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Classical Electrodynamics: Retarded Potentials and Power Emission by Accelerated Charges

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SOMMARIO. L'obiettivo di questo lavoro é quello di analizzare la potenza emessa da una carica elettrica accelerata. Saranno studiati due casi speciali: accelerazione lineare e accelerazione circolare. Queste sono le configurazioni piú frequenti e semplici da realizzare. Il primo passo consiste nel trovare un'espressione per il campo elettrico e il campo magnetico generati dalla carica. Questo sará reso possibile dallo studio della distribuzione di carica di una sorgente puntiforme e dei potenziali che la descrivono. Nel passo successivo verrá calcolato il vettore di Poynting per una tale carica. Useremo questo risultato per trovare la potenza elettromagnetica irradiata totale integrando su tutte le direzioni di emissione. Nell'ultimo capitolo, infine, faremo uso di tutto ció che é stato precedentemente trovato per studiare la potenza emessa da cariche negli acceleratori.

Abstract. This paper's goal is to analyze the power emitted by an accelerated electric charge. Two special cases will be scrutinized: the linear acceleration and the circular acceleration. These are the most frequent and easy to realize configurations. The first step consists of finding an expression for electric and magnetic field generated by our charge. This will be achieved by studying the charge distribution of a point-like source and the potentials that arise from it. The following step involves the computation of the Poynting vector. This will be used to calculate the total radiated electromagnetic power by integrating on all possible orientations. In the last chapter, we will combine the knowledge gathered thus far to study power emission in accelerators.

Contents

CHAPTER 1

Introduction

Electromagnetism is the study of electricity and magnetism. These two are profoundly related to one another, so much so that they are considered two aspects of the same phenomenon. It was not until the XIX century that they were treated as such. The first one who proposed this was J. C. Maxwell (1831 - 1879), whose revolutionary point of view still to this day proves very powerful and effective in describing of every classical electromagnetic phenomenon known to man. In 1905 A. Einstein (1879 - 1955) established the interconnection between electricity and magnetism even strongly with special relativity.

Classical electrodynamics studies the phenomena associated with moving electric charges and their interaction with electric and magnetic fields. The speeds involved are allowed to get arbitrarily close to the speed of light, as the theory ties in perfectly with special relativity.

Electrodynamics is formulated through fields, which permeate the entire universe. Electric charges couple to the field and respond to it with an interaction that forces the field itself to change. The fundamental problem is to study the interaction that arises between an electric charge moving in an electromagnetic field.

The concepts that will be discussed in this paper are standard, and can be found in any modern and classic book of electrodynamics.

CHAPTER 2

Electric and magnetic fields

2.1. Point-like charge approximation

In order to find an expression for the electric and magnetic fields $\boldsymbol{E}(\boldsymbol{x},t)$, $\boldsymbol{B}(\boldsymbol{x},t)$ generated by a moving charge, we first have to consider the electric charge density $\rho(\boldsymbol{x},t)$ and current density $\boldsymbol{j}(\boldsymbol{x},t)$ of such a charge q, which are known from classical electromagnetism to have the form

$$\rho(\boldsymbol{x},t) = \frac{q}{4\pi {r_0}^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right), \tag{2.1.1a}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = \frac{q}{4\pi {r_0}^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\dot{\boldsymbol{x}}(t), \tag{2.1.1b}$$

where r_0 is a constant with the dimensions of length, $\boldsymbol{x}(t)$ is the trajectory in three dimensional space evaluated at time t, $f(\xi)$ with $\xi \geq 0$ is a regular real valued function such that

$$\int_0^\infty \mathrm{d}\xi\, \xi^2 f(\xi) = 1, \tag{2.1.2a}$$

$$f(\xi) \to 0 \text{ for } \xi \gg 1. \tag{2.1.2b}$$

The function $f(\xi)$ bares information about the spacial distribution of the charge. Equation (2.1.2a) makes sure that the integral of the electric density $\rho(\boldsymbol{x},t)$ throughout space yields the total charge q, while (2.1.2b) specifies that the charge is mostly concentrated around a ball of radius r_0. This means that (2.1.1) describe a charged ball of radius r_0 rigidly moving through space on an arbitrary trajectory $\boldsymbol{x}(t)$.

The definitions (2.1.1) are consistent with the charge conservation equation

$$\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{x},t) = 0. \tag{2.1.3}$$

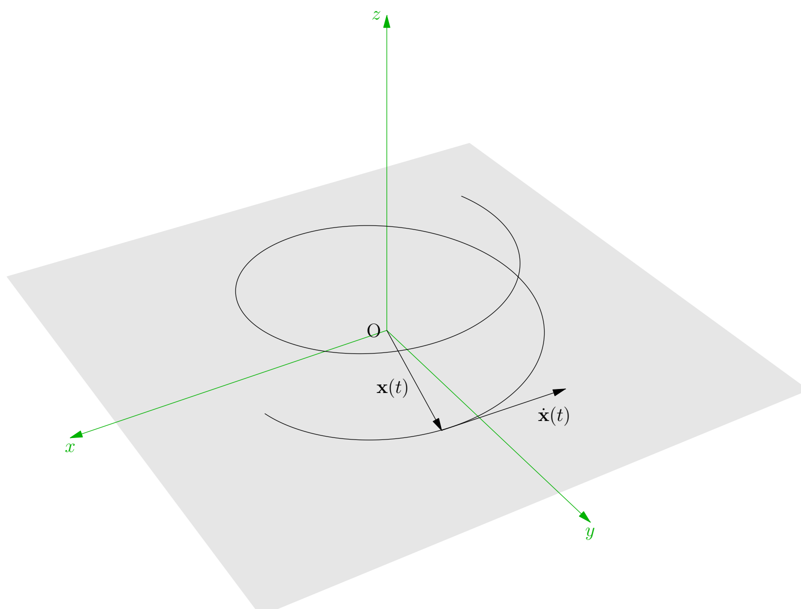

FIGURE 2.1.1. A moving charge.

Proof. The computation of the left hand side of (2.1.3) gives

$$\begin{aligned}\frac{\partial \rho(\boldsymbol{x},t)}{\partial t} &= \frac{\partial}{\partial t}\frac{q}{4\pi {r_0}^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right) \\ &= \frac{q}{4\pi {r_0}^3} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\frac{\partial}{\partial t}\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0} \\ &= -\frac{q}{4\pi {r_0}^4} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\frac{\boldsymbol{x}-\boldsymbol{x}(t)}{|\boldsymbol{x}-\boldsymbol{x}(t)|}\boldsymbol{\cdot}\dot{\boldsymbol{x}}(t),\end{aligned} \tag{2.1.4}$$

where the notation $f'(\xi)$ stands for the derivative of the function $f(\xi)$ with respect to its argument ξ. The right hand side is

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{j}(\boldsymbol{x},t) &= \boldsymbol{\nabla}\boldsymbol{\cdot}\frac{q}{4\pi {r_0}^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\dot{\boldsymbol{x}}(t) \\ &= \frac{q}{4\pi {r_0}^3} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\dot{\boldsymbol{x}}(t)\right) \\ &= \frac{q}{4\pi {r_0}^4} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{r_0}\right)\frac{\boldsymbol{x}-\boldsymbol{x}(t)}{|\boldsymbol{x}-\boldsymbol{x}(t)|}\boldsymbol{\cdot}\dot{\boldsymbol{x}}(t).\end{aligned} \tag{2.1.5}$$

This proves (2.1.3).

In most applications, the size r_0 of the charged particle is negligible compared to the distance $\boldsymbol{x}$ at which the effects of the particle are measured. We shall assume that the object generating the field is small enough that it can be treated as point-like. This approximation is satisfied more often than not, since the field source can be thought of an electron, whose radius is very small (its upper limit is around 10^{-22} m). This approximation is expressed through the condition

$$r_0 \ll |\boldsymbol{x} - \boldsymbol{x}(t)|, \tag{2.1.6a}$$

or, in other words,

$$r_0 \to 0. \tag{2.1.6b}$$

When condition (2.1.6a) is satisfied, the particle is geometrically point-like, meaning that the entire charge q is located at $\boldsymbol{x}(t)$ at all times t. From now on, we shall only consider such particles, which is to say we shall always assume condition (2.1.6a) satisfied. It can be shown that

$$\lim_{r_0 \to 0} \frac{q}{4\pi {r_0}^3} f\left(\frac{|\boldsymbol{x} - \boldsymbol{x}(t)|}{r_0}\right) = q\delta(\boldsymbol{x} - \boldsymbol{x}(t)), \tag{2.1.7}$$

where $\delta(\boldsymbol{\xi})$ is the three dimensional Dirac delta function. This is intuitively proved by noting that, if we allow the spacial distribution of the charge q to be a Gaussian-like function, condition (2.1.6a) makes this function more and more peaked around $\boldsymbol{x}(t)$ and, in the limit where $r_0 \to 0$, it becomes a delta-shaped function. Applying (2.1.7) to the equations for the density and current density (2.1.1) yields

$$\rho(\boldsymbol{x}, t) = q\delta(\boldsymbol{x} - \boldsymbol{x}(t)), \tag{2.1.8a}$$

$$\boldsymbol{j}(\boldsymbol{x}, t) = q\delta(\boldsymbol{x} - \boldsymbol{x}(t))\dot{\boldsymbol{x}}(t). \tag{2.1.8b}$$

Equations (2.1.8) represent the density and current density of a point-like charged particle. Indeed, if we think of the delta function as an infinitely peaked Gaussian function, we can easily understand how the charge q is exactly located at $\boldsymbol{x}(t)$, following exactly the trajectory of the particle.

2.2. Computation of electric and magnetic fields

In order to be able to compute the electric field $\boldsymbol{E}(\boldsymbol{x},t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ we need an expression for the electric potential $\Phi(\boldsymbol{x},t)$ and vector potential $\boldsymbol{A}(\boldsymbol{x},t)$. Once these potentials are known, the fields we are after are easily found through classical electromagnetism to be

$$\boldsymbol{E}(\boldsymbol{x},t) = -\boldsymbol{\nabla}\Phi(\boldsymbol{x},t) - \frac{1}{c}\frac{\partial \boldsymbol{A}(\boldsymbol{x},t)}{\partial t}, \tag{2.2.1a}$$

$$\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x},t). \tag{2.2.1b}$$

So we must find the electric potential $\Phi(\boldsymbol{x},t)$ and vector potential $\boldsymbol{A}(\boldsymbol{x},t)$ first. These can be computed through

$$\Phi(\boldsymbol{x},t) = \int \mathrm{d}^3\boldsymbol{x}' \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\rho\left(\boldsymbol{x}', t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right), \tag{2.2.2a}$$

$$\boldsymbol{A}(\boldsymbol{x},t) = \int \mathrm{d}^3\boldsymbol{x}' \frac{1}{c|\boldsymbol{x}-\boldsymbol{x}'|}\boldsymbol{j}\left(\boldsymbol{x}', t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right). \tag{2.2.2b}$$

These relations can be obtained once again through considerations of electromagnetic theory, but won't be proven here.

For the sake of compactness, it is useful to introduce two pieces of notation, namely

$$\boldsymbol{n}(\boldsymbol{x},t) = \frac{\boldsymbol{x}-\boldsymbol{x}(t)}{|\boldsymbol{x}-\boldsymbol{x}(t)|}, \tag{2.2.3a}$$

$$\boldsymbol{\beta}(t) = \frac{\dot{\boldsymbol{x}}(t)}{c}. \tag{2.2.3b}$$

With these definitions, $\boldsymbol{n}(\boldsymbol{x},t)$ is a unit vector (whose length is thus $|\boldsymbol{n}(\boldsymbol{x},t)| = 1$) in the direction that connects the trajectory $\boldsymbol{x}(t)$ to the observation point $\boldsymbol{x}$ and $\boldsymbol{\beta}(t)$ is the velocity of the particle in units equal to c, whose length is strictly less than 1 ($|\boldsymbol{\beta}(t)| < 1$).

Inserting the expression for $\rho(\boldsymbol{x},t)$ and $\boldsymbol{j}(\boldsymbol{x},t)$ (2.1.8) into (2.2.2) and solving the integrals yields

$$\Phi(\boldsymbol{x},t) = \left.\frac{q}{1-\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\right|_{t^*=t^*(\boldsymbol{x},t)}, \tag{2.2.4a}$$

$$\boldsymbol{A}(\boldsymbol{x},t) = \left.\frac{q}{1-\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)}\frac{\boldsymbol{\beta}(t^*)}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\right|_{t^*=t^*(\boldsymbol{x},t)}, \tag{2.2.4b}$$

where $t^* = t^*(\boldsymbol{x},t)$ is called the retarded time and is the unique solution of

$$t^* - t + \frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}{c} = 0. \tag{2.2.5}$$

It is also evident from (2.2.4) that

$$\boldsymbol{A}(\boldsymbol{x},t) = \Phi(\boldsymbol{x},t)\boldsymbol{\beta}(t^*)\big|_{t^*=t^*(\boldsymbol{x},t)}. \tag{2.2.6}$$

Proof. In order to prove expressions (2.2.4), we shall first utilize a standard delta function trick,

$$\rho(\boldsymbol{x},t) = \int \mathrm{d}t'\rho(\boldsymbol{x},t')\delta(t'-t), \tag{2.2.7}$$

where $\delta(\xi)$ is the one dimensional Dirac delta function. Using the trick above and substituting (2.1.8) into (2.2.2) yields

$$\begin{aligned}\Phi(\boldsymbol{x},t) &= \int \mathrm{d}^3\boldsymbol{x}'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\rho\left(\boldsymbol{x}',t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\ &= \int \mathrm{d}^3\boldsymbol{x}'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\int \mathrm{d}t'\rho(\boldsymbol{x}',t')\delta\left(t'-t+\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\ &= \int \mathrm{d}^3\boldsymbol{x}'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\int \mathrm{d}t' q\delta(\boldsymbol{x}'-\boldsymbol{x}(t'))\delta\left(t'-t+\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\ &= q\int \mathrm{d}t'\int \mathrm{d}^3\boldsymbol{x}'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\delta(\boldsymbol{x}'-\boldsymbol{x}(t'))\delta\left(t'-t+\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\ &= q\int \mathrm{d}t'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t')|}\delta\left(t'-t+\frac{|\boldsymbol{x}-\boldsymbol{x}(t')|}{c}\right),\end{aligned} \tag{2.2.8a}$$

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x},t) &= \int \mathrm{d}^3\boldsymbol{x}'\frac{1}{c|\boldsymbol{x}-\boldsymbol{x}'|}\boldsymbol{j}\left(\boldsymbol{x}',t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\ &= \int \mathrm{d}^3\boldsymbol{x}'\frac{1}{c|\boldsymbol{x}-\boldsymbol{x}'|}\int \mathrm{d}t'\boldsymbol{j}(\boldsymbol{x}',t')\delta\left(t'-t+\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\ &= \int \mathrm{d}^3\boldsymbol{x}'\frac{1}{c|\boldsymbol{x}-\boldsymbol{x}'|}\int \mathrm{d}t' q\delta(\boldsymbol{x}'-\boldsymbol{x}(t'))\dot{\boldsymbol{x}}(t')\delta\left(t'-t+\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\ &= \frac{q}{c}\int \mathrm{d}t'\int \mathrm{d}^3\boldsymbol{x}'\frac{\dot{\boldsymbol{x}}(t')}{|\boldsymbol{x}-\boldsymbol{x}'|}\delta(\boldsymbol{x}'-\boldsymbol{x}(t'))\delta\left(t'-t+\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)\end{aligned} \tag{2.2.8b}$$

$$= \frac{q}{c} \int \mathrm{d}t' \frac{\dot{\boldsymbol{x}}(t')}{|\boldsymbol{x} - \boldsymbol{x}(t')|} \delta \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t')|}{c} \right).$$

We shall now compute the integrals in t'. To do that, it is necessary to find the value of t' for which the argument of the delta function inside the integral vanishes. This is equivalent to finding the solutions t^* to equation (2.2.5). The quantity $|\boldsymbol{x} - \boldsymbol{x}(t^*)|/c$ is the time required for a light signal to travel from $\boldsymbol{x}(t^*)$ to $\boldsymbol{x}$. Since $|\dot{\boldsymbol{x}}(t')| < c$ and $t^* \leq t$, equation (2.2.5) has one and only one solution $t^* = t^*(\boldsymbol{x}, t)$. Indeed, a spherical front moving from infinity at time $-\infty$ converging to $\boldsymbol{x}$ at time t sweeps the whole space at speed c, therefore it will eventually meet the point-like charge at least once. Since $|\dot{\boldsymbol{x}}(t')| < c$, the front intersects the trajectory of the point charge only once. The front and the charge meet at time t^*, which satisfies precisely (2.2.5).

Now that we have established the uniqueness of t^*, we can solve the integrals. Although it won't be proven here, it can be shown that

$$\delta(f(t)) = \frac{\delta(t - t_0)}{|f'(t_0)|}, \tag{2.2.9}$$

if $f(t_0) = 0$, $f'(t_0) \neq 0$ and t_0 is the only value of t with such property. Therefore

$$\begin{aligned}
\delta \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t')|}{c} \right) &= \left| \frac{\mathrm{d}}{\mathrm{d}t'} \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t')|}{c} \right) \right|^{-1} \delta(t' - t^*(\boldsymbol{x}, t)) \qquad (2.2.10) \\
&= \left| 1 - \frac{\dot{\boldsymbol{x}}(t')}{c} \boldsymbol{\cdot} \frac{\boldsymbol{x} - \boldsymbol{x}(t')}{|\boldsymbol{x} - \boldsymbol{x}(t')|} \right|^{-1} \delta(t' - t^*(\boldsymbol{x}, t)) \\
&= \frac{1}{1 - \boldsymbol{\beta}(t') \boldsymbol{\cdot} \boldsymbol{n}(\boldsymbol{x}, t')} \delta(t' - t^*(\boldsymbol{x}, t)),
\end{aligned}$$

where the absolute value has been dropped in the last step because both $\boldsymbol{\beta}(t')$ and $\boldsymbol{n}(\boldsymbol{x}, t')$ both have lengths not exceeding 1. It is now straightforward to prove (2.2.4) by substituting (2.2.10) into (2.2.8).

Having an expression for the potentials allows us to calculate the electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ starting from (2.2.1), obtaining

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{q}{(1 - \boldsymbol{\beta}(t^*) \boldsymbol{\cdot} \boldsymbol{n}(\boldsymbol{x}, t^*))^3} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|^2} \tag{2.2.11a}$$

$$\left\{ \left(1 - |\boldsymbol{\beta}(t^*)^2|\right) \left(\boldsymbol{n}(\boldsymbol{x}, t^*) - \boldsymbol{\beta}(t^*)\right) + \frac{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}{c} \boldsymbol{n}(\boldsymbol{x}, t^*) \times \left[\left(\boldsymbol{n}(\boldsymbol{x}, t^*) - \boldsymbol{\beta}(t^*)\right) \times \dot{\boldsymbol{\beta}}(t^*)\right] \right\}\Bigg|_{t^*=t^*(\boldsymbol{x},t)},$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \frac{q}{(1 - \boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*))^3} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|^2} \boldsymbol{n}(\boldsymbol{x}, t^*) \times \left\{ \left(1 - |\boldsymbol{\beta}(t^*)^2|\right) \left(\boldsymbol{n}(\boldsymbol{x}, t^*) - \boldsymbol{\beta}(t^*)\right) + \frac{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}{c} \boldsymbol{n}(\boldsymbol{x}, t^*) \times \left[\left(\boldsymbol{n}(\boldsymbol{x}, t^*) - \boldsymbol{\beta}(t^*)\right) \times \dot{\boldsymbol{\beta}}(t^*)\right] \right\}\Bigg|_{t^*=t^*(\boldsymbol{x},t)}. \tag{2.2.11b}$$

Proof. There is more than one way of proving expressions (2.2.11). Here, we will prove it through variational calculus. Since this proof is rather complicated, we shall divide it in steps for the sake of clarity.

Step 1. First of all, we need to compute the electric potential's variation under infinitesimal variations of $\boldsymbol{x}$ and t. Therefore, we have to apply the variational operator δ (beware not to confuse it with the delta function) to the electric potential (2.2.4a). In order to keep the notation slightly more compact, we shall avoid repeating that $t^* = t^*(\boldsymbol{x}, t)$ at the end of every expression if there is no ambiguity in the meaning of t^*.

$$\begin{aligned} \delta\Phi(\boldsymbol{x}, t) &= \delta \left[\frac{q}{1 - \boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|} \right] \\ &= q \Bigg\{ - \frac{1}{1 - \boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|^2} \frac{\boldsymbol{x} - \boldsymbol{x}(t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|} \cdot \delta(\boldsymbol{x} - \boldsymbol{x}(t^*)) \\ &\quad + \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|} \frac{1}{(1 - \boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*))^2} \delta(\boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)) \Bigg\}. \end{aligned} \tag{2.2.12}$$

Step 2. Now, we need to compute the two variations in (2.2.12). The first one is easily carried out, since we can evaluate the variation of the sum as a sum of variations; then, keeping in mind that $\delta\boldsymbol{\xi}(t) = (\partial\boldsymbol{\xi}(t)/\partial t)\,\delta t$, we find

$$\begin{aligned} \delta(\boldsymbol{x} - \boldsymbol{x}(t^*)) &= \delta\boldsymbol{x} - \dot{\boldsymbol{x}}(t^*)\delta t^* \\ &= \delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*. \end{aligned} \tag{2.2.13}$$

The second one is slightly more complicated. For this step it is necessary to know how to deal with the variation of a unit vector. Given $\boldsymbol{n} = \boldsymbol{r}/r$ any vector of length 1, where $r = |\boldsymbol{r}|$ it can be shown that

$$\delta \boldsymbol{n} = \frac{\mathbb{1} - \boldsymbol{n}\boldsymbol{n}}{r} \cdot \delta \boldsymbol{r}, \tag{2.2.14}$$

where the product $\boldsymbol{n}\boldsymbol{n}$ represents the dyadic product of two vectors, which is known to yield a matrix. In fact,

$$\begin{aligned}
\delta \boldsymbol{n} = \delta \left(\frac{\boldsymbol{r}}{r}\right) &= \frac{\delta \boldsymbol{r}}{r} + \boldsymbol{r}\delta\left(\frac{1}{r}\right) \\
&= \frac{\delta \boldsymbol{r}}{r} - \boldsymbol{r}\frac{1}{r^2}\delta|\boldsymbol{r}| \\
&= \frac{\delta \boldsymbol{r}}{r} - \frac{\boldsymbol{r}}{r^2}\left(\frac{\boldsymbol{r}}{r}\cdot\delta\boldsymbol{r}\right) \\
&= \frac{\delta \boldsymbol{r}}{r} - \left(\frac{\boldsymbol{r}\boldsymbol{r}}{r^2}\right)\cdot\frac{\delta\boldsymbol{r}}{r} \\
&= \mathbb{1}\cdot\frac{\delta \boldsymbol{r}}{r} - (\boldsymbol{n}\boldsymbol{n})\cdot\frac{\delta\boldsymbol{r}}{r} \\
&= (\mathbb{1} - \boldsymbol{n}\boldsymbol{n})\cdot\frac{\delta\boldsymbol{r}}{r},
\end{aligned} \tag{2.2.15}$$

which proves (2.2.14). Therefore, using a Leibniz-like rule and result (2.2.14), it is easy to see that

$$\begin{aligned}
\delta(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)) &= \boldsymbol{\beta}(t^*)\cdot\delta\boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{n}(\boldsymbol{x},t^*)\cdot\delta\boldsymbol{\beta}(t^*) \\
&= \boldsymbol{\beta}(t^*)\cdot\frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\cdot\delta(\boldsymbol{x} - \boldsymbol{x}(t^*)) \\
&\quad + \boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\delta t^* \\
&= \boldsymbol{\beta}(t^*)\cdot\frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\cdot(\delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*) \\
&\quad + \boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\delta t^*,
\end{aligned} \tag{2.2.16}$$

where in the last step we have made use of (2.2.13) once again. Now we have an expression for $\delta\Phi(\boldsymbol{x}, t)$ in terms of the variation of its arguments $\delta\boldsymbol{x}$ and δt^*,

$$\begin{aligned}
\delta\Phi(\boldsymbol{x},t) = q\Bigg\{ &-\frac{1}{1 - \boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|^2}\frac{\boldsymbol{x} - \boldsymbol{x}(t^*)}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\cdot\delta(\boldsymbol{x} - \boldsymbol{x}(t^*)) \\
&+ \frac{1}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}\frac{1}{(1 - \boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^2}\delta(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))\Bigg\}
\end{aligned} \tag{2.2.17}$$

$$= \frac{q}{1-\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)} \frac{1}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|} \Bigg\{ -\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\boldsymbol{\cdot}(\delta\boldsymbol{x}-c\boldsymbol{\beta}(t^*)\delta t^*)$$
$$+\frac{1}{1-\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)}\Bigg[\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\frac{\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\boldsymbol{\cdot}(\delta\boldsymbol{x}-c\boldsymbol{\beta}(t^*)\delta t^*)$$
$$+\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\delta t^*\Bigg]\Bigg\},$$

obtained by substituting (2.2.13) and (2.2.16) into (2.2.12). In order to make this long expression easier to manipulate, we shall define

$$\alpha = 1-\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*), \tag{2.2.18a}$$

$$r = |\boldsymbol{x}-\boldsymbol{x}(t^*)|. \tag{2.2.18b}$$

With these pieces of notation, (2.2.4a) becomes

$$\Phi(\boldsymbol{x},t) = \frac{q}{\alpha r}, \tag{2.2.19}$$

and (2.2.17) becomes

$$\delta\Phi(\boldsymbol{x},t) = \frac{q}{\alpha r}\Bigg\{-\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r}\boldsymbol{\cdot}(\delta\boldsymbol{x}-c\boldsymbol{\beta}(t^*)\delta t^*) \tag{2.2.20}$$
$$+\frac{1}{\alpha}\Bigg[\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\frac{\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{r}\boldsymbol{\cdot}(\delta\boldsymbol{x}-c\boldsymbol{\beta}(t^*)\delta t^*)$$
$$+\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\delta t^*\Bigg]\Bigg\}.$$

Step 3. Then, it is possible to evaluate the gradient $\boldsymbol{\nabla}\Phi(\boldsymbol{x},t)$ by setting $\delta t = 0$ and formally "dividing" (2.2.20) by $\delta\boldsymbol{x}$. Indeed, one should recall that $\boldsymbol{\nabla}\Phi(\boldsymbol{x},t)\boldsymbol{\cdot}\delta\boldsymbol{x} = \delta\Phi(\boldsymbol{x},t)|_{t=\text{const}}$. However, (2.2.20) is expressed in terms of δt^*, not δt. It is necessary to find the relation between these two differentials. In order to achieve this, we can note that the retarded time is implicitly defined by (2.2.5). Therefore, applying the operator δ to differentiate expression (2.2.5) yields

$$0 = \delta\left(t^*-t+\frac{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}{c}\right) \tag{2.2.21}$$
$$= \delta t^*\left[1-\frac{\dot{\boldsymbol{x}}(t^*)}{c}\boldsymbol{\cdot}\frac{\boldsymbol{x}-\boldsymbol{x}(t^*)}{|\boldsymbol{x}-\boldsymbol{x}(t^*)|}\right]-\delta t+\delta\boldsymbol{x}\boldsymbol{\cdot}\frac{\boldsymbol{x}-\boldsymbol{x}(t^*)}{c|\boldsymbol{x}-\boldsymbol{x}(t^*)|}$$
$$= \alpha\delta t^*-\delta t+\frac{1}{c}\delta\boldsymbol{x}\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*).$$

Expression (2.2.21) tells us that when δt vanishes, δt^* does not. More precisely, setting $\delta t = 0$ into (2.2.21) yields

$$\delta t^* = -\frac{\delta \boldsymbol{x} \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha c}. \tag{2.2.22}$$

This is exactly the relation between the differentials we were looking for. So, setting $\delta t^* = -\left(\delta \boldsymbol{x} \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)\right) / (\alpha c)$ is equivalent to setting $\delta t = 0$. Keeping this in mind, (2.2.20) becomes

$$\begin{aligned}
\boldsymbol{\nabla}\Phi(\boldsymbol{x}, t) \cdot \delta \boldsymbol{x} = {} & \frac{q}{\alpha r}\Bigg\{ -\frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{r} \cdot \left(\delta \boldsymbol{x} + c\boldsymbol{\beta}(t^*) \frac{\delta \boldsymbol{x} \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha c} \right) \\
& + \frac{1}{\alpha}\Bigg[\boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)}{r} \cdot \left(\delta \boldsymbol{x} + c\boldsymbol{\beta}(t^*) \frac{\delta \boldsymbol{x} \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha c} \right) \\
& - \boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \dot{\boldsymbol{\beta}}(t^*) \frac{\delta \boldsymbol{x} \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha c} \Bigg] \Bigg\} \\
= {} & \frac{q}{\alpha r}\Bigg\{ -\frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{r} - \frac{1}{r}\boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*) \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha} \\
& + \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha r} \\
& + \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha r} \cdot \boldsymbol{\beta}(t^*) \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha} \\
& - \frac{\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \dot{\boldsymbol{\beta}}(t^*)}{\alpha} \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha c} \Bigg\} \cdot \delta \boldsymbol{x}.
\end{aligned} \tag{2.2.23}$$

Step 3. It is now possible to evaluate the gradient $\boldsymbol{\nabla}\Phi(\boldsymbol{x}, t)$. Since both sides of (2.2.23) show the term $\delta \boldsymbol{x}$, is is clear that

$$\begin{aligned}
\boldsymbol{\nabla}\Phi(\boldsymbol{x}, t) = {} & \frac{q}{\alpha r}\Bigg\{ -\frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{r} - \frac{1}{r}\boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*) \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha} \\
& + \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha r} \\
& + \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha r} \cdot \boldsymbol{\beta}(t^*) \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha} \\
& - \frac{\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \dot{\boldsymbol{\beta}}(t^*)}{\alpha} \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha c} \Bigg\}.
\end{aligned} \tag{2.2.24}$$

In this step, we will algebraically manipulate (2.2.24) in order to simplify its expression a little bit. Firstly, we need to deal with the dyadic products. So

$$\begin{aligned}
\boldsymbol{\beta}(t^*) \cdot \left(\mathbb{1} - \boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*) \right) & = \boldsymbol{\beta}(t^*) \cdot \mathbb{1} - \boldsymbol{\beta}(t^*) \cdot \left(\boldsymbol{n}(\boldsymbol{x}, t^*)\boldsymbol{n}(\boldsymbol{x}, t^*) \right) \\
& = \boldsymbol{\beta}(t^*) - \left(\boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*) \right) \boldsymbol{n}(\boldsymbol{x}, t^*)
\end{aligned} \tag{2.2.25a}$$

$$= \boldsymbol{\beta}(t^*) - (1-\alpha)\,\boldsymbol{n}(\boldsymbol{x},t^*),$$

and, therefore,

$$\begin{aligned}\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\left(\mathbb{1} - \boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\right)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*) &= \boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\left[\boldsymbol{\beta}(t^*) - (1-\alpha)\,\boldsymbol{n}(\boldsymbol{x},t^*)\right] \\ &= \boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*) - (1-\alpha)\left(\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*)\right) \\ &= |\boldsymbol{\beta}(t^*)|^2 - (1-\alpha)^2\,.\end{aligned} \tag{2.2.25b}$$

Inserting (2.2.25) into (2.2.24) yields

$$\begin{aligned}\boldsymbol{\nabla}\Phi(\boldsymbol{x},t) &= \frac{q}{\alpha r}\Bigg\{-\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r} - \frac{1}{r}\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha} \\ &\quad + \frac{\boldsymbol{\beta}(t^*) - (1-\alpha)\,\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r} \\ &\quad + \frac{|\boldsymbol{\beta}(t^*)|^2 - (1-\alpha)^2}{\alpha r}\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha} \\ &\quad - \frac{\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)}{\alpha}\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha c}\Bigg\} \\ &= -\frac{q}{\alpha^3 r^2}\Bigg\{\alpha^2\boldsymbol{n}(\boldsymbol{x},t^*) + \alpha\,(1-\alpha)\,\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad - \alpha\boldsymbol{\beta}(t^*) + \alpha\,(1-\alpha)\,\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) + (1-\alpha)^2\,\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad + \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{n}(\boldsymbol{x},t^*)\Bigg\} \\ &= -\frac{q}{\alpha^3 r^2}\Bigg\{\alpha^2\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad - \alpha\boldsymbol{\beta}(t^*) + 2\alpha\,(1-\alpha)\,\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) + (1-\alpha)^2\,\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\quad + \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{n}(\boldsymbol{x},t^*)\Bigg\}.\end{aligned} \tag{2.2.26}$$

We finally found the expression for the gradient.

Step 4. The purpose of this step is to calculate the partial derivative of the vector potential $\boldsymbol{A}(\boldsymbol{x},t)$ with respect to time. We shall approach this problem by first evaluating the differential of $\boldsymbol{A}(\boldsymbol{x},t)$. If one recalls (2.2.6), then it is clear that applying the differential operator δ yields

$$\begin{aligned}\delta \boldsymbol{A}(\boldsymbol{x},t) &= \delta\left(\Phi(\boldsymbol{x},t)\boldsymbol{\beta}(t^*)\right) \\ &= \Phi(\boldsymbol{x},t)\delta\boldsymbol{\beta}(t^*) + \boldsymbol{\beta}(t^*)\delta\Phi(\boldsymbol{x},t) \\ &= \Phi(\boldsymbol{x},t)\dot{\boldsymbol{\beta}}(t^*)\delta t^* + \boldsymbol{\beta}(t^*)\delta\Phi(\boldsymbol{x},t).\end{aligned} \tag{2.2.27}$$

Now, we have already carried out the calculation for the quantity $\delta\Phi(\boldsymbol{x},t)$, which is given by (2.2.20). Expression (2.2.27) can be used to compute $\partial\boldsymbol{A}(\boldsymbol{x},t)/\partial t^*$. However, $\partial\boldsymbol{A}(\boldsymbol{x},t)/\partial t$ is the quantity that we are after. Therefore, we need to find a relation between the two. This relation can be derived once we know the relationship between the differentials of time t and the retarded time t^*. From relation (2.2.21), setting consecutively $\delta\boldsymbol{x} = \boldsymbol{0}$ and $\delta t^* = 0$, one finds that

$$\frac{\partial t^*}{\partial t} = \frac{1}{\alpha}, \tag{2.2.28a}$$

$$\boldsymbol{\nabla} t^* = -\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha c}. \tag{2.2.28b}$$

The second one (2.2.28b) will be used later, while the first one (2.2.28a) is the relationship between the differentials of time t and the retarded time t^* we were looking for. Using this it is easy to see that

$$\begin{aligned}\frac{\partial\boldsymbol{A}(\boldsymbol{x},t)}{\partial t} &= \frac{\partial\boldsymbol{A}(\boldsymbol{x},t)}{\partial t^*}\frac{\partial t^*}{\partial t} \\ &= \frac{\partial\boldsymbol{A}(\boldsymbol{x},t)}{\partial t^*}\frac{1}{\alpha}.\end{aligned} \tag{2.2.29}$$

Next, we have to compute the variation of $\Phi(\boldsymbol{x},t)$ when $\delta\boldsymbol{x} = \boldsymbol{0}$. Recalling that $(\partial\Phi(\boldsymbol{x},t)/\partial t)\delta t = \delta\Phi(\boldsymbol{x},t)|_{\boldsymbol{x}=\text{const}}$, we can set $\delta\boldsymbol{x} = \boldsymbol{0}$ in (2.2.20). This results into

$$\begin{aligned}\delta\Phi(\boldsymbol{x},t) = \frac{q}{\alpha r}\Bigg\{ &-\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r}\cdot(-c\boldsymbol{\beta}(t^*)\delta t^*) \\ &+\frac{1}{\alpha}\Big[\boldsymbol{\beta}(t^*)\cdot\frac{\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)}{r}\cdot(-c\boldsymbol{\beta}(t^*)\delta t^*) \\ &+\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\delta t^*\Big]\Bigg\} \\ = \frac{q}{\alpha r}\Bigg\{ &\frac{c}{r}\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\boldsymbol{\beta}(t^*) \\ &-\frac{c}{\alpha r}\boldsymbol{\beta}(t^*)\cdot\left(\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\right)\cdot\boldsymbol{\beta}(t^*)\end{aligned} \tag{2.2.30}$$

$$+\frac{1}{\alpha}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\Big\}\delta t^*.$$

We now have all the ingredients needed to compute $\partial\boldsymbol{A}(\boldsymbol{x},t)/\partial t$. Indeed, we substitute into (2.2.27) the expression for the potential $\Phi(\boldsymbol{x},t)$ from (2.2.19) and for its differential $\delta\Phi(\boldsymbol{x},t)$ from (2.2.30), then we divide by δt^* and keep in mind the functional relationship between the differentials (2.2.29), obtaining

$$\begin{aligned}\frac{\partial\boldsymbol{A}(\boldsymbol{x},t)}{\partial t} &= \frac{1}{\alpha}\Big\{\frac{q}{\alpha r}\dot{\boldsymbol{\beta}}(t^*)+\boldsymbol{\beta}(t^*)\frac{q}{\alpha r}\Big[\frac{c}{r}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*) \\ &\quad -\frac{c}{\alpha r}\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\left(\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\right)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*)+\frac{1}{\alpha}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\Big]\Big\} \\ &= -\frac{qc}{\alpha^3r^2}\Big\{-\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*)-\frac{\alpha r}{c}\boldsymbol{\beta}\Big[\frac{c}{r}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*) \\ &\quad -\frac{c}{\alpha r}\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\left(\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\right)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*)+\frac{1}{\alpha}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\Big]\Big\},\end{aligned} \tag{2.2.31}$$

where in the last step, we have multiplied and divided by $-c/(\alpha r)$ for reasons that will become clear later. (2.2.31) can be simplified even further by computing the dyadic product, which we have already done through (2.2.25b). Substituting this in, (2.2.31) becomes

$$\begin{aligned}\frac{\partial\boldsymbol{A}(\boldsymbol{x},t)}{\partial t} &= -\frac{qc}{\alpha^3r^2}\Big\{-\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*)-\frac{\alpha r}{c}\boldsymbol{\beta}(t^*)\Big[\frac{c}{r}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*) \\ &\quad -\frac{c}{\alpha r}\left(|\boldsymbol{\beta}(t^*)|^2-(1-\alpha)^2\right)+\frac{1}{\alpha}\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\Big]\Big\} \\ &= -\frac{qc}{\alpha^3r^2}\Big\{-\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*)-\alpha\left(\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\boldsymbol{\beta}(t^*)\right)\boldsymbol{\beta}(t^*) \\ &\quad +|\boldsymbol{\beta}(t^*)|^2\boldsymbol{\beta}(t^*)-(1-\alpha)^2\boldsymbol{\beta}(t^*)-\frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{\beta}(t^*)\Big\}.\end{aligned} \tag{2.2.32}$$

Step 5. It is now the appropriate time to substitute back in $\alpha = 1-\boldsymbol{\beta}(t^*)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x},t^*)$. However, we shall only do that for the instances of α appearing in the curly brackets. First, let us commence with the gradient from (2.2.26):

$$\begin{aligned}\boldsymbol{\nabla}\Phi(\boldsymbol{x},t) = -\frac{q}{\alpha^3r^2}\Big\{&\alpha^2\boldsymbol{n}(\boldsymbol{x},t^*)-\alpha\boldsymbol{\beta}(t^*) \\ &+2\alpha\boldsymbol{n}(\boldsymbol{x},t^*)-2\alpha^2\boldsymbol{n}(\boldsymbol{x},t^*)-|\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) \\ &+\boldsymbol{n}(\boldsymbol{x},t^*)-2\alpha\boldsymbol{n}(\boldsymbol{x},t^*)+\alpha^2\boldsymbol{n}(\boldsymbol{x},t^*)+\frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\boldsymbol{\cdot}\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{n}(\boldsymbol{x},t^*)\Big\}\end{aligned} \tag{2.2.33}$$

$$
\begin{aligned}
&= -\frac{q}{\alpha^3 r^2}\Big\{ -\alpha\boldsymbol{\beta}(t^*) - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) \\
&\quad + \boldsymbol{n}(\boldsymbol{x},t^*) + \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{n}(\boldsymbol{x},t^*)\Big\} \\
&= -\frac{q}{\alpha^3 r^2}\Big\{ -\boldsymbol{\beta}(t^*) + \left(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)\boldsymbol{\beta}(t^*) \\
&\quad - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) \\
&\quad + \boldsymbol{n}(\boldsymbol{x},t^*) + \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{n}(\boldsymbol{x},t^*)\Big\}.
\end{aligned}
$$

Now, let us do the same thing with the derivative of the vector potential with respect to time from (2.2.32):

$$
\begin{aligned}
\frac{\partial \boldsymbol{A}(\boldsymbol{x},t)}{\partial t} &= -\frac{qc}{\alpha^3 r^2}\Big\{ -\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*) - \alpha\,(1-\alpha)\,\boldsymbol{\beta}(t^*) \\
&\quad + |\boldsymbol{\beta}(t^*)|^2\boldsymbol{\beta}(t^*) - (1-\alpha)^2\,\boldsymbol{\beta}(t^*) - \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{\beta}(t^*)\Big\} \qquad (2.2.34) \\
&= -\frac{qc}{\alpha^3 r^2}\Big\{ -\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*) - (1-\alpha)\,\boldsymbol{\beta}(t^*) \\
&\quad + |\boldsymbol{\beta}(t^*)|^2\boldsymbol{\beta}(t^*) - \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{\beta}(t^*)\Big\} \\
&= -\frac{qc}{\alpha^3 r^2}\Big\{ |\boldsymbol{\beta}(t^*)|^2\boldsymbol{\beta}(t^*) - \left(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)\boldsymbol{\beta}(t^*) \\
&\quad + \frac{r}{c}\left[\left(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)\dot{\boldsymbol{\beta}}(t^*) - \dot{\boldsymbol{\beta}}(t^*) - \left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{\beta}(t^*)\right]\Big\}.
\end{aligned}
$$

We finally found an expression for the derivative of the vector potential with respect to time $\partial \boldsymbol{A}(\boldsymbol{x},t)/\partial t$.

Step 6. Now we can clearly see the reason why we decided to keep that strange multiplying factor: by (2.2.1a) we have the final expression for the electric field, and both the quantities being summed have the same multiplying factor, making the summation much more straightforward. Indeed, inserting (2.2.33) and (2.2.34) into (2.2.1a) shows that

$$
\begin{aligned}
\boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{\alpha^3 r^2}\Big\{ &\left(1-|\boldsymbol{\beta}(t^*)|^2\right)\boldsymbol{n}(\boldsymbol{x},t^*) - \left(1-|\boldsymbol{\beta}(t^*)|^2\right)\boldsymbol{\beta}(t^*) \qquad (2.2.35) \\
&+ \frac{r}{c}\Big[\left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{n}(\boldsymbol{x},t^*) - \dot{\boldsymbol{\beta}}(t^*)
\end{aligned}
$$

$$\begin{aligned} &+ \left(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)\dot{\boldsymbol{\beta}}(t^*) - \Big(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Big)\,\boldsymbol{\beta}(t^*)\Big]\Big\} \\ = \frac{q}{\alpha^3 r^2}\Big\{ &\left(1-|\boldsymbol{\beta}(t^*)|^2\right)\left(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\right) \qquad (2.2.36) \\ &+ \frac{r}{c}\Big[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Big)\,\boldsymbol{n}(\boldsymbol{x},t^*) - \dot{\boldsymbol{\beta}}(t^*) \\ &+ \left(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)\dot{\boldsymbol{\beta}}(t^*) - \Big(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Big)\,\boldsymbol{\beta}(t^*)\Big]\Big\}. \end{aligned}$$

This expression seems very complicated since it shows a lot of terms. However, the term in square brackets being multiplied by r/c can be simplified into a much more recognizable form: a triple cross product. We can make use of a very well known property for which

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b})\,. \qquad (2.2.37)$$

In our particular case, $\boldsymbol{b}$ actually is $\boldsymbol{a}-\boldsymbol{b}$. Therefore, making use of (2.2.37), one has

$$\begin{aligned} \boldsymbol{a}\times\left[(\boldsymbol{a}-\boldsymbol{b})\times\boldsymbol{c}\right] &= (\boldsymbol{a}-\boldsymbol{b})\,(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}\left[\boldsymbol{a}\cdot(\boldsymbol{a}-\boldsymbol{b})\right] \qquad (2.2.38) \\ &= (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - |\boldsymbol{a}|^2\boldsymbol{c} + (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}. \end{aligned}$$

In our case, $\boldsymbol{a} = \boldsymbol{n}(\boldsymbol{x},t^*)$, $\boldsymbol{b} = \boldsymbol{\beta}(t^*)$ and $\boldsymbol{c} = \dot{\boldsymbol{\beta}}(t^*)$. Note that $|\boldsymbol{a}|^2 = 1$, since $|\boldsymbol{n}(\boldsymbol{x},t^*)|^2 = 1$. Keeping this in mind, (2.2.35) becomes

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{\alpha^3 r^2}\Big\{ &\left(1-|\boldsymbol{\beta}(t^*)|^2\right)\left(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\right) \qquad (2.2.39) \\ &+ \frac{r}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\times\Big[\left(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*)\right)\times\dot{\boldsymbol{\beta}}(t^*)\Big]\,. \end{aligned}$$

Equation (2.2.11a) is proved once we recall the definitions (2.2.18) of α and r.

Step 6. The last part of this proof is the computation of the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$. We don't need to do much work, because relations (2.2.1) show that there's a connection between the electric field $\boldsymbol{E}(\boldsymbol{x},t)$ which we have just calculated and the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$. In fact,

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{x},t) &= \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x},t) \qquad (2.2.40) \\ &= \boldsymbol{\nabla}\times\left(\Phi(\boldsymbol{x},t)\boldsymbol{\beta}(t^*)\right) \\ &= \Phi(\boldsymbol{x},t)\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*) - \boldsymbol{\beta}(t^*)\times\boldsymbol{\nabla}\Phi(\boldsymbol{x},t), \end{aligned}$$

where we used (2.2.6) in the second to last step and, in the last one, a well know calculus relation which allowed us to compute the curl of a vector factorized in a scalar function multiplied by a vectorial function.

In (2.2.40), the only factor we still need to calculate is $\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*)$. We can tackle this by first considering the mixed product

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*), \tag{2.2.41}$$

where $\boldsymbol{u}$ is a constant vector. By a well know triple product property, one has

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*) &= \boldsymbol{u}\times\boldsymbol{\nabla}\cdot\boldsymbol{\beta}(t^*) \\ &= \boldsymbol{u}\times\boldsymbol{\nabla}t^*\cdot\dot{\boldsymbol{\beta}}(t^*) \\ &= \boldsymbol{u}\cdot\boldsymbol{\nabla}t^*\times\dot{\boldsymbol{\beta}}(t^*). \end{aligned} \tag{2.2.42}$$

Therefore, since $\boldsymbol{u}$ is arbitrary,

$$\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*) = -\dot{\boldsymbol{\beta}}(t^*)\times\boldsymbol{\nabla}t^*. \tag{2.2.43}$$

The expression for the gradient $\boldsymbol{\nabla}t^*$ has the form (2.2.28b).

Now we can finally insert everything we know into (2.2.40) and find the answer we were looking for: the electric potential $\Phi(\boldsymbol{x},t)$ is given by (2.2.19), the curl $\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*)$ is given by (2.2.43), the gradient $\boldsymbol{\nabla}\Phi(\boldsymbol{x},t)$ is given by (2.2.33). Substituting it all into (2.2.40) yields

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{x},t) &= \Phi(\boldsymbol{x},t)\boldsymbol{\nabla}\times\boldsymbol{\beta}(t^*) - \boldsymbol{\beta}(t^*)\times\boldsymbol{\nabla}\Phi(\boldsymbol{x},t) \\ &= \frac{q}{\alpha r}\dot{\boldsymbol{\beta}}(t^*)\times\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha c} + \boldsymbol{\beta}(t^*)\times\frac{q}{\alpha^3 r^2}\bigg\{ -\boldsymbol{\beta}(t^*) + (\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))\,\boldsymbol{\beta}(t^*) \\ &\qquad - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{n}(\boldsymbol{x},t^*) + \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{n}(\boldsymbol{x},t^*)\bigg\} \\ &= \frac{q}{\alpha^3 r^2}\bigg\{\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*)\times\boldsymbol{n}(\boldsymbol{x},t^*) - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{\beta}(t^*)\times\boldsymbol{n}(\boldsymbol{x},t^*) \\ &\qquad + \boldsymbol{\beta}(t^*)\times\boldsymbol{n}(\boldsymbol{x},t^*) + \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{\beta}(t^*)\times\boldsymbol{n}(\boldsymbol{x},t^*)\bigg\} \\ &= \frac{q}{\alpha^3 r^2}\boldsymbol{n}(\boldsymbol{x},t^*)\times\bigg\{ -\frac{\alpha r}{c}\dot{\boldsymbol{\beta}}(t^*) + |\boldsymbol{\beta}(t^*)|^2\boldsymbol{\beta}(t^*) \\ &\qquad - \boldsymbol{\beta}(t^*) - \frac{r}{c}\left(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\right)\boldsymbol{\beta}(t^*)\bigg\} \end{aligned} \tag{2.2.44}$$

$$= \frac{q}{\alpha^3 r^2}\boldsymbol{n}(\boldsymbol{x},t^*)\times\Bigg\{|\boldsymbol{\beta}(t^*)|^2\boldsymbol{\beta}(t^*) - \boldsymbol{\beta}(t^*) + \frac{r}{c}\left[-\dot{\boldsymbol{\beta}}(t^*) + \left(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)\dot{\boldsymbol{\beta}}(t^*) - \Big(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Big)\boldsymbol{\beta}(t^*)\right]\Bigg\},$$

where in the last step we have made use of the fact that $\alpha = 1 - \boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)$. If one compares this result with what we found for the electric field $\boldsymbol{E}(\boldsymbol{x},t)$ in (2.2.35), one cannot fail to notice that the two expressions are strikingly similar. Inside the curly bracket in (2.2.44) every single term appears to be present in (2.2.35) too except the ones which are proportional to $\boldsymbol{n}(\boldsymbol{x},t^*)$. Furthermore, the factors inside the curly brackets in (2.2.44) are all multiplied by $\boldsymbol{n}(\boldsymbol{x},t^*)$ through a cross product. Since the cross product of a vector with itself vanishes identically ($\boldsymbol{a}\times\boldsymbol{a} \equiv \boldsymbol{0}$), we can add whatever quantity we want inside the curly bracket in (2.2.44) without changing the result as long as that quantity is proportional to $\boldsymbol{n}(\boldsymbol{x},t^*)$. As a consequence, we can say that

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x},t) = \frac{q}{\alpha^3 r^2}\boldsymbol{n}(\boldsymbol{x},t^*)\times\Bigg\{ &\left(1-|\boldsymbol{\beta}(t^*)|^2\right)\boldsymbol{n}(\boldsymbol{x},t^*) - \left(1-|\boldsymbol{\beta}(t^*)|^2\right)\boldsymbol{\beta}(t^*) \\ &+ \frac{r}{c}\Bigg[\Big(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Big)\boldsymbol{n}(\boldsymbol{x},t^*) - \dot{\boldsymbol{\beta}}(t^*) \\ &+ \left(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)\dot{\boldsymbol{\beta}}(t^*) - \Big(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Big)\boldsymbol{\beta}(t^*)\Bigg]\Bigg\},\end{aligned} \tag{2.2.45}$$

since all the factors that have been added are proportional to $\boldsymbol{n}(\boldsymbol{x},t^*)$. This proves (2.2.11b) and concludes the proof.

The expressions (2.2.11) for the electric field $\boldsymbol{E}(\boldsymbol{x},t)$ and the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ are rather long and complicated, and so is the proof leading up to those results. However, the physical implications of such relations are outstanding and teach us a lot about how nature behaves. First of all, we note that

$$\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{n}(\boldsymbol{x},t^*)\big|_{t^*=t^*(\boldsymbol{x},t)}\times\boldsymbol{E}(\boldsymbol{x},t), \tag{2.2.46}$$

and, therefore,

$$\boldsymbol{B}(\boldsymbol{x},t)\cdot\boldsymbol{E}(\boldsymbol{x},t) = 0, \tag{2.2.47}$$

showing that the electric and magnetic field emitted by a moving point-like charge are always perpendicular to each other. Then, the electromagnetic wave associated to these fields moves in the direction identified by $\boldsymbol{n}(\boldsymbol{x}, t^*)$ (which is in general different from $\boldsymbol{n}(\boldsymbol{x}, t)$), while the fields $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ rotate in a very complicated way, while always maintaining themselves perpendicular to each other.

The fact that both the fields depend on the retarded time t^* tells us that the potentials in a point of space $\boldsymbol{x}$ calculated at time t does not depend on the position $\boldsymbol{x}(t)$ of the source at time t, but on the position that the source was in when the electromagnetic field was emitted. This position is exactly $\boldsymbol{x}(t^*)$. Therefore, the existence of a retarded time tells us that the velocity at which the electromagnetic wave travels is finite because, if it was infinite, any change in the source would instantaneously affect all points throughout space.

2.3. Approximations and regimes

In many physical situations, the velocity of the source is small compared to the rapidity of updating of the generated electromagnetic field. The latter is precisely the speed of light in vacuum $c = 299\,792\,458\,\mathrm{m/s}$, while the former is $|\dot{\boldsymbol{x}}(t^*)|$. In such a condition, every relativistic effect is completely negligible.

It is convenient to define the non relativistic regime when the condition

$$1 \gg \frac{|\dot{\boldsymbol{x}}(t^*)|}{c} \tag{2.3.1}$$

is satisfied. If one recalls definition (2.2.3b), one can understand that condition (2.3.1) translates to

$$1 \gg |\boldsymbol{\beta}(t^*)|. \tag{2.3.2}$$

The electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ generated in the non relativistic regime are given by

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x}, t) = \frac{q}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|^2} \Bigg\{ &\boldsymbol{n}(\boldsymbol{x}, t^*) \qquad\qquad (2.3.3\text{a})\\ &+ \frac{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}{c} \boldsymbol{n}(\boldsymbol{x}, t^*) \times \Big(\boldsymbol{n}(\boldsymbol{x}, t^*) \times \dot{\boldsymbol{\beta}}(t^*) \Big) \Bigg\} \Bigg|_{t^* = t^*(\boldsymbol{x}, t)},\end{aligned}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \frac{q}{|\boldsymbol{x} - \boldsymbol{x}(t^*)|^2} \left\{ \boldsymbol{\beta}(t^*) + \frac{|\boldsymbol{x} - \boldsymbol{x}(t^*)|}{c} \dot{\boldsymbol{\beta}}(t^*) \right\} \times \boldsymbol{n}(\boldsymbol{x}, t^*) \Bigg|_{t^* = t^*(\boldsymbol{x}, t)}. \tag{2.3.3b}$$

Proof. For sake of compactness, we shall treat $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, $\boldsymbol{n}$ and t^* as independent variables. In order to prove (2.3.3), we need to evaluate expressions (2.2.11) as $|\boldsymbol{\beta}| \to 0$. First of all, it is clear that

$$1 - \boldsymbol{\beta} \cdot \boldsymbol{n} \approx 1, \qquad |\boldsymbol{\beta}| \to 0, \tag{2.3.4a}$$

$$\left(1 - |\boldsymbol{\beta}|^2\right) (\boldsymbol{n} - \boldsymbol{\beta}) \approx (\boldsymbol{n} - \boldsymbol{\beta}) \approx \boldsymbol{n}, \qquad |\boldsymbol{\beta}| \to 0. \tag{2.3.4b}$$

These approximations are enough to prove (2.3.3a). Secondly, using (2.2.37), one has

$$\boldsymbol{n} \times \left\{ \boldsymbol{n} \times \left[(\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\} = \left\{ \left[(\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \times \boldsymbol{n} \right\} \times \boldsymbol{n} \tag{2.3.5}$$

$$= \left\{ \boldsymbol{n} \times \left(\dot{\boldsymbol{\beta}} \times \boldsymbol{n} \right) - \boldsymbol{\beta} \times \left(\dot{\boldsymbol{\beta}} \times \boldsymbol{n} \right) \right\} \times \boldsymbol{n}$$
$$= \left\{ \dot{\boldsymbol{\beta}} - \left(\boldsymbol{n} \cdot \dot{\boldsymbol{\beta}} \right) \boldsymbol{n} - \left(\boldsymbol{\beta} \cdot \boldsymbol{n} \right) \dot{\boldsymbol{\beta}} + \left(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} \right) \boldsymbol{n} \right\} \times \boldsymbol{n}$$
$$= \left(1 - \boldsymbol{\beta} \cdot \boldsymbol{n} \right) \dot{\boldsymbol{\beta}} \times \boldsymbol{n},$$

where in the last step we have eliminated every term proportional to $\boldsymbol{n}$ since they vanish once the cross product is computed. Thus,

$$\boldsymbol{n} \times \left\{ \boldsymbol{n} \times \left[\left(\boldsymbol{n} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right\} \approx \dot{\boldsymbol{\beta}} \times \boldsymbol{n}, \qquad |\boldsymbol{\beta}| \to 0. \tag{2.3.6}$$

Lastly, one has

$$\boldsymbol{n} \times \left\{ \left(1 - |\boldsymbol{\beta}|^2 \right) \left(\boldsymbol{n} - \boldsymbol{\beta} \right) \right\} = \left(1 - |\boldsymbol{\beta}|^2 \right) \boldsymbol{\beta} \times \boldsymbol{n}, \tag{2.3.7}$$

thus obtaining

$$\boldsymbol{n} \times \left\{ \left(1 - |\boldsymbol{\beta}|^2 \right) \left(\boldsymbol{n} - \boldsymbol{\beta} \right) \right\} \approx \boldsymbol{\beta} \times \boldsymbol{n}, \qquad |\boldsymbol{\beta}| \to 0. \tag{2.3.8}$$

Using (2.3.4a), (2.3.6) and (2.3.8) to evaluate expression (2.2.11b) as $|\boldsymbol{\beta}| \to 0$ leads readily to (2.3.3b).

Another common situation is represented by the quasistatic regime. In this approximation, one assumes that, during the time necessary for a light signal to travel from $\boldsymbol{x}(t)$ (the trajectory of the particle, indicating the point the source is in at time t) to $\boldsymbol{x}$ (the point the fields are measured at), the point charge moves very little compared to the distance $|\boldsymbol{x} - \boldsymbol{x}(t)|$ traveled by the light signal, so much so that the variation in each derivative of $\boldsymbol{x}(t)$ is smaller and smaller. This can be expressed more rigorously by saying that the quasistatic regime is satisfied when the condition

$$|\boldsymbol{x} - \boldsymbol{x}(t)| \gg \frac{|\boldsymbol{x} - \boldsymbol{x}(t)|}{c} |\dot{\boldsymbol{x}}(t)| \gg \left[\frac{|\boldsymbol{x} - \boldsymbol{x}(t)|}{c} \right]^2 |\ddot{\boldsymbol{x}}(t)| \gg \dots \tag{2.3.9}$$

is satisfied. The quasistatic regime implies the non relativistic regime (2.3.1). One can understand the physical interpretation given above by noting that, when (2.3.9) holds, one has

$$\boldsymbol{x} \left(t + \frac{|\boldsymbol{x} - \boldsymbol{x}(t)|}{c} \right) - \boldsymbol{x}(t) \approx \frac{|\boldsymbol{x} - \boldsymbol{x}(t)|}{c} \dot{\boldsymbol{x}}(t), \tag{2.3.10a}$$

$$\dot{\boldsymbol{x}}\left(t+\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{c}\right)-\dot{\boldsymbol{x}}(t)\approx\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{c}\ddot{\boldsymbol{x}}(t), \tag{2.3.10b}$$

$$\ddot{\boldsymbol{x}}\left(t+\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{c}\right)-\ddot{\boldsymbol{x}}(t)\approx\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{c}\dddot{\boldsymbol{x}}(t),\ldots, \tag{2.3.10c}$$

obtained by truncating the Taylor expansions at the first order. Since the source is slow moving, the effect of the retardation is negligible. By (2.3.9), $\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \ldots$ vary little in a time equal to the difference of t and t^* as the Taylor series of $\boldsymbol{x}(t^*)-\boldsymbol{x}(t)$ is rapidly convergent. Hence, the following equations (2.3.11) hold.

$$\boldsymbol{\beta}(t^*)\approx\boldsymbol{\beta}(t), \tag{2.3.11a}$$

$$\dot{\boldsymbol{\beta}}(t^*)\approx\dot{\boldsymbol{\beta}}(t). \tag{2.3.11b}$$

Approximations (2.3.11) can be used to compute the electric field $\boldsymbol{E}(\boldsymbol{x},t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ generated in the quasistatic regime, which have a very simple expression and are given by

$$\boldsymbol{E}(\boldsymbol{x},t)=\frac{q}{|\boldsymbol{x}-\boldsymbol{x}(t)|^2}\boldsymbol{n}(\boldsymbol{x},t), \tag{2.3.12a}$$

$$\boldsymbol{B}(\boldsymbol{x},t)=\frac{q}{|\boldsymbol{x}-\boldsymbol{x}(t)|^2}\boldsymbol{\beta}(t)\times\boldsymbol{n}(\boldsymbol{x},t). \tag{2.3.12b}$$

Proof. Since the quasistatic regime implies the non relativistic regime, we can assume expressions (2.3.3) to be a valid approximation for the electric and the magnetic fields. Equations (2.3.12) are easily derived once the limit $|\boldsymbol{x}-\boldsymbol{x}(t)|/c\to 0$ is carried out in (2.3.3).

Lastly, there is another regime worth considering: the radiation regime. This is defined by

$$|\dot{\boldsymbol{x}}(t)|\ll\frac{|\boldsymbol{x}-\boldsymbol{x}(t)|}{c}|\ddot{\boldsymbol{x}}(t)|. \tag{2.3.13}$$

Upon inspecting the right hand side of condition (2.3.13), one can see that condition (2.3.13) is satisfied either when the motion of the source is strongly accelerated or when there is a sufficiently large distance between the trajectory of the particle and the observation point.

In the radiation regime, the electric field $\boldsymbol{E}(\boldsymbol{x},t)$ and the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ are given by

$$\boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{\left(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)^3}\frac{1}{c\,|\boldsymbol{x}-\boldsymbol{x}(t^*)|} \boldsymbol{n}(\boldsymbol{x},t^*)\times\left[(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)\right]\Big|_{t^*=t^*(\boldsymbol{x},t)}, \tag{2.3.14a}$$

$$\boldsymbol{B}(\boldsymbol{x},t) = \frac{q}{\left(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)\right)^3}\frac{1}{c\,|\boldsymbol{x}-\boldsymbol{x}(t^*)|} \boldsymbol{n}(\boldsymbol{x},t^*)\times\left\{\boldsymbol{n}(\boldsymbol{x},t^*)\times\left[(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)\right]\right\}\Big|_{t^*=t^*(\boldsymbol{x},t)}. \tag{2.3.14b}$$

In this regime, $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$ satisfy

$$\boldsymbol{n}(\boldsymbol{x},t^*)\big|_{t^*=t^*(\boldsymbol{x},t)}\cdot\boldsymbol{E}(\boldsymbol{x},t) = 0, \tag{2.3.15a}$$

$$\boldsymbol{n}(\boldsymbol{x},t^*)\big|_{t^*=t^*(\boldsymbol{x},t)}\cdot\boldsymbol{B}(\boldsymbol{x},t) = 0, \tag{2.3.15b}$$

while still being perpendicular to one another by (2.2.47).

In the following sections, we shall always assume the radiation regime, therefore condition (2.3.13) is assumed to be satisfied throughout.

CHAPTER 3

Poynting vector and power emission

3.1. Poynting vector

The Poynting vector field $\boldsymbol{S}(\boldsymbol{x},t)$ of a moving charge in the radiation regime is given by

$$\boldsymbol{S}(\boldsymbol{x},t) = \frac{c}{4\pi}|\boldsymbol{E}(\boldsymbol{x},t)|^2\, \boldsymbol{n}(\boldsymbol{x},t^*)\big|_{t^*=t^*(\boldsymbol{x},t)}\,. \tag{3.1.1}$$

Proof. Keeping in mind the definition of the Poynting vector

$$\boldsymbol{S}(\boldsymbol{x},t) = \frac{c}{4\pi}\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{B}(\boldsymbol{x},t), \tag{3.1.2}$$

and equations (2.2.46), (2.2.37), (2.2.47), one has

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{x},t) &= \frac{c}{4\pi}\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{B}(\boldsymbol{x},t) \\ &= \frac{c}{4\pi}\boldsymbol{E}(\boldsymbol{x},t)\times\left[\boldsymbol{n}(\boldsymbol{x},t^*)\times\boldsymbol{E}(\boldsymbol{x},t)\right]_{t^*=t^*(\boldsymbol{x},t)} \\ &= \frac{c}{4\pi}\left[|\boldsymbol{E}(\boldsymbol{x},t)|^2\boldsymbol{n}(\boldsymbol{x},t^*) - \boldsymbol{n}(\boldsymbol{x},t^*)\cdot\boldsymbol{E}(\boldsymbol{x},t)\boldsymbol{E}(\boldsymbol{x},t)\right]_{t^*=t^*(\boldsymbol{x},t)} \\ &= \frac{c}{4\pi}\,|\boldsymbol{E}(\boldsymbol{x},t)|^2\boldsymbol{n}(\boldsymbol{x},t^*)\big|_{t^*=t^*(\boldsymbol{x},t)}\,.\end{aligned} \tag{3.1.3}$$

This proves (3.1.1)

The Poynting vector is electromagnetic energy current density; its magnitude is the amount of energy flowing through a unit area transverse to the energy flow in a unit time. Its direction is that of the energy flow. It cannot be interpreted as a photon packet as $\boldsymbol{E}$ and $\boldsymbol{B}$ are not interpretable as photon wave functions. Keeping in mind that Gaussian unit system are used throughout, dimensional analysis shows that the Poynting vector's units are energy per unit area per unit time (SI: erg/ (cm^{2}s)).

Equation (3.1.1) shows that the energy flow in the point $\boldsymbol{x}$ at time t is not perpendicular to $\boldsymbol{n}(\boldsymbol{x}, t)$, but rather it points toward $\boldsymbol{n}(\boldsymbol{x}, t^*)$. This is due to the retardation effect. All points in space are not instantly connected, therefore the energy acts as if it was being generated by a fictitious particle located in $\boldsymbol{x}(t^*)$ at time t, rather than the actual physical particle, which is located at $\boldsymbol{x}(t)$. Thus, every particle does not react to the universe around it in its current configuration (i.e. at time t), but rather it responds to an earlier configuration of it, because every physically meaningful quantity needs to be evaluated at time $t^* < t$. As a consequence, it seems more natural to express the Poynting vector in terms of the retarded time t^* instead of the time t. To this end, consider

$$\boldsymbol{S}^*(\boldsymbol{x}, t^*) = \boldsymbol{S}(\boldsymbol{x}, t)|_{t=t(\boldsymbol{x},t^*)} \frac{\partial t}{\partial t^*}(\boldsymbol{x}, t^*), \tag{3.1.4}$$

where $t = t(\boldsymbol{x}, t^*)$ is the inverse function of $t^* = t^*(\boldsymbol{x}, t)$. This way, the independent time variable is no longer t, but t^*.

3.2. Total radiated power

In Chapter 2 we established that a moving charge generates an electromagnetic field. Since the latter carries energy, the charge must transform part of its kinetic energy into electromagnetic energy. At time t^*, the energy radiated in $\boldsymbol{x}$ by a moving charge through a solid angle δo along the direction $\boldsymbol{n}(\boldsymbol{x}, t^*)$ in a time δt^* is

$$\delta W = \boldsymbol{S}^*(\boldsymbol{x}, t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*) \left|\boldsymbol{x} - \boldsymbol{x}(t^*)\right|^2 \delta o \, \delta t^*. \tag{3.2.1}$$

The power emitted by the moving charge per unit solid angle is

$$\frac{\mathrm{d}P^*}{\mathrm{d}o}(\boldsymbol{x}, t^*) = \boldsymbol{S}^*(\boldsymbol{x}, t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*) \left|\boldsymbol{x} - \boldsymbol{x}(t^*)\right|^2. \tag{3.2.2}$$

The quantity $\mathrm{d}P^*/\,\mathrm{d}o(\boldsymbol{x}, t^*)$ in (3.2.2) is called differential radiated power. Equation (3.1.4) allows us to write the differential radiated power as

$$\begin{aligned}\frac{\mathrm{d}P^*}{\mathrm{d}o}(\boldsymbol{x}, t^*) = \frac{q^2}{4\pi c} & \frac{1}{\left(1 - \boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)\right)^5} \\ & \left\{ \left[\left(\boldsymbol{n}(\boldsymbol{x}, t^*) - \boldsymbol{\beta}(t^*)\right) \times \dot{\boldsymbol{\beta}}(t^*) \right]^2 - \left[\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \boldsymbol{\beta}(t^*) \times \dot{\boldsymbol{\beta}}(t^*) \right]^2 \right\}.\end{aligned} \tag{3.2.3}$$

Proof. From (3.2.2), (3.1.4), (3.1.1), one has

$$\begin{aligned}\frac{\mathrm{d}P^*}{\mathrm{d}o}(\boldsymbol{x}, t^*) &= \boldsymbol{S}(\boldsymbol{x}, t)|_{t = t(\boldsymbol{x}, t^*)} \frac{\partial t}{\partial t^*}(\boldsymbol{x}, t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*) \left|\boldsymbol{x} - \boldsymbol{x}(t^*)\right|^2 \\ &= \frac{c}{4\pi} |\boldsymbol{E}(\boldsymbol{x}, t)|^2 \, |_{t^* = t^*(\boldsymbol{x}, t)} \left|\boldsymbol{x} - \boldsymbol{x}(t^*)\right|^2 \frac{\partial t}{\partial t^*}(\boldsymbol{x}, t^*).\end{aligned} \tag{3.2.4}$$

From (2.2.28a) and (2.2.18a), it is clear that

$$\frac{\partial t}{\partial t^*}(\boldsymbol{x}, t^*) = 1 - \boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*). \tag{3.2.5}$$

We now have everything we need to prove (3.2.3): the expression for $|\boldsymbol{E}(\boldsymbol{x}, t)|^2$ is given by (2.3.14a), while the one for $\partial t / \partial t^*(\boldsymbol{x}, t^*)$ is given by (3.2.5). Inserting it all into (3.2.4) yields exactly (3.2.3).

Upon inspecting equation (3.2.3), it is evident that the only way $\mathrm{d}P^*/\,\mathrm{d}o(\boldsymbol{x}, t^*)$ depends on $\boldsymbol{x}$ is through $\boldsymbol{n}(\boldsymbol{x}, t^*)$. As a consequence

$$\frac{\mathrm{d}P^*}{\mathrm{d}o}(\boldsymbol{x}, t^*) = \frac{\mathrm{d}P^*}{\mathrm{d}o}(\boldsymbol{n}(\boldsymbol{x}, t^*), t^*). \tag{3.2.6}$$

We can now integrate on every possible orientation of emission $\boldsymbol{n}$ and obtain the total radiated power $P^*(t^*)$. Defining

$$P^*(t^*) \equiv \int \mathrm{d}^2\boldsymbol{n} \frac{\mathrm{d}P^*}{\mathrm{d}o}(\boldsymbol{n}, t^*), \tag{3.2.7}$$

one has

$$P^*(t^*) = \frac{2q^2}{3c} \frac{1}{(1 - |\boldsymbol{\beta}(t^*)|^2)^3} \left[|\dot{\boldsymbol{\beta}}(t^*)|^2 - |\boldsymbol{\beta}(t^*) \times \dot{\boldsymbol{\beta}}(t^*)|^2 \right]. \tag{3.2.8}$$

Often times, the acceleration $\dot{\boldsymbol{\beta}}$ of the charge is broken up into its components along the direction parallel and perpendicular to its speed $\boldsymbol{\beta} = |\boldsymbol{\beta}|\hat{\boldsymbol{\beta}}$. These are identified by

$$\dot{\boldsymbol{\beta}}_\parallel = \dot{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{\beta}} \hat{\boldsymbol{\beta}}, \tag{3.2.9a}$$

$$\dot{\boldsymbol{\beta}}_\perp = \dot{\boldsymbol{\beta}} - \dot{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{\beta}} \hat{\boldsymbol{\beta}}. \tag{3.2.9b}$$

Clearly, one has

$$\dot{\boldsymbol{\beta}} = \dot{\boldsymbol{\beta}}_\parallel + \dot{\boldsymbol{\beta}}_\perp, \tag{3.2.10a}$$

$$\dot{\boldsymbol{\beta}}_\parallel \cdot \dot{\boldsymbol{\beta}}_\perp = 0. \tag{3.2.10b}$$

With these definitions, it is possible to express the total radiated power in terms of $\dot{\boldsymbol{\beta}}_\parallel$ and $\dot{\boldsymbol{\beta}}_\perp$, obtaining

$$P^*(t^*) = \frac{2q^2}{3c} \frac{1}{(1 - |\boldsymbol{\beta}(t^*)|^2)^3} \left[|\dot{\boldsymbol{\beta}}_\parallel(t^*)|^2 + \left(1 - |\boldsymbol{\beta}(t^*)|^2\right) |\dot{\boldsymbol{\beta}}_\perp(t^*)|^2 \right]. \tag{3.2.11}$$

Proof. Here, we shall neglect the t^* dependence. For any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the cross product and the dot product are related by

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2. \tag{3.2.12}$$

From (3.2.10) and (3.2.12), one has

$$|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2 = |\dot{\boldsymbol{\beta}}_\parallel + \dot{\boldsymbol{\beta}}_\perp|^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}_\perp|^2, \qquad \text{as } \boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}_\parallel = \mathbf{0}, \tag{3.2.13}$$

$$= |\dot{\boldsymbol{\beta}}_\parallel|^2 + |\dot{\boldsymbol{\beta}}_\perp|^2 - |\boldsymbol{\beta}|^2|\dot{\boldsymbol{\beta}}_\perp|^2, \qquad \text{as } \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}_\perp = 0,$$

$$= |\dot{\boldsymbol{\beta}}_\parallel|^2 + \left(1 - |\boldsymbol{\beta}|^2\right)|\dot{\boldsymbol{\beta}}_\perp|^2.$$

Substituting (3.2.13) into (3.2.8) immediately yields (3.2.11).

It is also convenient to write the total radiated power in terms of the relativistic momentum of the charge

$$\boldsymbol{p}(t^*) = \frac{mc\boldsymbol{\beta}(t^*)}{(1 - |\boldsymbol{\beta}(t^*)|^2)^{1/2}}. \tag{3.2.14}$$

Similarly to the case of the acceleration $\dot{\boldsymbol{\beta}}$, we can break the quantity $\dot{\boldsymbol{p}}$ into its components along the direction parallel and perpendicular to its speed $\boldsymbol{\beta}$. These are

$$\dot{\boldsymbol{p}}_\parallel = \dot{\boldsymbol{p}}\cdot\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}, \tag{3.2.15a}$$

$$\dot{\boldsymbol{p}}_\perp = \dot{\boldsymbol{p}} - \dot{\boldsymbol{p}}\cdot\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}. \tag{3.2.15b}$$

Clearly, one has

$$\dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}_\parallel + \dot{\boldsymbol{p}}_\perp, \tag{3.2.16a}$$

$$\dot{\boldsymbol{p}}_\parallel\cdot\dot{\boldsymbol{p}}_\perp = 0. \tag{3.2.16b}$$

One can now express the total radiated power in terms of $\dot{\boldsymbol{p}}_\parallel$ and $\dot{\boldsymbol{p}}_\perp$, obtaining

$$P^*(t^*) = \frac{2q^2}{3m^2c^3}\left[|\dot{\boldsymbol{p}}_\parallel(t^*)|^2 + \frac{|\dot{\boldsymbol{p}}_\perp(t^*)|^2}{1 - |\boldsymbol{\beta}(t^*)|^2}\right]. \tag{3.2.17}$$

Proof. We shall neglect the dependence on t^*. From (3.2.14), (3.2.10a), (3.2.9a), a simple computation shows that

$$\dot{\boldsymbol{p}} = mc\left[\frac{\dot{\boldsymbol{\beta}}}{(1 - |\boldsymbol{\beta}|^2)^{1/2}} + \frac{\dot{\boldsymbol{\beta}}\cdot\boldsymbol{\beta}\boldsymbol{\beta}}{(1 - |\boldsymbol{\beta}|^2)^{3/2}}\right] \tag{3.2.18}$$

$$= mc \left[\frac{\dot{\boldsymbol{\beta}}_\parallel + \dot{\boldsymbol{\beta}}_\perp}{(1 - |\boldsymbol{\beta}|^2)^{1/2}} + \frac{|\boldsymbol{\beta}|^2 \dot{\boldsymbol{\beta}}_\parallel}{(1 - |\boldsymbol{\beta}|^2)^{3/2}} \right]$$

$$= mc \left[\frac{\dot{\boldsymbol{\beta}}_\perp}{(1 - |\boldsymbol{\beta}|^2)^{1/2}} + \frac{\dot{\boldsymbol{\beta}}_\parallel}{(1 - |\boldsymbol{\beta}|^2)^{3/2}} \right].$$

From (3.2.18), we read off the expressions of $\dot{\boldsymbol{p}}_\parallel$ and $\dot{\boldsymbol{p}}_\perp$ in terms of $\dot{\boldsymbol{\beta}}_\parallel$ and $\dot{\boldsymbol{\beta}}_\perp$ as

$$\dot{\boldsymbol{p}}_\parallel = \frac{mc\dot{\boldsymbol{\beta}}_\parallel}{(1 - |\boldsymbol{\beta}|^2)^{3/2}}, \tag{3.2.19a}$$

$$\dot{\boldsymbol{p}}_\perp = \frac{mc\dot{\boldsymbol{\beta}}_\perp}{(1 - |\boldsymbol{\beta}|^2)^{1/2}}. \tag{3.2.19b}$$

Inverting these yields

$$\dot{\boldsymbol{\beta}}_\parallel = \frac{\left(1 - |\boldsymbol{\beta}|^2\right)^{3/2} \dot{\boldsymbol{p}}_\parallel}{mc}, \tag{3.2.20a}$$

$$\dot{\boldsymbol{\beta}}_\perp = \frac{\left(1 - |\boldsymbol{\beta}|^2\right)^{1/2} \dot{\boldsymbol{p}}_\perp}{mc}. \tag{3.2.20b}$$

Substituting (3.2.20) into (3.2.11) immediately yields (3.2.17).

It is interesting to compare (3.2.11) and (3.2.17). According to (3.2.11), a longitudinal acceleration $\dot{\boldsymbol{\beta}}_\parallel$ involves an energy loss by radiation $(1 - |\boldsymbol{\beta}|^2)^{-1}$ times larger than a transversal acceleration $\dot{\boldsymbol{\beta}}_\perp$ of the same magnitude. However, according to (3.2.17), a longitudinal force $\dot{\boldsymbol{p}}_\parallel$ involves an energy loss by radiation $1 - |\boldsymbol{\beta}|^2$ times smaller than a transversal force $\dot{\boldsymbol{p}}_\perp$ of the same magnitude. The contradiction is only apparent. It is related to the fact that, relativistically, the mass of a particle depends on its velocity according to

$$m(t^*) = \frac{m_0}{(1 - |\boldsymbol{\beta}(t^*)|^2)^{1/2}}. \tag{3.2.21}$$

In the following chapter, we shall analyze two specific cases: linear acceleration and circular acceleration. It is clear that

$$\text{linear:} \qquad \dot{\boldsymbol{\beta}} = \dot{\boldsymbol{\beta}}_\parallel, \quad \dot{\boldsymbol{\beta}}_\perp = \mathbf{0}, \tag{3.2.22a}$$

$$\text{circular:} \qquad \dot{\boldsymbol{\beta}} = \dot{\boldsymbol{\beta}}_\perp, \quad \dot{\boldsymbol{\beta}}_\parallel = \mathbf{0}. \tag{3.2.22b}$$

CHAPTER 4

Two common cases: linear and circular accelerations

4.1. Introduction

The main experimental setups in elementary particle physics involve accelerating particles, boosting their kinetic energy and having them collide to form new particles. It is even possible to create different particles with different masses, proving that energy and mass are two ways of calling the same thing. This strange property of nature is described perfectly by special relativity, that provides us with the exact relation between energy, mass and momentum. If the particle being accelerated carries electric charge, it will emit electromagnetic power. Therefore, the problem of studying, measuring and computing the radiation energy loss in accelerators has both a practical and a theoretical interest.

Historically, the first accelerators were linear. At the beginning of the XX century, technology allowed physicists to accelerate electrons up to a few MeV. Today, we can reach much higher energies with circular accelerators, up to several TeV. In the following sections, we shall analyze these two main types of accelerators.

4.2. Linear acceleration

Consider a particle of charge q moving on a straight trajectory $\boldsymbol{x}(t)$. We shall measure its motion along the frame such that the particle goes through the origin at time $t = 0$. Therefore

$$\boldsymbol{x}(t) = s(t)\boldsymbol{e}, \tag{4.2.1}$$

where $s(t)$ is the linear ascissa of the particle and $\boldsymbol{e}$ is the unit vector in the direction of motion. From (3.2.3), it follows that the differential radiated power of the charge is given by

$$\frac{\mathrm{d}P^*}{\mathrm{d}o}(\boldsymbol{x}, t^*) = \frac{q^2}{4\pi c} \frac{(\boldsymbol{n}(\boldsymbol{x}, t^*)\times\boldsymbol{e})^2\, \dot\beta(t^*)^2}{(1 - \beta(t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)\cdot\boldsymbol{e})^5}, \tag{4.2.2}$$

where

$$\beta(t) = \frac{\dot s(t)}{c}. \tag{4.2.3}$$

Proof. For sake of compactness, we shall treat $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, $\boldsymbol{n}$, β and $\dot\beta$ as independent variables. From (2.2.3b), (4.2.1) and (4.2.3) one has

$$\boldsymbol{\beta} = \beta\boldsymbol{e}, \tag{4.2.4a}$$

$$\dot{\boldsymbol{\beta}} = \dot\beta\boldsymbol{e}, \tag{4.2.4b}$$

so that

$$\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} = \boldsymbol{0}. \tag{4.2.5}$$

Upon inspecting equation (3.2.3), we only have to compute two expressions. The first one is

$$1 - \boldsymbol{\beta}\cdot\boldsymbol{n} = 1 - \beta\boldsymbol{n}\cdot\boldsymbol{e}, \tag{4.2.6a}$$

while the second one is easily computed keeping (4.2.5) in mind,

$$\begin{aligned}\Big((\boldsymbol{n} - \boldsymbol{\beta}) \times\dot{\boldsymbol{\beta}}\Big)^2 - \Big(\boldsymbol{n}\cdot\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\Big)^2 &= \Big(\boldsymbol{n}\times\dot{\boldsymbol{\beta}}\Big)^2 \\ &= \dot\beta^2\,(\boldsymbol{n}\times\boldsymbol{e})^2\,.\end{aligned} \tag{4.2.6b}$$

Inserting equations (4.2.6) into (3.2.3) readily yields (4.2.2).

Upon inspecting (4.2.2), it is evident that $\boldsymbol{n}(\boldsymbol{x}, t^*)$ and $\boldsymbol{e}$ only appear in scalar or vector products. Therefore, the physically meaningful quantity is neither $\boldsymbol{n}(\boldsymbol{x}, t^*)$ nor $\boldsymbol{e}$, but rather the angle between them. We shall denote by $\theta(\boldsymbol{x}, t^*)$ the latter and view θ as an independent variable. Then, the differential radiated power becomes a function of θ,

$$\frac{\mathrm{d}P^*}{\mathrm{d}o}(\theta, t^*) = \frac{q^2}{4\pi c}\frac{\dot{\beta}(t^*)^2 \sin^2\theta}{(1-\beta(t^*)\cos\theta)^5}. \tag{4.2.7}$$

The angular dependence of (4.2.7) is governed by the function

$$f(\theta;\beta) = \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}, \tag{4.2.8}$$

where β is treated as an independent variable such that $0 \leq \beta < 1$ and $0 \leq \theta < 2\pi$. The function $f(\theta;\beta)$ has two minima for $\theta = 0, \pi$ and a maximum for $\theta = \theta(\beta)$, where

$$\theta(\beta) = \cos^{-1}\left[\frac{(1+15\beta^2)^{1/2}-1}{3\beta}\right]. \tag{4.2.9}$$

The minimum and maximum values are, respectively,

$$f_{\min}(\beta) = f(0;\beta) = f(2\pi;\beta) = 0, \tag{4.2.10a}$$

$$f_{\max}(\beta) = f(\theta(\beta);\beta) = \frac{54}{\beta^2}\,\frac{-1-3\beta^2+(1+15\beta^2)^{1/2}}{\left(4-(1+15\beta^2)^{1/2}\right)^5}. \tag{4.2.10b}$$

These complicated expressions simplify in the non relativistic limit $\beta \to 0$ and in the relativistic limit $\beta \to 1$,

$$\theta(\beta) \approx \frac{\pi}{2}, \qquad \beta \to 0, \tag{4.2.11a}$$

$$\theta(\beta) \approx \frac{(1-\beta^2)^{1/2}}{2}, \qquad \beta \to 1. \tag{4.2.11b}$$

Furthermore,

$$f_{\max}(\beta) \approx 1, \qquad \beta \to 0, \tag{4.2.12a}$$

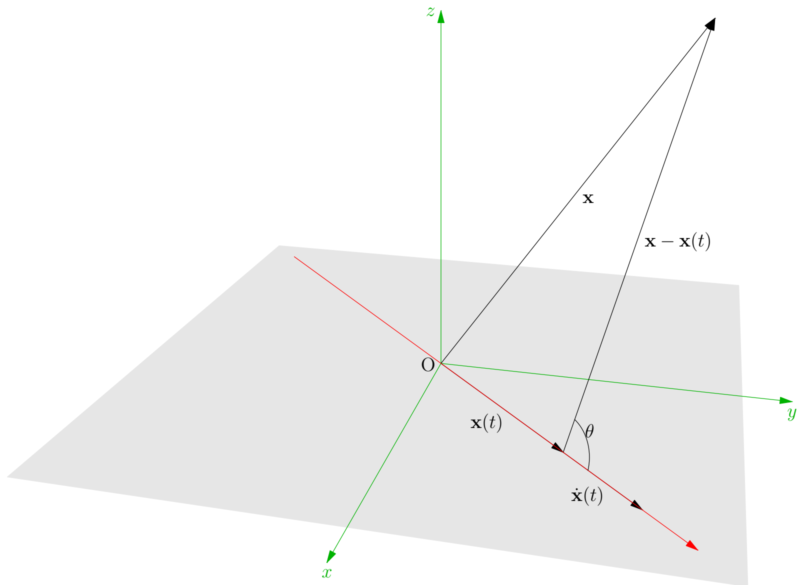

FIGURE 4.2.1. A linear accelerator.

$$f_{\max}(\beta) \approx \frac{1}{4}\left(\frac{8}{5}\right)^5 \frac{1}{(1-\beta^2)^4}, \qquad \beta \to 1. \tag{4.2.12b}$$

Proof. A simple computation shows that

$$\frac{\partial f}{\partial \theta}(\theta;\beta) = \frac{3\beta\cos^2\theta + 2\cos\theta - 5\beta}{(1-\beta\cos\theta)^6}\sin\theta. \tag{4.2.13}$$

The values of θ corresponding to the extrema of $f(\theta;\beta)$ are those for which $\partial f/\partial\theta(\theta;\beta) = 0$. Hence they are the solutions of either equations

$$\sin\theta = 0, \tag{4.2.14a}$$

$$3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0, \tag{4.2.14b}$$

which can be written as

$$\sin\theta = 0, \tag{4.2.15a}$$

$$\cos\theta = \frac{\pm(1+15\beta^2)^{1/2} - 1}{3\beta}. \tag{4.2.15b}$$

The solution to (4.2.15b) with the $-$ sign is not physically acceptable because it diverges to $-\infty$ as $\beta \to 0$. This is not mathematically consistent, as $|\cos\theta| \le 1$ for any real θ.

Therefore, the only extrema of $f(\theta;\beta)$ are

$$\theta = 0, \pi, \theta(\beta), \tag{4.2.16}$$

where $\theta(\beta)$ is the solution to (4.2.15b) with the + sign, given by (4.2.9). Now, since $f(\theta;\beta) \geq 0$ by definition and $f(\theta;\beta) = 0$ for $\theta = 0, \pi$, we can conclude that $\theta = 0, \pi$ must be two minima for $f(\theta;\beta)$. As a consequence, since $f(\theta;\beta)$ is continuous, $\theta = \theta(\beta)$ must be a maximum. A straightforward computation shows that the value of the maximum $f(\theta(\beta);\beta)$ is exactly given by (4.2.10b).

A simple Taylor expansion shows that

$$\frac{\left(1+15\beta^2\right)^{1/2}-1}{3\beta} = O(\beta), \qquad \beta \to 0, \tag{4.2.17a}$$

$$\begin{aligned} \frac{\left(1+15\beta^2\right)^{1/2}-1}{3\beta} &= 1 - \frac{1-\beta}{4} + O\left((1-\beta)^2\right), \qquad &(4.2.17b) \\ &= 1 - \frac{1-\beta^2}{8} + O\left((1-\beta)^2\right), \qquad \beta \to 1, \end{aligned}$$

where in the last step we used the fact that

$$\begin{aligned} \frac{1-\beta}{4} + O\left((1-\beta)^2\right) &= \frac{1-\beta^2}{4\,(1+\beta)} + O\left((1-\beta)^2\right) \qquad &(4.2.18) \\ &= \frac{1-\beta^2}{4}\frac{1}{2-(1-\beta)} + O\left((1-\beta)^2\right) \\ &\approx \frac{1-\beta^2}{8} + O\left((1-\beta)^2\right). \qquad &(4.2.19) \end{aligned}$$

Using that $\cos^{-1} x \approx \pi/2$ as $x \to 0$ and $\cos^{-1}(1+x) \approx (-2x)^{1/2}$ as $x \to 0^-$, we get (4.2.11).

By evaluating the limit of $f_{\max}(\beta)$ as $\beta \to 0$ from (4.2.10b), we get (4.2.12a) after a quite lengthy but straightforward computation.

Lastly, in order to compute (4.2.12b), we first note that by a Taylor expansion in the variable β^2 around $\beta^2 = 1$, we find

$$4 - \left(1+15\beta^2\right)^{1/2} = \frac{15}{8}(1-\beta^2) + O\left((1-\beta^2)^2\right), \tag{4.2.20a}$$

$$-1 - 3\beta^2 + \left(1+15\beta^2\right)^{1/2} = \frac{9}{8}(1-\beta^2) + O\left((1-\beta^2)^2\right). \tag{4.2.20b}$$

By substituting (4.2.20) into (4.2.10b), we easily get (4.2.12b).

The total radiated power of our charge is given by

$$P^*(t^*) = \frac{2q^2}{3c} \frac{\dot{\beta}(t^*)^2}{(1 - \beta(t^*)^2)^3}. \tag{4.2.21}$$

Proof. This result follows readily from (3.2.8) upon taking (4.2.5) into account.

Here are the angular plot of $f(\theta; \beta)$ for some β values.

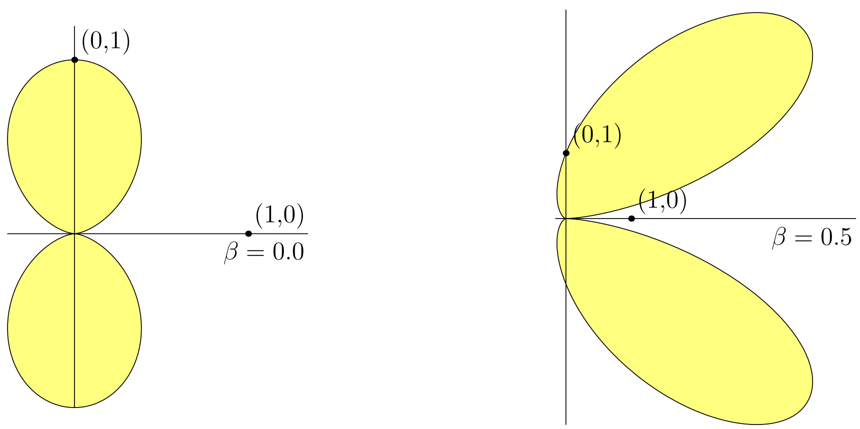

FIGURE 4.2.2. Angular plot of $f(\theta; \beta)$ for $\beta = 0.0, 0.5$.

4.3. Circular acceleration

Consider a charge q moving on a circular orbit $\boldsymbol{x}(t)$ of radius r_0 with constant angular velocity $\boldsymbol{\omega}$. The origin of the frame used to measure the particle's motion lies on the center of the circumference. Given the symmetry of the problem, it is convenient to employ a cylindrical coordinate system whose axis is determined by the versor of the angular velocity, so that

$$\boldsymbol{\omega} = \omega \boldsymbol{e}_z. \tag{4.3.1}$$

The trajectory of the orbit is given by

$$\boldsymbol{x}(t) = r_0 \boldsymbol{e}_\rho(\omega t). \tag{4.3.2}$$

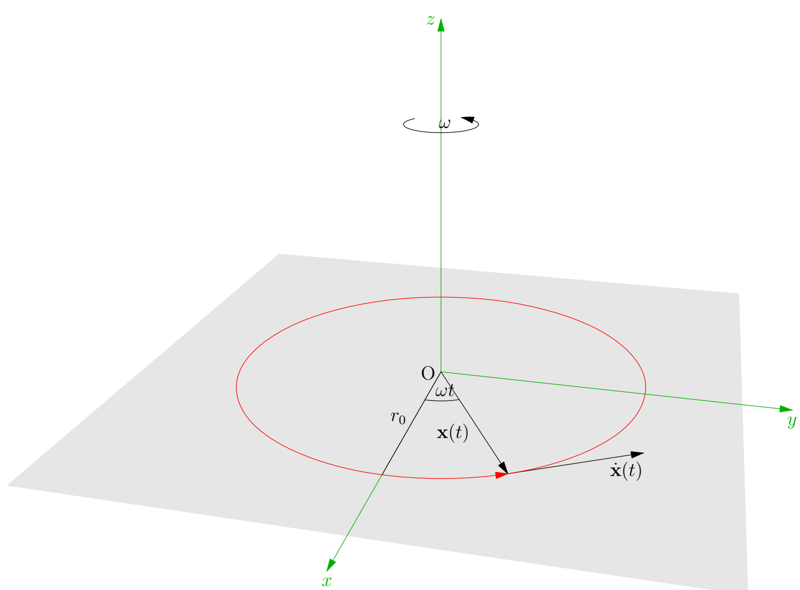

FIGURE 4.3.1. A circular accelerator.

From (3.2.3), it follows that the differential radiated power of the charge is given by

$$\frac{\mathrm{d}P^*}{\mathrm{d}o}(\boldsymbol{x}, t^*) = \frac{(q\omega\beta)^2}{4\pi c} \frac{\left(1 - \beta \boldsymbol{n}(\boldsymbol{x}, t^*)\cdot\boldsymbol{e}_\phi(\omega t^*)\right)^2 - (1-\beta^2)\left(\boldsymbol{n}(\boldsymbol{x}, t^*)\cdot\boldsymbol{e}_\rho(\omega t^*)\right)^2}{\left(1 - \beta \boldsymbol{n}(\boldsymbol{x}, t^*)\cdot\boldsymbol{e}_\phi(\omega t^*)\right)^5}, \tag{4.3.3}$$

where

$$\beta = \frac{\omega r_0}{c}. \tag{4.3.4}$$

Proof. For sake of compactness, we shall treat $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, $\boldsymbol{n}$, $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_z$ as independent vector variables. From (2.2.3b), (4.3.2) one has

$$\boldsymbol{\beta} = \frac{r_0 \dot{\boldsymbol{e}}_\rho}{c}. \tag{4.3.5}$$

Now, keeping in mind that the cylindrical versors have the following property,

$$\frac{\mathrm{d}\boldsymbol{e}_\rho}{\mathrm{d}t}(\omega t) = \omega \boldsymbol{e}_\phi(\omega t), \tag{4.3.6}$$

and the definition (4.3.4), one has

$$\boldsymbol{\beta} = \beta \boldsymbol{e}_\phi. \tag{4.3.7}$$

Furthermore, since the analogous property to (4.3.6) for $\boldsymbol{e}_\phi$ is

$$\frac{\mathrm{d}\boldsymbol{e}_\phi}{\mathrm{d}t}(\omega t) = -\omega \boldsymbol{e}_\rho(\omega t), \tag{4.3.8}$$

one has

$$\dot{\boldsymbol{\beta}} = -\omega\beta \boldsymbol{e}_\rho. \tag{4.3.9}$$

The following properties also hold

$$\boldsymbol{e}_z \times \boldsymbol{e}_\phi = -\boldsymbol{e}_\rho, \tag{4.3.10a}$$

$$\boldsymbol{e}_\phi \times \boldsymbol{e}_\rho = -\boldsymbol{e}_z. \tag{4.3.10b}$$

Using (4.3.7), (4.3.9), (4.3.1) and (4.3.10) one verifies easily that

$$\dot{\boldsymbol{\beta}} = \boldsymbol{\omega} \times \boldsymbol{\beta}, \tag{4.3.11a}$$

$$\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = \beta^2 \boldsymbol{\omega}. \tag{4.3.11b}$$

Upon inspecting equation (3.2.3), we only have to compute two expressions. The first one is

$$1 - \boldsymbol{\beta} \cdot \boldsymbol{n} = 1 - \beta \boldsymbol{n} \cdot \boldsymbol{e}_\phi. \tag{4.3.12}$$

The second one requires more effort. Using (4.3.11), one has

$$\xi = \Big((\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \Big)^2 - \Big(\boldsymbol{n} \cdot \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} \Big)^2 = \big(\boldsymbol{n} \times (\boldsymbol{\omega} \times \boldsymbol{\beta}) - \beta^2 \boldsymbol{\omega} \big)^2 - \big(\boldsymbol{n} \cdot \left(\beta^2 \boldsymbol{\omega} \right) \big)^2. \tag{4.3.13}$$

Now, computing the triple product with the rule (2.2.37), one has

$$\begin{aligned}\xi &= \left(\left(\boldsymbol{n}\cdot\boldsymbol{\beta}-\beta^2\right)\boldsymbol{\omega}-\boldsymbol{n}\cdot\boldsymbol{\omega}\boldsymbol{\beta}\right)^2-\beta^4\left(\boldsymbol{n}\cdot\boldsymbol{\omega}\right)^2 \\ &= \left(\boldsymbol{n}\cdot\boldsymbol{\beta}-\beta^2\right)^2\omega^2+\left(\boldsymbol{n}\cdot\boldsymbol{\omega}\right)^2\beta^2-2\left(\boldsymbol{n}\cdot\boldsymbol{\beta}-\beta^2\right)\left(\boldsymbol{n}\cdot\boldsymbol{\omega}\right)\boldsymbol{\omega}\cdot\boldsymbol{\beta}-\beta^4\left(\boldsymbol{n}\cdot\boldsymbol{\omega}\right)^2.\end{aligned} \tag{4.3.14}$$

From (4.3.1) and (4.3.7), one has

$$\boldsymbol{\omega}\cdot\boldsymbol{\beta}=\omega\beta\boldsymbol{e}_z\cdot\boldsymbol{e}_\phi=0, \tag{4.3.15}$$

since the versors are perpendicular to one another by definition. Therefore, the double product term in (4.3.14) vanishes. Now, using (4.3.1) and (4.3.7) once again, one has

$$\xi=\beta^2\omega^2\left[\left(\boldsymbol{n}\cdot\boldsymbol{e}_\phi-\beta\right)^2+\left(1-\beta^2\right)\left(\boldsymbol{n}\cdot\boldsymbol{e}_z\right)^2\right]. \tag{4.3.16}$$

Adding and subtracting $\beta^2\omega^2[(1-\beta\boldsymbol{n}\cdot\boldsymbol{e}_\phi)^2]$ in the right side of (4.3.16) and rearranging some terms yields

$$\xi=\beta^2\omega^2\left[\left(1-\beta\boldsymbol{n}\cdot\boldsymbol{e}_\phi\right)^2-\left(1-\beta^2\right)\left(1-\left(\boldsymbol{n}\cdot\boldsymbol{e}_\phi\right)^2-\left(\boldsymbol{n}\cdot\boldsymbol{e}_z\right)^2\right)\right]. \tag{4.3.17}$$

Using the fact that

$$\left(\boldsymbol{n}\cdot\boldsymbol{e}_\rho\right)^2+\left(\boldsymbol{n}\cdot\boldsymbol{e}_\phi\right)^2+\left(\boldsymbol{n}\cdot\boldsymbol{e}_z\right)^2=|\boldsymbol{n}|^2=1, \tag{4.3.18}$$

one has

$$\left(\left(\boldsymbol{n}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right)^2-\left(\boldsymbol{n}\cdot\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)^2=\beta^2\omega^2\left[\left(1-\beta\boldsymbol{n}\cdot\boldsymbol{e}_\phi\right)^2-\left(1-\beta^2\right)\left(\boldsymbol{n}\cdot\boldsymbol{e}_\rho\right)^2\right]. \tag{4.3.19}$$

Inserting equations (4.3.19) and (4.3.12) into (3.2.3) readily yields (4.3.3).

Upon inspecting (4.3.3), it appears that the dependence of $\mathrm{d}P^*/\,\mathrm{d}o(\boldsymbol{x},t^*)$ on $\boldsymbol{x}$ and t^* is described by two angles: the angle $\theta(\boldsymbol{x},t^*)$ between $\boldsymbol{n}(\boldsymbol{x},t^*)$ and $\boldsymbol{e}_\phi(\omega t^*)$ and the angle $\psi(\boldsymbol{x},t^*)$ between $\boldsymbol{n}(\boldsymbol{x},t^*)$ and $\boldsymbol{e}_z$. Viewing θ and ψ as independent variables, the differential radiated power is given by

$$\frac{\mathrm{d}P^*}{\mathrm{d}o}(\theta,\psi,t^*)=\frac{(q\omega\beta)^2}{4\pi c}\frac{(1-\beta\cos\theta)^2-(1-\beta^2)(\sin^2\psi-\cos^2\theta)}{(1-\beta\cos\theta)^5}. \tag{4.3.20}$$

Analogously to the linear case, the angular dependence of (4.3.20) is governed by the function

$$f(\theta,\psi;\beta) = \frac{(1-\beta\cos\theta)^2 - (1-\beta^2)(\sin^2\psi - \cos^2\theta)}{(1-\beta\cos\theta)^5}, \tag{4.3.21}$$

where $0 \leq \beta < 1$, $0 \leq \theta < \pi$ and $0 \leq \psi < \pi$. For any fixed ψ, the function $f(\theta,\psi;\beta)$ (as a function of θ) has two maxima for $\theta = 0, \pi$, and a minimum for $\theta = \theta(\psi;\beta)$, where

$$\theta(\psi;\beta) = \cos^{-1}\left\{\frac{1-\beta^2}{\beta}\left[\frac{1}{1-\beta^2} - \frac{4}{3} + \frac{4}{3}\left(1 - \frac{15\left(1-\beta^2\sin^2\psi\right)}{16\left(1-\beta^2\right)}\right)^{1/2}\right]\right\}. \tag{4.3.22}$$

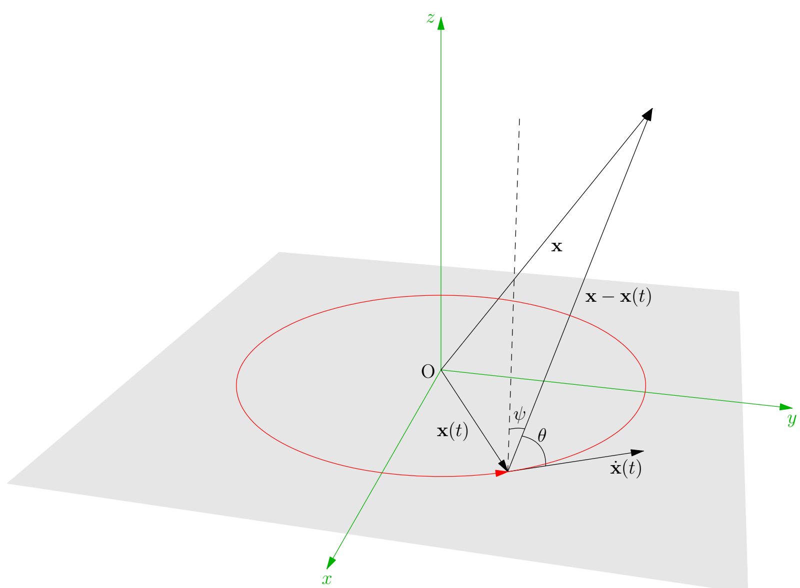

FIGURE 4.3.2. The angles θ and ψ.

The most interesting case is represented by $\psi = \pi/2$. In this case, a simple calculation shows that

$$\theta(\pi/2;\beta) = \cos^{-1}\beta. \tag{4.3.23}$$

Furthermore, the function $f(\theta,\psi = \pi/2;\beta)$ evaluated at the stationary points yields

$$f(0,\pi/2;\beta) = \frac{1}{(1-\beta)^3}, \tag{4.3.24a}$$

$$f(\theta(\pi/2;\beta),\pi/2;\beta) = 0, \tag{4.3.24b}$$

$$f(\pi,\pi/2;\beta) = \frac{1}{(1+\beta)^3}. \tag{4.3.24c}$$

Proof. A simple computation shows that one can write $f(\theta,\psi;\beta)$ in two other equivalent forms:

$$f(\theta,\psi;\beta) = \frac{1-\beta^2}{\beta^2}\left[\frac{1}{1-\beta^2}\frac{1}{(1-\beta\cos\theta)^3} - \frac{2}{(1-\beta\cos\theta)^4} + \frac{1-\beta^2\sin^2\psi}{(1-\beta\cos\theta)^5}\right] \tag{4.3.25a}$$

$$= \frac{1}{(1-\beta\cos\theta)^5}\left[(\beta-\cos\theta)^2 + \left(1-\beta^2\right)\cos^2\psi\right]. \tag{4.3.25b}$$

From (4.3.25b), it is clear that $f(\theta,\psi;\beta) \geq 0$. From (4.3.25a), one easily computes

$$\frac{\partial f}{\partial\theta}(\theta,\psi;\beta) = -\frac{\left(1-\beta^2\right)\sin\theta}{\beta\,(1-\beta\cos\theta)^6}\left[\frac{3}{1-\beta^2}(1-\beta\cos\theta)^2 - 8\,(1-\beta\cos\theta) + 5\left(1-\beta^2\sin^2\psi\right)\right]. \tag{4.3.26}$$

For fixed ψ, the values of θ corresponding to the extrema of $f(\theta,\psi;\beta)$ are those for which $\partial f/\partial\theta(\theta,\psi;\beta) = 0$. Hence they are the solutions to either equations

$$\sin\theta = 0, \tag{4.3.27a}$$

$$\frac{3}{1-\beta^2}(1-\beta\cos\theta)^2 - 8\,(1-\beta\cos\theta) + 5\left(1-\beta^2\sin^2\psi\right) = 0. \tag{4.3.27b}$$

Upon solving the second equation in the unknown $1-\beta\cos\theta$, equations (4.3.27) can be written as

$$\sin\theta = 0, \tag{4.3.28a}$$

$$1-\beta\cos\theta = \frac{4\left(1-\beta^2\right)}{3}\left[1 \pm \left(1 - \frac{15\left(1-\beta^2\sin^2\psi\right)}{16\left(1-\beta^2\right)}\right)^{1/2}\right]. \tag{4.3.28b}$$

In order to choose the correct solution in (4.3.28b), it is necessary to evaluate both expressions when $\beta \to 0$. A simple computation yields

$$\frac{4\left(1-\beta^2\right)}{3}\left[1 + \left(1 - \frac{15\left(1-\beta^2\sin^2\psi\right)}{16\left(1-\beta^2\right)}\right)^{1/2}\right] \to \frac{5}{3}, \qquad \beta \to 0, \tag{4.3.29a}$$

$$\frac{4\left(1-\beta^2\right)}{3}\left[1-\left(1-\frac{15\left(1-\beta^2\sin^2\psi\right)}{16\left(1-\beta^2\right)}\right)^{1/2}\right]\to 1,\qquad \beta\to 0. \tag{4.3.29b}$$

Since obviously $1-\beta\cos\theta\to 1$ as $\beta\to 0$, the solution (4.3.29a) with the + sign must be rejected. Thus, the values of θ corresponding to the extrema of $f(\theta,\psi;\beta)$ for fixed ψ are

$$\theta = 0, \pi, \theta(\psi;\beta), \tag{4.3.30}$$

where $\theta(\psi;\beta)$ is given by (4.3.22).

It is easy to see that

$$(\beta-\cos\theta)^2 \leq (1-\beta\cos\theta)^2\,. \tag{4.3.31}$$

A quick computation shows that

$$f(0,\psi;\beta) = \frac{1-\beta+(1+\beta)\cos^2\psi}{(1-\beta)^4}. \tag{4.3.32}$$

A comparison between (4.3.21) and (4.3.32) shows that, taking (4.3.31) into account,

$$f(\theta,\psi;\beta)\leq f(0,\psi;\beta). \tag{4.3.33}$$

Therefore, $\theta=0$ is a maximum value for $f(\theta,\psi;\beta)$. Repeating the same steps for $\theta=\pi$ will show that $\theta=\pi$ is also a maximum value, while $\theta=\theta(\psi;\beta)$ is a minimum value since $f(\theta,\psi;\beta)$ is continuous.

The rest of the proof (equations (4.3.23) and (4.3.24)) is easily derived by consecutively substituting $\psi=\pi/2$ into the correct equations.

The total radiated power of our charge is given by

$$P^*(t^*) = \frac{2\left(q\omega\beta\right)^2}{3c}\frac{1}{\left(1-\beta^2\right)^2}. \tag{4.3.34}$$

Proof. Using (4.3.11) first, followed by (3.2.12), one has

$$\begin{aligned}|\dot{\boldsymbol\beta}|^2-|\boldsymbol\beta\times\dot{\boldsymbol\beta}|^2 &= |\boldsymbol\omega\times\boldsymbol\beta|^2-|\beta^2\boldsymbol\omega|^2 \\ &= |\boldsymbol\omega|^2|\boldsymbol\beta|^2-(\boldsymbol\omega\cdot\boldsymbol\beta)^2-|\beta^2\boldsymbol\omega|^2\end{aligned} \tag{4.3.35}$$

$$= \omega^2 \beta^2 \left(1 - \beta^2\right),$$

where in the last step we have made use of (4.3.15). Using this relation in (3.2.8), we get immediately (4.3.34).

Here are the angular plot of $f(\theta, \psi; \beta)$ for some ψ, β values.

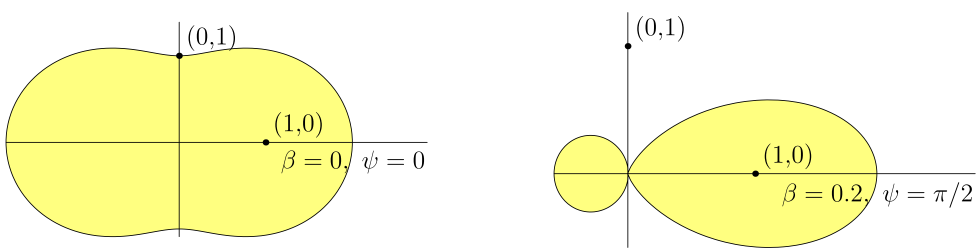

Figure 4.3.3. Angular plot of the angular function $f(\theta, \psi; \beta)$ for $\beta = 0.0, 0.2$ and $\psi = 0, \pi/2$.

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