Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea in Matematica Generale-Applicativa

The Representation of Symmetric Groups in Two-Dimensional Persistent Homology

Tesi di Laurea in Matematica Generale-Applicativa

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III Sessione
Anno Accademico 2016/2017

Abstract. It is well-known that two-dimensional persistent homology can be reduced to the persistent homology of a family of parameterized real functions, and that this reduction is related to a phenomenon of monodromy, which happens when a loop in the parameter space does not translate into loops of the elements of the persistence diagram. Instead these elements switch their places in a functorial way. In this thesis we introduce the corresponding functor and show that any symmetric group S^n can be seen as the monodromy group of a suitable filtering function.

Keywords: Persistence diagrams, topological persistence, monodromy, symmetric group.

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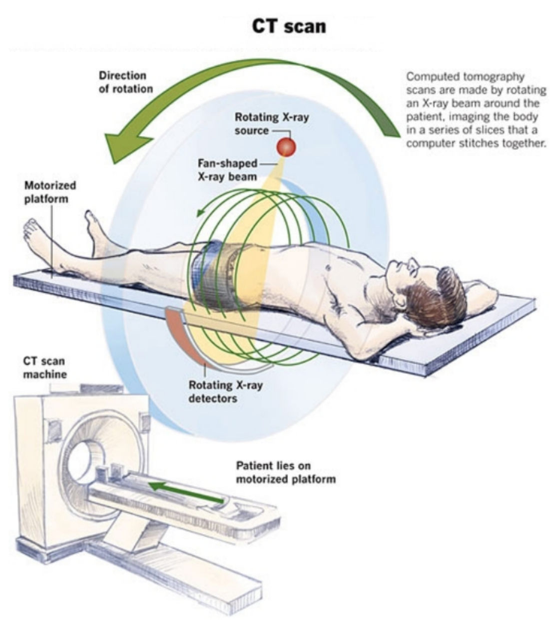
Introduzione

In visione artificiale e in computer grafica, un modello di forma è una struttura multidimensionale, caratterizzata in primo luogo da un forma o da una spazialità, con un'importante quantità di informazione visuale e semantica.
La comparazione, la classificazione ed il recupero di modelli di forma sono problemi complessi e si è notato uno spostamento della ricerca dai metodi che convogliano informazione quantitativa, che rappresenta un oggetto, a metodi che convogliano informazione qualitativa, che descrivono l'oggetto.
La differenza di una descrizione da una rappresentazione è che l'informazione contenuta non ci permette di approssimare l'oggetto, ma identifica l'oggetto come appartenente a qualche classe.
Data la crescente necessità di analizzare grandi quantitativi di dati, i metodi che sfruttano l'informazione qualitativa sono diventati di uso comune, e la geometria computazionale e l'analisi topologica di dati hanno applicazioni in computer imaging [16] e in analisi di forma [2].
In particolare, l'idea della *persistenza omologica* monodimensionale è di ridurre significativamente la complessità dei dati studiando la persistenza dell'omologia degli insiemi di sottolivello X_τ di una *funzione filtrante* $\varphi\colon X \to \mathbb{R}$ [17, 18]. Detto altrimenti, le proprietà che sono risultate rilevanti per la riduzione del rumore e la comparazione di forme sono le durate delle vite delle componenti connesse, o dei tunnel, o dei buchi k-dimensionali.
La persistenza topologica appare per la prima volta in [23], dove vengono introdotti i concetti di *funzione di taglia* e *pseudo-distanza naturale*.
Le funzioni di taglia sono equivalenti ai numeri di Betti persistenti di grado 0. Precisamente, se $\phi\colon X \to \mathbb{R}$ è una funzione continua, la funzione di taglia di ϕ calcolata in (u, v) è il numero di componenti connesse di X_v che contengono almeno un punto di X_u.
La pseudo-distanza naturale rappresenta la somiglianza tra due forme. In questo contesto, la forma consiste in una coppia data da uno spazio topologico e una funzione che riporta le proprietà rilevanti. La somiglianza, invece, è espressa attraverso gli omeomorfismi che preservano le proprietà delle funzioni. Ad esempio, se siamo interessati a preservare la metrica, consideriamo solo le isometrie. La pseudo-distanza naturale è infine definita come l'estremo inferiore della variazione delle due funzioni quando ci muoviamo da uno spazio all'altro attraverso omeomorfismi.
L'introduzione dell'idea di persistenza topologica ha condotto alla realizzazione di algoritmi e applicazioni [21, 22, 11].

Nel 1999, Vanessa Robins ha studiato l'immagine dell'omomorfismo indotto dall'inclusione in uno spazio campione [25]. Nel 2000, Edelsbrunner e collaboratori, proposero la teoria della persistenza omologica, presentarono un efficiente algoritmo per la computazione e introdussero i *diagrammi di persistenza* come un modo per descrivere la persistenza omologica [18]. Poco dopo, Carlsson e altri diedero una rappresentazione equivalente in termini di *barcodes* [6].
In questa tesi vengono utilizzati diagrammi di persistenza che sono multi-insiemi composti da *punti d'angolo*, i quali descrivono in maniera esaustiva l'omologia persistente [10], ed un insieme fissato Δ, contato come un unico elemento. Uno dei motivi di questa scelta risiede nel fatto che i diagrammi di persistenza possono essere inseriti in uno spazio metrico, la cui distanza li rende stabili rispetto al rumore [10].

In anni recenti si è assistito ad una rapida crescita dell'interesse per la *persistenza multidimensionale* come in [4], che ha portato ad un'estensione della teoria e all'elaborazione di nuovi metodi computazionali per l'analisi di funzioni a valore vettoriale, con potenziali applicazioni, ad esempio, alla diagnostica per immagini in campo medico.



Un'applicazione dell'analisi topologica di dati.

Uno di questi metodi consiste nella riduzione del caso n-dimensionale al caso 1-dimensionale attraverso una famiglia di funzioni parametriche $\{\varphi_{\alpha,\beta} \colon X \to \mathbb{R}\}$ ottenuta da φ. Questo approccio permette di risalire all'omologia persistente multidimensionale attraverso i diagrammi persistenza parametrici [1, 3].

È stata proposta una nuova idea di distanza tra due funzioni filtranti chiamata distanza coerente di matching, basata sull'impiego di matching che cambiano con continuità, sfruttando le proprietà topologiche dello spazio dei parametri [9].

Questa definizione deve affrontare un problema di *monodromia* [8], che si manifesta quando un laccio attorno ad uno o più *punti singolari* nello spazio dei parametri non si traduce in cammini chiusi degli elementi del diagramma di persistenza. I punti singolari sono parametri in cui molteplici punti d'angolo hanno le stesse coordinate, e quando si attraversano tali punti non è possibile seguire il moto dei punti d'angolo.
Quando i parametri cambiano in maniera continua, il movimento di un diagramma persistente ci dà un'idea di identità dei propri elementi. Ciò nonostante, questa idea ci impedisce di stabilire qualsiasi relazione geometrica tra un punto d'angolo e la *nascita* o la *morte* di una componente connessa.

Il principale scopo di questa tesi è misurare le discrepanze tra l'approccio intuitivo e il vero percorso dei punti d'angolo *propri*, che scaturisce nella permutazione dei punti d'angolo.

Per farlo abbiamo dimostrato formalmente l'esistenza della monodromia, sfruttando la geometria della funzione filtrante: associamo la nascita o la morte di una componente connessa ad una linea e osserviamo come questa linea interagisce con $\varphi_{\alpha,\beta}$.

Ne risulta che, a differenza di altri fenomeni di monodromia, le due coordinate di questi elementi si scambiano in due punti precisi ed è possibile provare l'esistenza di un laccio che determina una trasposizione. Inoltre, generalizziamo la funzione filtrante in modo che abbia n punti d'angolo e troviamo $n-1$ lacci che corrispondono a $n-1$ trasposizioni. Queste trasposizioni generano il gruppo n-simmetrico S_n, e il quale è rappresentato funtorialmente da ciò che chiamiamo *gruppo di monodromia*.

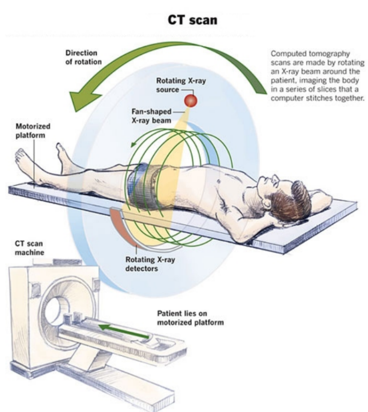
Si noti come in questa tesi sia stata presa in considerazione soltanto l'omologia di grado 0, ma in realtà è possibile adattare le funzioni filtranti per ottenere esempi di monodromia anche in altri gradi.

Questa tesi è strutturata come segue. Nella Sezione 2 sono presentate le principali definizioni ed idee di pseudo-distanza naturale e si enunciano alcuni teoremi. Nella Sezione 3 si fa riferimento alle definizioni di omologia persistente bidimensionale ed ai risultati necessari alla nostra esposizione, e viene mostrato il legame tra la pseudodistanza naturale e la teoria dell'omologia persistente. Nella Sezione 4 viene presentata una dimostrazione dell'esistenza della monodromia, che fungerà da base per la dimostrazione del teorema conclusivo. Nella Sezione 5 si dimostra che è possibile rappresentare funtorialmente ogni gruppo simmetrico come un gruppo di monodromia.

Introduction

In computer vision and computer graphics a shape model is a multidimensional medium primarily characterized by form or spatial extent, with a considerable amount of visual and semantic information.
Shape comparison, classification and retrieval are challenging issues, and we have seen a shift of research interests from methods conveying quantitative information, which represent an object, to methods conveying qualitative information, which describe the object.
What makes a description different from an object representation, is that the information contained does not allow us to approximate the object, but it does identify the object as a member of some class.
With the growing need to analyse large amounts of data, methods that study qualitative information have become of common use, and computational geometry and topological data analysis find their application in computer imaging [16] and shape analysis [2].
Notably, the idea of one-dimensional *homological persistence* is to deeply reduce the complexity of data by studying the persistence of the homology of the sublevel sets X_τ of a *filtering function* $\varphi\colon X \to \mathbb{R}$ [17, 18]. In other words, the relevant properties for denoising and shape comparison turn out to be the durations of the lives of the k-dimensional holes.
The study of persistence topology starts with [23], where the concepts of *size function* and *natural pseudo-distance* are introduced for the first time.
Size functions are equivalent to persistent Betti numbers in degree 0. Precisely, if $\phi\colon X \to \mathbb{R}$ is a continuous function, the size function of ϕ computed at (u, v) is the number of connected components of X_v that contain at least one point of X_u.
The natural pseudo-distance represents the similarity between two shapes. In this setting, the shape consists in a pair given by a topological space and a function conveying the relevant properties. The similarity, instead, is expressed by the homeomorphisms that preserve the properties of the functions. For example, if we are interested in preserving the distance, we would consider only the isometries. The natural pseudo-distance is then defined as the infimum of the variation of the functions when we move from one space to the other through homeomorphisms.
This also led to the realization of algorithms and applications [21, 22, 11].
In 1999 Vanessa Robins studied the image of the homomorphism induced by the inclusion into a sampled space [25].

In 2000 Edelsbrunner and collaborators proposed the theory of persistent homology, presented an efficient algorithm for the computation, and introduced the *persistence diagram* as a way to describe persistent homology [18]. Shortly after, Carlsson et al. gave an equivalent representation in terms of *barcodes* [6].
In our thesis we work with persistence diagrams, which are multisets containing *cornerpoints*, that completely describe persistent homology [10], and a fixed set Δ, counted as a single element. One reason for this choice is that they can be embedded into a metric space, whose distance makes them stable with respect to noise [10].
In recent years the interest in *multidimensional persistence* has rapidly increased, leading to an extension of the theory and to new computational methods for analysis of vector-valued functions (see [4]).



An application of Topological Data Analysis

One of these methods consists in the reduction of the n-dimensional case to the 1-dimensional setting by deriving a suitable parameterized family of functions $\{\varphi_{\alpha,\beta}\colon X \to \mathbb{R}\}$. This approach allows for recovering multidimensional persistent homology through the parameterized persistence diagrams [1, 3].
A new idea of a distance between two filtering functions has been proposed, which is called *coherent matching distance* and is based on the use of continuously changing matchings, exploiting the topological properties of the space of the parameters [9].
This definition has to manage a problem of *monodromy* [8], which occurs

when a loop around *singular points* in the parameter space does not translate into loops of the elements of the persistence diagram. The singular points are parameters where different cornerpoints have the same coordinates and it becomes impossible to keep track of their path.
When the parameters change continuously, the motion of a persistence diagram provides us with an idea of identity of its elements. However, that idea hinders us from establishing any geometric relation between a cornerpoint and the *birth* or *death* of a connected component.
The main purpose of this thesis is to measure the discrepancies between the intuitive approach and the actual path of the *proper* cornerpoints, resulting in a permutation of the cornerpoints.
In order to do that, we present a formal proof of the existence of monodromy, exploiting the geometry of the filtering function: we associate the birth or death of a connected component to a line and we study how these lines interact with $\varphi_{\alpha,\beta}$.
We find that, differently from other monodromy phenomena, the two coordinates of these elements switch at two precise points and we prove the existence of a loop leading to a transposition. Moreover, we generalize the filtering function to have n cornerpoints and we find $n-1$ loops resulting in $n-1$ transpositions. These transpositions generate the n-symmetric group S_n, and thus S_n is functorially represented as what we called a *monodromy group*.
We note that throughout this thesis we have only considered the 0^{th}-degree homology, but it is actually very possible to adjust the filtering functions to have examples of monodromy in other degrees.
This thesis is organized as follows. In Section 1 the main definitions and ideas concerning the natural pseudo-distance are introduced and we mention some theorems. In Section 2 we recall the definitions of two-dimensional persistent homology as well as results needed in our exposition, and we show the link between the natural pseudo-distance and the theory of homological persistence. In Section 3 a formal proof of the existence of monodromy is presented, which will serve as a base for the proof of the concluding theorem. In Section 4 we prove that it is possible to functorially represent any symmetric group as a monodromy group.

1 Natural pseudodistance

Topological data analysis (TDA) is based on the assumption that objects can only be studied indirectly and all we know about them is deduced by means of the functions representing the measurement data.
These functions are then interpreted by an observer who is entitled to decide about shape similarity. The observer then chooses an action on these functions that reflects the invariances that are relevant to him.
For example, if we have to define the shape of a letter, we would consider a letter as a function from a plane to a grey scale, and we would include other functions obtained just by expanding or contracting the coordinates of the plane; at the same time we would exclude rotations that would make 'Z' and 'N' equivalent.

In this approach we consider the set ϕ of *filtering functions*, which are continuous functions $\varphi\colon X \to \mathbb{R}^k$ from a finitely triangulable space X, together with a subgroup $G \subseteq \mathsf{Homeo}(X)$ that acts on ϕ.
The natural pseudo-distance $d_G\colon C^0(X, \mathbb{R}^k) \times C^0(X, \mathbb{R}^k) \to \mathbb{R}$ between two filtering function φ, ψ, is defined by setting

$$d_G(\varphi, \psi) = \inf_{g \in G} \sup_{x \in X} \parallel \varphi(x) - \psi(g(x)) \parallel .$$

We say that $d_G(\varphi, \psi)$ is a pseudo-distance, meaning that we allow $\varphi \neq \psi$ and $d_G(\varphi, \psi) = 0$.
We could think of $d_G(\varphi, \psi)$ as the distance associated with the best correspondence between the two functions φ and ψ by means of homeomorphisms in G.

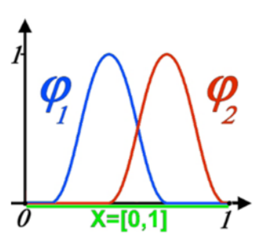


Figure 1: The natural pseudo-distance of these two functions is 0, because we can squeeze and stretch the domain by a suitable homeomorphism.

When $G = \mathsf{Homeo}(X)$, the set of all homeomorphisms, the theory leads to some relevant results (all concerning real one-dimensional functions) that we will mention [20, 13, 14, 15].

Theorem 1.1 (For manifolds)**.** *Assume X is a closed manifold of class C^1 and that $\varphi, \psi \colon X \to \mathbb{R}$ are two C^1 functions. Then at least one of the following properties holds:*

- *a positive odd integer m exists, such that $m \cdot d_G$ equals the distance between a critical value of φ and a critical value of ψ.*
- *a positive even integer m exists, such that $m \cdot d_G$ equals the distance between either two critical values of φ or two critical value of ψ.*

We could get some information about m by restricting ourselves in the case where X is a surface.

Theorem 1.2 (For surfaces)**.** *Assume X is a closed surface of class C^1 and that $\varphi, \psi \colon X \to \mathbb{R}$ are two C^1 functions. Then at least one of the following properties holds:*

- *d_G equals the distance between a critical value of φ and a critical value of ψ.*
- *$2d_G$ equals the distance between either two critical values of φ or two critical values of ψ.*
- *$3d_G$ equals the distance between a critical value of φ and a critical value of ψ.*

Even simpler is the case where X is a curve.

Theorem 1.3 (For curves)**.** *Assume X is a closed curve of class C^1 and that $\varphi, \psi \colon X \to \mathbb{R}$ are two C^1 functions. Then at least one of the following properties holds:*

- *d_G equals the distance between a critical value of φ and a critical values of ψ.*
- *$2d_G$ equals the distance between either two critical values of φ or two critical value of ψ.*

Generally, this pseudo-distance is not easily calculable and the brute force methods have a very high computational cost. The strategy to overcome this obstacle has been to find lower bounds and, in some cases, upper bounds [24] of $d_G(\varphi, \psi)$ and is very related to persistent homology.
The research is now focusing on proper subgroups of $\mathsf{Homeo}(X)$ (the rotations of S^1 are a classical example) and the interested reader is referred to [19] for further details.

2 Persistent homology

2.1 Preliminaries

In this section we recall basic definitions and some results of multidimensional persistent homology.
By using the standard notation, Δ^+ will be the open set $\{(u, v) \in \mathbb{R} \times \mathbb{R} \colon u < v\}$ and Δ will be the diagonal $\{(u, v) \in \mathbb{R} \times \mathbb{R} \colon u = v\}$. Δ^* is defined as the union of Δ^+ and the "points at infinity" of the form (u, ∞) with $u \in \mathbb{R}$.
Given a triangulable topological space X and a continuous map $\varphi \colon X \to \mathbb{R}$ for every couple of real numbers $(u, v) \in \Delta^+$, we define the sublevel sets $X_u = \{x \in X \colon \varphi(x) \leq u\}$ and $X_v = \{x \in X \colon \varphi(x) \leq v\}$.

Definition 2.1. In the standard notation, given $k \in \mathbb{N}$, the k^{th} persistent homology group $H_k^{(u,v)}(X, \varphi)$ is the image of the k^{th} homology group of X_u into the k^{th} homology group of X_v through the homomorphism induced by the inclusion map.

Definition 2.2 (Persistent Betti Numbers Function)**.** The persistent Betti numbers function of φ, briefly PBN, is the function $b_{\varphi,k} \colon \Delta^+ \to \mathbb{N} \cup \{\infty\}$ defined by

$$b_{\varphi,k}(u, v) = \dim H_k^{(u,v)}(X, \varphi).$$

To simplify the notation in the next definitions we write $b_{\varphi,k}$ as b_φ and any reference to the degree will be omitted.
Intuitively, u is related to the birth of connected components or holes and v to their death when they join or fill up.
Although this concept might be useful at first, it may cause ambiguity when two different components join, and throughout the thesis we will see that this will have some counterintuitive consequences.

Definition 2.3 (Multiplicity)**.** The multiplicity $\mu_\varphi(u, v)$ of $(u, v) \in \Delta^+$ is a non-negative number given by

$$\min_{\substack{\epsilon>0 \\ u+\epsilon<v-\epsilon}} b_\varphi(u+\epsilon, v-\epsilon) - b_\varphi(u-\epsilon, v-\epsilon) - b_\varphi(u+\epsilon, v+\epsilon) + {}_\varphi(u-\epsilon, v+\epsilon).$$

The multiplicity $\mu_\varphi(u, \infty)$ of a point at infinity (u, ∞) is similarly given by

$$\min_{\substack{\epsilon>0 \\ u+\epsilon<v}} b_\varphi(u+\epsilon, v) - b_\varphi(u-\epsilon, v).$$

We make some observations that will be helpful later.

Remark 1. Let us consider a point $(u, v) \in \mathbb{R}^2$ with positive multiplicity and the Equation 2.3. We have that $\forall \epsilon > 0$

$$-b_\varphi(u+\epsilon, v+\epsilon) + b_\varphi(u-\epsilon, v+\epsilon) \leq 0$$

because the PBS is increasing in the first variable and

$$b_\varphi(u+\epsilon, v-\epsilon) - b_\varphi(u-\epsilon, v-\epsilon) > 0$$

because the multiplicity is positive.
If we are talking about the 0^{th}-persistent Betti Number function (which will be our focus), this implies that $H_0^{(u+\epsilon,v-\epsilon)}(X,\varphi)/H_0^{(u-\epsilon,v-\epsilon)}(X,\varphi)$ is not trivial, and there exists a connected component of $X_{u+\epsilon}$ that has its image there through the map induced by the inclusion. It is understandable to refer to this phenomenon as a birth.
For similar reasons, we have that

$$b_\varphi(u+\epsilon, v-\epsilon) - b_\varphi(u+\epsilon, v+\epsilon) > 0$$

and there exist, at least, two different connected components in $X_{u+\epsilon}$, having distinct image in $H_0^{(u+\epsilon,v-\epsilon)}(X,\varphi)$, that have the same image in $H_0^{(u+\epsilon,v+\epsilon)}(X,\varphi)$. This corresponds to the fact that they join in X_v.

2.2 Persistence diagram

The following definition not only helps represent the PBN, but is also key to this thesis.

Definition 2.4 (Persistence Diagram)**.** The persistence diagram $\mathrm{Dgm}(\varphi)$ is the multiset of all points $p \in \Delta^*$ such that $\mu_\varphi(p) > 0$, called cornerpoints, counted with their multiplicity, and of the set Δ, seen as a single point, counted with infinite multiplicity.

Clearly, we will speak of cornerpoints at infinity when they are points at infinity, and proper cornerpoints otherwise.
The distance usually defined on the persistence diagrams is the so-called *bottleneck distance*, whose definition is deferred for now; nevertheless it is essential to provide a topology to the space where the diagrams are embedded.

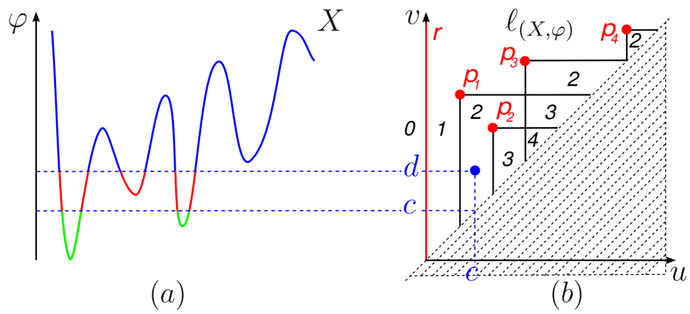


Figure 2: In (a) the filtering function is the height of a graph in $\mathbb{R}^2$ and in (b) there is its persistence diagram.

In order to do so, we will consider the pseudo-distance on $\Delta^* \cup \{\Delta\}$, upon which the *bottleneck distance* is built.

$$d((u,v),(u',v')) = \min\left\{\max\{|u-u'|,|v-v'|\}, \max\{\frac{v-u}{2}, \frac{v'-u'}{2}\}\right\} \quad (1)$$

for every $(u,v),(u',v') \in \Delta^* \cup \{\Delta\}$, using the convention that $\infty - \infty = 0$.

This distance measures the *cost* of taking a point p to a point p' as the minimum between the cost of moving one point directly onto the other and the cost of moving them both to Δ.

2.3 Multidimensional persistent homology

Now we want to extend the definition of the Persistent Betti Numbers to filtering function $\varphi = (\varphi_1, \varphi_2)\colon X \to \mathbb{R}^2$. It has been proved [10] that the information we get is equivalent to the one enclosed in the set of the following filtering function:

$$\varphi_{(\alpha,\beta)}(x) = \min\{\alpha, 1-\alpha\} \cdot \max\left\{\frac{\varphi_1(x)-\beta}{\alpha}, \frac{\varphi_2(x)+\beta}{1-\alpha}\right\}$$

where $x \in X$ and (α, β) is an admissible pair, i.e. they belong to the set $Adm_2 =]0,1[\times\mathbb{R}$.
Intuitively, each admissible pair corresponds to a line of $\mathbb{R}^2$, denoted $r_{(\alpha,\beta)}$, the points of which can be written as $w = \binom{\alpha}{1-\alpha}\tau + \binom{\beta}{-\beta}, \quad \tau \in \mathbb{R}$.
Every point is then associated to a sublevel set $X_w = \{x \in X \mid \varphi_1 \leq w_1, \varphi_2 \leq w_2\}$ which forms a filtration equivalent to the one obtained by $\varphi_{(\alpha,\beta)}$.

2.4 Moving cornerpoints

Clearly, when the parameters $(\alpha, \beta) \in Adm_2$ change, the cornerpoints of $\mathrm{Dgm}(\varphi_{(\alpha,\beta)})$ start moving in the topological space $\Delta^* \cup \{\Delta\}$ and we will present a relevant result proven in [8], but before we can state it, we have to restrict the parameter space, so that it is actually possible to follow the cornerpoints without having problems.
There is no way to identify the path of a cornerpoint after it crosses another cornerpoint and it is impossible to tell which one of the cornerpoints is the original one; we will say *singular for proper cornerpoints* for φ for ﬄthe pair $(\alpha, \beta) \in Adm_2$ when there exists a cornerpoint in $\mathrm{Dgm}(\varphi_{(\alpha,\beta)}))$ that has multiplicity greater than one; *regular for proper cornerpoints* for every other pair. The function φ is said to be *normal* if the set of regular pairs of φ, which is called Adm_2^*, is discrete.
Now we can finally enunciate the theorem.

Theorem 2.1. *Let $\varphi\colon X \to \mathbb{R}^2$ be a normal filtering function. For every continuous path $\gamma\colon I = [0,1] \to Adm_2^*(\varphi)$ and every proper cornerpoint $p \in Dgm(\varphi_{\gamma(0)})$, there exists a continuous function $c\colon I \to \Delta^* \cup \{\Delta\}$ such that $c(0) = p$ and $c(t) \in Dgm(\varphi_{\gamma(t)})\, \forall t \in I$. Furthermore, if there is no $t \in I$ such that $c(t) = \Delta$, c is the only continuous function to have this property.*

2.5 Matching Distance

Here we introduce the mentioned bottleneck distance, a distance between persistence diagrams which can be proven to be stable [10, 12]. The idea is to use the pseudo-distance, defined in (1) in subsection 2.2, to measure the cost of moving the cornerpoints of a persistence diagram into the nearest cornerpoints of another persistence diagram.
In order to do so we consider every bijection $\sigma\colon \mathrm{Dgm}_1 \to \mathrm{Dgm}_2$ between the two persistence diagrams and we define:

$$d_B(\mathrm{Dgm}_1, \mathrm{Dgm}_2) = \min_{\sigma} \max_{p \in \mathrm{Dgm}_1} d(p, \sigma(p)).$$

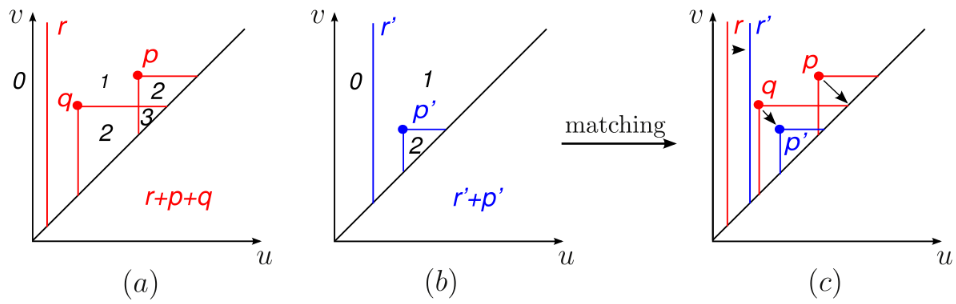


Figure 3: Here the best correspondence between the two diagrams takes r and p to respectively r' and q', and send p to Δ

The concept of stability is to be referred to small changes in the considered functions and has been formalized as follows [10, 12]:

Theorem 2.2. *Let $\varphi, \psi \colon X \to \mathbb{R}$, be two continuous function. Then*

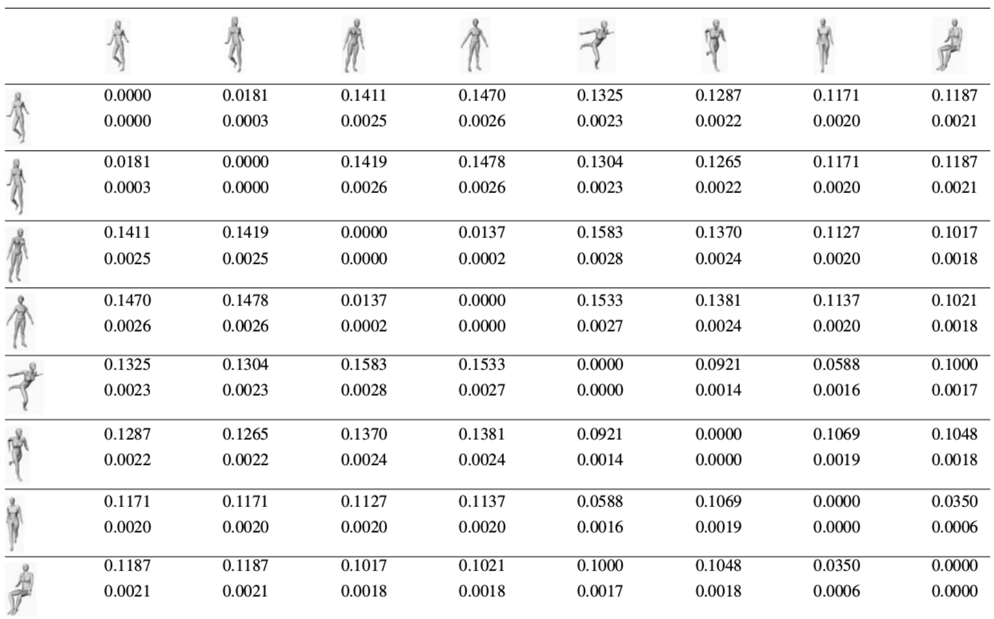
$$d_B(Dgm(\varphi), Dgm(\psi)) \leq \| \varphi - \psi \|_\infty .$$

The bottleneck distance can be easily transposed to the two-dimensional background with the following definition.

Definition 2.5 (Two-dimensional matching distance)**.** Let $\varphi, \psi \colon X \to \mathbb{R}^2$ be two continuous functions from a triangulable space X, and be $\varphi_{(\alpha,\beta)}(x)$ and $\psi_{(\alpha,\beta)}(x)$ like in subsection 2.3. The two-dimensional matching distance is:

$$D_{\text{match}}(b_\varphi, b_\psi) = \sup_{(\alpha,\beta) \in Adm^n} d_B(\text{Dgm}(\varphi_{(\alpha,\beta)}), \text{Dgm}(\psi_{(\alpha,\beta)})),$$

where we recall that b_φ, b_ψ are the PBNs of φ and ψ.



	0.0000 0.0000	0.0181 0.0003	0.1411 0.0025	0.1470 0.0026	0.1325 0.0023	0.1287 0.0022	0.1171 0.0020	0.1187 0.0021
	0.0181 0.0003	0.0000 0.0000	0.1419 0.0026	0.1478 0.0026	0.1304 0.0023	0.1265 0.0022	0.1171 0.0020	0.1187 0.0021
	0.1411 0.0025	0.1419 0.0025	0.0000 0.0000	0.0137 0.0002	0.1583 0.0028	0.1370 0.0024	0.1127 0.0020	0.1017 0.0018
	0.1470 0.0026	0.1478 0.0026	0.0137 0.0002	0.0000 0.0000	0.1533 0.0027	0.1381 0.0024	0.1137 0.0020	0.1021 0.0018
	0.1325 0.0023	0.1304 0.0023	0.1583 0.0028	0.1533 0.0027	0.0000 0.0000	0.0921 0.0014	0.0588 0.0016	0.1000 0.0017
	0.1287 0.0022	0.1265 0.0022	0.1370 0.0024	0.1381 0.0024	0.0921 0.0014	0.0000 0.0000	0.1069 0.0019	0.1048 0.0018
	0.1171 0.0020	0.1171 0.0020	0.1127 0.0020	0.1137 0.0020	0.0588 0.0016	0.1069 0.0019	0.0000 0.0000	0.0350 0.0006
	0.1187 0.0021	0.1187 0.0021	0.1017 0.0018	0.1021 0.0018	0.1000 0.0017	0.1048 0.0018	0.0350 0.0006	0.0000 0.0000

Figure 4: The table above shows the values of the two-dimensional matching distance between some filtering functions $\varphi^k = (\varphi_1^k, \varphi_2^k)$ describing different shapes of women. The values are compared with the maximum among the one-dimensional matching distance between φ_1^k and the one-dimensional matching distance between φ_2^k. The two-dimensional matching distance is significantly greater, and this highlights the fact that it yields more information.

An important application of the matching distance is to find a lower bound of the natural pseudo-distance. The following theorem, that we enunciate in the two-dimensional case, has been proved in [7].

Theorem 2.3. *Let X be a (finitely) triangulable space and let $\varphi, \psi \colon X \to \mathbb{R}^2$ be two continuous function. Then*

$$D_{match}(b_\varphi, b_\psi) \leq d_G(\varphi, \psi).$$

3 Monodromy of two points

The following filtering function has been considered in [8] to show the existence of the monodromy phenomenon.
Let us consider $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\varphi_1(x,y) = x$ and φ_2 initially defined over some lines:

$$\varphi_2(x,y) = \begin{cases} -x & \text{if } y = 0, \\ -x+1 & \text{if } y = 1, \\ -2x & \text{if } y = 2, \\ -2x + \frac{5}{4} & \text{if } y = 3, \end{cases}$$

The lines that are the graph of the four functions $\varphi_2(x,i)$, $i \in \{0,1,2,3\}$ are indicated as r_i.
For every x the function is linearly extended in the segments joining $(x,0)$ with $(x,1)$, $(x,1)$ with $(x,2)$ and $(x,2)$ with $(x,3)$.
Outside, where $y \leq 0$ or $y \geq 3$, $\varphi_2(x,y)$ is constructed so that, fixed an x_0, $\varphi_2(x_0,y)$ has slope -1 in the variable y.

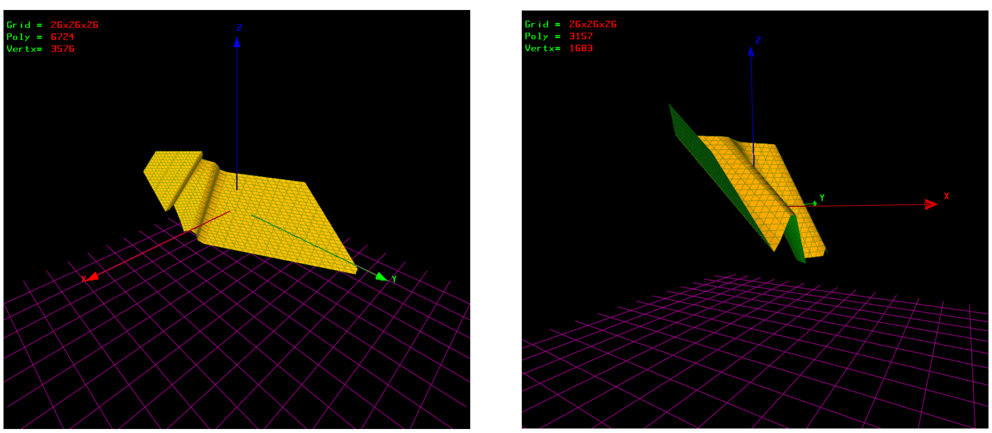


Figure 5: Graph of φ_2 (centered)

The filtering function we are considering is:

$$\varphi_{(\alpha,\beta)}(x,y) = \min\{\alpha, 1-\alpha\} \cdot \max\left\{\frac{\varphi_1(x,y)-\beta}{\alpha}, \frac{\varphi_2(x,y)+\beta}{1-\alpha}\right\}$$

where $(\alpha,\beta) \in Adm_2 =]0,1[\times\mathbb{R}$.

Since $(\alpha, 1-\alpha)$, provided with the norm $\| \quad \|_1$, is a unit vector, we will keep that norm (and its relative distance).
Here the line $r_{(\alpha,\beta)}$, that is the set of the points $\{w \in \mathbb{R}^2 \mid w = \tau\left(\begin{smallmatrix}\alpha \\ 1-\alpha\end{smallmatrix}\right) + \left(\begin{smallmatrix}\beta \\ -\beta\end{smallmatrix}\right), \quad \tau \in \mathbb{R}\}$, has the equation given by $z = \frac{(1-\alpha)(x-\beta)}{\alpha} - \beta$, where $w_2 = z$ and we have indentified w_1 with x.

Remark 2. $\varphi(x,y)$ is differentiable outside $\{y = i\}, \quad i \in \{0,1,2,3\}$ and, if $(x,y) \neq (-1,y)$ with $1 < y < 2$, its differential $D_\varphi(x,y)$ is surjective.

Remark 3. A more intuitive way to illustrate this function is to see it as the projection π on the xz-plane of the graph of $\varphi_2(x,y)$ (for we have $\varphi_1(x,y) = x$), as in Figure 5.
Since $h = (x, y, \varphi_2(x,y))$ is a homeomorphism and $\pi \circ h = \varphi$, the persistent homology of π is completely analogous to the one of φ, and we will make an extensive use of this figurative model without further distinctions.

Remark 4. The theory of persistent homology usually regards a filtering function over a compact manifold. The reason why this function was defined over $\mathbb{R}^2$ was merely practical. Since this example is about proper cornerpoints, the cornerpoints at infinity were avoided. Of course we could restrict this function to an appropriate compact and still have the same phenomena we will study on φ.

Now we want to find a way to describe the connection between the point where $r_{(\alpha,\beta)}$ intersects a line r_i, the coordinates of a cornerpoint, and what we will often associate with time: the signed lengths of the segment connecting that point with $b = \left(\begin{smallmatrix}\beta \\ -\beta\end{smallmatrix}\right)$.

Definition 3.1. Given $(\tilde{\alpha}, \tilde{\beta}) \in Adm_2^*(\varphi)$ and a proper cornerpoint $(u,v) \in \mathrm{Dgm}(\varphi_{\tilde{\alpha},\tilde{\beta}})$ we will say that (u,v) is born at the line r_i if $\exists\, u'$ such that

$$\left(\begin{smallmatrix}\tilde{\alpha}u' + \tilde{\beta} \\ (1-\tilde{\alpha})u' - \tilde{\beta}\end{smallmatrix}\right) \in r_i \quad \text{and}$$

$$u = \min\{\alpha, 1-\alpha\} \cdot \max\left\{\frac{(\tilde{\alpha}u' + \tilde{\beta}) - \tilde{\beta}}{\tilde{\alpha}}, \frac{(1-\tilde{\alpha})u' - \tilde{\beta} + \tilde{\beta}}{1-\tilde{\alpha}}\right\} = \min\{\alpha, 1-\alpha\}u';$$

similarly we will say that it dies at the line r_j if $\exists\, v'$ such that

$$\left(\begin{smallmatrix}\tilde{\alpha}v' + \tilde{\beta} \\ (1-\tilde{\alpha})v' - \beta\end{smallmatrix}\right) \in r_j \quad \text{and} \quad \min\{\alpha, 1-\alpha\}v' = v.$$

For simplicity purpose, we will sometimes refer directly to (u', v'), where the relation $\min\{\alpha, 1-\alpha\}(u', v') = (u,v)$ is understood.

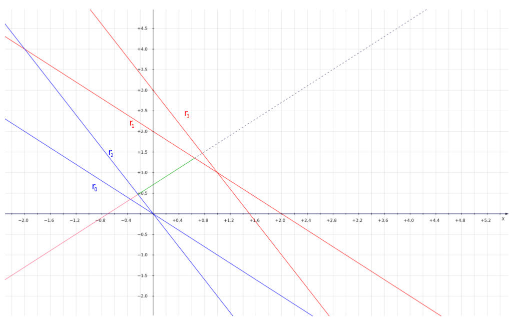


Figure 6: $r_{(\tilde{\alpha},\tilde{\beta})}$ in the codomain of φ

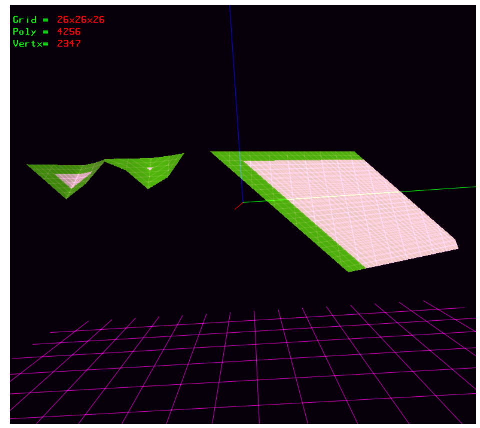


Figure 7: $X_u(\tilde{\alpha},\tilde{\beta})$ e $X_v(\tilde{\alpha},\tilde{\beta})$

Example 3.1. Let us take $(\tilde{\alpha},\tilde{\beta}) = (\frac{1}{2}, -\frac{1}{2})$ and see $r_{(\tilde{\alpha},\tilde{\beta})}$ in Figure 6. The blue lines are r_0 and r_2, while the red ones are r_1 and r_3. The points of $r_{(\tilde{\alpha},\tilde{\beta})}$ can be written as $\left(\begin{smallmatrix}\tilde{\alpha}\tau+\tilde{\beta}\\ (1-\tilde{\alpha})\tau-\beta\end{smallmatrix}\right)$ and are coloured in pink if $\tau \leq u$, in green if $u' < \tau \leq v'$ and dotted if $v' < \tau$, where u' and v' are respectively the distance between b and the intersection with r_2 and r_1.
Since $r_{(\tilde{\alpha},\tilde{\beta})}$ intersects r_0 for a smaller value of τ than r_2, we will say that $r_{(\tilde{\alpha},\tilde{\beta})}$ meets r_0 before r_2. Analogously it meets r_1 before r_3
In Figure 7 we see $X_u(\tilde{\alpha},\tilde{\beta})$ coloured in pink and the difference set $X_v(\tilde{\alpha},\tilde{\beta}) \setminus X_u(\tilde{\alpha},\tilde{\beta})$ coloured in green. The figure shows the birth of the central connected component and its death when it connects to the left one. As we will show next, (u,v) is a proper cornerpoint, it is born at r_2 and it dies at r_1.

Proposition 3.1. *Let $(\tilde{\alpha},\tilde{\beta}) \in Adm_2^*(\varphi)$. Then every cornerpoint (u,v) of $Dgm(\varphi_{\tilde{\alpha},\tilde{\beta}})$ is born either at r_0 or at r_2, and dies at either r_1 or at r_3.*

Proof. Le us recall that $X_u(\tilde{\alpha},\tilde{\beta}) = \{(x,y) \in \mathbb{R}^2 \colon \varphi_1(x,y) \leq \tilde{\alpha}u' + \tilde{\beta} \wedge \varphi_2(x,y) \leq (1-\tilde{\alpha})u' - \tilde{\beta})\}$.
Since (u,v) is a cornerpoint there is a connected component C of X_u that is born there (see the Remark 1 in subsection 2.1).

Let us take a point (x,y) in $\varphi^{-1}_{(\tilde{\alpha},\tilde{\beta})}(u) \cap C$ and suppose first that φ is differentiable in (x,y). As we observed in Remark 2, if $(x,y) \neq (-1,y)$

with $1 < y < 2$, $ImD\varphi = \mathbb{R}^2$ and therefore it should exist a point (x', y') in $C \cap X_{u-\epsilon}(\tilde{\alpha}, \tilde{\beta})$, which would be absurd for how we have chosen C. In other words, what we are looking for is some local minima of the function $\varphi_{(\tilde{\alpha},\tilde{\beta})}(x, y)$.
We have to exclude $y = 1$ and $y = 3$ (for every x, φ_2 is increasing for $0 < y < 1 \vee 2 < y < 3$ and we would have the same absurd as before) as well as the case $(x, y) = (-1, y)$ with $1 < y < 2$, since $\varphi(-1, y) = \varphi(-1, 1)$; so y is equal to 0 or 2 .
Therefore, $\varphi(x, y)$ belongs to one of the two lines r_0 o r_2 and u' has to be equal (or opposite if u' is negative) to the length of the segment in $r_{(\tilde{\alpha},\tilde{\beta})}$, joining one of the two lines to $\tilde{b} = \left(\begin{smallmatrix} \tilde{\beta} \\ -\tilde{\beta} \end{smallmatrix}\right)$. In the same way it is possible to prove that v is determined by the lenght of the segment joining $\tilde{b}$ to r_1 or r_3.
□

Remark 5. What we found is that the intersection of $r_{(\alpha,\beta)}$ with r_i does not always imply the existence of a cornerpoint which is born or dies there. If $\alpha + \beta \geq -1$, $r_{(\alpha,\beta)}$ intersects r_1 before r_2 (or at the same moment), and there are no cornerpoints that have the u necessary to be born there according to the definition 3.1. That is because you could easily find a curve in $X_u(\tilde{\alpha}, \tilde{\beta})$, not decreasing in both coordinates of φ, connecting the two lines $y = 2$ and $y = 1$ (see Figure 8) and, for the same reasons of the previous proof, there are no births.
Conversely, if $\alpha + \beta < -1$, both r_0 and r_2 are crossed before r_1 and r_3; if u' the signed distance between b and, for example r_2, then $X_u(\alpha, \beta) \cap \{y = 2\}$ consists in exactly one point that is a local minimum of $\varphi_{(\alpha,\beta)}(x, y)$. Thus, in this case, there are either two proper cornerpoints or a *singularity*, a cornerpoint of multiplicity 2, when $r_{(\alpha,\beta)}$ meets both the intersection of r_0 and r_2 and the intersection of r_1 and r_3.

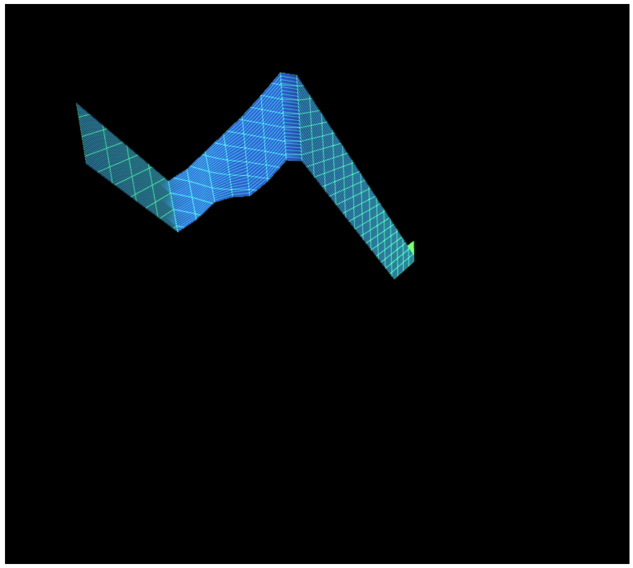


Figure 8: Level sets when $\frac{\alpha+\beta}{1-\alpha} \geq 2$

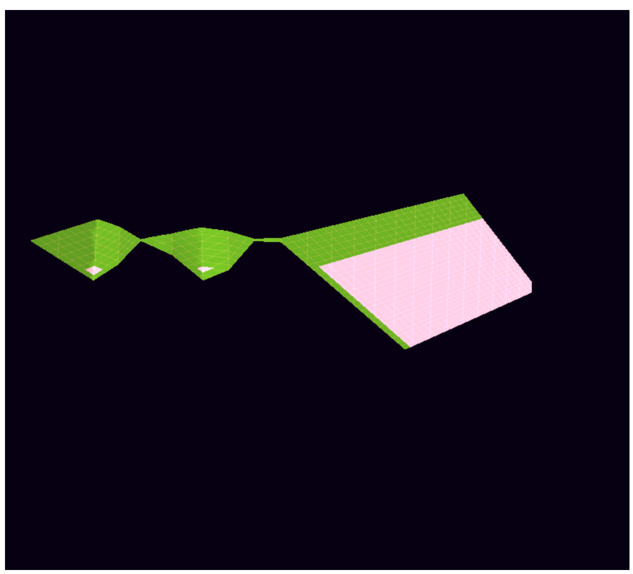


Figure 9: Level sets at the singularity

Definition 3.2. From now on $Adm^2(\varphi)$ will be the subset of $Adm_2^*(\varphi)$ of the parameters $(\tilde{\alpha}, \tilde{\beta})$, such that $\mathrm{Dgm}(\varphi_{\tilde{\alpha},\tilde{\beta}})$ has exactly 2 proper cornerpoints.

Proposition 3.2. $Adm^2(\varphi) = \{(\alpha, \beta) \in Adm_2^*(\varphi) \mid \alpha + \beta < -1\}$

The following definition is the main tool of this thesis, and it is based on the ideas of the subsection 2.4.

Definition 3.3. Let us set $\mathrm{Dgm}(\varphi_{\alpha,\beta})^* := \mathrm{Dgm}(\varphi_{\alpha,\beta}) \setminus \{\Delta\}$. We call F_φ or simply F the covariant functor associating $\gamma(t) : I \to Adm^2(\varphi)$, loop starting from $(\tilde{\alpha}, \tilde{\beta}) \in Adm^2(\varphi)$, to a permutation $\sigma : \mathrm{Dgm}(\varphi_{\tilde{\alpha},\tilde{\beta}})^* \to \mathrm{Dgm}(\varphi_{\tilde{\alpha},\tilde{\beta}})^*$ in the following way: for every $p \in \mathrm{Dgm}(\varphi_{\tilde{\alpha},\tilde{\beta}})^*$ we consider the path c_p starting in p, as in the Theorem 2.1, and we define $\sigma(c_p(0)) = c_p(1))$.

The reason why we restricted Adm_2^* into a smaller space is that, for that definition to work, we need to change continuously the parameters, thus creating a path for every cornerpoint, making sure that those paths do not end, nor start, at Δ. It is easy to verify that, keeping a fixed number n of proper cornerpoints of $\mathrm{Dgm}(\varphi_{\alpha,\beta})$, is a sufficient condition.

Definition 3.4 (Monodromy Group)**.** The monodromy group of φ is the subgroup of $(S_2, \circ)$ that is the image of F_φ.

Definition 3.5. For an arbitrary $\tau \in \mathbb{R}$, X_τ has a single not bounded connected component, to which we will refer as the *base*.

Remark 6. We observe that $\forall \tau\, X_\tau(\tilde{\alpha}, \tilde{\beta}) \cap \{y = i\} \quad i \in \{0, 1, 2, 3\}$ consists in just one single connected component or is void. This implies that two points $(x_1, 2)$ and $(x_2, y_i)\, y_i > 2$ are connected in $X_\tau(\tilde{\alpha}, \tilde{\beta})$ if and only if they are connected in $X_\tau(\tilde{\alpha}, \tilde{\beta}) \cap \{(x, y) \in \mathbb{R}^2 \mid y \geq 2\}$

The following proposition proves that, as we anticipated, it is possible to switch the cornerpoints, which can be locally distinguished by at least one coordinate.

Proposition 3.3. *Suppose $\gamma \in Adm^2(\varphi)$ is simple and turns around the point $(\frac{1}{4}, 0)$, then $F_\varphi(\gamma)$ generates S_2.*

Proof. Assume $\beta > 0$ so that $r_{(\alpha,\beta)}$ meets r_0 first, and say (u_1, v_1) is born there. Now this gives two connected components in $X_{u_1}(\alpha, \beta)$, one of which is a point of $y = 0$ and the other is the base, which join (see the Remark 1 in subsection 2.1) when both $\{y = 1\}$ and $\{y = 3\}$ intersect X_{v_1}. Therefore, (u_1, v_1) will die at the last line $r_{(\alpha,\beta)}$ will intersect. In particular, if $\beta < 1/4$ and $0 < \alpha < \frac{1}{4} - \beta$, we will have that (u_1, v_1) dies at r_3; otherwise, it will die at r_1.

Similarly, if $\beta < 0$ $u_1 \neq u_2$. Since the $r_{(\alpha,\beta)}$ meets r_2 first, if let's say $u_1 < u_2$, then (u_1, v_1) is born there. This implies that $X_{u_1}(\alpha, \beta)$ consists in two connected components, one of which is a point of $\{y = 2\}$ and the other is the base. The cornerpoint can't die at r_1 because of the remark 6 so it has to die at r_3.

When $\frac{1}{4} - \beta < \alpha < 1$, $r_{(\alpha,\beta)}$ meets r_1 last and $v_1 \neq v_2$. If, let's say $v_1 < v_2$, this implies that, for an arbitrarily small $\epsilon > 0$, $X_{v_1 - \epsilon}(\alpha, \beta)$ is composed by exactly two connected components, separated by $\{y = 1\}$, one of which is the base, and the other contains a point of $\{y = 0\}$. The base by definition is not bounded, so (u_1, v_1) has to be born at $r = 0$.

Similarly for $0 < \alpha < \frac{1}{4} - \beta$ $r_{(\alpha,\beta)}$ meets r_3 last. We suppose $v_1 < v_2$ once again, and we remark that $X_{v_1 - \epsilon}(\alpha, \beta)$ is composed by the base and another connected component, which contains points from both $\{y = 0\}$ and $\{y = 2\}$. In this case $X_{u_1 - \epsilon}$ has to be contained entirely in the base, and this means that (u_1, v_1) is born when $r_{(\alpha,\beta)}$ meets the first line. That line is r_2 if $\beta > 0$, r_0 otherwise.

Since every simple loop around $(\frac{1}{4}, 0)$ has to pass an odd number of times on $V := \{(\alpha, \beta) \| \alpha = \frac{1}{4} - \beta, \beta > 0\}$ and $U := \{(\alpha, \beta) \| 0 < \alpha < \frac{1}{4} - \beta, \beta = 0\}$, at the end of a simple loop one point has to be born and dies where at the start the other point used to. $\square$

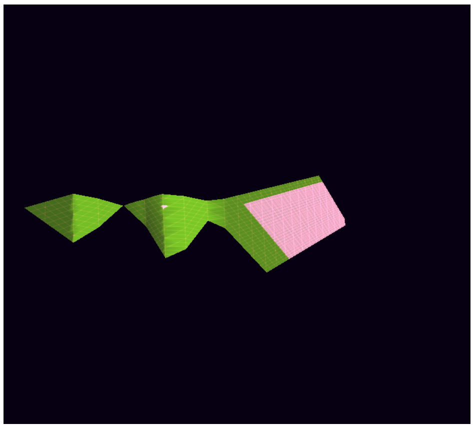


Figure 10: $\beta > 0$, $\frac{1}{4} - \beta < \alpha < 1$

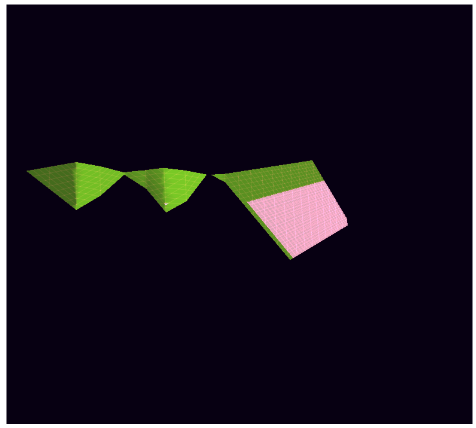


Figure 11: $\beta > 0$, $0 < \alpha < \frac{1}{4} - \beta$

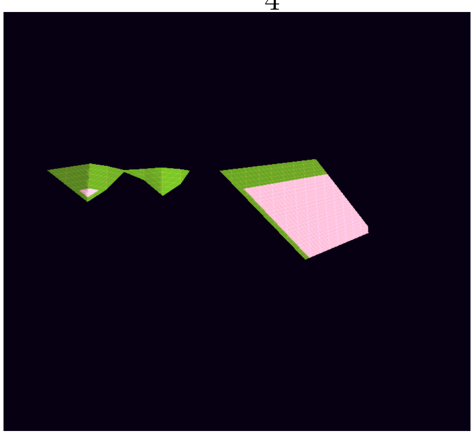


Figure 12: $\beta < 0$, $0 < \alpha < \frac{1}{4} - \beta$

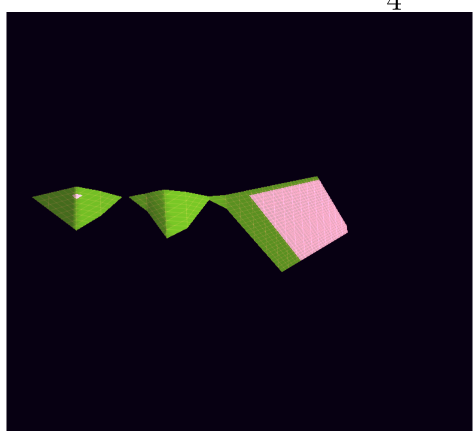


Figure 13: $\beta < 0$, $\frac{1}{4} - \beta < \alpha < 1$

4 Monodromy of n points

Now that we have found a way to represent the Symmetric group S_2 of two elements, via a filtering function having at most two cornerpoints, we want to find another filtering function having at most n cornerpoints, whose *Monodromy Group*, which will be defined in analogous way, is S_n.
The idea of the generalization is quite natural, requiring some minor precautions, and we will follow the steps that led to the previous proof. All of the remarks and propositions made in the last section find here their equivalent, and the most evident ones will not be repeated.
We are still considering a filtering function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\varphi_1^n(x, y) = x$ and so that:

$$\varphi_2^n(x,y) = \begin{cases} -x & \text{if } y = 0, \\ -x + \dfrac{1}{n} + n & \text{if } y = 1, \\ \dots & \\ -(i+1)x + \dfrac{i^2}{n} & \text{if } y = 2i, \\ -(i+1)x + \dfrac{(i+1)^2}{n} + n & \text{if } y = 2i+1, \\ \dots & \\ -nx + \dfrac{(n-1)^2}{n} & \text{if } y = 2n-2, \\ -nx + 2n & \text{if } y = 2n-1, \end{cases}$$

The costruction follows as before, linearly joining the lines and prolonging outside in order to have a slope of -1 in the variable y.
We will call r_k the straight lines of equation $z = \varphi_2(x, k)$, $k \in \{0, \dots, 2n-1\}$.

Remark 7. $\varphi(x, y)$ is, quite clearly, differentiable outside $\{y = k\}$, $k \in \{0, \dots, 2n-1\}$ but its differential $D_{\varphi^n}(x, y)$ is not always surjective. Notably, $(\varphi^n)^{-1}(r_{2i+2} \cap r_{2i+1})$, $i \in \{0, \dots, n-2\}$ is a segment (because of our definition of φ^n). The abscissa of its points can be found like this:

$$-(i+2)x + \frac{(i+1)^2}{n} = -(i+1)x + \frac{(i+1)^2}{n} + n \Rightarrow x = -n. \tag{2}$$

We have that if $(-n, y)$ with $2i+1 < y < 2i+2$, the differential has rank one, otherwise has rank two.

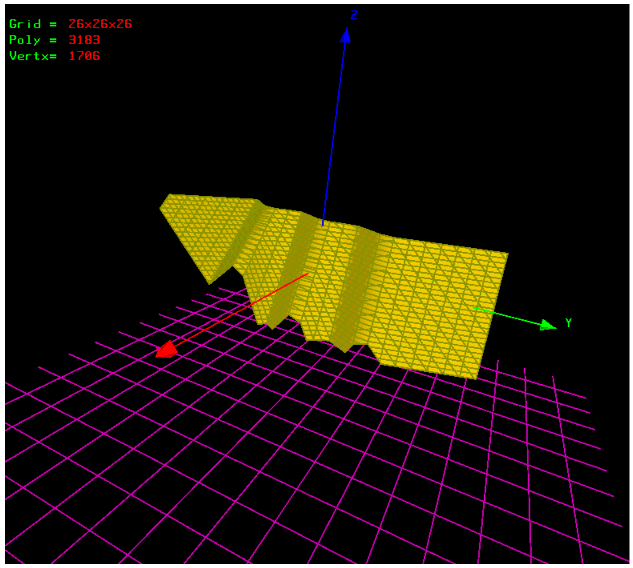


Figure 14: Graph of $\varphi_2^3(x, y)$ (centered)

The definition of birth and death in the lines r_i of φ, still works for φ^n, as well as the next proposition that can be proven in the same way of the two-dimensional case.

Proposition 4.1. *Let us consider $(\tilde{\alpha}, \tilde{\beta}) \in Adm_2^*(\varphi)$, then an arbitrary cornerpoint of $Dgm(\varphi^n_{\tilde{\alpha},\tilde{\beta}})$ is born at r_{2i}, $i \in \{0, \ldots, n\}$ and dies at r_{2j+1}, $j \in \{0, \ldots, n\}$.*

Definition 4.1. Like before, $Adm^n(\varphi^n)$ is defined as the subset of $Adm_2^*(\varphi^n)$ of the parameters $(\tilde{\alpha}, \tilde{\beta})$, such that $\mathrm{Dgm}(\varphi^n_{\tilde{\alpha},\tilde{\beta}})$ has exactly n proper cornerpoints.

Proposition 4.2. $Adm^n(\varphi_n) = \{(\alpha, \beta) \in Adm_2^*(\varphi^n) \mid (n + \frac{1}{n})\alpha + \beta > -n\}$

Proof. A point in the lines $\{y = 2i\}, \quad i \in \{1, \ldots, n-1\}$ in the domain is a minimum of $\varphi_{(\alpha,\beta)}(x, y)$ if and only if $r_{(\alpha,\beta)}$ meets r_{2i} before r_{2i-1}. We have to avoid the parameters whose line passes above the intersection of these lines. Because they have the same abscissa, as we have shown in equation (2), it is sufficient that $r_{(\alpha,\beta)}$ passes under the first one which is $(-n, 2n + \frac{1}{n})$.

$$\frac{1-\alpha}{\alpha}(-n - \beta) < 2n + \frac{1}{n} + \beta \Rightarrow (n + \frac{1}{n})\alpha + \beta > -n$$

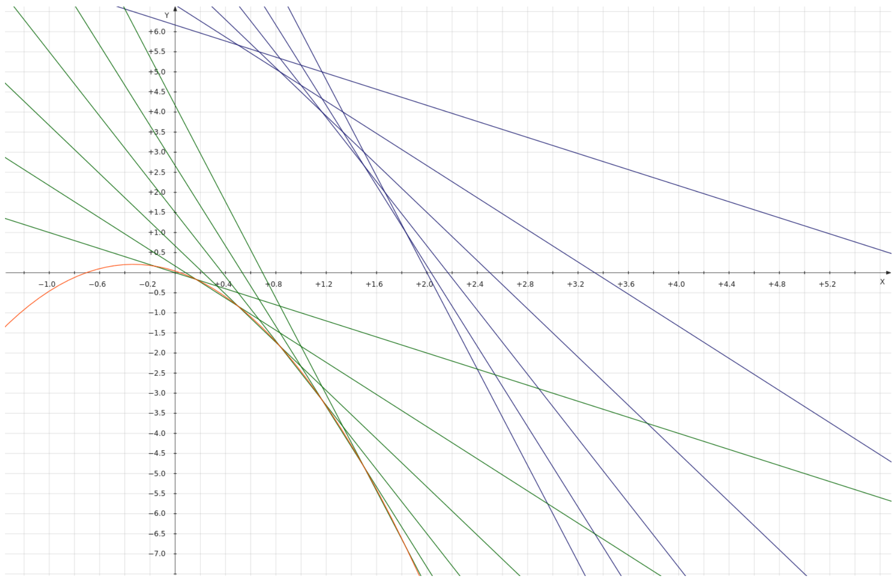


Figure 15: Codomain of φ^n

$\square$

In order to make the final proof easier we split the parameter space. We want that $\alpha, \beta \in A_i$, $i \in \{0, \dots, n-1\}$ if $r_{(\alpha,\beta)}$ meets r_{2i} or r_{2i+2} first. The point of the intersection of the two lines is: $\frac{1}{n}\left(\begin{smallmatrix} 2i+1 \\ -(i^2+3i+1) \end{smallmatrix}\right)$.

All of these points belong in the parabola $z = -\dfrac{nx^2}{4} - x + \dfrac{1}{4n}$.

Since every line r_{2i} intersects exactly two times this parabola at the points of its intersection with r_{2i-2} and r_{2i+2} (if they exist), to find when $r_{(\alpha,\beta)}$ meets r_{2i} as the first line it is sufficient that it passes between the two intersections, because every other line will be met outside of the parabola.

Definition 4.2. More precisely, $A_0 = \left\{(\alpha, \beta) \in Adm^n(\varphi) \mid (\beta > \dfrac{2}{n}\alpha + \dfrac{3}{n}\right\}$,
$A_{n-2} = \left\{(\alpha, \beta) \in Adm^n(\varphi) \mid \dfrac{(n^2-5n+6)}{n}\alpha + \dfrac{2n-5}{n} < \beta\right\}$
and $\forall i \in \{1, \dots, n-3\}$:

$$A_i = \left\{(\alpha, \beta) \in Adm^n(\varphi) \mid \frac{(+i^2-i)}{n}\alpha + \frac{2i-1}{n} < \beta < \frac{(+i^2+3i+2)}{n}\alpha + \frac{2i+3}{n}\right\}.$$

Let us consider now the intersection between r_{2i+1} and r_{2n-1}, which is $\begin{pmatrix} \frac{n+i+1}{n} \\ n-i-1 \end{pmatrix}$.

Definition 4.3. The line joining the intersection between r_{2i+1} and r_n and the intersection between r_{2i} and r_{2i+2} with $i \in \{0, \dots, n-2\}$, is parameterized by $s_i = (s_i^1, s_i^2)$, where $s_i^1 = \dfrac{n-i}{n^2 - in + i^2 + 2i + 1}$ and $s_i^2 = \dfrac{(2i+1)n^2 - i^2 n + i^3 + 4i^2 + 4i + 1}{(n^2 - in + i^2 + 2i + 1)(n)}$.

Remark 8. We need to verify that $0 < s_i^1 < 1$, but it is actually simple because $\forall i \in \{0, \dots, n-2\}, \quad s_i^1 < s_{i+1}^1$ and $s_0 = \dfrac{n}{n^2+1}$ while $s_{n-1} = \dfrac{1}{n^2+n}$.

We repeat the definition of the functor F and of the monodromy group, this time for φ^n.

Definition 4.4. Let us set $\mathrm{Dgm}(\varphi_{\alpha,\beta})^* := \mathrm{Dgm}(\varphi_{\alpha,\beta}) \setminus \{\Delta\}$. We call F_φ^n the covariant functor associating $\gamma(t) : I \to Adm^n(\varphi)$, loop starting from $(\tilde{\alpha}, \tilde{\beta}) \in Adm^n(\varphi)$, to a permutation $\sigma : \mathrm{Dgm}(\varphi^n_{\tilde{\alpha},\tilde{\beta}})^* \to \mathrm{Dgm}(\varphi^n_{\tilde{\alpha},\tilde{\beta}})^*$ in the following way: for every $p \in \mathrm{Dgm}(\varphi^n_{\tilde{\alpha},\tilde{\beta}})^*$ we consider the path c_p, as in the subsection 2.4, starting in p and we define $\sigma(c_p(0)) = c_p(1))$.

Definition 4.5 (Monodromy Group)**.** The monodromy group of φ_n is the subgroup of $(S_n, \circ)$ that is the image of F_φ^n.

Now we find a loop that exchanges two cornerpoints in a very similar way to the two-dimensional case. Also, we will refer again to the only unbounded connected component as the base.

Proposition 4.3. *Let (α_0, β_0) be such that $(n^2 - ni + i + 1)\alpha_0 + n\beta_0 \neq n + i + 1$ (so that $r_{(\alpha_0,\beta_0)}$ does not intersect $r_{2n-1} \cap r_{2i-1}$), then it exists a loop γ_i around s^i in A_i, $i \in \{0, \dots, n-1\}$ with $\gamma_i(0) = \gamma_i(1) = (\alpha_0, \beta_0)$ such that $F_\varphi^n(\gamma) = (i\, n)$, where i will be associated to the cornerpoint dying at $t = 0$ in r_{2i-1}, and n to the one dying in r_{2n-1}.*

Proof. Assume $r_{(\alpha,\beta)}$ meets r_{2i} first, and say (u, v) is born there. Now this gives two connected components in $X_u(\alpha, \beta)$, one of which is a point of $\{y = 2i\}$ and the other is the base, which joins when both $\{y = 2j + 1\}$, $i \leq$

$j \leq n-1$ intersect X_{v_1}. Therefore, (u, v) will die at the last of those line $r_{(\alpha,\beta)}$ will intersect. In particular, (u, v) can die either at r_{2i+1} or at r_{2n-1}.

Similarly, if $r_{(\alpha,\beta)}$ meets r_{2i+2} first and (u, v) is born there, this implies that $X_u(\alpha, \beta)$ consists in two connected components, one of which is a point of $\{y = 2i + 2\}$ and the other is the base. The cornerpoint cannot die at r_{2i} because of the same reasoning of the remark 6, and it has to die at r_{2n-1}.

When $r_{(\alpha,\beta)}$ meets r_{2i+1} after r_{2i+3} and let us say (u, v) dies there, this implies that, for an arbitrarily small $\epsilon > 0$, $X_{v-\epsilon}(\alpha, \beta)$ is composed by exactly $i + 2$ connected components, separated by $\{y = 2j + 1\}$, $0 \leq j \leq i$, one of which is the base containing a point of $\{y = 2i + 2\}$. The base by definition is not bounded, and since we need the connected component to join the base in $X_v(\alpha, \beta)$, (u, v) has to be born in $r = 2i$.

Similarly, for if $r_{(\alpha,\beta)}$ meets r_{2i+3} after r_{2i+1}, we suppose (u, v) dies there once again, and we remark that $X_{v-\epsilon}(\alpha, \beta)$ is composed by again $i + 2$ connected components, but this time the base does not intersect $\{y = 2i + 2\}$, and there is a connected component with points from $\{y = 2i + 2\}$ and $\{y = 2i\}$. This means that (u, v) is born when $r_{(\alpha,\beta)}$ meets the first line.

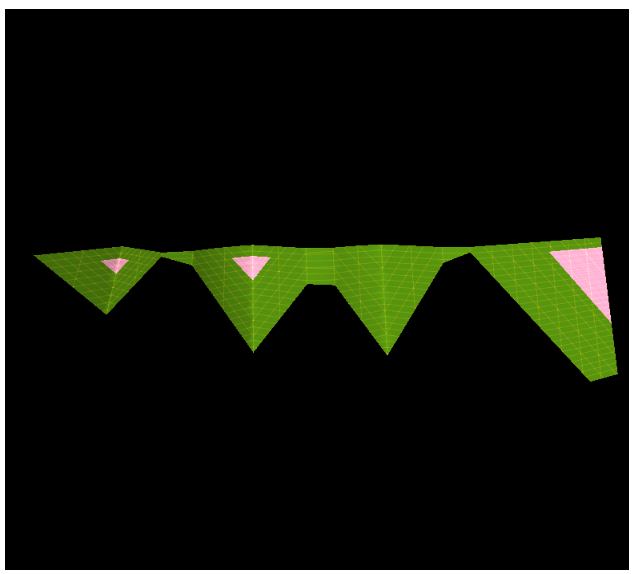


Figure 16: s_i when $n = 3$

Now it is clear that s_i is a singularity, and we could consider a short loop γ_i in A_i around it, making sure that there is no point $(\alpha_0, \beta_0) \in \gamma_i$ such that $r_{(\alpha_0,\beta_0)}$ passes through any other intersections of the lines r_{2j}. This would prevent any other cornerpoints from switching.

If we also suppose γ_i to be simple, for the same reasoning of the proof for $n = 2$, at the end of the loop the two mentioned cornerpoints would exchange their coordinates. □

Theorem 4.4. *The Monodromy Group of φ^n is S_n.*

Proof. Given a point $(\alpha_0, \beta_0) \in Adm^n$ and $(\alpha_i, \beta_i) \in A_i$ we can find a path l in Adm^n connecting them. If γ_i, $\gamma_i(0) = (\alpha_i, \beta_i)$, is a loop such that $F^n_\varphi(\gamma_i) = (i\, n)$ so will $l^{-1} \circ \gamma_i \circ l$. Now, since F^n_φ is a covariant functor, and $\langle (1\, n), \ldots, ((n-1)\, n) \rangle = S_n$, we have that every permutation of $\mathrm{Dgm}(\varphi^n_\alpha)$ can be obtained through composition of loops of the form $l^{-1} \circ \gamma_i \circ l$. □

5 Conclusions

The purpose of this thesis was to study the monodromy phenomenon of a filtering function with two proper cornerpoints and prove the existence of a non-trivial functor associating the loops in the parameter space to the permutations of the two cornerpoints. Furthermore we have generalized this functor for a suitable filtering function with a maximum of n proper cornerpoints to represent an n-symmetric group.
The mathematical tool this functor uses gives us the path of a cornerpoint when the parameters change continuously, but, if the cornerpoint reaches the diagonal, this path turns out to be constant. To avoid that, we restricted the parameter space in order to have exactly n cornerpoints at every point of the loops. This approach prevents us from finding a representation of a proper subgroup of S_n, because we are not taking into consideration the loops with a varying number of proper cornerpoints.
One idea to solve this issue is to develop a tool that follows the cornerpoints even after passing through the diagonal and to define a more general functor. Another idea is to work with a filtering function defined over different connected components. That could help us generalize our theorem and could be a subject of a further study.

Acknowledgements

A special thank you goes to the developers of the modeling software MathMod, which has proved to be very useful in understanding, verifying and displaying some examples of monodromy.

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