motion of average spin in the proximity of the singularity.

3.2.5 Equations of motion of spin.

In the previous section we have obtained asymptotic expressions for the velocities of a particle moving in a double Kasner universe. In this section we use these expressions to obtain asymptotic expressions near $t \to 0$ for the angular velocities of a spinning particle in this universe. We consider (3.181) ÷ (3.186) and substitute results (3.205) and (3.206) in them; first we observe that

$$\frac{\gamma}{1+\gamma} \xrightarrow[t\to 0]{} 1 \; ,$$

as a consequence of (3.206) and of the positivity of p_3. The angular velocities then read

$$\Omega^{\hat{x}}_{(1)} = 0 \; ; \tag{3.207}$$

$$\Omega^{\hat{y}}_{(1)} \underset{t\to 0}{\sim} -e^{(q_0-q_1)x} t^{-p_1} e^{q_0 x} \left(q_2 - q_0\right) \underset{t\to 0}{\sim} 0 \; , \tag{3.208}$$

because $-p_1$ is positive;

$$\begin{aligned}\Omega^{\hat{z}}_{(1)} &\underset{t\to 0}{\sim} e^{(q_0-q_1)x} t^{-p_1} \frac{C_2}{C_3} e^{(q_0+q_3-q_2)x} t^{p_3-p_2} \left(q_0 - q_2\right) = \\ &= \frac{C_2}{C_3} \left(q_0 - q_2\right) e^{(2q_0+q_3-q_1-q_2)x} t^{p_3-p_2-p_1} \; ,\end{aligned}$$

and by evaluating explicitly the exponent we have

$$p_3 - p_2 - p_1 = \frac{u^2+u-u-1+u}{u^2+u+1} = \frac{u^2+u-1}{u^2+u+1} > 0 \; ,$$

and therefore

$$\Omega^{\hat{z}}_{(1)} \underset{t\to 0}{\sim} 0 \; ; \tag{3.209}$$

$$\begin{aligned}\Omega^{\hat{x}}_{(2)} &\underset{t\to 0}{\sim} \frac{C_2}{C_3} e^{(q_0+q_3-q_2)x} t^{p_3-p_2} e^{q_0 x} \left(\frac{p_2-p_3}{t}\right) = \\ &= \frac{C_2}{C_3} \left(p_2 - p_3\right) e^{(2q_0+q_3-q_2)x} t^{p_3-p_2-1} \; ,\end{aligned}$$

and by evaluating explicitly the exponent we have

$$p_3 - p_2 - 1 = \frac{u^2+u-u-1-u^2-u-1}{u^2+u+1} = \frac{-u-2}{u^2+u+1} < 0 \; ,$$

so $\Omega^{\hat{x}}_{(2)}$ survives in the vicinity of the singularity;

$$\Omega^{\hat{y}}_{(2)} \underset{t\to 0}{\sim} \frac{1}{C_3} e^{(q_3+q_1+q_0)x} t^{p_3-p_1} e^{q_0 x} \left(\frac{p_3-p_1}{t}\right) = \frac{p_3-p_1}{C_3} e^{(q_3+q_1+2q_0)x} t^{p_3-p_1-1} \; ,$$

and by evaluating explicitly the exponent we have

$$p_3 - p_1 - 1 = \frac{u^2 + u + u - u^2 - u - 1}{u^2 + u + 1} = \frac{u - 1}{u^2 + u + 1} > 0 \ ,$$

if we assume $u \neq 1$ for now, so that we have

$$\Omega^{\hat{y}}_{(2)} \underset{t \to 0}{\sim} 0 \ ; \tag{3.210}$$

$$\begin{aligned}\Omega^{\hat{z}}_{(2)} \underset{t \to 0}{\sim} & \frac{C_2}{C_3} e^{(q_0 + q_3 - q_2)x} t^{p_3 - p_2} \frac{1}{C_3} e^{(q_3 + q_1 + q_0)x} t^{p_3 - p_1} \left(\frac{p_1 - p_2}{t} \right) = \\ & = \frac{C_2}{C_3^2} \left(p_1 - p_2 \right) e^{(2q_0 + 2q_3 + q_1 - q_2)x} t^{2p_3 - p_2 - p_1 - 1} \ ,\end{aligned}$$

and by evaluating explicitly the exponent we have

$$2p_3 - p_2 - p_1 - 1 = \frac{2u^2 + 2u - u - 1 + u - u^2 - u - 1}{u^2 + u + 1} = \frac{u^2 + u - 2}{u^2 + u + 1} > 0 \ ,$$

remembering that $u \neq 1$, coherently with the previous assumption, so that we have

$$\Omega^{\hat{z}}_{(2)} \underset{t \to 0}{\sim} 0 \ . \tag{3.211}$$

We observe that the only component of angular velocity surviving in the vicinity of singularity $t \to 0$ is $\Omega^{\hat{x}}_{(2)}$. System (3.162) is therefore simplified as

$$\begin{cases} \dfrac{ds^{\hat{x}}}{dt} \underset{t \to 0}{\sim} 0 \\ \dfrac{ds^{\hat{y}}}{dt} \underset{t \to 0}{\sim} -\dfrac{C_2}{C_3} \left(p_2 - p_3 \right) e^{(2q_0 + q_3 - q_1)x} t^{p_3 - p_2 - 1} s^{\hat{z}} \\ \dfrac{ds^{\hat{z}}}{dt} \underset{t \to 0}{\sim} \dfrac{C_2}{C_3} \left(p_2 - p_3 \right) e^{(2q_0 + q_3 - q_1)x} t^{p_3 - p_2 - 1} s^{\hat{y}} \end{cases} . \tag{3.212}$$

By setting $A = \dfrac{C_2}{C_3} \left(p_2 - p_3 \right) e^{(2q_0 + q_3 - q_1)x}$, the asymptotic solution of the system is straightforward:

$$\begin{cases} s^{\hat{x}} \underset{t \to 0}{\sim} s^{\hat{x}}_0 \\ s^{\hat{y}} \underset{t \to 0}{\sim} s_0 \cos \left(\dfrac{A t^{p_3 - p_2}}{p_3 - p_2} + \phi \right) \\ s^{\hat{z}} \underset{t \to 0}{\sim} s_0 \sin \left(\dfrac{A t^{p_3 - p_2}}{p_3 - p_2} + \phi \right) \end{cases} , \tag{3.213}$$

where s_0 and ϕ are integration constants representing an amplitude and a phase respectively. We observe that the exponent of t in (3.213) is explicitly given by

$$p_3 - p_2 = \frac{u^2-1}{u^2+u+1} > 0 \ ,$$

in continuity with hypothesis $u \neq 1$ used before. This implies that the singularity is integrable, because in limit $t \to 0$ the average spin $\mathbf{s}$ tends to a fixed direction in a regular way, despite the angular velocity diverging near $t \to 0$.

3.2.6 Equations of motion of spin: $\mathbf{p} = (\mathbf{-1/3,2/3,2/3})$.

Now let's consider the case we excluded before, that is $u = 1$; this implies that the p_i's assume the values

$$p_1 = -\frac{1}{3} \ , \ p_2 = \frac{2}{3} \ , \ p_3 = \frac{2}{3} \ .$$

We want to solve the equations of motion of spin in this particular case, so we evaluate the components of the angular velocity; as in the previous general case, the components of $\mathbf{\Omega}_{(1)}$ are asymptotically equal to 0, as easily seen from their expressions evaluated previously. The components of $\mathbf{\Omega}_{(1)}$ read explicitly

$$\Omega^{\hat{x}}_{(2)} \underset{t\to 0}{\sim} \frac{C_2}{C_3}\left(p_2 - p_3\right) e^{(2q_0+q_3-q_2)x} t^{p_3-p_2-1} = 0 \ , \tag{3.214}$$

because p_2 and p_3 are equal;

$$\Omega^{\hat{y}}_{(2)} \underset{t\to 0}{\sim} \frac{p_3 - p_1}{C_3} e^{(q_3+q_1+2q_0)x} t^{p_3-p_1-1} = \frac{1}{C_3} e^{(q_3+q_1+2q_0)x} \ ; \tag{3.215}$$

$$\Omega^{\hat{z}}_{(2)} \underset{t\to 0}{\sim} \frac{C_2}{C_3^2}\left(p_1 - p_2\right) e^{(2q_0+2q_3+q_1-q_2)x} t^{2p_3-p_2-p_1-1} = -\frac{C_2}{C_3^2} e^{(2q_0+2q_3+q_1-q_2)x} \ . \tag{3.216}$$

In this particular case we have obtained that the angular velocity is constant (in time). System (3.162) then becomes

$$\begin{cases} \dfrac{ds^{\hat{x}}}{dt} = \dfrac{1}{C_3} e^{(q_3+q_1+2q_0)x} s^{\hat{z}} + \dfrac{C_2}{C_3^2} e^{(2q_0+2q_3+q_1-q_2)x} s^{\hat{y}} \\ \dfrac{ds^{\hat{y}}}{dt} = -\dfrac{C_2}{C_3^2} e^{(2q_0+2q_3+q_1-q_2)x} s^{\hat{x}} \\ \dfrac{ds^{\hat{z}}}{dt} = -\dfrac{1}{C_3} e^{(q_3+q_1+2q_0)x} s^{\hat{x}} \end{cases} , \tag{3.217}$$

and can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} s^{\hat{x}} \\ s^{\hat{y}} \\ s^{\hat{z}} \end{pmatrix} = \begin{pmatrix} 0 & -\Omega^{\hat{z}}_{(2)} & \Omega^{\hat{y}}_{(2)} \\ \Omega^{\hat{z}}_{(2)} & 0 & 0 \\ -\Omega^{\hat{y}}_{(2)} & 0 & 0 \end{pmatrix} \begin{pmatrix} s^{\hat{x}} \\ s^{\hat{y}} \\ s^{\hat{z}} \end{pmatrix} \ . \tag{3.218}$$

System (3.218) is analogous to (3.96), solved for the Melvin space-time, for the substitutions $\Omega^{\hat{z}}_{(1)} \leftrightarrow \Omega^{\hat{z}}_{(2)}$ and $\Omega^{\hat{\phi}}_{(1)} \leftrightarrow \Omega^{\hat{y}}_{(2)}$ for the matrix elements, and the substitution $\mathbf{\Sigma} \leftrightarrow \mathbf{s}$ for the functions of the differential equations. The solution of (3.218) is then obtained formally in the same way we obtained (3.101), so we can use (3.101) and apply the aforementioned substitutions:

$$\begin{pmatrix} s^{\hat{x}} \\ s^{\hat{y}} \\ s^{\hat{z}} \end{pmatrix} = \begin{pmatrix} s_1 \cos\left[\omega t + \phi\right] \\ s_2 \frac{\Omega^{\hat{y}}_{(2)}}{\Omega^{\hat{z}}_{(2)}} + s_1 \frac{\Omega^{\hat{z}}_{(2)}}{\omega} \sin\left[\omega t + \phi\right] \\ s_2 - s_1 \frac{\Omega^{\hat{y}}_{(2)}}{\omega} \sin\left[\omega t + \phi\right] \end{pmatrix} , \tag{3.219}$$

where $\omega = \sqrt{\Omega^{\hat{y}2}_{(2)} + \Omega^{\hat{z}2}_{(2)}}$ and s_1, s_2 and ϕ are integration constants. Even in this case, the average spin vector tends in a regular way to a fixed direction, but differently from the general case the precession velocity does not diverge.

3.2.7 Cosmic jets.

In [24] the authors study some examples of "*cosmic jets*", as in particles accelerated to the speed of light by gravitational fields. In particular, they study examples of "Kasner-like" space-times, where one direction collapses while the other two expand, the same thing that happens in the standard Kasner universe. In [11] the authors take the double-Kasner space-time into consideration, but in solving the geodesics equations they consider only particular cases for parameters $\mathbf{p_i}$'s. In section (3.2.4) we have generalized their work, finding an asymptotic solution for geodesics equations leaving the Kasner parameters free.

Our results are in complete agreement with the results reached by the authors, and the asymptotic analysis could be easily extended in an analogous way to the limit $t \to \infty$, which is of no interest to us. Moreover, our result is also in agreement with the solution for the standard Kasner space-time [24] , [3].

3.2.8 Double Kasner space-time from Einstein equations.

In a previous section we demonstrated that metric (3.135) satisfying (3.132) $\div$ (3.134) is a solution of the Einstein equations in vacuum. In this section we want to demonstrate that the double Kasner metric emerges naturally by solving the Einstein equations in vacuum using as few assumptions as possible. This is analogous to what we did in appendix A, where we demonstrated that starting from a Bianchi-I universe we obtain the Kasner space-time by solving the Einstein equations.

We start from a diagonal line element whose coefficients depend only on the variables x and t:

$$ds^2 = a^2(t,x)dt^2 - \left[b^2(t,x)dx^2 + c^2(t,x)dy^2 + d^2(t,x)dz^2\right] \ ; \tag{3.220}$$

we can perform a time rescaling so that the coefficient a of dt^2 depends only on the spacial coordinate x [9]. The new time coordinate t represents what is called *cosmological time*. If we leave the name t unchanged, line element (3.220) becomes

$$ds^2 = a^2(x)dt^2 - \left[b^2(t,x)dx^2 + c^2(t,x)dy^2 + d^2(t,x)dz^2\right] \ . \tag{3.221}$$

Now we want to solve the Einstein field equations in vacuum $R_{\mu\nu} = 0$, and in order to do this we first calculate the coefficients of affine connection $\Gamma^\lambda_{\mu\nu}$. Denoting the derivative with respect to time t with a dot $\cdot$, and the derivative with respect to space x with an apostrophe $'$, we have explicitly:

$$\Gamma^0_{\mu\nu} = \frac{1}{2}g^{00}\left(g_{\mu 0,\nu} + g_{\nu 0,\mu} - g_{\mu\nu,0}\right) = \frac{1}{2a^2}\begin{pmatrix} 0 & 2aa' & 0 & 0 \\ 2aa' & 2b\dot{b} & 0 & 0 \\ 0 & 0 & 2c\dot{c} & 0 \\ 0 & 0 & 0 & 2d\dot{d} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \frac{a'}{a} & 0 & 0 \\ \frac{a'}{a} & \frac{b\dot{b}}{a^2} & 0 & 0 \\ 0 & 0 & \frac{c\dot{c}}{a^2} & 0 \\ 0 & 0 & 0 & \frac{d\dot{d}}{a^2} \end{pmatrix} \ ; \tag{3.222}$$

$$\Gamma^1_{\mu\nu} = \frac{1}{2}g^{11}\left(g_{\mu 1,\nu} + g_{\nu 1,\mu} - g_{\mu\nu,1}\right) = -\frac{1}{2b^2}\begin{pmatrix} -2aa' & -2b\dot{b} & 0 & 0 \\ -2b\dot{b} & -2bb' & 0 & 0 \\ 0 & 0 & 2cc' & 0 \\ 0 & 0 & 0 & 2dd' \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{aa'}{b^2} & \frac{\dot{b}}{b} & 0 & 0 \\ \frac{\dot{b}}{b} & \frac{b'}{b} & 0 & 0 \\ 0 & 0 & -\frac{cc'}{b^2} & 0 \\ 0 & 0 & 0 & -\frac{dd'}{b^2} \end{pmatrix} \ ; \tag{3.223}$$

$$\Gamma^2_{\mu\nu} = \frac{1}{2}g^{22}\left(g_{\mu 2,\nu} + g_{\nu 2,\mu}\right) = -\frac{1}{2c^2}\begin{pmatrix} 0 & 0 & -2c\dot{c} & 0 \\ 0 & 0 & -2cc' & 0 \\ -2c\dot{c} & -2cc' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & \frac{\dot c}{c} & 0 \\ 0 & 0 & \frac{c'}{c} & 0 \\ \frac{\dot c}{c} & \frac{c'}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; ; \tag{3.224}$$

$$\Gamma^3_{\mu\nu} = \frac{1}{2} g^{33} \left(g_{\mu 3,\nu} + g_{\nu 3,\mu} \right) = -\frac{1}{2d^2} \begin{pmatrix} 0 & 0 & 0 & -2d\dot d \\ 0 & 0 & 0 & -2dd' \\ 0 & 0 & 0 & 0 \\ -2d\dot d & -2dd' & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \frac{\dot d}{d} \\ 0 & 0 & 0 & \frac{d'}{d} \\ 0 & 0 & 0 & 0 \\ \frac{\dot d}{d} & \frac{d}{d} & 0 & 0 \end{pmatrix} \; . \tag{3.225}$$

Now, using (3.222) $\div$ (3.225), we can evaluate the components of the Ricci tensor and explicitly obtain:

$$R_{00} = \Gamma^\rho_{00,\rho} - \Gamma^\rho_{0\rho,0} + \Gamma^\rho_{00}\Gamma^\lambda_{\rho\lambda} - \Gamma^\lambda_{0\rho}\Gamma^\rho_{0\lambda} = \frac{a'^2}{b^2} + \frac{aa''}{b^2} - \frac{2aa'b'}{b^3} - \frac{\ddot b}{b} + \frac{\dot b^2}{b^2} - \frac{\ddot c}{c^2} + \frac{\dot c^2}{c^2} -$$

$$-\frac{\ddot d}{d} + \frac{\dot d^2}{d^2} + \frac{a'^2}{b^2} + \frac{aa'b'}{b^3} + \frac{aa'c'}{cb^2} + \frac{aa'd'}{db^2} - \frac{a'^2}{b^2} - \frac{a'^2}{b^2} - \frac{\dot b^2}{b^2} - \frac{\dot c^2}{c^2} - \frac{\dot d^2}{d^2} =$$

$$= \frac{aa''}{b^2} - \frac{aa'b'}{b^3} - \frac{\ddot b}{b} - \frac{\ddot c}{c} - \frac{\ddot d}{d} + \frac{aa'c'}{cb^2} + \frac{aa'd'}{db^2} \; ;$$

$$R_{11} = \Gamma^\rho_{11,\rho} - \Gamma^\rho_{1\rho,1} + \Gamma^\rho_{11}\Gamma^\lambda_{\rho\lambda} - \Gamma^\lambda_{1\rho}\Gamma^\rho_{1\lambda} = \frac{b\ddot b}{a^2} + \frac{\dot b^2}{a^2} - \frac{a''}{a} + \frac{a'^2}{a^2} - \frac{c''}{c} + \frac{c'^2}{c^2} - \frac{d''}{d} + \frac{d'^2}{d^2} +$$

$$+\frac{\dot b^2}{a^2} + \frac{b\dot b\dot c}{ca^2} + \frac{b\dot b\dot d}{da^2} + \frac{b'a'}{ba} + \frac{b'^2}{b^2} + \frac{b'c'}{bc} + \frac{b'd}{bd} - \frac{a'^2}{a^2} - \frac{\dot b^2}{a^2} - \frac{\dot b^2}{a^2} - \frac{b'^2}{b^2} - \frac{c'^2}{c^2} - \frac{d'^2}{d^2} =$$

$$= \frac{b\ddot b}{a^2} - \frac{a''}{a} - \frac{c''}{c} - \frac{d''}{d} + \frac{b\dot b\dot c}{ca^2} + \frac{b\dot b\dot d}{da^2} + \frac{b'a'}{ba} + \frac{b'c'}{bc} + \frac{b'd'}{bd} \; ;$$

$$R_{22} = \Gamma^\rho_{22,\rho} + \Gamma^\rho_{22}\Gamma^\lambda_{\rho\lambda} - \Gamma^\lambda_{2\rho}\Gamma^\rho_{2\lambda} = \frac{c\ddot c}{a^2} + \frac{\dot c^2}{a^2} - \frac{cc''}{b^2} - \frac{c'^2}{b^2} + \frac{2cc'b'}{b^3} + \frac{c\dot c\dot b}{ba^2} + \frac{\dot c^2}{a^2} + \frac{c\dot c\dot d}{da^2} -$$

$$-\frac{cc'a'}{ab^2} - \frac{cc'b'}{b^3} - \frac{c'^2}{b^2} - \frac{cc'd'}{db^2} - \frac{\dot c^2}{a^2} + \frac{c'^2}{b^2} - \frac{c'^2}{a^2} + \frac{c'^2}{b^2} =$$

$$= \frac{c\ddot c}{a^2} - \frac{cc''}{b^2} + \frac{cc'b'}{b^3} + \frac{c\dot c\dot b}{ba^2} + \frac{c\dot c\dot d}{da^2} - \frac{cc'a'}{ab^2} - \frac{cc'd'}{db^2} \; ;$$

$$R_{33} = \Gamma^\rho_{33,\rho} + \Gamma^\rho_{33}\Gamma^\lambda_{\rho\lambda} - \Gamma^\lambda_{3\rho}\Gamma^\rho_{3\lambda} = \frac{d\ddot d}{a^2} + \frac{\dot d^2}{a^2} - \frac{dd''}{b^2} - \frac{d'^2}{b^2} + \frac{2dd'b'}{b^3} + \frac{d\dot d\dot b}{ba^2} + \frac{d\dot d\dot c}{ca^2} + \frac{\dot d^2}{a^2} -$$

$$-\frac{dd'a'}{ab^2}-\frac{dd'b'}{b^3}-\frac{dd'c'}{cb^2}-\frac{d'^2}{b^2}-\frac{\dot{d}^2}{a^2}+\frac{d'^2}{b^2}-\frac{\dot{d}^2}{a^2}+\frac{d'^2}{b^2}=$$

$$=\frac{d\ddot{d}}{a^2}-\frac{dd''}{b^2}+\frac{dd'b'}{b^3}+\frac{d\dot{d}\dot{b}}{ba^2}+\frac{d\dot{d}\dot{c}}{ca^2}-\frac{dd'a'}{ab^2}-\frac{dd'c'}{cb^2}\ ;$$

$$R_{01}=\Gamma^{\rho}_{01,\rho}-\Gamma^{\rho}_{0\rho,1}+\Gamma^{\rho}_{01}\Gamma^{\lambda}_{\rho\lambda}-\Gamma^{\lambda}_{0\rho}\Gamma^{\rho}_{1\lambda}=\frac{\dot{b}'}{b}-\frac{\dot{b}b'}{b^2}-\frac{\dot{b}'}{b}+\frac{\dot{b}b'}{b^2}-\frac{\dot{c}'}{c}+\frac{\dot{c}c'}{c^2}-\frac{\dot{d}'}{d}+\frac{\dot{d}d'}{d^2}+$$

$$+\frac{a'\dot{b}}{ab}+\frac{a'\dot{c}}{ac}+\frac{a'\dot{d}}{ad}+\frac{a'\dot{b}}{ab}+\frac{\dot{b}b'}{b^2}+\frac{\dot{b}c'}{bc}+\frac{\dot{d}\dot{b}}{bd}-\frac{a'\dot{b}}{ab}-\frac{a'\dot{b}}{ab}-\frac{b'\dot{b}}{b^2}-\frac{\dot{c}c'}{c^2}-\frac{\dot{d}d'}{d^2}=$$

$$=-\frac{\dot{c}'}{c}-\frac{\dot{d}'}{d}+\frac{a'\dot{c}}{ac}+\frac{a'\dot{d}}{ad}+\frac{\dot{b}c'}{bc}+\frac{\dot{b}d'}{bd}\ ;$$

$$R_{02}=\Gamma^{\rho}_{02,\rho}+\Gamma^{\rho}_{02}\Gamma^{\lambda}_{\rho\lambda}-\Gamma^{\lambda}_{0\rho}\Gamma^{\rho}_{2\lambda}=0\ ;$$

$$R_{03}=\Gamma^{\rho}_{03,\rho}+\Gamma^{\rho}_{03}\Gamma^{\lambda}_{\rho\lambda}-\Gamma^{\lambda}_{0\rho}\Gamma^{\rho}_{2\lambda}=0\ ;$$

$$R_{12}=\Gamma^{\rho}_{12,\rho}+\Gamma^{\rho}_{12}\Gamma^{\lambda}_{\rho\lambda}-\Gamma^{\lambda}_{1\rho}\Gamma^{\rho}_{2\lambda}=0\ ;$$

$$R_{13}=\Gamma^{\rho}_{13,\rho}+\Gamma^{\rho}_{13}\Gamma^{\lambda}_{\rho\lambda}-\Gamma^{\lambda}_{1\rho}\Gamma^{\rho}_{3\lambda}=0\ ;$$

$$R_{23}=\Gamma^{\rho}_{23,\rho}+\Gamma^{\rho}_{23}\Gamma^{\lambda}_{\rho\lambda}-\Gamma^{\lambda}_{2\rho}\Gamma^{\rho}_{3\lambda}=0\ .$$

The Einstein equations in vacuum $R_{\mu\nu}=0$ therefore become

$$\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\ddot{d}}{d}=\frac{a}{b^2}\left(a''-\frac{a'b'}{b}+\frac{a'c'}{c}+\frac{a'd'}{d}\right)\ ; \tag{3.226}$$

$$\frac{b}{a^2}\left(\ddot{b}+\frac{\dot{b}\dot{c}}{c}+\frac{\dot{b}\dot{d}}{d}\right)=\frac{a''}{a}+\frac{c''}{c}+\frac{d''}{d}-\frac{b'a'}{ba}-\frac{b'c'}{bc}-\frac{b'd'}{bd}\ ; \tag{3.227}$$

$$\frac{c}{a^2}\left(\ddot{c}+\frac{\dot{c}\dot{b}}{b}+\frac{\dot{c}\dot{d}}{d}\right)=\frac{c}{b^2}\left(c''+\frac{c'a'}{a}+\frac{c'd'}{d}-\frac{c'b'}{b}\right)\ ; \tag{3.228}$$

$$\frac{d}{a^2}\left(\ddot{d}+\frac{\dot{d}\dot{b}}{b}+\frac{\dot{d}\dot{c}}{c}\right)=\frac{d}{b^2}\left(d''+\frac{d'a'}{a}+\frac{d'c'}{c}-\frac{d'b'}{b}\right)\ ; \tag{3.229}$$

$$\frac{\dot{c}'}{c}+\frac{\dot{d}'}{d}-\frac{a'\dot{c}}{ac}-\frac{a'\dot{d}}{ad}-\frac{\dot{b}c'}{bc}-\frac{\dot{b}d'}{bd}=0\ . \tag{3.230}$$

In (3.226) $\div$ (3.229) the parts involving derivatives with respect to t and those involving derivatives with respect to x are explicitly separated, so it is natural to look for solutions involving factorized functions. This means that if we consider a function of x and t, $f(t,x)$, it is true that

$$f(t,x)=f_t(t)\cdot f_x(x)\ . \tag{3.231}$$

This implies

$$\frac{\dot{f(t,x)}}{f(t,x)} = \frac{\dot{f_t(t)}f_x(x)}{f_t(t)f_x(x)} = \frac{\dot{f_t(t)}}{f_t(t)} \quad \text{and} \quad \frac{f'(t,x)}{f(t,x)} = \frac{f_t(t)f_x'(x)}{f_t(t)f_x(x)} = \frac{f_x'(x)}{f_x(x)} \ . \tag{3.232}$$

We can use (3.232) to factorise $b(t,x)$, $c(t,x)$ and $d(t,x)$. It is easy to verify that in (3.226) $\div$ (3.229) the terms on the left depend only on coordinate t, and the terms on the right depend only on coordinate x. This means that we can consider the right and the left terms separately and put the equal to a common constant. For the sake of simplicity, and without losing generality, we can put this constant equal to 0. In appendix C we show that this choice is sufficient and necessary in order to obtain double Kasner metric. To solve the system we start considering the left terms of (3.227) $\div$ (3.229), which after the factorisation become

$$\frac{\ddot{b}_t}{b_t} + \frac{\dot{b}_t\dot{c}_t}{b_tc_t} + \frac{\dot{b}_t\dot{d}_t}{b_td_t} = 0 \ , \tag{3.233}$$

$$\frac{\ddot{c}_t}{c_t} + \frac{\dot{c}_t\dot{b}_t}{b_tc_t} + \frac{\dot{c}_t\dot{d}_t}{c_td_t} = 0 \ , \tag{3.234}$$

$$\frac{\ddot{d}_t}{d_t} + \frac{\dot{d}_t\dot{b}_t}{b_td_t} + \frac{\dot{d}_t\dot{c}_t}{c_td_t} = 0 \ . \tag{3.235}$$

This set of equations is analogous to set (A.6) $\div$ (A.8) solved in appendix A, so we omit the calculations that have already been shown there. The solution is therefore

$$b_t(t) = b_0t^{p_1} \ , \ c_t(t) = c_0t^{p_2} \ , \ d_t(t) = d_0t^{p_3} \ ,$$

$$\text{with} \ \ p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \ . \tag{3.236}$$

For the temporal factor of functions b, c, and d, we have found exactly the Kasner solution as we expected.

To find the remaining spacial factors, we start by considering the right term of (3.226):

$$\frac{a_x''}{a_x} - \frac{a_x'b_x'}{a_xb_x} + \frac{a_x'c_x'}{a_xc_x} + \frac{a_x'd_x'}{a_xd_x} = 0 \ .$$

By adding and subtracting $\dfrac{a_x'^2}{a_x^2}$, the equation becomes

$$\frac{a_x''}{a_x} - \frac{a_x'^2}{a_x^2} + \frac{a_x'}{a_x}\left(\frac{a_x'}{a_x} - \frac{b_x'}{b_x} + \frac{c_x'}{c_x} + \frac{d_x'}{d_x}\right) = 0 \ ,$$

and by using Leibniz's rule we obtain

$$\left(\frac{a'_x}{a_x}\right)' + \frac{a'_x}{a_x}\left(\frac{a'_x}{a_x} - \frac{b'_x}{b_x} + \frac{c'_x}{c_x} + \frac{d'_x}{d_x}\right) = 0 \; .$$

If we divide by $\frac{a'_x}{a_x}$ and use the logarithmic derivative, we obtain

$$\left(\ln \frac{a'_x}{a_x}\right)' = \left(\ln \frac{b_x}{a_x c_x d_x}\right)' \; ,$$

that is easily integrated to

$$\frac{a'_x}{a_x} = \frac{A b_x}{a_x c_x d_x} \; . \tag{3.237}$$

In a completely analogous way, from the right terms of (3.228) and (3.229) we obtain

$$\frac{c'_x}{c_x} = \frac{C b_x}{a_x c_x d_x} \tag{3.238}$$

and

$$\frac{d'_x}{d_x} = \frac{D b_x}{a_x c_x d_x} \; . \tag{3.239}$$

Now we want to show that $\frac{b_x}{a_x c_x d_x}$ is a constant, and in order to do this we consider the right term of (3.227):

$$\frac{a''_x}{a_x} + \frac{c''_x}{c_x} + \frac{d''_x}{d_x} - \frac{b'_x a'_x}{b_x a_x} - \frac{b'_x c'_x}{b_x c_x} - \frac{b'_x d'_x}{b_x d_x} = 0 \; .$$

Thanks to Leibniz's rule we have

$$\frac{a''_x}{a_x} = \left(\frac{a'_x}{a_x}\right)' + \frac{a'^2_x}{a^2_x} \; .$$

Using the previous relation and (3.237) $\div$ (3.239), we obtain

$$\left(\frac{A b_x}{a_x c_x d_x}\right)' + \left(\frac{C b_x}{a_x c_x d_x}\right)' + \left(\frac{D b_x}{a_x c_x d_x}\right)' + \frac{(A^2 + C^2 + D^2)\, b_x^2}{a_x^2 c_x^2 d_x^2} - \\ - \frac{b'_x}{b_x}\left(\frac{(A + C + D)\, b_x}{a_x c_x d_x}\right) = 0 \; ;$$

finally, we obtain

$$\frac{b'_x}{b_x} = \frac{A^2 + C^2 + D^2}{A + C + D} \frac{b_x}{a_x c_x d_x} + \frac{\left(\frac{b_x}{a_x c_x d_x}\right)'}{\frac{b_x}{a_x c_x d_x}} \ . \tag{3.240}$$

It is obvious that metric (3.221) is invariant for permutations of the coordinates $x \leftrightarrow y \leftrightarrow z$, so it is reasonable to assume that functions b_x, c_x and d_x all have the same functional form. This implies that the last term in (3.240) must be equal to 0. This is true if and only if $\frac{b_x}{a_x c_x d_x}$ is constant. So, we have demonstrated that $\frac{b_x}{a_x c_x d_x}$ is constant, and we also found the equation for b_x. By using these results in (3.237) $\div$ (3.240), we obtain

$$a'_x = q_0 a_x \ \Rightarrow \ a_x = \tilde{A} e^{q_0 x} \ , \tag{3.241}$$

$$b'_x = q_1 b_x \ \Rightarrow \ b_x = \tilde{B} e^{q_1 x} \ , \tag{3.242}$$

$$c'_x = q_2 c_x \ \Rightarrow \ c_x = \tilde{C} e^{q_2 x} \ , \tag{3.243}$$

$$d'_x = q_3 d_x \ \Rightarrow \ d_x = \tilde{D} e^{q_3 x} \ . \tag{3.244}$$

It is easy to verify that the parameters q satisfy the relations of the double Kasner metric; in fact, by substituting the explicit expressions for a, b, c and d into the left term of (3.226), we obtain

$$q_0 \left(q_0 - q_1 + q_2 + q_3\right) = 0 \ \Rightarrow \ q_0 - q_1 + q_2 + q_3 = 0 \ . \tag{3.245}$$

We would have obtained the same result if we had used (3.228) or (3.229). Starting from (3.227) we obtain

$$q_0^2 + q_2^2 + q_3^2 - q_1 \left(q_0 + q_2 + q_3\right) = 0 \ ,$$

and by using (3.245) we obtain

$$q_0^2 - q_1^2 + q_2^2 + q_3^2 = 0 \ . \tag{3.246}$$

Finally, starting from (3.230) we obtain a relation that links the parameters p_i and q_j:

$$\begin{aligned} & p_2 q_2 + p_3 q_3 - p_2 q_0 - p_3 q_0 - p_1 q_2 - p_1 q_3 = 0 \ \Rightarrow \\ & \Rightarrow \ q_0 \left(p_2 + p_3\right) = q_2 \left(p_2 - p_1\right) + q_3 \left(p_3 - p_1\right) \ . \end{aligned} \tag{3.247}$$

Putting all these results in (3.221) and reparametrizing the coordinates so that we can absorb all the constants, we obtain

$$ds^2 = e^{2q_0 x} dt^2 - \left(t^{2p_1} e^{2q_1 x} dx^2 + t^{2p_2} e^{2q_2 x} dy^2 + t^{2p_3} e^{2q_3 x} dz^2\right) \ , \tag{3.248}$$

which is in complete accordance with (3.130). So, we have obtained the double Kasner metric in a totally general way by solving the Einstein equations.

Conclusions.

In the first chapter I have studied the mathematical tools necessary to describe fermions in a general manifold. First I have discussed the tetrads [4], used to define an alternative formalism to describe general relativity, and subsequently I have described the Foldy-Wouthuysen transformation and the advantages that derive from in this representation [5].

In the second chapter I have applied the previous mathematical tools to give an exhaustive description of fermions in a general gravitational field [2]. In particular I have described the way spin couples with the gravitational field, reaching the equations of motion of the spin operator in a general background.

In the third chapter I have applied the formalism developed in the previous chapters in order to analyse two particular anisotropic cosmological models. First, I described in depth the $Melvin\ space-time$ [8] and I showed that it is possible to find the Melvin solution by solving the Einstein field equations starting from simple assumptions. Then, I derived the expressions for the angular velocity operators of fermions (3.32) , (3.37) and (3.38). To obtain the final expressions for the operators, I had to evaluate the velocities of a particle in the Melvin space-time and to do this I solved the geodesics equations. I solved these equations both by direct calculation, and by using Killing vector fields and I found results in complete agreement. With the final expressions for angular velocity operators I solved the equation of motion of the polarisation operator and I found that the operator precedes in a regular way near the early universe. Finally I described the transition to the $Melvin\ Cosmology$ [10] and evaluated the polarisation operator in this cosmology finding complete agreement with Melvin space-time near the early universe. It is quite interesting to observe that the results obtained are in complete agreement with the semiclassical ones, derived in this chapter.

I have then applied the same mathematical apparatus to another cosmological model, the $double\ Kasner\ space - time$ [11]. I have asymptotically solved the geodesics equations near the early universe for general values of the parameters involved in metric (3.130). I then asymptotically solved the semiclassical equations of average spin near the early universe and I have found complete agreement with previous results [3] for the standard Kasner model. I found that near the singularity $t \to 0$ the average spin precedes in a regular way despite the divergence of the angular velocity. Finally I showed that the double Kasner space-time can be obtained as a vacuum solution of the Einstein field equations by using a diagonal, anisotropic and inhomogeneous metric.

The results obtained provide an exhaustive description of the spin-gravitational field coupling in these cosmological models. The study of anisotropic models might be important because the fundamental Friedmann–Lemaître–Robertson–Walker describes the isotropic and homogeneous universe, but the effects of anisotropy and inhomogeneity may be essential to describe some phenomena near the early universe such as structure formations or baryon anti-baryon asymmetries. In particular a description of spin precession near the early universe may be used to describe the helicity flip of massive Dirac neutrinos: neutrinos with left chirality may change their helicity because of the interaction with the gravitational field, so right-handed neutrinos could be produced. These are sterile particles that interact gravitationally only; this could be an interesting characteristic to study, because these sterile neutrinos may be a contribution to dark matter [3].

Appendix A

Kasner space-time.

In [22] and in previous articles, E. Kasner demonstrates some theorems about the *Cosmological equations*, which are Einstein equations in vacuum. In particular, he shows that a solution of the Einstein equations in vacuum where potentials involve only one variable is

$$ds^2 = t^{-2}dt^2 - x_1^2\left(dx_1^2 + dx_2^2 + dx_3^2\right) \ , \tag{A.1}$$

and this can be further reduced to

$$ds^2 = t^{2a_1}dt^2 - t^{2a_2}dx_1^2 - t^{2a_3}dx_2^2 - t^{2a_4}dx_3^2$$

$$\text{where} \quad a_2 + a_3 + a_4 = 1 + a_1 \ , \quad \text{and} \quad a_2^2 + a_3^2 + a_4^2 = (1 + a_1)^2 \ . \tag{A.2}$$

In a reference frame that uses *cosmological time* $a_1 = 0$, and the Kasner line element becomes

$$ds^2 = dt^2 - t^{2p_1}dx_1^2 - t^{2p_2}dx_2^2 - t^{2p_3}dx_3^2$$

$$\text{where} \quad p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2 \ . \tag{A.3}$$

In many textbooks, such as in [9], the Kasner solution is introduced as an assumption and it is then verified that it satisfies the Einstein equations. Here we generalize this process by showing how to obtain the Kasner metric starting from a Bianchi-I type cosmology [9]

$$ds^2 = dt^2 - \left(a(t)^2dx^2 + b(t)^2dy^2 + c(t)^2dz^2\right) \ . \tag{A.4}$$

In other words, we demonstrate that the vacuum solution of the Einstein field equations that represents a Bianchi-I cosmology is the Kasner metric. It is easy to demonstrate that the Einstein equations in vacuum are

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0 \ , \tag{A.5}$$

$$\frac{\dot{a}\dot{b}}{ab}+\frac{\dot{a}\dot{c}}{ac}+\frac{\ddot{a}}{a}=0\ , \tag{A.6}$$

$$\frac{\dot{a}\dot{b}}{ab}+\frac{\dot{b}\dot{c}}{bc}+\frac{\ddot{b}}{b}=0\ , \tag{A.7}$$

$$\frac{\dot{a}\dot{c}}{ac}+\frac{\dot{b}\dot{c}}{bc}+\frac{\ddot{c}}{c}=0\ . \tag{A.8}$$

(A.6) may be written as

$$\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}+\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)=0\ \Rightarrow\ \frac{d}{dt}\left(\frac{\dot{a}}{a}\right)+\frac{\dot{a}}{a}\frac{d}{dt}(\ln abc)=0\ \Rightarrow$$

$$\Rightarrow\ \frac{d}{dt}\left(\ln\frac{\dot{a}}{a}\right)+\frac{d}{dt}(\ln abc)=0\ \Rightarrow\ \frac{\dot{a}}{a}=\frac{A}{abc}\quad\text{where}\quad A=const.\ . \tag{A.9}$$

Analogously, we obtain

$$\frac{\dot{b}}{b}=\frac{B}{abc}\quad\text{and}\quad\frac{\dot{c}}{c}=\frac{C}{abc}\quad\text{where}\quad B,C=const.\ . \tag{A.10}$$

Now we demonstrate that abc is linearly dependent on t:

$$\frac{d^2}{dt^2}(abc)=\frac{d}{dt}\left(\dot{a}bc+a\dot{b}c+ab\dot{c}\right)=\ddot{a}bc+a\ddot{b}c+ab\ddot{c}+2\dot{a}\dot{b}c+2\dot{a}b\dot{c}+2a\dot{b}\dot{c}\ ,$$

and by dividing by abc we obtain (A.6) + (A.7) + (A.8). So,

$$\frac{d^2}{dt^2}(abc)=0\ . \tag{A.11}$$

Thanks to this linearity we can set $abc=vt$ and, by using $\frac{A}{v}=p_1$, $\frac{B}{v}=p_2$ and $\frac{C}{v}=p_3$, we can obtain

$$\frac{\ddot{a}}{a}=\frac{p_1^2-p_1}{t^2}\ ,\ \frac{\ddot{b}}{b}=\frac{p_2^2-p_2}{t^2}\ ,\ \frac{\ddot{c}}{c}=\frac{p_3^2-p_3}{t^2}\ . \tag{A.12}$$

By using (A.12) in (A.5) we obtain

$$p_1+p_2+p_3=p_1^2+p_2^2+p_3^2\ , \tag{A.13}$$

and substituting (A.12) in (A.6) + (A.7) + (A.8) we obtain

$$(p_1+p_2+p_3)(p_1+p_2+p_3-1)=0\ .$$

The first solution $p_1 + p_2 + p_3 = 0$, that combined with (A.13) gives $p_1 = p_2 = p_3 = 0$, reproduces the Minkowski space-time; so, we keep the other solution, $p_1 + p_2 + p_3 = 1$. By putting all these results together we find the Kasner solution:

$$a(t) = a_0 t^{p_1} \ , \ b(t) = b_0 t^{p_2} \ , \ c(t) = c_0 t^{p_3} \ ,$$
$$\text{with} \ \ p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \ . \tag{A.14}$$

In order to study the properties of the Kasner indices, it is often useful to introduce the $Lifshitz - Khalatnikov$ parametrization [23]:

$$p_1 = -\frac{u}{1+u+u^2} \ , \ p_2 = \frac{u+1}{1+u+u^2} \ , \ p_3 = \frac{u\,(u+1)}{1+u+u^2} \ , \ u \in [1;+\infty) \ , \tag{A.15}$$

where we have assumed, without loss of generality, $p_1 < p_2 < p_3$. It is easy to obtain

$$-\frac{1}{3} \leq p_1 \leq 0 \ , \ 0 \leq p_2 \leq \frac{2}{3} \ , \ \frac{2}{3} \leq p_3 \leq 1 \ , \tag{A.16}$$

therefore two exponents are positive and one is negative. This means, in physical terms, that the universe undergoes an expansion along two directions and a contraction along the third.

(A.3) represents the standard Kasner solution in timelike form; it can be written in spacelike form as

$$ds^2 = x_1^{2p_1} dt^2 - dx_1^2 - x_1^{2p_2} dx_2^2 - x_1^{2p_3} dx_3^2 \ . \tag{A.17}$$

It follows that the double-Kasner metric studied in chapter 3 is a nonlinear superposition of the timelike and the spacelike forms of the Kasner metric.

Now we will show that if one of the three parameters p_i is equal to 1, then metric (A.3) is nothing more than a different parametrization of the Minkowski metric. If we consider the limit $u \to \infty$, we obtain

$$p_1 = p_2 = 0 \ , \ p_3 = 1 \ \Rightarrow \ ds^2 = dt^2 - dx^2 - dy^2 - t^2 dz^2 \ . \tag{A.18}$$

Let's consider the Minkowski line element with signature -2:

$$ds^2 = d\tau^2 - dx^2 - dy^2 - d\xi^2 \ ; \tag{A.19}$$

now we change variables by setting

$$\tau = t \cosh z \ , \ \xi = t \sinh z \ .$$

By differentiating the two expressions we obtain

$$d\tau = dt\cosh z + t\sinh z dz \ , \ d\xi = dt\sinh z + t\cosh z dz \ ,$$

and substituting in (A.19) we obtain

$$\begin{aligned} ds^2 = dt^2\cosh^2 z + t^2\sinh^2 z dz^2 + 2t\cosh z\sinh z dtdz - dx^2 - dy^2 - dt^2\sinh^2 z - \\ -t^2\cosh^2 z - 2t\cosh z\sinh z dtdz = dt^2\left(\cosh^2 z - \sinh^2 z\right) - dx^2 - dy^2 - \\ -t^2 dz^2\left(\cosh^2 z - \sinh^2 z\right) = dt^2 - dx^2 - dy^2 - t^2 dz^2 \ , \end{aligned} \tag{A.20}$$

that is exactly (A.18).

Appendix B

Killing vector fields.

First we have to remember a couple of definitions [12]:

Def. B.1 (Isometry) : let X and Y be two metric spaces with distances respectively d_X and d_Y, an application $f : X \to Y$ is called an *isometry* if it preserves the distance, that is

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) \ . \tag{B.1}$$

Def. B.2 (Killing vector field) : a *Killing vector field* V is a field with respect to which the metric is invariant, that is

$$L_V g = 0 \ , \tag{B.2}$$

where L_V denotes the Lie derivative along V. (B.2) can be rewritten as

$$\nabla_{(\nu} V_{\mu)} = V_{(\mu;\nu)} = 0 \ , \tag{B.3}$$

and (B.3) is the *Killing equation*.

Killing vector fields are the generators of infinitesimal isometries, and here we will show how. Starting from condition (B.1) it follows that

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = g_{\mu\nu}(x') \ ,$$

that is the isometry condition for the metric. We have

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = g'_{\rho\sigma}(x') dx'^\rho dx'^\sigma = g'_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} dx^\mu dx^\nu \ ,$$

and by using the isometry condition we obtain

$$g_{\mu\nu}(x) = g_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \ . \tag{B.4}$$

If we consider the infinitesimal transformations

$$x^\mu \to x'^\mu = x^\mu + \epsilon \xi^\mu + O\left(\epsilon^2\right) \quad \text{where} \quad |\epsilon| \ll 1 \ , \tag{B.5}$$

we can study (B.4) at the first order in ϵ. We have

$$\frac{\partial x'^\rho}{\partial x^\mu} = \delta^\rho_\mu + \epsilon \frac{\partial \xi^\rho}{\partial x^\mu}(x) + O\left(\epsilon^2\right) \ ,$$

$$g_{\rho\sigma}(x') = g_{\rho\sigma}(x) + \epsilon \frac{\partial g_{\rho\sigma}(x)}{\partial x^\lambda} \xi^\lambda + O\left(\epsilon^2\right) \ ,$$

and by substituting the latter expressions into (B.4) we obtain at first order

$$g_{\mu\nu}(x) = \left(\delta^\rho_\mu + \epsilon \frac{\partial \xi^\rho}{\partial x^\mu}(x)\right)\left(\delta^\sigma_\nu + \epsilon \frac{\partial \xi^\sigma}{\partial x^\nu}(x)\right)\left(g_{\rho\sigma}(x) + \epsilon \frac{\partial g_{\rho\sigma}(x)}{\partial x^\lambda}\xi^\lambda\right) \ \Rightarrow$$

$$\Rightarrow \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda}\xi^\lambda(x) + \frac{\partial \xi^\lambda(x)}{\partial x^\mu}(x) g_{\lambda\nu}(x) + \frac{\partial \xi^\lambda(x)}{\partial x^\nu}(x) g_{\lambda\mu}(x) = 0 \ , \tag{B.6}$$

that is (B.2) in a coordinate basis. We have demonstrated that Killing vector fields are the generators of infinitesimal isometries; this is particularly useful when the metric does not depend on one of the coordinates, in which case a Killing vector field is simply the generator of translations for that coordinate.

Now we show that the scalar $u^\mu \xi_\mu$ is a constant of motion along the geodesics:

$$\frac{d}{d\lambda}\left(u^\mu \xi_\mu\right) = \frac{D}{D\lambda}\left(u^\mu \xi_\mu\right) = \frac{Du^\mu}{D\lambda}\xi_\mu + u^\mu \frac{D\xi_\mu}{D\lambda} = u^\mu \frac{D\xi_\mu}{D\lambda} = \xi_{\mu;\nu} u^\mu u^\nu = 0 \ , \tag{B.7}$$

where $\dfrac{D}{D\lambda}$ is the covariant derivative along the geodesic. In the first equality of (B.7) we have used the fact that $u^\mu \xi_\mu$ is a scalar, in the third equality we have used the geodesic definition and in the last equality we have used the antisymmetry of $\xi_{\mu;\nu}$, which is a consequence of (B.3).

This consideration is particularly useful because it simplifies the solution of the geodesic equations in the case of Killing vectors obtained from the independence of the metric of some coordinates.

Appendix C

Observations upon the double Kasner metric.

In section 3.2.8 we demonstrated that it is possible to obtain the double Kasner metric by solving the Einstein field equations in vacuum assuming metric (3.220) as a starting point. As we were solving system (3.226) $\div$ (3.230) we obtained a solution for the metric coefficients factorised in a space-dependent only term and a time-dependent only term. This factorisation implied that each equation in the set (3.226) $\div$ (3.229) was split in a pair of equations, one with time derivatives only, one with space derivatives only. Each equation of the pairs involves an arbitrary constant that we put equal to 0 for the sake of simplicity. In this appendix we discuss the necessariness of putting this constant equal to 0 in order to find a solution which is in accordance with the Kasner metric.

Let's start from (A.6), that is the temporal part of (3.227). We consider the case of a non-zero constant K:

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = -\frac{d}{dt}(\ln abc)\,\frac{\dot{a}}{a} + K\ . \tag{C.1}$$

(C.1) is a linear differential equation with respect to the function $\frac{\dot{a}}{a}$, so we can find a solution by using standard formulas of mathematical analysis:

$$\frac{\dot{a}}{a} = e^{-\int \frac{d}{du}(\ln abc)du}\left[A + \int K e^{\int \frac{d}{d\tau}(\ln abc)d\tau} du\right]\ , \tag{C.2}$$

where A is an integration constant. (C.2) becomes then

$$\frac{\dot{a}}{a} = \frac{1}{abc}\left[A + K\int abc\, du\right]\ . \tag{C.3}$$

If we had started from (A.7) and (A.8) we would have obtained analogous equations with different constants. The set of three equations analogous to (C.3) is a set of integro-differential equations, and it is not easy to solve it analytically. To proceed with calculations we use the hypothesis that abc is a linear function of t, so we can put $abc = vt$. This hypothesis is reasonable because it is verified in the standard Kasner case as shown in (A.11). Therefore (C.3) becomes

$$\frac{\dot{a}}{a} = \frac{1}{abc}\left[A + K\int abc\, du\right] = \frac{A}{vt} + \frac{1}{vt}\frac{Kv}{2}t^2 = \frac{A}{vt} + \frac{K}{2}t \ , \tag{C.4}$$

and the other two analogous equations are

$$\frac{\dot{b}}{b} = \frac{1}{abc}\left[B + L\int abc\, du\right] = \frac{B}{vt} + \frac{1}{vt}\frac{Lv}{2}t^2 = \frac{B}{vt} + \frac{L}{2}t \ , \tag{C.5}$$

$$\frac{\dot{c}}{c} = \frac{1}{abc}\left[C + M\int abc\, du\right] = \frac{C}{vt} + \frac{1}{vt}\frac{Mv}{2}t^2 = \frac{C}{vt} + \frac{M}{2}t \ , \tag{C.6}$$

(C.4), (C.5) and (C.6) are differential equation that can be easily solved through separation of variables, and they lead to

$$\ln a = \frac{A}{v}\ln t + \frac{K}{4}t^2 \ \Rightarrow \ a = t^{\frac{A}{v}} e^{\frac{K}{4}t^2} \ , \tag{C.7}$$

and analogously

$$b = t^{\frac{B}{v}} e^{\frac{L}{4}t^2} \ , \tag{C.8}$$

$$c = t^{\frac{C}{v}} e^{\frac{M}{4}t^2} \ . \tag{C.9}$$

The power-like parts of a, b and c is analogous to what we derived in Appendix A, and in particular (A.14) holds. The exponential parts of a, b and c are not in agreement with the hypothesis $abc = vt$ unless the constants K, L and M are all equal to 0. So we obtained that a necessary and sufficient condition in order to reproduce a result in accordance with the Kasner metric is that the constants emerging from the factorisation of (3.226) $\div$ (3.229) are all equal to 0.

Another formulation for these observations starts from considering (C.4) $\div$ (C.6) without the assumption of $abc = vt$:

$$\dot{a}bc = A + K\int abc\, du \ , \tag{C.10}$$

$$a\dot{b}c = B + L\int abc\, du \ , \tag{C.11}$$

$$ab\dot{c} = C + M \int abc\, du \ . \tag{C.12}$$

If we consider (C.10) + (C.11) + (C.12) we obtain

$$(a\dot{b}c) = U + V \int abd\, du \ , \tag{C.13}$$

where $U = A + B + C$ and $V = K + L + M$. If $V = 0$ we obtain that abc is a linear function of t and in particular it is necessary that $K = L = M = 0$ to obtain expressions reproducing the Kasner solution.

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