Alma Mater Studiorum · Università di Bologna

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI

Corso di Laurea Magistrale in Matematica

A COMPACTNESS THEOREM IN GROUP INVARIANT PERSISTENT HOMOLOGY

Tesi di Laurea in Topologia Computazionale

Relatore:
Chiar.mo Prof.
Patrizio Frosini

Presentata da:
Nicola Quercioli

Sessione Unica
Anno Accademico 2015-2016

Contents

Chapter 1

Introduction

In topological data analysis datasets are frequently represented by $\mathbb{R}^m$-valued continuous functions defined on a topological space X. As simple examples among many others, these functions can describe the coloring of 3D objects, the coordinates of the points in a planar curve, or the grey-levels in X-ray CT images. Therefore, we want to compare these datasets and see if they are similar and how much they are similar. For this purpose, we can use two mathematical tools: the natural pseudo-distance and group invariant persistent homology. Let Φ be a set of functions from X to $\mathbb{R}^m$ and G a subgroup of the group $Homeo(X)$ of all self-homeomorphisms of X. We assume that the group G acts on Φ by composition on the right. Hence, we can define the *natural pseudo-distance* d_G on Φ. In plain words, the definition of this pseudo-metric is based on the attempt of finding the best correspondence between two functions of Φ. The natural pseudo-distance represents our ground truth. Unfortunately, in many cases d_G is difficult to compute. This is also a consequence of the fact that we can easily find subgroups G of $Homeo(X)$ that cannot be approximated with arbitrary precision by finite subgroups of G. Nevertheless, d_G can be approximated with arbitrary precision by means of a dual approach based on persistent homology and group invariant non-expansive operators. This approach is known as *group invariant persistent homology*.

Persistent homology consists in the study of the properties of filtered topological spaces. Each continuous function $\varphi : X \to \mathbb{R}^m$ is called a *filtering function* and naturally induces a (multi)filtration on X, made by the sublevel sets of φ. Hence, we analyse the data represented by a filtering function examining how much the topological properties of its sublevel sets "persist" when we go through the filtration. The main mathematical tool to perform this analysis is given by persistent homology. This theory describes the birth and the death of k-dimensional holes when we move along the considered filtration of the space X. When the filtering function is real-valued we can look at it as a time, and the distance between the times of birth and death of a hole is defined to be its *persistence*. The more persistent is a hole, the more important it is for shape comparison, since holes with small persistence are usually due to noise. Moreover, if the function is real-valued, persistent homology is described by suitable collections of points called *persistence diagrams*. These diagrams can be compared by a suitable metric d_{match}, called *bottleneck* (or *matching*) distance. We observe that d_{match} gives us a lower bound for the natural pseudo-distance. For the sake of simplicity, in the rest of this thesis we will assume that filtering functions are real-valued. An important property of classical persistent homology consists in the fact that if a self-homeomorphism g is given, then the filtering functions $\varphi, \varphi \circ g$ cannot be distinguished from each other by computing the persistent homology of the filtrations induced by φ and $\varphi \circ g$. This is a relevant issue in the applications where the functions $\varphi, \varphi \circ g$ cannot be considered equivalent. At first we present a possible solution for the previously described problem. It consists in changing the direct study of the group G into the study of how the operators that are invariant under the action of G act on classical persistent homology. These operators are functors between suitable categories. The objects of those categories are the filtering functions and their arrows are the self-homeomorphisms in G. This change of perspective allows us to treat G as a variable in our applications. The use of operators allows to combine persistent homology and the invariance with respect to the group G. In particular, the main result of this thesis is the proof of a compactness theorem for the set of these operators, under suitable hypotheses. Moreover, this theorem has a corollary that allows us to find an arbitrary good approximation for the natural pseudo-distance.

Outline of the thesis

Our work is organized as follows. In the second chapter we introduce the mathematical setting that will be used in the thesis. Moreover, we recall some basic concepts about persistent homology. In the third chapter we prove the compactness theorem and some related results.

Introduzione

Negli ultimi anni si è affermata in analisi topologica dei dati l'esigenza di studiare insiemi di funzioni continue a valori reali su uno spazio topologico. Tale esigenza nasce dal fatto che spesso nelle applicazioni abbiamo solo misurazioni dello spazio topologico X che possono essere interpretate come funzioni continue a valori reali o più in generale a valori in un opportuno $\mathbb{R}^m$. Semplici esempi possono essere: le funzioni che descrivono la colorazione degli oggetti 3D, le coordinate dei punti di una curva piana oppure i livelli di grigio di una immagine tomografica ottenuta tramite raggi X. Alla luce di tutto ciò, vogliamo poter confrontare questi insiemi di dati, capire se sono simili e quanto sono simili. A tal fine i due strumenti utilizzati nella tesi sono: la pseudo-distanza naturale e l'omologia persistente invariante per gruppi.
Se Φ è un insieme di funzioni, su cui agisce per composizione a destra un sottogruppo G del gruppo $Homeo(X)$ di tutti gli omeomorfismi di X, possiamo definire la *pseudo-distanza naturale* d_G su Φ. Tale distanza nasce dall'idea di trovare l'omeomorfismo di "minor costo" che trasformi una nell'altra due funzioni di Φ. Questa pseudo-metrica è lo strumento scelto per stabilire la similitudine fra due funzioni, ma sfortunatamente in molti casi è difficile da calcolare. Ciò è conseguenza del fatto che possiamo facilmente trovare sottogruppi di $Homeo(X)$ che non possono essere arbitrariamente approssimati da sottogruppi finiti.
In realtà, ed è proprio questo lo scopo della tesi, possiamo approssimare d_G con arbitraria precisione tramite un approccio duale basato su omologia persistente e operatori non-espansivi invarianti per gruppi. Chiameremo tale

approccio *omologia persistente invariante per gruppi.*
La persistenza omologica consiste nello studio delle proprietà delle filtrazioni degli spazi topologici. Ogni funzione continua $\varphi : X \to \mathbb{R}^m$ è chiamata *funzione filtrante* e induce naturalmente una (multi)filtrazione su X, costituita dagli insiemi di sottolivello di φ. Dunque andiamo ad analizzare i dati rappresentati da ogni funzione filtrante esaminando quanto le proprietà topologiche dei suoi insiemi di sottolivello "persistono" quando procediamo con la filtrazione. Lo strumento matematico principale per questa analisi è l'omologia persistente. Questa teoria ci descrive la nascita e la morte delle classi omologiche k-dimensionali quando ci muoviamo lungo la filtrazione dello spazio X presa in considerazione. Nel caso in cui la funzione filtrante prenda valori reali, possiamo guardare ad essa come se fosse il tempo e la distanza tra la nascita di una classe omologica e la sua morte è definita come la sua *persistenza.* Tanto più è persistente una classe, quanto è più importante ai fini del confronto di forma, dato che solitamente le classi con piccola persistenza sono dovute al rumore. Inoltre se la funzione filtrante prende valori reali, l'omologia persistente può essere descritta tramite un'opportuna collezione di punti chiamata *diagramma di persistenza.* Tali diagrammi possono essere messi a confronto tramite una metrica d_{match} che viene detta *distanza di matching.* È interessante osservare che la distanza di matching fornisce una limitazione inferiore per la pseudo-distanza naturale. Per semplicità, per il resto della tesi considereremo solo funzioni filtranti a valori reali.
Un'importante proprietà dell'omologia persistente è l'invarianza sotto l'azione di omeomorfismi, cioè dati un omeomorfismo $g : X \to X$ e una funzione filtrante φ, si ha che l'omologia persistente calcolata sulle filtrazioni indotte da φ e $\varphi \circ g$ è la stessa. Nelle applicazioni ciò diventa un problema se φ e $\varphi \circ g$ non possono essere considerate equivalenti. Quindi in primo luogo cerchiamo di fornire una possibile risposta a tale problema, che consiste nel cambiare l'oggetto di studio. Passiamo, dunque, all'analisi di come gli operatori invarianti rispetto a un certo sottogruppo G del gruppo degli $Homeo(X)$ degli omeomorfismi agiscono sulla persistenza omologica classica. Gli operatori

che andremo a utilizzare, saranno più propriamente funtori definiti sulle categorie che hanno per oggetti le funzioni filtranti ammissibili e come frecce gli omeomorfismi del gruppo G. Questo cambio di prospettiva, ci permette inoltre di utilizzare G come una variabile nelle nostre applicazioni. L'utilizzo dei suddetti operatori ci consente di combinare l'omologia persistente classica con l'invarianza rispetto a G. In particolare, il risultato principale della tesi è la dimostrazione di un teorema di compattezza per l'insieme di tali operatori, sotto opportune ipotesi. Tale teorema ci permetterà anche di fornire sotto opportune condizioni una approssimazione arbitrariamente precisa della pseudo-distanza naturale.

Chapter 2

Mathematical settings

Let us consider a set $X \neq \emptyset$ and a compact topological subspace Φ of the set of all bounded functions from X to $\mathbb{R}$ (equipped with the Euclidean topology), denoted by $\mathbb{R}_b^X$ and endowed with the topology induced by the sup-norm $\|\cdot\|_\infty$. Since Φ is compact, we have that the normed real vector space Φ is bounded, i.e., there exists a positive real value L, such that $||\varphi||_\infty < L$ for every $\varphi \in \Phi$. Moreover, we suppose that every positive constant function c, such that there exists $\varphi \in \Phi$ with $c \leq ||\varphi||_\infty$, belongs to Φ.
Now we endow X with the initial topology with respect to Φ. We recall the definition of initial topology [8]:

Definition 2.0.1. Let X be a set, I be an indexing set.
Let $((Y_i, \mu_i))_{i \in I}$ be an indexed family of topological spaces indexed by I.
Let $(f_i : X \to Y_i)_{i \in I}$ be an indexed family of mappings indexed by I.
Let μ be the coarsest topology on X such that each $f_i : X \to Y_i$ is (μ, μ_i)-continuous. Then τ is known as the *initial topology* on X with respect to $(f_i)_{i \in I}$.

A base for X with this topology is

$$\mathcal{B} = \{\bigcap_{i \in I} \varphi_i^{-1}(U_i),\ \forall I \text{ finite}, \varphi_i \in \Phi,\ \forall\ U_i \subseteq \mathbb{R} \text{ open set}\}.$$

We can define a function

$$d(x_1, x_2) = \sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)|$$

from $X \times X$ to $\mathbb{R}$.

Proposition 2.0.2. *The function d is a pseudo-metric on X.*

Remark 2.0.3. We recall that a pseudo-metric is just a distance d without the property: if $d(a, b) = 0$, then $a = b$.

Proof.

- The value $d(x_1, x_2)$ is finite for every $x_1, x_2 \in X$, because Φ is bounded. Indeed, a finite constant L exists such that $\|\varphi\|_\infty \leq L$ for every $\varphi \in \Phi$. Hence, $|\varphi(x_1) - \varphi(x_2)| \leq \|\varphi\|_\infty + \|\varphi\|_\infty \leq 2L$ for any $\varphi \in \Phi$ and any $x_1, x_2 \in X$. This implies that $d(x_1, x_2) \leq 2L$ for every $x_1, x_2 \in X$.
- d is obviously symmetrical.
- The definition of d immediately implies that $d(x, x) = 0$ for any $x \in X$.
- The triangle inequality holds, since

$$\begin{aligned} d(x_1, x_2) &= \sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)| \\ &\leq \sup_{\varphi \in \Phi} (|\varphi(x_1) - \varphi(x_3)| + |\varphi(x_3) - \varphi(x_2)|) \\ &\leq \sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_3)| + \sup_{\varphi \in \Phi} |\varphi(x_3) - \varphi(x_2)| \\ &= d(x_1, x_3) + d(x_3, x_2) \end{aligned}$$

  for any $x_1, x_2, x_3 \in X$.

□

Moreover, every pseudometric space (X, d) can be considered as a topological space by choosing as a base $\mathcal{B}_d$ the set of all the sets

$$B(x, \varepsilon) = \{x' \in X : d(x, x') < \varepsilon\}$$

where $\varepsilon > 0$ and $x \in X$ (see, [6]).

We can find a method to approximate d:

Proposition 2.0.4. *Let $x_1, x_2 \in X$. Then for any $\delta > 0$ there exists Φ_δ, finite subset of Φ, such that*

$$|\sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)| - \sup_{\varphi \in \Phi_\delta} |\varphi(x_1) - \varphi(x_2)|| \leq 2\delta$$

Proof. Let fix $x_1, x_2 \in X$. Since Φ is compact, we can find a finite subset $\Phi^\delta = \{\varphi_1, \ldots, \varphi_n\}$ such that for each $\varphi \in \Phi$ there exists $\varphi_i \in \Phi^\delta$, for which $||\varphi - \varphi_i||_\infty \leq \delta$ and it follows that for any $x \in X$, $|\varphi(x) - \varphi_i(x)| \leq \delta$. Because of the definition of supremum of a subset of the set $\mathbb{R}^+$ of all positive real numbers, for any $\varepsilon > 0$ we can choose a $\bar{\varphi} \in \Phi$ such that

$$\sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)| - |\bar{\varphi}(x_1) - \bar{\varphi}(x_2)| \leq \varepsilon.$$

Now, if we take an index i, for which $||\bar{\varphi} - \varphi_i||_\infty \leq \delta$, we have that:

$$\begin{aligned} |\bar{\varphi}(x_1) - \bar{\varphi}(x_2)| &= |\bar{\varphi}(x_1) - \varphi_i(x_1) + \varphi_i(x_1) - \varphi_i(x_2) + \varphi_i(x_2) - \bar{\varphi}(x_2)| \\ &= |\bar{\varphi}(x_1) - \varphi_i(x_1)| + |\varphi_i(x_1) - \varphi_i(x_2)| + |\varphi_i(x_2) - \bar{\varphi}(x_2)| \\ &\leq |\varphi_i(x_1) - \varphi_i(x_2)| + 2\delta \\ &\leq \sup_{\varphi_j \in \Phi^\delta} |\varphi_j(x_1) - \varphi_j(x_2)| + 2\delta \end{aligned}$$

Hence,

$$\sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)| - \varepsilon \leq |\bar{\varphi}(x_1) - \bar{\varphi}(x_2)| \leq \sup_{\varphi_j \in \Phi^\delta} |\varphi_j(x_1) - \varphi_j(x_2)| + 2\delta.$$

Finally, as ε goes to zero, we have that

$$\sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)| \leq \sup_{\varphi_j \in \Phi^\delta} |\varphi_j(x_1) - \varphi_j(x_2)| + 2\delta.$$

On the other hand, since $\Phi^\delta \subseteq \Phi$:

$$\sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)| \geq \sup_{\varphi_j \in \Phi^\delta} |\varphi_j(x_1) - \varphi_j(x_2)|.$$

Therefore we proved the statement.

□

Now we want to say something more about the connection between these two topologies on X.

Proposition 2.0.5. *The initial topology μ on X with respect to Φ is finer than the topology τ on X induced by the pseudo-metric $d(x_1, x_2) = \sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)|$.*

Proof. Let $B = B(x, \varepsilon)$ be an open ball in τ. It is sufficient to show that B is an open set in the initial topology. This is true if we prove that $B = \bigcup(\bigcap_{i \in I} \varphi_i^{-1}(U_i))$, where I is a finite set.
By the definition of an open set in τ, $\forall y \in B$, there exist a positive δ and a positive $\varepsilon' < \varepsilon - 2\delta$ such that $B(y, \varepsilon') \subseteq B$. Let us consider a finite set $\Phi^\delta \subseteq \Phi$ so that for all $\varphi \in \Phi$, there exists a $\varphi_i \in \Phi^\delta$ for which $||\varphi - \varphi_i||_\infty \leq \delta$. Moreover, $\forall x,\ x' \in X$:

$$\sup_{\varphi' \in \Phi^\delta} |\varphi'(x) - \varphi'(x')| \leq \sup_{\varphi \in \Phi} |\varphi(x) - \varphi(x')|.$$

Hence, we can define

$$B^\delta(y, \varepsilon') = \{\bar{x} \in X | \sup_{\varphi \in \Phi^\delta} |\varphi(y) - \varphi(\bar{x})| < \varepsilon'\} \subseteq B(y, \varepsilon' + 2\delta).$$

We claim that $U_y = \bigcap_{\widetilde{\varphi} \in \Phi^\delta} \widetilde{\varphi}^{-1}(\ ]\widetilde{\varphi}(y) - \varepsilon', \widetilde{\varphi}(y) + \varepsilon'[\ ) \subseteq B^\delta(y, \varepsilon')$. Surely, $U_y \neq \emptyset$ because at least y belongs to the set.
If $z \in U_y$, then $|\widetilde{\varphi}(z) - \widetilde{\varphi}(y)| < \varepsilon'$ for every $\widetilde{\varphi} \in \Phi^\delta$. Therefore, $z \in B^\delta(y, \varepsilon')$. Therefore, $U_y \subseteq B^\delta(y, \varepsilon') \subseteq B(y, \varepsilon) \subseteq B$, because of Proposition 2.0.4. Hence, $\bigcup_y U_y = B$.
In conclusion, we wrote $B = B(x, \varepsilon)$ as an open set in the initial topology.

□

Proposition 2.0.6. *The topology τ on X induced by the pseudo-metric $d(x_1, x_2) = \sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)|$ is finer than the initial topology μ on X with respect to Φ.*

Proof. It is sufficient to show that $\varphi_1(U_1) \cap \varphi_2(U_2) \cap \cdots \cap \varphi_t(U_t)$ is an open set in the pseudometric topology τ, where $t \in \mathbb{N}$, $\varphi_i \in \Phi$ for all $i = 1, \ldots, t$ and $U_i \subset \mathbb{R}$ is an open set for all $i = 1, \ldots, t$. This is true if we prove that $\varphi_i^{-1}(U_i) = \bigcup_j B_j(x_j, r_j)$.
Without loss of generality, we can consider $\varphi_1^{-1}(U_1)$, but $U_1 = \bigcup_i]a_i, b_i[,\ a_i, b_i \in \mathbb{R}$ for any i and $\varphi_1^{-1}(\bigcup_i]a_i, b_i[) = \bigcup_i (\varphi_1^{-1}(\ ]a_i, b_i[\ )$, so we can focus on showing that $\varphi_1^{-1}(\ ]a, b[\ ) = \bigcup_k B_k(x_k, r_k)$. If $y \in \varphi_1^{-1}(\ ]a, b[\ )$, then there exists $c \in]a, b[$ such that $\varphi(y) = c$.
Now let us consider $m_y = \min\{c - a, b - c\}$.
Let $U_y = B(y, m_y)$; $U_y \neq \emptyset$ because at least $y \in U_y$. For any $y' \in U_y$ it holds that $\sup_{\varphi \in \Phi} |\varphi(y) - \varphi(y')| < m_y$, and hence $|\varphi_1(y) - \varphi_1(y')| < m_y$, that is

$$a < \varphi_1(y) - m_y < \varphi_1(y') < \varphi_1(y) + m_y < b.$$

Therefore, $U_y \subseteq \varphi_1^{-1}(\ ]a, b[\ )$. As a consequence, $\varphi_1^{-1}(\ ]a, b[\ ) = \bigcup_y U_y$ and our statement is proved. □

Consequently, we can claim the following result:

Theorem 2.0.7. *The topology τ induced by the pseudo-metric d and the initial topology are equivalent.*

Remark 2.0.8. (X, τ) could not be a T_0-space. For example, if $\Phi = \{f | f$ constant $, a \leq f \leq b,\ a, b \in \mathbb{R}\}$, then the elements of X are not distinct points for their images through those function are the same.

Remark 2.0.9. In general X is not compact. In fact, if X is a real interval and $\Phi = \{id : X \to X\}$, the induced topology is simply the Euclidean topology.

From now on we suppose that X is compact.
Let us consider $\mathcal{S}_X = \{g : X \to X, g \text{ bijective}\}$ the group with respect to the composition whose elements are bijective functions. Let G be a subgroup of

$\mathcal{S}_X$, such that $\forall\ g \in G$, $\forall\ \varphi \in \Phi$ we have that $\varphi \circ g \in \Phi$. We observe that at least the identity group verifies the property we request.

Proposition 2.0.10. *G is a subgroup of the group Homeo(X) of all homeomorphisms from X onto X.*

Proof. If all $g \in G$ are continuous functions, then all g^{-1} are continuous (since G is a group), so they are homeomorphisms. Hence, it's sufficient to prove that every g is continuous.
Let x_n be a sequence in X converging to $x \in X$.
If $g \in G$

$$\begin{aligned} d(g(x_n), g(x)) &= \sup_{\varphi \in \Phi} |\varphi(g(x_n)) - \varphi(g(x))| \\ &= \sup_{\varphi \in \Phi} |(\varphi \circ g)(x_n) - (\varphi \circ g)(x)| \\ &= \sup_{\psi \in \Phi} |\psi(x_n) - \psi(x)| = d(x_n, x) \end{aligned}$$

Since x_n converges to x, g is continuous. $\square$

We do not require G to be a proper subgroup of $Homeo(X)$, so the equality $G = Homeo(X)$ can possibly hold.
Now, we can define the function

$$d_1(g_1, g_2) = \sup_{\varphi \in \Phi} \|\varphi \circ g_1 - \varphi \circ g_2\|_\infty$$

from $G \times G$ to $\mathbb{R}$.

Proposition 2.0.11. *The function d_1 is a pseudo-metric on G.*

Proof.

- The value $d_1(g_1, g_2)$ is finite for every $g_1, g_2 \in G$, because Φ is compact and hence bounded. Indeed, a finite constant L exists such that $\|\varphi\|_\infty \leq L$ for every $\varphi \in \Phi$. Hence, $\|\varphi \circ g_1 - \varphi \circ g_2\|_\infty \leq \|\varphi\|_\infty + \|\varphi\|_\infty \leq 2L$ for any $\varphi \in \Phi$ and any $g_1, g_2 \in G$, since $\varphi \circ g_1, \varphi \circ g_2 \in \Phi$. This implies that $d_1(g_1, g_2) \leq 2L$ for every $g_1, g_2 \in G$.

- d_1 is obviously symmetrical.
- The definition of d_1 immediately implies that $d_1(g,g)=0$ for any $g \in G$.
- The triangle inequality holds, since

$$d_1(g_1,g_2) = \sup_{\varphi\in\Phi} \|\varphi\circ g_1 - \varphi\circ g_2\|_\infty$$

$$\leq \sup_{\varphi\in\Phi}(\|\varphi\circ g_1 - \varphi\circ g_3\|_\infty + \|\varphi\circ g_3 - \varphi\circ g_2\|_\infty)$$

$$\sup_{\varphi\in\Phi} \|\varphi\circ g_1 - \varphi\circ g_3\|_\infty + \sup_{\varphi\in\Phi} \|\varphi\circ g_3 - \varphi\circ g_2\|_\infty$$

$$= d_1(g_1,g_3) + d_1(g_3,g_2)$$

  for any $g_1, g_2, g_3 \in G$.

$\square$

Now, d_∞ in defined by setting $d_\infty(\varphi_1,\varphi_2) := ||\varphi_1 - \varphi_2||_\infty$ and we can prove the following property:

Theorem 2.0.12. *Homeo*(X) *is a topological group with respect to the pseudo-metric topology.*

Proof. It will suffice to prove that if $f = \lim_{i\to+\infty} f_i$ and $g = \lim_{i\to+\infty} g_i$ with respect of the pseudo-metric topology, then $g\circ f = \lim_{i\to+\infty} g_i \circ f_i$ and $f^{-1} = \lim_{i\to+\infty} f_i^{-1}$.
We have that

$$d_1(g_i\circ f_i, g\circ f) \leq d_1(g_i\circ f_i, g\circ f_i) + d_1(g\circ f_i, g\circ f) =$$

$$= \sup_{\varphi\in\Phi} \|\varphi\circ(g_i\circ f_i) - \varphi\circ(g\circ f_i)\|_\infty + \sup_{\varphi\in\Phi} \|\varphi\circ(g\circ f_i) - \varphi\circ(g\circ f)\|_\infty$$

$$= \sup_{\varphi\in\Phi}\ \sup_{x\in X} |\varphi(g_i(f_i(x)) - \varphi(g(f_i(x))| + \sup_{\varphi\in\Phi}\ \sup_{x\in X} |\varphi(g(f_i(x)) - \varphi(g(f(x))|$$

$$= \sup_{\varphi \in \Phi} \sup_{y \in X} |\varphi(g_i(y)) - \varphi(g(y))| + \sup_{\varphi \in \Phi} \sup_{x \in X} |\varphi(g(f_i(x)) - \varphi(g(f(x))|$$

$$\leq \sup_{\varphi \in \Phi^\delta} \sup_{y \in X} |\varphi(g_i(y)) - \varphi(g(y))| + \sup_{\varphi \in \Phi^\delta} \sup_{x \in X} |\varphi(g(f_i(x)) - \varphi(g(f(x))| + 4\delta$$

because of compactness of Φ and Proposition 2.0.4. Since $g = \lim_{i \to +\infty} g_i$, the limit of the first addend as i approaches infinity is 0; given that g is a uniformly continuous function and $f = \lim_{i \to +\infty} f_i$, the second addend converges to 0, too.
Therefore, $g \circ f = \lim_{i \to +\infty} g_i \circ f_i$.
We want to prove that $f^{-1} = \lim_{i \to +\infty} f_i^{-1}$. By contradiction, if we had not that $\lim_{i \to \infty} d_1(f_i^{-1}, f^{-1}) = 0$, then there would exist a constant $c > 0$ and a subsequence (f_{i_j}) of (f_i) such that $d_1(f_{i_j}^{-1}, f^{-1}) \geq c > 0$, $\forall j$. Indeed we should still have $\lim_{j \to \infty} d_1(f_{i_j}, f) = 0$ because (f_{i_j}) is a subsequence of (f_i).
$d_1(f_i^{-1}, f^{-1}) \geq c > 0$, $\forall j$ would imply the existence, for any j, of $\varphi_j \in \Phi$ such that $||\varphi_j \circ f_{i_j}^{-1} - \varphi_j \circ f^{-1}||_\infty \geq c > 0$.
Because of compactness of Φ, it would not be restrictive to assume (possibly by considering subsequences) the existence of the following limits: $\bar{\varphi} = \lim_{j \to \infty} \varphi_j$ and $\hat{\varphi} = \lim_{j \to \infty} \varphi_j \circ f_{i_j}^{-1}$.
Obviously we would have:

$$d_\infty(\hat{\varphi}, \bar{\varphi} \circ f^{-1}) = d_\infty(\lim_{j \to \infty} \varphi_j \circ f_{i_j}^{-1}, \lim_{j \to \infty} \varphi_j \circ f^{-1})$$

$$= \lim_{j \to \infty} d_\infty(\varphi_j \circ f_{i_j}^{-1}, \varphi_j \circ f^{-1}) \geq c > 0$$

so that $\hat{\varphi} \neq \bar{\varphi} \circ f^{-1}$.

On the other hand, we should have

$$d_\infty(\hat{\varphi}\circ f, \bar{\varphi}) = d_\infty((\lim_{j\to\infty}\varphi_j \circ f_{i_j}^{-1})\circ f, \lim_{j\to\infty}\varphi_j)$$

$$= \lim_{j\to\infty} d_\infty((\varphi_j \circ f_{i_j}^{-1})\circ f, (\varphi_j \circ f_{i_j}^{-1})\circ f_{i_j})$$

$$\leq \lim_{j\to\infty} d_1(f_{i_j}, f) = 0.$$

Finally $\hat{\varphi}\circ f = \bar{\varphi}$, that is absurd since we know that $\hat{\varphi} \neq \bar{\varphi}\circ f^{-1}$. We just proved that $\lim_{i\to\infty} f_i^{-1} = f^{-1}$ □

Proposition 2.0.13. *The action of G on Φ through right composition is continuous.*

Proof. We have to prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $||\varphi - \psi||_\infty \leq \delta$ and $d_1(f,g) \leq \delta$, then $||\varphi\circ f - \psi\circ g||_\infty \leq \varepsilon$.
Now,

$$||\varphi\circ f - \psi\circ g||_\infty \leq ||\varphi\circ f - \varphi\circ g||_\infty + ||\varphi\circ g - \psi\circ g||_\infty.$$

Since $d_1(f,g) = \sup_{\varphi'\in\Phi} ||\varphi'\circ f - \varphi'\circ g||_\infty$, it follows that

$$||\varphi\circ f - \varphi\circ g||_\infty \leq \sup_{\varphi'\in\Phi} ||\varphi'\circ f - \varphi'\circ g||_\infty \leq \delta.$$

Moreover,

$$||\varphi\circ g - \psi\circ g||_\infty = \sup_{x\in X} |\varphi(g(x)) - \psi(g(x))|$$

and, setting $y = g(x)$:

$$\sup_{x\in X} |\varphi(g(x)) - \psi(g(x))| = \sup_{y\in X} |\varphi(y) - \psi(y)| = ||\varphi - \psi||_\infty \leq \delta.$$

Hence, $\forall\varepsilon > 0$, there exists $0 < \delta < \varepsilon/2$:

$$||\varphi\circ f - \psi\circ g||_\infty \leq ||\varphi\circ f - \varphi\circ g||_\infty + ||\varphi\circ g - \psi\circ g||_\infty \leq 2\delta \leq \varepsilon.$$

□

We can consider the natural pseudo-distance d_G on the space Φ:

Definition 2.0.14. The pseudo-distance $d_G : \Phi \times \Phi \to \mathbb{R}$ is defined by setting

$$d_G(\varphi_1, \varphi_2) = \inf_{g \in G} \max_{x \in X} |\varphi_1(x) - \varphi_2(g(x))|.$$

It is called the *natural pseudo-distance* associated with the group G acting on Φ.

By the definition, when $G = \{Id : x \mapsto x\}$, then d_G equals the sup-norm distance d_∞ on Φ. If G_1 and G_2 are subgroups of $Homeo(X)$ that preserve Φ and $G_1 \subseteq G_2$, then the definition of d_G implies that $d_{G_2}(\varphi_1, \varphi_2) \leq d_{G_1}(\varphi_1, \varphi_2)$ for every $\varphi_1,\ \varphi_2 \in \Phi$. Therefore, it follows that

$$d_{Homeo(X)}(\varphi_1, \varphi_2) \leq d_G(\varphi_1, \varphi_2) \leq d_\infty(\varphi_1, \varphi_2)$$

for every $G \subseteq Homeo(X)$ that preserves Φ and every $\varphi_1,\ \varphi_2 \in \Phi$.

In order to proceed, we define a category $\mathcal{F}_X^{\Phi,G}$, whose objects are the elements of Φ, which is a compact subspace of $\mathbb{R}_b^X$, and arrows are the elements of G, a topological subgroup of $Homeo(X)$ that preserves Φ by composition on the right. It's easy to check that this category is well defined and we call it a *perception category.*

Definition 2.0.15. Assume that $\mathcal{F}_X^{\Phi,G}$, $\mathcal{F}_Y^{\Psi,H}$ are two perception categories. Each functor $F : \mathcal{F}_X^{\Phi,G} \to \mathcal{F}_Y^{\Psi,H}$ is called a *Group Invariant Non-expansive Operator (GINO)* if:

1. F is Group Invariant: $F(\varphi \circ g) = F(\varphi) \circ F(g),\ \forall \varphi,\ \forall g$;

2. F is non-expansive on Φ: $\|F(\varphi_1) - F(\varphi_2)\|_\infty \leq \|\varphi_1 - \varphi_2\|_\infty,\ \forall \varphi_1, \varphi_2$;

3. F is non-expansive on G: $d_1^H(F(g_1), F(g_2)) \leq d_1^G(g_1, g_2),\ \forall g_1, g_2$.

This simple statement holds:

Proposition 2.0.16. *For every $F \in \mathcal{W}$ and every $\varphi \in \Phi$: $||F(\varphi)||_\infty \leq ||\varphi||_\infty + ||F(\boldsymbol{0})||_\infty$, where **0** denotes the function taking the value 0 everywhere.*

Proof. Since F is non-expansive, we have that

$$\begin{aligned}||F(\varphi)||_\infty &= ||F(\varphi) - F(\mathbf{0}) + F(\mathbf{0})||_\infty \\ &\leq ||F(\varphi) - F(\mathbf{0})||_\infty + ||F(\mathbf{0})||_\infty \\ &\leq ||\varphi - \mathbf{0}||_\infty + ||F(\mathbf{0})||_\infty = ||\varphi||_\infty + ||F(\mathbf{0})||_\infty\end{aligned}$$

□

In order to proceed, we consider the set $\mathcal{W}$ of all operators that verify the previous properties.
Let $\mathcal{F}_X^{\Phi,G}$, $\mathcal{F}_Y^{\Psi,H}$ be two perception categories; if $\mathcal{W}' \neq \emptyset$ is a subset of $\mathcal{W}$, by recalling Φ is compact and hence bounded with respect to d_∞, then we can consider the function

$$d_{\mathcal{W}'}(F_1, F_2) := \sup_{\varphi \in \Phi} \|F_1(\varphi) - F_2(\varphi)\|_\infty$$

from $\mathcal{W}' \times \mathcal{W}'$ to $\mathbb{R}$.

Remark 2.0.17. We observe that $d_{\mathcal{W}'} = \sup_{g \in G} \sup_{\varphi \in \Phi} \|F_1(\varphi) \circ F_1(g) - F_2(\varphi) \circ F_2(g)\|_\infty$, since $\Phi \circ G = \Phi$.

Proposition 2.0.18. *If $\mathcal{W}'$ is a non-empty subset of $\mathcal{W}$, then $d_{\mathcal{W}'}$ is a pseudo-distance on $\mathcal{W}'$.*

Proof.

- The value $d_{\mathcal{W}'}(F_1, F_2)$ is finite for every $F_1, F_2 \in \mathcal{W}$, because Ψ is compact and hence bounded. Indeed, a finite constant L exists such that $\|\psi\|_\infty \leq L$ for every $\psi \in \Psi$. Hence, $\|F_1(\varphi) - F_2(\varphi)\|_\infty \leq \|F_1(\varphi)\|_\infty + \|F_2(\varphi)\|_\infty \leq 2L$ for any $\varphi \in \Phi$, since $F_1(\varphi),\ F_2(\varphi) \in \Psi$. This implies that $d_{\mathcal{W}'}(F_1, F_2) \leq 2L$ for every $F_1, F_2 \in \mathcal{W}$.
- $d_{\mathcal{W}'}$ is obviously symmetrical.
- The definition of $d_{\mathcal{W}'}$ immediately implies that $d_{\mathcal{W}'}(F, F) = 0$ for any $F \in \mathcal{W}'$.

- The triangle inequality holds, since

$$
\begin{aligned}
d_{\mathcal{W}'}(F_1, F_2) &= \sup_{\varphi \in \Phi} \|F_1(\varphi) - F_2(\varphi)\|_\infty \\
&\leq \sup_{\varphi \in \Phi} (\|F_1(\varphi) - F_3(\varphi)\|_\infty + \|F_3(\varphi) - F_2(\varphi)\|_\infty) \\
&\leq \sup_{\varphi \in \Phi} \|F_1(\varphi) - F_3(\varphi)\|_\infty + \sup_{\varphi \in \Phi} \|F_3(\varphi) - F_2(\varphi)\|_\infty \\
&= d_{\mathcal{W}'}(F_1, F_3) + d_{\mathcal{W}}(F_3, F_2)
\end{aligned}
$$

for any $F_1, F_2, F_3 \in \mathcal{W}'$.

□

Remark 2.0.19. In general G is not compact. In fact, if $X = \mathrm{S}^1$ and $\Phi = \{f(x, y) = x\cos(\alpha) + y\mathrm{sen}(\alpha) : 0 \leq \alpha \leq 2\pi\}$ (which is compact), $G = \{\rho_{2\pi q} \in \mathbb{Q}, q > 0\}$ (where $\rho_{2\pi q}$ is the rotation of angle $2\pi q$) is a subgroup of $Homeo(X)$ but it is not sequentially compact and so it is not compact.

2.1 Persistent Homology

Before proceeding, we recall some basic definitions and facts in persistent homology. For further and more detailed information, we refer the reader to [1, 2, 5].

Let φ be a real-valued continuous function on a topological space X. We can say that persistent homology represents the changes of the homology groups of the sub-level set $X_t = \varphi^{-1}((-\infty, t])$ varying t in $\mathbb{R}$. We can see the parameter t as an increasing time, whose changes produce the birth and the death of k-dimensional holes in the sub-level set X_t. For example the number of 0-dimensional holes equals the number of the connected components of X, 1-dimensional holes refer to tunnels and 2-dimensional holes to voids. Persistent homology can be introduced in different ways and settings. In this thesis, we chose to consider the topological settings and the simplicial and singular homology functor H. The reader can find an elementary introduction

to singular homology in [7].
Now we can define the persistent homology group with respect to the function $\varphi : X \to \mathbb{R}$ and some related concepts. We will refer to singular homology.

Definition 2.1.1. If $u, v \in \mathbb{R}$ and $u < v$, we can consider the inclusion i of X_u into X_v. Such an inclusion induces a homomorphism $i_k : H_k(X_u) \to H_k(X_v)$ between the homology groups of X_u and X_v in degree k. The group $PH_k^{\varphi}(u, v) := i_k(H_k(X_u))$ is called the kth *persistent homology group* with respect to the function $\varphi : X \to \mathbb{R}$, computed at the point (u, v).

Moreover, the rank $r_k(\varphi)(u, v)$ of $PH_k^{\varphi}(u, v)$ is said the kth *persistent Betti number function* with respect to the function $\varphi : X \to \mathbb{R}$, computed at the point (u, v).

Remark 2.1.2. Let X and Y be two homeomorphic spaces and let $h : Y \to X$ be a homeomorphism. Then the persistent homology group with respect to the function $\varphi : X \to \mathbb{R}$ and the persistent homology group with respect to the function $\varphi \circ h : Y \to \mathbb{R}$ are isomorphic at each point (u, v) in the domain. More precisely, the isomorphism is the one that maps each homology class $[c = \sum_{i=1}^{r} a_i \cdot \sigma_i] \in PH_k^{\varphi}(u, v)$ to the homology class $[c' = \sum_{i=1}^{r} a_i \cdot (h^{-1} \circ \sigma_i)] \in PH_k^{\varphi \circ h}(u, v)$, where each σ_i is a singular simplex involved in the representation of the cycle c. Therefore we can say that the persistent homology groups and the persistent Betti number functions are invariant under the action of $Homeo(X)$.

Now we want to present a classical description of persistent Betti number functions given by multisets called *persistence diagrams.*
The kth persistence diagram is the multiset of all the pairs $p_j = (b_j, d_j)$, where b_j and d_j are the times of birth and death of the jth k-dimensional hole, respectively. When a hole never dies, we set its time to death equal to ∞. The multiplicity $m(p_j)$ says how many holes share both the time of birth b_j and the time of death d_j. For technical reasons, the points (t, t) are added to each persistence diagram, each one with infinite multiplicity.
We can compare persistence diagram by means of the *bottleneck distance* or

matching distance δ_{match}. We are recalling its formal definition later.
Each persistence diagram D can contain an infinite number of points, and the multiplicity of each point is $m(p) \geq 1$. Now we set:

- $\Delta := \{(x, y) \in \mathbb{R}^2 \ : \ x = y\}$;
- $\Delta^+ := \{(x, y) \in \mathbb{R}^2 \ : \ x < y\}$;
- $\bar{\Delta}^+ := \{(x, y) \in \mathbb{R}^2 \ : \ x \leq y\}$;
- $\Delta^* := \Delta^+ \cup \{(x, \infty) : x \in \mathbb{R}\}$;
- $\bar{\Delta}^* := \bar{\Delta}^+ \cup \{(x, \infty) : x \in \mathbb{R}\}$.

For every $q \in \Delta^*$, the equality $m(q) = 0$ means that q does not belong to the persistence diagram D. We define on $\bar{\Delta}^*$ a pseudo-metric as follows

$$d^*((x, y), (x', y')) := \min\left\{\max\{|x - x'|, |y - y'|\}, \max\left\{\frac{y - x}{2}, \frac{y' - x'}{2}\right\}\right\}$$

by agreeing that $\infty - y = \infty$, $y - \infty = -\infty$ for $y \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\pm\infty| = \infty$, $\min\{\infty, c\} = c$, $\max\{\infty, c\} = \infty$.
The pseudo-metric d^* between two points p and p' takes the smaller value between the cost of moving p to p' and the cost of moving p' and p onto Δ. Obviously, $d^*(p, p') = 0$ for every $p, p' \in \Delta$. If $p \in \Delta^+$ and $p' \in \Delta$, then $d^*(p, p')$ equals the distance, endowed by the max-norm, between p and Δ. Points at infinity have a finite distance only to the other points at infinity, and their distance equals the Euclidean distance between abscissas.

Definition 2.1.3. Let D, D' be two persistence diagrams. We define the *bottleneck distance* δ_{match} between D and D' by setting

$$\delta_{\text{match}}(D, D') := \inf_{\sigma} \sup_{x \in D} d^*(x, \sigma(x)),$$

where $\sigma : D \to D'$ is a bijection.

For further informations about persistence diagrams and bottleneck distance, we refer the reader to [5, 4]. Each persistent Betti number function is associated with exactly one persistence diagram. Then the metric δ_{match} induces a pseudo-metric d_{match} on the sets of the persistent Betti number functions. For more details, we refer the reader to [3]. The following result shows the stability of the pseudo-distance d_{match} with respect to d_∞ and $d_{Homeo(X)}$.

Theorem 2.1.4. *If k is a natural number and $\varphi_1, \varphi_2 \in \Phi$, then*

$$d_{match}(r_k(\varphi_1), r_k(\varphi_2)) \leq d_{Homeo(X)}(\varphi_1, \varphi_2) \leq d_\infty(\varphi_1, \varphi_2)$$

The proof of the first inequality $d_{\text{match}}(r_k(\varphi_1), r_k(\varphi_2)) \leq d_{Homeo(X)}(\varphi_1, \varphi_2)$ in Theorem 2.1.4 can be found in [3]. Instead, the second inequality $d_{Homeo(X)}(\varphi_1, \varphi_2) \leq d_\infty(\varphi_1, \varphi_2)$ follows from the definition of $d_{Homeo(X)}$.

2.2 Strongly Group-invariant Comparison of Filtering Functions Via Persistent Homology

Let us consider a subset $\mathcal{W}' \neq \emptyset$ of $\mathcal{W}$. For every fixed k, we can consider the following pseudo-metric $\mathcal{D}^{\mathcal{W}',k}_{\text{match}}$ on Φ:

$$\mathcal{D}^{\mathcal{W}',k}_{\text{match}}(\varphi_1, \varphi_2) := \sup_{F \in \mathcal{W}'} d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2)))$$

for every $\varphi_1 \varphi_2 \in \Phi$, where $r_k(\varphi)$ denotes the kth persistent Betti number function with respect to the function $\varphi : X \to \mathbb{R}$.
In this work, we will say that a pseudo-metric $\hat{d}$ on Φ is *strongly G-invariant* if it is invariant under the action of G with respect to each variable, that is, if $\hat{d}(\varphi_1, \varphi_2) = \hat{d}(\varphi_1 \circ g, \varphi_2) = \hat{d}(\varphi_1, \varphi_2 \circ g) = \hat{d}(\varphi_1 \circ g, \varphi_2 \circ g)$ for every $\varphi_1, \varphi_2 \in \Phi$ and every $g \in G$.

Remark 2.2.1. It is easily seen that the natural pseudo-distance d_G is strongly G-invariant.

Proposition 2.2.2. *$\mathcal{D}^{\mathcal{W}',k}_{match}$ is a strongly G-invariant pseudo-metric on Φ.*

Proof. Theorem 2.1.4 and the non-expansivity of every $F \in \mathcal{W}'$ imply that

$$\begin{aligned} d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2))) &\leq ||F(\varphi_1) - F(\varphi_2)||_\infty \\ &\leq ||\varphi_1 - \varphi_2||_\infty. \end{aligned}$$

Therefore $\mathcal{D}^{\mathcal{W}',k}_{\text{match}}$ is a pseudo-metric, since it is the supremum of a family of pseudometrics that are bounded at each pair (φ_1, φ_2). Moreover, for every $\varphi_1, \varphi_2 \in \Phi$ and every $g \in G$

$$\begin{aligned} \mathcal{D}^{\mathcal{W}',k}_{\text{match}}(\varphi_1, \varphi_2 \circ g) &:= \sup_{F \in \mathcal{W}'} d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2 \circ g))) \\ &= \sup_{F \in \mathcal{W}'} d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2) \circ F(g))) \\ &= \sup_{F \in \mathcal{W}'} d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2)) \\ &= \mathcal{D}^{\mathcal{W}',k}_{\text{match}}(\varphi_1, \varphi_2) \end{aligned}$$

because $F(\varphi \circ g) = F(\varphi) \circ F(g)$ for every $\varphi \in \Phi$ and every $g \in G$ and the invariance of persistent homology under the action of the homeomorphisms(metterci il label del remark su sta cosa). Due to the fact the function $\mathcal{D}^{\mathcal{W}',k}_{\text{match}}$ is symmetric, thi is sufficient to guarantee that $\mathcal{D}^{\mathcal{W}',k}_{\text{match}}$ is strongly G-invariant. □

2.3 Approximating $\mathcal{D}^{\mathcal{W}',k}_{\text{match}}$

We give a method to approximate $\mathcal{D}^{\mathcal{W}',k}_{\text{match}}$ as follows:

Proposition 2.3.1. *Let $\mathcal{W}^* = \{F_1, \dots, F_m\}$ be a finite subset of $\mathcal{W}' \subseteq \mathcal{W}$. If for every $F \in \mathcal{W}'$ at least one index $i \in \{1, \dots, m\}$ exists, such that $d_{\mathcal{W}'}(F_i, F) \leq \varepsilon$, then*

$$|\mathcal{D}^{\mathcal{F}^*,k}_{match}(\varphi_1, \varphi_2) - \mathcal{D}^{\mathcal{W}',k}_{match}(\varphi_1, \varphi_2)| \leq 2\varepsilon$$

for every $\varphi_1, \varphi_2 \in \Phi$.

Proof. Let us assume $F \in \mathcal{W}'$ and $d_{\mathcal{W}'} \leq \varepsilon$. Because of the definition of $d_{\mathcal{W}'}$, for any $\varphi_1, \varphi_2 \in \Phi$ we have the inequalities $||F_i(\varphi_1) - F(\varphi_1)||_\infty \leq \varepsilon$ and $||F_i(\varphi_2) - F(\varphi_2)||_\infty \leq \varepsilon$. Hence

$$d_{\text{match}}(r_k(F_i(\varphi_1)), r_k(F(\varphi_1)) \leq \varepsilon$$

and

$$d_{\text{match}}(r_k(F_i(\varphi_2)), r_k(F(\varphi_2)) \leq \varepsilon$$

because of the stability of the persistent homology (Theorem 2.1.4).
It follows that

$$|d_{\text{match}}(r_k(F_i(\varphi_1)), r_k(F_i(\varphi_2)) - d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2))| \leq 2\varepsilon.$$

The statement immediately follows from the definitions of $\mathcal{D}_{\text{match}}^{\mathcal{W}',k}$ and $\mathcal{D}_{\text{match}}^{\mathcal{F}^*,k}$. □

Therefore, if we can cover $\mathcal{W}'$ by a finite set of balls of radius ε, centered at points of $\mathcal{W}'$ the approximation of $\mathcal{D}_{\text{match}}^{\mathcal{W}',k}(\varphi_1, \varphi_2)$ can be reduced to the computation of the maximum of a finite set of bottleneck distances between persistence diagrams, which are well-known to be computable by means of efficient algorithms.
This fact lead us to study the properties of the topological space $\mathcal{W}$.

Chapter 3

Main results

At first we want to show that the pseudo-metric $\mathcal{D}_{\text{match}}^{\mathcal{W}',k}$ is stable with respect to both the natural pseudo-distance associated with group G and the sup-nom.

Theorem 3.0.1. *If $\mathcal{W}'$ is a non-empty subset of $\mathcal{W}$, then*

$$\mathcal{D}_{match}^{\mathcal{W}',k} \leq d_G \leq d_\infty$$

Proof. For every $F \in \mathcal{D}_{\text{match}}^{\mathcal{W}',k}$, every $g \in G$ and every $\varphi_1, \varphi_2 \in \Phi$, we have that

$$\begin{aligned} d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2))) &= d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2) \circ F(g))) \\ &= d_{\text{match}}(r_k(F(\varphi_1)), r_k(F(\varphi_2 \circ g))) \\ &\leq ||F(\varphi_1) - F(\varphi_2 \circ g)||_\infty \leq ||\varphi_1 - \varphi_2 \circ g||_\infty. \end{aligned}$$

The first equality follows from the invariance of persistent homology under action of $Homeo(X)$ (see Remark 2.1.2), and the second equality follows from the fact F is a Group-invariant operator. The first inequality follows from the stability of persistent homology (Theorem 2.1.4), while the second inequality follows from the non-expansivity of F.
It follows that, if $\mathcal{W}' \subseteq \mathcal{W}$, then for every $g \in G$ and every $\varphi_1, \varphi_2 \in \Phi$

$$\mathcal{D}_{\text{match}}^{\mathcal{W}',k}(\varphi_1, \varphi_)) \leq ||\varphi_1 - \varphi_2 \circ g||_\infty.$$

Hence,

$$\begin{aligned}\mathcal{D}_{\text{match}}^{\mathcal{W}',k}(\varphi_1,\varphi_2) &\le \inf_{g\in G}||\varphi_1-\varphi_2\circ g||_\infty\\ &\le ||\varphi_1-\varphi_2||_\infty = d_\infty(\varphi_1,\varphi_2)\end{aligned}$$

for every $\varphi_1,\varphi_2\in\Phi$. □

The definitions of the natural pseudo-distance d_G and the pseudo-distance $\mathcal{D}_{\text{match}}^{\mathcal{W}',k}$ come from different theoretical concepts. The former is based on a variation approach involving the set of all homeomorphisms G, while the latter refers only to a comparison of persistent homologies depending on a family of Group-invariant operators. Given those comments, the next result may appear unexpected.

Theorem 3.0.2. *Let us consider* $\mathcal{W}=\{F:\mathcal{F}_X^{\Phi,G}\to\mathcal{F}_X^{\Phi,G}, F$ *is a GINO*$\}$. *Then* $\mathcal{D}_{match}^{\mathcal{W},k}=d_G$.

Proof. For every $\psi\in\Phi$ let us consider the operator $F_\psi:\mathcal{F}_X^{\Phi,G}\to\mathcal{F}_X^{\Phi,G}$ defined by setting $F_\psi(\varphi)$ equal to the constant function taking everywhere the value $d_G(\varphi,\psi)$ for every $\varphi\in\Phi$ (i.e., $F_\psi(\varphi)(x)=d_G(\varphi,\psi)$ for any $x\in X$) and $F_\psi(g)=g$ for every $g\in G$.
We observe that

1. F_ψ is a Group-invariant operator on Φ, because the strong invariance of the natural pseudo-distance d_G with respect to the group G (Remark 2.2.1) implies that if $\varphi\in\Phi$ and $g\in G$, then $F_\psi(\varphi\circ g)(x)=d_G(\varphi\circ g,\psi)=F_\psi(\varphi)(g(x))=(F_\psi(\varphi)\circ g)(x)=(F_\psi(\varphi)\circ F_\psi(g))(x)$, for every $x\in X$.

2. F_ψ is non-expansive on Φ, because for every $\varphi_1,\varphi_2\in\Phi$

$$\begin{aligned}||F_\psi(\varphi_1)-F_\psi(\varphi_2)||_\infty &= |d_G(\varphi_1,\psi)-d_G(\varphi_2,\psi)|\\ &\le d_G(\varphi_1,\varphi_2)\le||\varphi_1-\varphi_2||_\infty\end{aligned}$$

3. F_ψ is non-expansive on G: in fact for every $g_1, g_2 \in G$

$$d_1(F_\psi(g_1), F_\psi(g_2)) = d_1(g_1, g_2)$$

Therefore, $F_\psi \in \mathcal{W}$.
For every $\varphi_1, \varphi_2, \psi \in \Phi$ we have that

$$d_{\text{match}}(r_k(F_\psi(\varphi_1)), r_k(F_\psi(\varphi_2))) = |d_G(\varphi_1, \psi) - d_G(\varphi_2, \psi)|.$$

Indeed, apart from the trivial points on the line $\{(u, v) \in \mathbb{R}^2 \ : \ u = v\}$, the persistence diagram associated with $r_k(F_\psi(\varphi_1)$ contains only the point $(d_G(\varphi_1, \psi), \infty)$, while the persistence diagram associated with $r_k(F_\psi(\varphi_2)$ contains only the point $(d_G(\varphi_2, \psi), \infty)$. Both the points have the same multiplicity, which equals the (non-null) k-th Betti number of X.
Setting $\psi = \varphi_2$, we have that

$$d_{\text{match}}(r_k(F_\psi(\varphi_1)), r_k(F_\psi(\varphi_2))) = d_G(\varphi_1, \varphi_2).$$

As a consequence, we have that

$$\mathcal{D}^{\mathcal{W},k}_{\text{match}}(\varphi_1, \varphi_2) \geq d_G(\varphi_1, \varphi_2)$$

. By applying Theorem 3.0.1, we get

$$\mathcal{D}^{\mathcal{W},k}_{\text{match}}(\varphi_1, \varphi_2) = d_G(\varphi_1, \varphi_2)$$

for every φ_1, φ_2. □

Now we are ready to expose the main results of the thesis:

Theorem 3.0.3. *Let $\mathcal{F}^{\Phi,G}_X$, $\mathcal{F}^{\Psi,H}_Y$ be two perception categories; if Φ, G , Ψ, H are compact with respect to their topologies, then $\mathcal{W}$ is compact with respect to the pseudo-metric topology endowed by $d_{\mathcal{W}}$.*

Proof. Because of our hypothesis, we have that $(\mathcal{W}, d_{\mathcal{W}})$ is a pseudo-metric space. Therefore it will suffice to prove that $\mathcal{W}$ is sequentially compact.
In order to do this, let us assume that a sequence (F_i) in $\mathcal{W}$ is given.

Given that $\varPhi$ is a compact (and hence separable) metric space, we can find a countable and dense subset $\varPhi^* = \{\varphi_j\}_{j\in\mathbb{N}}$. In the same way, we can find a countable and dense subset $G^* = \{g_k\}_{k\in\mathbb{N}}$ of G.

Now we want to extract a subsequence (F_{i_h}) from (F_i), such that for every pair of indeces (j,k) the sequence $(F_{i_h}(\varphi_j))$ converges to a function in Ψ with respect to the sup-norm and the sequence $(F_{i_h}(g_k))$ converges to a homeomorphism in H with respect to d_1. We will show it as follows. Since Ψ is compact, the sequence $(F_i(\varphi_1))$ admits a subsequence $(F_i^{(1)}(\varphi_1))$ that converges in Ψ. Again, since Ψ is compact, the sequence $(F_i(\varphi_2))$ admits a subsequence $(F_i^{(2)}(\varphi_2))$ that converges in Ψ. Recursively, we can build a family of subsequences $(F_i^{(k)})_{i\in\mathbb{N}},\ k\in\mathbb{N}$ such that $(F_i^{(k+1)})_{i\in\mathbb{N}}$ is a subsequence of $(F_i^{(k)})_{i\in\mathbb{N}}$ and $(F_i^{(k)}(\varphi_k))$ converges in Ψ for every $k\in\mathbb{N}$.

Now we set $F_{i_h} = F_h^{(h)}$. It results that $F_{i_h} = F_h^{(h)} \in \{F_m^{(k)} \mid m\in\mathbb{N}\}$ for every $h\geq k$. Therefore, $(F_{i_h})_{h\geq k}$ is a subsequence of $(F_m^{(k)})$, for every $k\in\mathbb{N}$. Since $(F_i^{(k)}(\varphi_k))$ converges for every k, $F_{i_h}(\varphi_k)$ converges for every k.

Similarly, (F_{i_h}) admits a subsequence $(F_{i_{h_t}})$ that converges in G, because G is compact. For the sake of simplicity, we set $F_{i_{h_t}} = F_{i_h}$.

Now, let us consider the functor $\bar{F} : \mathcal{F}_X^{\varPhi,G} \to \mathcal{F}_Y^{\varPsi,H}$ defined as follow.

We define $\bar{F}$ on $\varPhi^*$ by setting $\bar{F}(\varphi_j) := \lim_{h\to\infty} F_{i_h}(\varphi_j)$ for each $\varphi_j\in\varPhi^*$; in a similar way, we can define on G^* $\bar{F}(g_k) := \lim_{h\to\infty} F_{i_h}(g_k)$. Then we want to extend $\bar{F}$ to $\varPhi$ and G.

First we extend $\bar{F}$ to $\varPhi$ as follows. $\forall\ \varphi\in\varPhi$ we choose a sequence (φ_{j_r}) in $\varPhi^*$, converging to $\varphi\in\varPhi$, and set $\bar{F} := \lim_{r\to\infty}\bar{F}(\varphi_{j_r})$. We claim that such a limit exists in Ψ and does not depend on the sequence that we have chosen, converging to $\varphi\in\varPhi$. In order to prove that the previous limit exists, we observe that for every $r,\ s\in\mathbb{N}$

$$\begin{aligned}
||\bar{F}(\varphi_{j_r}) - \bar{F}(\varphi_{j_s})||_\infty &= ||\lim_{h\to\infty} F_{i_h}(\varphi_{j_r}) - \lim_{h\to\infty} F_{i_h}(\varphi_{j_s})||_\infty \\
&= \lim_{h\to\infty} ||F_{i_h}(\varphi_{j_r}) - F_{i_h}(\varphi_{j_s})||_\infty \\
&\leq \lim_{h\to\infty} ||\varphi_{j_r} - \varphi_{j_s}||_\infty = ||\varphi_{j_r} - \varphi_{j_s}||_\infty,
\end{aligned}$$

because each functor F_{i_h} is non-expansive.
Since the sequence (φ_{j_r}) converges to $\varphi \in \Phi$, it follows that $(\bar{F}(\varphi_{j_r}))$ is a Cauchy sequence. The compactness of Ψ implies that $(\bar{F}(\varphi_{j_r}))$ converges in Ψ.
If another sequence (φ_{k_r}) in given in Φ^*, converging to $\varphi \in \Phi$, then for every index $r \in \mathbb{N}$

$$\begin{aligned}||\bar{F}(\varphi_{j_r}) - \bar{F}(\varphi_{k_r})||_\infty &= || \lim_{h\to\infty} F_{i_h}(\varphi_{j_r}) - \lim_{h\to\infty} F_{i_h}(\varphi_{k_r})||_\infty \\ &= \lim_{h\to\infty} ||F_{i_h}(\varphi_{j_r}) - F_{i_h}(\varphi_{k_r})||_\infty \\ &\le \lim_{h\to\infty} ||\varphi_{j_r} - \varphi_{k_r}||_\infty \\ &= ||\varphi_{j_r} - \varphi_{k_r}||_\infty .\end{aligned}$$

Since both (φ_{j_r}) and (φ_{k_r}) converge to φ it follows that $\lim_{r\to\infty} \bar{F}(\varphi_{j_r}) = \lim_{r\to\infty} \bar{F}(\varphi_{k_r})$. Therefore the definition of $\bar{F}(\varphi)$ does not depend on the sequence (φ_{j_r}) that we have chosen, converging to φ.
Just as we have done, we extend $\bar{F}$ to G.
For each $g \in G$, we choose a sequence (g_{j_r}) in G^*, converging to $g \in G$, and set $\bar{F} := \lim_{r\to\infty} \bar{F}(g_{j_r})$. We claim that such a limit exists in G and does not depend on the sequence that we have chosen, converging to $g \in G$. In order to prove that the previous limit exists, we observe that for every $r,\ s \in \mathbb{N}$

$$\begin{aligned}d_1^H(\bar{F}(g_{j_r}), \bar{F}(g_{j_s})) &= \sup_{\psi\in\Psi} ||\psi \circ (\lim_{h\to\infty} F_{i_h}(g_{j_r})) - \psi \circ (\lim_{h\to\infty} F_{i_h}(g_{j_s}))||_\infty \\ &= \lim_{h\to\infty} \sup_{\psi\in\Psi} ||\psi \circ F_{i_h}(g_{j_r}) - \psi \circ F_{i_h}(g_{j_s})||_\infty \\ &\le \lim_{h\to\infty} \sup_{\varphi\in\Phi} ||\varphi \circ g_{j_r} - \varphi \circ g_{j_s}||_\infty \\ &= \sup_{\varphi\in\Phi} ||\varphi \circ g_{j_r} - \varphi \circ g_{j_s}||_\infty = d_1^G(g_{j_r}, g_{j_s})\end{aligned}$$

because each functor F_{i_h} is non-expansive.
Since the sequence (g_{j_r}) converges to $g \in G$, it follows that $(\bar{F}(g_{j_r}))$ is a

Cauchy sequence. The compactness of H implies that $(\bar{F}(g_{j_r}))$ converges in H.

If another sequence (g_{k_r}) is given in G^*, converging to $g \in G$, then for every index $r \in \mathbb{N}$

$$
\begin{aligned}
d_1^H(\bar{F}(g_{j_r}), \bar{F}(g_{k_r})) &= \sup_{\psi\in\Psi} ||\psi \circ (\lim_{h\to\infty} F_{i_h}(g_{j_r})) - \psi \circ (\lim_{h\to\infty} F_{i_h}(g_{k_r}))||_\infty \\
&= \lim_{h\to\infty} \sup_{\psi\in\Psi} ||\psi \circ F_{i_h}(g_{j_r}) - \psi \circ F_{i_h}(g_{k_r})||_\infty \\
&\le \lim_{h\to\infty} \sup_{\varphi\in\Phi} ||\varphi \circ g_{j_r} - \varphi \circ g_{k_r}||_\infty \\
&= \sup_{\varphi\in\Phi} ||\varphi \circ g_{j_r} - \varphi \circ g_{k_r}||_\infty = d_1^G(g_{j_r}, g_{k_r}).
\end{aligned}
$$

Since both (g_{j_r}) and (g_{k_r}) converge to G it follows that $\lim_{r\to\infty} \bar{F}(g_{j_r}) = \lim_{r\to\infty} \bar{F}(g_{k_r})$. Therefore the definition of $\bar{F}(g)$ does not depend on the sequence (g_{j_r}) that we have chosen, converging to g.

Now we have to prove that $\bar{F} \in \mathcal{W}$, i.e., that $\bar{F}$ verifies the three properties defining this set of functors.

We have already seen that $\bar{F} : \mathcal{F}_X^{\Phi,G} \to \mathcal{F}_Y^{\Psi,H}$.

$\forall\ \varphi,\ \varphi'$ we can consider two sequences (φ_{j_r}), (φ_{k_r}) in Φ^*, converging to φ and φ', respectively. Due to the fact that the functors F_{i_h} are non-expansive, we have that

$$
\begin{aligned}
||\bar{F}(\varphi) - \bar{F}(\varphi')||_\infty &= || \lim_{r\to\infty} \bar{F}(\varphi_{j_r}) - \lim_{r\to\infty} \bar{F}(\varphi_{k_r})||_\infty \\
&= || \lim_{r\to\infty} \lim_{h\to\infty} F_{i_h}(\varphi_{j_r}) - \lim_{r\to\infty} \lim_{h\to\infty} F_{i_h}(\varphi_{k_r})||_\infty \\
&= \lim_{r\to\infty} \lim_{h\to\infty} ||F_{i_h}(\varphi_{j_r}) - F_{i_h}(\varphi_{k_r})||_\infty \\
&\le \lim_{r\to\infty} \lim_{h\to\infty} ||\varphi_{j_r} - \varphi_{k_r}||_\infty \\
&= \lim_{r\to\infty} ||\varphi_{j_r} - \varphi_{k_r}||_\infty \\
&= ||\varphi - \varphi'||_\infty.
\end{aligned}
$$

Therefore, $\bar{F}$ is non-expansive in Φ. As a consequence, it is also continuous in Φ.
Similarly, we want to prove that the functor $\bar{F}$ is non expansive in G.
$\forall\ g,\ g'$ we can consider two sequences (g_{j_r}), (g_{k_r}) in G^*, converging to g and g', respectively. Due to the fact that the functors F_{i_h} are non-expansive, we have that

$$
\begin{aligned}
d_1^H(\bar{F}(g), \bar{F}(g')) &= \sup_{\psi \in \Psi} ||\psi \circ (\lim_{r\to\infty} \lim_{h\to\infty} F_{i_h}(g_{j_r})) - \psi \circ (\lim_{r\to\infty} \lim_{h\to\infty} (F_{i_h}(g_{k_r}))||_\infty \\
&= \lim_{r\to\infty} \lim_{h\to\infty} \sup_{\psi \in \Psi} ||\psi \circ F_{i_h}(g_{j_r}) - \psi \circ F_{i_h}(g_{k_r})||_\infty \\
&\leq \lim_{r\to\infty} \lim_{h\to\infty} \sup_{\varphi \in \Phi} ||\varphi \circ g_{j_r} - \varphi \circ g_{k_r}||_\infty \\
&= \lim_{r\to\infty} \sup_{\varphi \in \Phi} ||\varphi \circ g_{j_r} - \varphi \circ g_{k_r}||_\infty \\
&= \sup_{\varphi \in \Phi} ||\varphi \circ g - \varphi \circ g'||_\infty \\
&= d_1^G(g, g').
\end{aligned}
$$

Therefore, $\bar{F}$ is non-expansive in G. As a consequence, it is also continuous in G.
Now we can prove that the sequence (F_{i_h}) converges to $\bar{F}$ with respect to $d_{\mathcal{W}}$.
Let us consider an arbitrary small $\varepsilon > 0$. Since Φ is compact and Φ^* is dense in Φ, we can find a finite subset $\{\varphi_{j_1}, \ldots, \varphi_{j_n}\}$ of Φ^* such that $\forall \varphi \in \Phi$, there exists an index $r \in \{1, \ldots, n\}$, for which $||\varphi - \varphi_{j_r}||_\infty \leq \varepsilon$.
Since the sequence (F_{i_h}) converges pointwise to $\bar{F}$ on the set Φ^*, an index h', such that $||\bar{F}(\varphi_{j_r}) - F_{i_h}(\varphi_{j_r})||_\infty \leq \varepsilon$ for any $h \geq h'$ and any $r \in \{1, \ldots, n\}$.
Therefore, for every $\varphi \in \Phi$ we can find an index $r \in \{1, \ldots, n\}$ such that $||\varphi - \varphi_{j_r}||_\infty \leq \varepsilon$ and the following inequalities hold for every index $h \geq h'$,

because of the non-expansivity of $\bar{F}$ and F_{i_h}:

$$||\bar{F}(\varphi) - F_{i_h}(\varphi)||_\infty$$
$$\leq ||\bar{F}(\varphi) - \bar{F}(\varphi_{j_r})||_\infty + ||\bar{F}(\varphi_{j_r}) - F_{i_h}(\varphi_{j_r})||_\infty + ||F_{i_h}(\varphi_{j_r}) - F_{i_h}(\varphi)||_\infty$$
$$\leq ||\varphi - \varphi_{j_r}||_\infty + ||\bar{F}(\varphi_{j_r}) - F_{i_h}(\varphi_{j_r})||_\infty + ||\varphi_{j_r} - \varphi||_\infty \leq 3\epsilon.$$

We observe that h' does not depend on φ, but only on ϵ and on the set $\{\varphi_{j_1}, \ldots, \varphi_{j_n}\}$. It follows that $||\bar{F}(\varphi) - F_{i_h}(\varphi)||_\infty \leq 3\varepsilon$ for every $\varphi \in \Phi$ and every $h \geq h'$.
Hence, $\sup_{\varphi\in\Phi} ||\bar{F}(\varphi) - F_{i_h}(\varphi)||_\infty \leq 3\varepsilon$ for every $h \geq h'$ Therefore, the sequence (F_{i_h}) converges to $\bar{F}$ with respect to $d_{\mathcal{W}}$.
The last thing that we have to show is that $\bar{F}$ is a group-invariant functor. Let us consider a $\varphi \in \Phi$, a sequence (φ_{j_r}) in Φ^* converging to φ in Φ and a $g \in G$. Obviously, the sequence $(\varphi_{j_r} \circ g)$ converges to $\varphi \circ g$ in Φ. We recall that the right action of G on Φ is continuous, $\bar{F}$ is continuous and each F_{i_h} is a group-invariant functor. Hence, given that the sequence (F_{i_h}) converges to $\bar{F}$ with respect to $d_{\mathcal{W}}$:

$$\begin{aligned}
\bar{F}(\varphi \circ g) &= \bar{F}(\lim_{r\to\infty}(\varphi_{j_r} \circ g)) \\
&= \lim_{r\to\infty} \bar{F}(\varphi_{j_r} \circ g) \\
&= \lim_{r\to\infty} \lim_{h\to\infty} F_{i_h}(\varphi_{j_r} \circ g) \\
&= \lim_{r\to\infty} \lim_{h\to\infty} F_{i_h}(\varphi_{j_r}) \circ F_{i_h}(g) \\
&= \lim_{r\to\infty} \bar{F}(\varphi_{j_r}) \circ \bar{F}(g) \\
&= \bar{F}(\varphi) \circ \bar{F}(g).
\end{aligned}$$

This proves that $\bar{F}$ is a group-operator.
In conclusion, $\bar{F} \in \mathcal{W}$.

From the fact that the sequence F_{i_h} converges to $\bar{F}$ with respect to $d_{\mathcal{W}}$, it follows that $(\mathcal{W}, d_{\mathcal{W}})$ is sequentially compact. □

Corollary 3.0.4. *Let $\mathcal{W}'$ be a non-empty subset of $\mathcal{W}$. For every $\varepsilon > 0$, a finite subset $\mathcal{W}^*$ of $\mathcal{W}'$ exists, such that*

$$|\mathcal{D}_{match}^{\mathcal{W}^*,k}(\varphi_1, \varphi_2) - \mathcal{D}_{match}^{\mathcal{W}',k}(\varphi_1, \varphi_2)| \leq \varepsilon$$

for every $\varphi_1, \varphi_2 \in \Phi$.

Proof. Let us consider the closure $\bar{\mathcal{W}}'$ of $\mathcal{W}'$ in $\mathcal{W}$. Let us also consider the covering $\mathcal{U}$ of $\bar{\mathcal{W}}'$ obtained by taking all the open balls of radius $\frac{\varepsilon}{2}$ centered at points of $\mathcal{W}'$. Theorem 3.0.3 guarantees that $\mathcal{W}$ is compact, henve also $\bar{\mathcal{W}}'$ is compact. Therefore we can extract a finite covering $\{B_1, \ldots, B_m\}$ of $\mathcal{W}'$ from $\mathcal{U}$. We can set $\mathcal{W}^*$ equal to the set of all the centered balls $B_1, \ldots, B_m$. The statement of our corollary immediately follows from proposition 2.3.1. □

The previous corollary shows that, under suitable hypotheses, the computation of the $\mathcal{D}_{\mathrm{match}}^{\mathcal{W},k}$ can be reduced to the computation of the maximum of a finite set of bottleneck distances between persistence diagrams, for every $\varphi_1, \varphi_2 \in \Phi$.

Bibliography

[1] S. Biasotti, L. De Floriani, B. Falcidieno, P. Frosini, D. Giorgi, C. Landi, L. Papaleo, and M. Spagnuolo. Describing shapes by geometrical-topological properties of real functions. *ACM Comput. Surv.*, 40(4):12:1–12:87, October 2008.

[2] Gunnar Carlsson and Afra Zomorodian. The theory of multidimensional persistence. *Discrete Comput. Geom.*, 42(1):71–93, 2009.

[3] Andrea Cerri, Barbara Di Fabio, Massimo Ferri, Patrizio Frosini, and Claudia Landi. Betti numbers in multidimensional persistent homology are stable functions. *Math. Methods Appl. Sci.*, 36(12):1543–1557, 2013.

[4] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Stability of persistence diagrams. *Discrete Comput. Geom.*, 37(1):103–120, 2007.

[5] Herbert Edelsbrunner and John Harer. Persistent homology—a survey. In *Surveys on discrete and computational geometry*, volume 453 of *Contemp. Math.*, pages 257–282. Amer. Math. Soc., Providence, RI, 2008.

[6] Steven A. Gaal. *Point set topology.* Pure and Applied Mathematics, Vol. XVI. Academic Press, New York-London, 1964.

[7] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[8] Stephen Willard. *General Topology.* Courier Corporation, 1970.