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RENORMALIZATION GROUP FLOWS BETWEEN NON-UNITARY CONFORMAL MODELS

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Every particle is divine and possesses information about the whole.
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Abstract

Recentemente sono stati valutati come fisicamente consistenti diversi modelli non-hermitiani sia in meccanica quantistica che in teoria dei campi. La classe dei modelli pseudo-hermitiani, infatti, si adatta ad essere usata per la descrizione di sistemi fisici dal momento che, attraverso un opportuno operatore metrico, risulta possibile ristabilire una struttura hermitiana ed unitaria. I sistemi $\mathcal{PT}$-simmetrici, poi, sono una categoria particolarmente studiata in letteratura. Gli esempi riportati sembrano suggerire che anche le cosiddette teorie conformi non-unitarie appartengano alla categoria dei modelli $\mathcal{PT}$-simmetrici, e possano pertanto adattarsi alla descrizione di fenomeni fisici. In particolare, si tenta qui la costruzione di determinate lagrangiane Ginzburg-Landau per alcuni modelli minimali non-unitari, sulla base delle identificazioni esistenti per quanto riguarda i modelli minimali unitari. Infine, si suggerisce di estendere il dominio del noto teorema c alla classe delle teorie di campo $\mathcal{PT}$-simmetriche, e si propongono alcune linee per una possibile dimostrazione dell'ipotizzato teorema c_{eff}.

Abstract

Recently many non-hermitian systems have been evaluated as physically sensible both in a quantum mechanical context and in quantum field theory. The class of pseudo-hermitian models can indeed be used to describe physical phenomena, since hermiticity and unitarity relations can be recoiled through the aid of a metric operator. $\mathcal{PT}$-symmetric models are a sub-class of them, studied in the literature. The studied examples seem to suggest that even the so-called non-unitary conformal field theories are part of this class, and as such can be adapted to the description of physical systems. In this thesis the construction of Ginzburg Landau lagrangian for some non-unitay minimal model is proposed, basing the suggestion on the identifications already existing in the unitary case. Finally, the possibility to extend the well known c-theorem to the class of $\mathcal{PT}$-symmetric field theories is sketched, as well as some possible lines of reasoning to demonstrate it. This could eventually lead to the so-called c_{eff}-theorem.

Contents

Introduzione

Hermiticity of the hamiltonian has come to be one of the fundamental axyoms of any quantum theory, in order to work with real energy values, and to have a probabilistic sensible interpretation of the described phenomena. In recent years it has been undestood, though, that a more general requirement can accomplish the task as well, namely pseudo-hermiticity. Pseudo-hermiticity has been shown to generate appropriate physical theories, through a suitable redefinition of the inner product; so that hamiltonians which have always been discarded as non-physical, can now be re-evaluated in a new context. For example, some hamiltonians with complex or unbounded potentials have been bent to usage. $\mathcal{PT}$-symmetric theories constitute a sub-class of pseudo-hermitian models, whose treatement is pretty well understood. In the context of conformal field theory (CFT), in particular, the so-called non-unitary models were known to possess ghost states, being non-unitary, but at the same time they fit in the description of some physical phenomena, as quantum Hall effect, percolation, polymers structure, turbulent flows in magneto-hydrodynamics, self-organizing systems, etc. Moreover some results on S-matrix of non-unitary CFT have been obtained, showing the same duality: the residue calculated at the pole shows the expected non-unitarity, while on the other hand fundamental unitarity conditions are respected. The situation suggests then to analyze both concepts of non-unitarity in the sense of norms, and pseudo-hermiticity. This thesis follows this suggestion and analyzes some problematic aspects related to it. In particular the identification of non-unitary minimal models with some Ginzburg-Landau (GL) theory is attempted, based on previous work on the minimal models and on the Yang-Lee model. Allowing the coupling parameter of GL lagrangians to take complex values seems to be the way to produce this identification; moreover, renormalization group (RG) flows between minimal models are analyzed, and interpreted in this wider picture. And when considering the RG theory, one of its main results in the context of (1+1) quantum field theory is the well known c-theorem, proved by A.B. Zamolodchikov in the case of unitary models. The c-theorem has a deep meaning, showing the RG flow as a sort of irreversible process, where information is lost along the RG way; this interpretation is validated also by more recent formulation of the theorem, where entanglement entropy is considered. The theorem does not hold instead for non-unitary theories, as various case studies have explicitly shown. Anyway this is not a surprise, because when studying non-unitary CFT, it is found that the free energy is dominated by terms proportional to $c_{eff} = c - 12d$, instead of c as in the unitary case, which shows the presence of a ground state different from the conformal vacuum, whose conformal dimension is $d < 0$, which breaks the conformal symmetry. This means that the physically relevant quantity in non-unitary conformal models is c_{eff} itself. The possibility to extend the theorem to non-unitary theories has been imagined, but for technical difficulties a c_{eff}-theorem has never been proved. In this thesis we have considered these difficulties, and proposed some way to overcome them. We have organized the work as follows:

Chapter 1 We recollect the general theory and main results about pseudo-hermitian systems, with the special focus on the $\mathcal{PT}$-symmetric case. Metric operator are introduced. These concepts are joined with the analysis of non-unitary CFT, and with basic ideas about integrability and S-matrix theory.

Chapter 2 Having exposed the conceptual basis we focus on non-unitary conformal systems, and we specialize on non-unitary minimal models, and the RG flows between them. On the basis of similar procedures for the unitary minimal models and the Yang-Lee non-unitary minimal model, we attempt the construction of some GL theories for the simpler non-unitary minimal models.

Chapter 3 From the models themself the focus now shifts to the RG flows between them. Zamolodchikov c-theorem and Casini and Huerta c-theorem are critically analyzed, with special attention to the underlying ideas on unitarity. Some ideas about a possible extension to non-unitary cases are proposed, where the main quantity to analyze would be the so-called effective central charge c_{eff}.

Chapter 1

$\mathcal{PT}$-symmetry

The assumption of hermiticity of the hamiltonian comes along with the need of having real energy eiegenvalues, together with the associated time-evolution unitarity. Non-hermitian hamiltonians are then ususally descarded as non-physical. A particular class of non-hermitian systems, though, have been proven to produce real eigenvalues, namely pseudo-hermitian hamiltonians. They can also be translated to some hermitian equivalent hamiltonian through a similarity transformation, i.e. they are quasi-hermitian. The equivalence operator, though, is neither unique nor always local. $\mathcal{PT}$-symmetric systems are a sub-class among pseudo-hermitians, which can be treated, when the $\mathcal{PT}$-symmetry is unbroken, through the construction of a $\mathcal{C}$ operator. Moreover, the related concepts of unitary evolution in these cases can be reconstructed via a redefinition of the inner product. A review of such topics can be found in [1].

1.1 Pseudo-hermitian systems

Among all the non-hermitian hamiltonians, the class of $pseudo - hermitians$ is of interest, since with them one can have a physical theory as well, meaning real energy eigenvalues. They are defined by the existence of some positive definite hermitian operator η such that

$$\eta H \eta^{-1} = H^\dagger \tag{1.1}$$

This operator is, in general, neither universal nor unique. In presence of a pseudo-hermitian theory a positive definite inner product is defined such that hermiticity and unitarity relations -with respect to this product- are recovered.

$$\langle \cdot | \cdot \rangle_\eta = \langle \cdot \eta | \cdot \rangle \tag{1.2}$$

Since this operator re-establishes a positive metric in Hilbert space, it is called *metric operator*. Moreover, an equivalent hermitian theory can be found through the similarity transformation

$$h = \sqrt{\eta} H \sqrt{\eta}^{-1}. \tag{1.3}$$

which in some case could be non-local. This last relation defines this system as *quasi-hermitian*. Diagonalizing the two hamiltonians the following relations hold:

$$H|\Psi_n\rangle = E_n|\Psi_n\rangle \qquad \langle\tilde{\Psi}_n|H = \langle\tilde{\Psi}_n|E_n \tag{1.4}$$

$$h|\psi_n\rangle = E_n|\psi_n\rangle \qquad \langle\psi_n|h = \langle\psi_n|E_n \tag{1.5}$$

where the notation $\langle\tilde{\Psi}| = \langle\Psi|\eta$ has been adopted. Comparing them, we find

$$|\psi_n\rangle = \sqrt{\eta}|\Psi_n\rangle \tag{1.6}$$

$$\langle\psi_n| = \langle\tilde{\Psi}_n|\sqrt{\eta}^{-1} \tag{1.7}$$

Respectively, the density matrices are

$$\tilde{\varrho}_n = |\Psi_n\rangle\langle\tilde{\Psi}_n| \qquad \rho_n = |\psi_n\rangle\langle\psi_n| \tag{1.8}$$

and their relation is

$$\tilde{\varrho}_n = \sqrt{\eta}^{-1}\rho_n\sqrt{\eta} \tag{1.9}$$

Analizing, for example, the scalar product of the hermitian equivalent theoy, where $|a_n\rangle$ is its basis, we can see how it can be expressed in the pseudo-hermitian basis $|A_n\rangle$. Horthogonality then reads:

$$\delta_{n,m} = \langle a_n|a_m\rangle = \langle\tilde{A}_n|A_m\rangle = \langle A_n\eta|A_m\rangle. \tag{1.10}$$

In this way, we are dealing with a proper quantum system. Slightly enlarging the notation we could also introduce the pseudo-hermitian conjugate

$$H^{\ddagger} = \eta^{-1}H^{\dagger}\eta = H \tag{1.11}$$

$$|A\rangle^{\ddagger} = \langle\tilde{A}| = \langle A|\eta \tag{1.12}$$

so that

$$\langle\tilde{0}|H|0\rangle^{\ddagger} = \langle\tilde{0}|H|0\rangle. \tag{1.13}$$

The main problem of quasi-hermitian systems is then to explicitly find metric operators. In the literature some methods are used. (1) If the eigenfunctions can be found explicitly, and so do their biorthonormal dual functions, then the metric operator is constructed as

$$\eta = \sum_n |\Psi_n\rangle\langle\Psi_n|. \tag{1.14}$$

otherwise (2) the entwining relation (1.1) can be solved directly, but often it can just be done perturbatively.

1.1.1 $\mathcal{PT}$-symmetric systems

In many works of Bender et al. the special case of $\mathcal{PT} - symmetric$ hamiltoninans with discrete spectrum are treated in great detail. We report here a review of the principal results, mainly considering [2]. For this class of systems

$$[\mathcal{PT}, H] = 0 \tag{1.15}$$

holds and a $\mathcal{C}$ operator can be constructed, such that

$$[\mathcal{PT}, \mathcal{C}] = 0 \qquad [\mathcal{C}, H] = 0 \qquad \mathcal{C}^2 = 1 \tag{1.16}$$

Moreover the $\mathcal{PT}$-symmetry is considered unbroken, which means that for eigenvectors:

$$\mathcal{PT}|\Psi\rangle = |\Psi\rangle. \tag{1.17}$$

With the aid of this operator, a positive definite inner product can be defined

$$\langle\cdot|\cdot\rangle_{\mathcal{CPT}} \tag{1.18}$$

such that norms are preserved and unitarity is restored. The $\mathcal{CPT}$ operator can be thus considered as a metric operator. Exponential representation has been given to $\mathcal{C}$:

$$\mathcal{C} = e^{\mathcal{Q}}\mathcal{P} \tag{1.19}$$

such that

$$\mathcal{CPT} = (e^{\mathcal{Q}}\mathcal{P})\mathcal{PT} = \eta\mathcal{T} \tag{1.20}$$

where η is also a metric operator, which relates the hamiltonian to its hermitian counterpart. This means that

$$\langle f|g\rangle_{\mathcal{CPT}} = \langle f|\mathcal{C}|g\rangle \tag{1.21}$$

It was a main result in [1] to proove that any quasi-hermitian system with discrete spectrum is also η-pseudo-hermitian, and that a generalized $\mathcal{CPT}$ structure ca be built for them. It is from this perspective that non-unitary minimal models are in this thesis considered.

1.1.2 From $\mathcal{PT}$-symetric quantum mechanics to $\mathcal{PT}$-symmetric quantum field theory

Research on non-hermitian operators have been developed in the past years also in Bologna, by mathematicians, with mathematical rigour and richness of results. Being far from the intention of this thesis to dirime delicate issues on intellectual rights, it seems to the author at least respectful to report the presence of these studies. We propose for example to focus on an article [3], where a pair of quantum problems have been compared: the spherically symmetric anharmonic oscillator

$$H(g,d) = \frac{1}{2}(-\nabla^2 + \mathbf{x}^2) + g^2(\mathbf{x}^2)^2 \tag{1.22}$$

defined in $L^2(\mathbf{R^d})$ and the double well with linear symmetry breaking

$$H(g,j) = -\frac{d^2}{dx^2} + x^2(gx-1)^2 - j(gx - \frac{1}{2}). \tag{1.23}$$

Denoting with $\tilde{E}_{j,n}$ the eigenvalues of the first hamiltonian, and with $E_{j,n}$ the eigenvalues of the second one, we report here one main result of the article: when considering the anharmonic oscillator in $d = 1$ and suitably complex-translated

$$H(ig,j)_\epsilon = \frac{1}{2}[-\frac{d^2}{dr^2} + \frac{j^2-1}{4(r-i\epsilon)^4} + (r-i\epsilon)^2] - g^2(r-i\epsilon)^4 \tag{1.24}$$

the identity of the energy levels of the two systems holds

$$\tilde{E}_{j,n}(ig) = E_{j,n}(g). \tag{1.25}$$

The reachness of the work, for which we remand to the original papers, consists in its mathematical rigour in presenting the above-mentioned equivalences among non-hermitian systems and some other hermitian ones. Another relevant paper is [4], where a $\mathcal{PT}$-symmetric quantum particle on a segment was considered and solved exactly for the metric. What characterizes this work is the nature of non-unitarity of the model, which is encoded in Robin-type boundary conditions

$$\psi'(0) = -i\alpha\psi(0) \qquad \psi'(d) = -i\alpha\psi(d) \tag{1.26}$$

instead that in the hamiltonian, which is just

$$H_\alpha = -\frac{d^2}{dx^2} \tag{1.27}$$

with Hilbert space $\mathcal{H} = L^2((0,d))$. The metric for this system is

$$\eta_\alpha = 1 + \phi^0_\alpha(\phi^0_\alpha, \cdot) + \eta_0 + i\alpha\eta_1 + \alpha^2\eta_2 \tag{1.28}$$

where

$$\phi^0_\alpha(x) = \sqrt{\frac{1}{d}}e^{i\alpha x} \tag{1.29}$$

and the operators η_0, η_1, η_2 acts in $\mathcal{H}$ as

$$(\eta_0\psi)(x) = -\frac{1}{d}(J\psi)(d) \tag{1.30}$$

$$(\eta_1\psi)(x) = 2(J\psi)(x) - \frac{x}{d}(J\psi)(d) - \frac{1}{d}(J^2\psi)(d) \tag{1.31}$$

$$(\eta_2\psi)(x) = -(J^2\psi)(x) - \frac{x}{d}(J^2\psi)(d) \tag{1.32}$$

with

$$(J\psi)(x) = \int_0^x \psi. \tag{1.33}$$

It can be noted, as expected, that the metric reduce to the identity operator when $\alpha = 0$. Models of $\mathcal{PT}$-symmetric quantum mechanics have been analyzed, among others, in [5]. The model

$$H = \frac{p^2}{2} + e^{2ix} \tag{1.34}$$

was studied, and compared in [6] with the imaginary Liouville field theory, with the explicit proposal to consider it as a $\mathcal{PT}$-symmetric model. In the quanto-mechanical case the authors of papers [6] solved the problem of finding the metric through deformation quantizaion techniques, transforming the entwining relation $H\eta^{-1} = \eta^{-1}H^\dagger$ in a partial differential equation (PDE). They adopted a phase space dual metric $\tilde{G}(x,p)$ and its Weyl kernel

$$G^{-1}(x', p') = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\ \tilde{G}(x,p)e^{i\tau(p'-p)+i\sigma(x'-x)} \tag{1.35}$$

together with the associative Groenewold star product

$$\star = \exp(\frac{i}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} - \frac{i}{2}\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}) \tag{1.36}$$

to transform the pseudo-hermiticity relation into

$$H(x,p) \star \tilde{G}(x,p) = \tilde{G}(x,p) \star \overline{H(x,p)} \tag{1.37}$$

which boils down to the PDE

$$p\frac{\partial}{\partial x}G(\tilde{x},p) = \sin(2x)\tilde{G}(x, p-1). \tag{1.38}$$

As solution it is found

$$\tilde{G}(x,p) = \frac{(\sin^2 x)^p}{(\Gamma(p+1))^2}. \tag{1.39}$$

Another quanto-mechanical system studied by the same authors has been the ix^3 potential,

$$H = \frac{p^2}{2} + i\epsilon x^3. \tag{1.40}$$

Using the same procedure they arrived at the entwining relation in the form of a PDE

$$(p\partial_x - 2x^3 + \frac{3}{2}x\partial_p^2)\tilde{G}(x,p) = 0 \tag{1.41}$$

which possesses the real general solution

$$\tilde{G}(x,p) = \int dtdz\ F(2iz + t^2)e^{ixt - ipz + \frac{3}{4}itz^2 + \frac{1}{2}t^3 z - \frac{1}{10}it^5} \tag{1.42}$$

with F some appropriate function to be chosen. These results join in the literature with the ones of [7], where the massive $i\varphi^3$ theory is also analyzed by Bender et al.

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\nabla^2\varphi + \frac{1}{2}m^2\varphi^2 + i\epsilon\varphi^3. \tag{1.43}$$

Considering the parity operator

$$\mathcal{P} = e^{i\frac{\pi}{2}\int\ dx[\varphi^2(x,t) - \pi^2(x,t) - 1]} \tag{1.44}$$

and assuming the fields as pseudo-scalars

$$\mathcal{P}\varphi(x,t)\mathcal{P} = -\varphi(-x,t) \tag{1.45}$$

$$\mathcal{P}\pi(x,t)\mathcal{P} = -\pi(-x,t) \tag{1.46}$$

they calculated iteratively the first terms of the expansion

$$\mathcal{C} = e^{\epsilon\mathcal{Q}_1 + \epsilon^3\mathcal{Q}_3 + \ldots}. \tag{1.47}$$

What they found is that

$$Q_1 = \int\int\int dx\ dy\ dz\ [M_{(xyz)}\pi_x\pi_y\pi_z + N_{x(yz)}\varphi_y\pi_x\varphi_z] \tag{1.48}$$

where the notation for M and N means that those functions are symmetric in their arguments closed in parethesis. We report here just the results for the $(1+1)$ dimensional case,

$$M_{(xyz)} = -\frac{1}{\pi\sqrt{3}m^2}K_0(mr) \tag{1.49}$$

$$N_{x(yz)} = -\frac{3\sqrt{3}}{4\pi}[1 - \frac{(y-z)^2}{r^2}]k_0(mr) + \frac{\sqrt{3}}{\pi}[1 - 3\frac{(y-z)^2}{2r^2}]\frac{k_0(mr)}{mr} + \tag{1.50}$$

$$\frac{1}{m^2}[1 - 3\frac{(y-z)^2}{r^2}]\delta(x-y)\delta(x-z) \tag{1.51}$$

where

$$r^2 = \frac{1}{2}[(x-y)^2 + (y-z)^2 + (z-x)^2] \tag{1.52}$$

and K_0 is a Bessel function. This is an example where the metric is non-local, and that would introduce non-locality in the hermitian equivalent system. The collection of these results found in the literature suggests to regard the so-called non-unitary conformal theories as pseudo-hermitian models.

1.2 Towards $\mathcal{PT}$-symmetric CFT

One of the firsts conformal field theory ever studied has been the massless Thirring model. It is interesting that it is (in its massive extension) one of the first $\mathcal{PT}$-symmetric quantum field theory ever presented [2]. In particular the model

$$\mathcal{L}_{Th} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi), \tag{1.53}$$

when supplemented with a $\mathcal{PT}$-symmetric term

$$\mathcal{L} = \mathcal{L}_{Th} + \varepsilon m\bar{\psi}\gamma^5\psi \tag{1.54}$$

belongs to the class of non-hermitian $\mathcal{PT}$-symmetric models. This can be seen expliciting the fact that in $(1+1)$ dimensions the Clifford algebra is composed by

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.55}$$

plus

$$\gamma_5 = \gamma_0\gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.56}$$

The action of the parity and time-reversal operators are

$$\mathcal{P}\psi(x,t)\mathcal{P} = \gamma_0\psi(-x,t) \tag{1.57}$$

$$\mathcal{T}\psi(x,t)\mathcal{T} = \gamma_0\psi(x,-t) \tag{1.58}$$

and through them the lagrangian density (1.53) has been proven to be equivalent to the Thirring model itself, but with a modified mass given by

$$\mu^2 = m^2(1-\varepsilon). \tag{1.59}$$

The similarity transformation is given by the metric operator $\eta = e^Q\mathcal{P}$, where Q has been calculated to be

$$Q = -\tanh^{-1}\varepsilon \int \psi^\dagger\gamma_5\psi. \tag{1.60}$$

We point out that when $\varepsilon = 1$, this non-hermitian model is (obviously) equivalent to a unitary CFT, namely the massless Thirring model. Together with the above system another one seems to be related to CFT, in ways yet to be fully established. The system in analysis is a modification of the usual Sine-Gordon model

$$\mathcal{L}_{SG} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{m^2}{\gamma}(\cos\sqrt{\gamma}\varphi - 1) \tag{1.61}$$

which, as it is known, is dual to the Thirring model. Since the latter presents a $\mathcal{PT}$-symmetric nature, the first would do it as well.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\phi + \frac{m^2}{\gamma}(\cos\sqrt{\gamma}\varphi - 1 + i\epsilon\,\sin\sqrt{\gamma}\varphi) = \mathcal{L}_{SG} + i\epsilon\phi_{pert.} \tag{1.62}$$

It has been shown by Bender (always in [2]) to be spectrally equivalent to

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\varphi\partial^\mu\phi + \frac{m^2(1-\epsilon^2)}{\gamma}\cos\sqrt{\gamma}\varphi \tag{1.63}$$

in the sense that

$$\eta\mathcal{L}\eta^{-1} = \mathcal{L}'. \tag{1.64}$$

with

$$\eta = e^{-Q} \tag{1.65}$$

$$Q = \frac{2\tanh^{-1}\epsilon}{\sqrt{\gamma}}\int dx\;\pi(x) \tag{1.66}$$

$$\pi = \partial_0\varphi. \tag{1.67}$$

Again, the limit case $\epsilon = 1$ makes contact with CFT: the $\mathcal{PT}$-symmetric lagrangian would be the imaginary Liouville theory:

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\varphi\partial^\mu\phi + \frac{m^2}{\gamma}(e^{i\sqrt{\gamma}\varphi} - 1) \tag{1.68}$$

and its entwined counterpart would coincide with the free boson. Maybe just out of curiosity we notice that the same equivalence coul be reached by substituting the equation of motion in the energy-momentum tensor. To express this result we use Smirnov renormalization scheme [8]:

$$T_{\mu\mu} = \frac{M_1^2}{4\sin\xi}\;(\cos\sqrt{\gamma}\varphi + i\epsilon\sin\sqrt{\gamma}\varphi) \tag{1.69}$$

$$\Box\varphi = \frac{\sqrt{2}M_1^2}{\sqrt{\gamma}\sin\xi}\xi\;(\sin\sqrt{\gamma}\varphi - i\epsilon\cos\sqrt{\gamma}\varphi) \tag{1.70}$$

This would lead to

$$T_{\mu\mu} = \frac{M_1^2(1-\epsilon^2)}{4\sin\xi}\;\cos\sqrt{\gamma}\varphi + i\epsilon 2^{-5/2}\frac{\sqrt{\gamma}}{\xi}\Box\varphi \tag{1.71}$$

wich is the energy-momentum tensor associated to Bender's lagrangian plus total derivative. As mentioned above, Curtrigth et al. compared the reported quantum-mechanical models to their field theoretic extensions. In particular, it seems very interesting to us that the imaginary Liouville theory -written in the Schrödinger wave-functional formalism-

$$\mathcal{H}[\varphi] = -\frac{\delta}{\delta^2\varphi^2} + (\partial_x\varphi)^2 + e^{2i\varphi} \tag{1.72}$$

is showed to be entwined with the free theory

$$\mathcal{H}[\phi] = -\frac{\delta}{\delta^2\phi^2} + (\partial_x\phi)^2. \tag{1.73}$$

which supports the results obtained by Bender et al. The relationsihp between them found in [6] is encoded in

$$\mathcal{H}[\varphi]e^{iF} = \mathcal{H}[\phi]e^{iF} + 2i\partial_x[(e^{i\varphi}\cos\phi)e^{iF}] \tag{1.74}$$

where

$$F[\varphi,\phi] = \int dx\ \varphi\partial_x\phi + me^{i\varphi}\sin\phi. \tag{1.75}$$

This idea could in fact shed light on the so-far misterious connection among interacting (1 + 1) field theories and Coulomb gas models. These ideas together suggests us to consider the non-unitary CFTs as generalized $\mathcal{PT}$-symmetric models, as supported by the fact that they posses real eigenvalues, even if a rigorous proof is still missing. Some more clue on the possibility of considering non-unitary CFTs as pseudo-hermitian models is found in an article [9], where Cardy and Mussardo wrote some comments about non-unitarity in the scaling Lee-Yang model, whose Landau-Ginzburg like hamiltonian is

$$H = \int d^2x[\frac{1}{2}(\partial\varphi)^2 + i(h-h_c)\varphi + ig\varphi^3] \tag{1.76}$$

It is interesting how the authors suggested a so-called C-operator to heal the non-unitarity of this model, and it is interesting because it is strongly reminiscent of the general $\mathcal{CPT}$-formalism, even if they do not explicitly express the contact with the above mentioned ideas. In fact they defined a charge conjugate operator C such that

$$C^2 = 1 \tag{1.77}$$

and

$$C\varphi C = -\varphi \tag{1.78}$$

and so that

$$CH^\dagger C = H. \tag{1.79}$$

Moreover, they assert that the multiparticle states of the Fock space, defined through the iterate action of the field φ on the vacuum state:

$$|N\rangle = \lim_{t_1,\ldots,t_N\to-\infty} \varphi(t_1,x_1)...\varphi(t_N,x_N)|0\rangle \tag{1.80}$$

are eigenvectors of such an operator, with eigenvalue $(-1)^N$:

$$C|N\rangle = (-1)^N|N\rangle. \tag{1.81}$$

This would lead us to think in the same way at the critical point $h = h_c$: the system is here a conformal one, and namely is individuated to coincide with the non-unitary minimal model $M_{2,5}$, where the above mentioned states are

$$|N\rangle = \lim_{|z|_1\ldots,|z|_N\to 0} \varphi(z_1,z_1)...\varphi(z_N,\bar{z}_N)|0\rangle \tag{1.82}$$

even if here it is a known fact that asymptotic states are not really free. Anyway, we will say more on that in next sections. The formalism adopted by Cardy and Mussardo is strongly reminiscent of the general one based in $\mathcal{PT}$-symmetry, but the precise relationship among these theories is still to be made explicit. Further and even more explicit comparisons among $\mathcal{PT}$-symmetric quantum mechanics and conformal field theory, finally, have been drawn in [10]. It would be desirable to have a general treatment to heal non-unitary CFTs from their non-unitarity, and it could be argued that, if ever present, this could be found in studying the Coulomb gas formalism, which we will explore in next sections.

1.2.1 Non-unitary CFT

To better compare $\mathcal{PT}$-symmetry and general non-unitarity in a CFT context we collect here some known facts about non-unitary CFT, following the scheme of [11], which of course is rooted in the seminal paper [12]. A CFT is said unitary when the inner product in the space of states is positive definite, that is when ghost states are not present. The Hilbert space of CFT have the following structure: above the vacuum $|0\rangle$, which is associated to the lowest eigenvalues of the Hamiltonian, highest weight vectors are built applying primary fields to the vacuum

$$|\phi\rangle = \phi(0)|0\rangle. \tag{1.83}$$

The level of a state is n if its L_0 eigenvalues is $h + n$ and higher level states are constructed by

$$L_{-k_1}...L_{-k_m}|\phi\rangle$$

with $n \geq 1$, $k_1 \geq ... \geq k_m > 0$ and $\sum_i k_i = n$. Every tower of states built on a given highest weight vector forms a representation of Virasoro algebra $\mathcal{V}(c,h)$. If a ghost is found at any level of a representation, it is non-unitary. It was proven by Kac the general formula for the determinant of the matrix of inner products of the n-level states:

$$\det M_{(n)}(c,h) = A \prod_{pq\leq n} [h - h_{p,q}(c)]^{P(n-pq)} \tag{1.84}$$

where A is a positive constant and $P(N)$ is the number of ways of writing N as a sum of positive integers, i.e. the partition of N. $h_{p,q}(c)$ then gives the scaling dimension $d = h + \bar{h}$ of the fields $\phi_{p,q}$. If the Kac determinant is positive definite, then the theory is unitary. The values of unitarity have been recognized to be

$$c \geq 1 \qquad h \geq 0 \tag{1.85}$$

and

$$c = 1 - \frac{6}{m(m+1)} \qquad h_{p,q}(c) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \tag{1.86}$$

where $m = 2, 3, 4, ...$, $p = 1, 2, ..., m-1$, $q = 1, 2, ..., p$. All other values correspont to non-unitary representations. In unitary CFT the hermiticity relation

$$[T(1/\bar{z})d(1/\bar{z})^2]^\dagger = T(z)dz^2 \tag{1.87}$$

holds, or equivalently, in terms of Virasoro generators

$$L_n^\dagger = L_{-n}, \tag{1.88}$$

and so it does for their conjugates. In particular, since the general CFT hamiltonian can be written as

$$H = L_0 + \bar{L}_0 - \frac{c}{12} \tag{1.89}$$

and the energy-momentum tensor as

$$T(z) = \sum_{n\in\mathbf{Z}} z^{-n-2} L_n \tag{1.90}$$

we might expect that in the non-unitary cases at least some of the Virasoro generators are not hermitian. Anyway the spectrum of non-unitary CFT is still real, and this suggests us to stay with the theory of $\mathcal{PT}$-symmetric quantum field theory. So we might expect the possibilty to construct a certain η operator, such that

$$\eta L_n^\dagger \eta^{-1} = L_{-n} \tag{1.91}$$

and

$$\det M_{(n,\eta)}(c,h) = A \prod_{pq\leq n} [h - h_{p,q}(c)]^{P(n-pq)} \tag{1.92}$$

is positive definite. Non-unitary CFTs have already been proposed in various physical contexts, as turbulent two-dimensional magnetohydrodynamics [13], quantum Hall effect, percolation theory, polymers description, self-organizing systems.

Free boson formalism In trying to understand the situation conduct an exercise in the free boson formalism, where, in the context of Coulomb gas, the hope is to find a general solution for the problem of unitarity. A free massless boson on the cylinder is expressed by the hamiltonian

$$H = \frac{2\pi}{L}\sum_{k\neq 0}(a_{-k}a_k + \bar{a}_{-k}\bar{a}_k) + \frac{\pi_0^2}{2gL}. \tag{1.93}$$

that can be re-expressed through the usual bosonic -plus the zero modes operators- via the transformation

$$a_k = -i\sqrt{k}\tilde{a}_k \qquad \bar{a}_k = -i\sqrt{k}\tilde{a}_{-k} \qquad k > 0 \tag{1.94}$$

$$a_k = i\sqrt{-k}\tilde{a}^\dagger_{-k} \qquad \bar{a}_k = i\sqrt{-k}\tilde{a}^\dagger_k \qquad k < 0 \tag{1.95}$$

giving

$$H = \frac{2\pi}{L}\sum_{\vec{k}\neq 0}(2|k|\tilde{a}^\dagger_k\tilde{a}_k + 2|k|) + \frac{\pi_0^2}{2gL}. \tag{1.96}$$

Inserting the definition of frequence

$$\omega_k = \frac{2\pi|k|}{L} \tag{1.97}$$

it coincides with

$$H = \sum_{|k|\neq 0} 2\omega_k(n_k + 1) + \frac{\pi_0^2}{2gL}. \tag{1.98}$$

We test the action of the $(-1)^{\mathcal{N}}$ operator, inspired by the Cardy and Mussardo article: where the number operator $\mathcal{N}$ is defined as

$$\mathcal{N} = \sum_{|k|\neq 0} n_k. \tag{1.99}$$

Of course the hypothesis follows just an analogy, because the nature of particles in the sense of Ginzburg-Landau theory and in free field formalism is different, anyway it could be useful to try the following expression. Virasoro operators are expressed as well in terms of bosonic ones:

$$L_0 = \sum_{k>0} k\tilde{a}^\dagger_k\tilde{a}_k + \frac{a_0^2}{2} = \sum_{k>0} kn_k + \frac{a_0^2}{2} \tag{1.100}$$

$$L_n = \frac{1}{2}\sum_{k<0}\sqrt{k(k-n)}\tilde{a}^\dagger_{-k}\tilde{a}_{n-k} - \frac{1}{2}\sum_{0<k<n}\sqrt{k(n-k)}\tilde{a}_{n-k}\tilde{a}_k + \frac{1}{2}\sum_{k>n}\sqrt{k(k-n)}\tilde{a}^\dagger_{k-m}\tilde{a}_k - i\sqrt{n}\tilde{a}_n a_0$$

$$L_{-n} = \frac{1}{2}\sum_{k<-n}\sqrt{k(k+n)}\tilde{a}^\dagger_{-k}\tilde{a}_{-n-k} - \frac{1}{2}\sum_{-n<k<0}\sqrt{-k(n+k)}\tilde{a}^\dagger_{n+k}\tilde{a}^\dagger_{-k} + \frac{1}{2}\sum_{k>0}\sqrt{k(k+n)}\tilde{a}^\dagger_{k+n}\tilde{a}_k$$
$$+ i\sqrt{n}\tilde{a}^\dagger_n a_0$$

with

$$L_n^\dagger = L_{-n}. \tag{1.101}$$

So that the state, expressed in Fock base, is

$$L_{-n}|\alpha\rangle = \frac{1}{2}\sum_{-n<k<0} -\sqrt{(-k)(n+k)}|n+k,-k\rangle + i\sqrt{n}\alpha|n\rangle. \tag{1.102}$$

The number operator acts on it like

$$\mathcal{N}L_{-n}|\alpha\rangle = \sum_{-n<k<0} -\sqrt{(-k)(n+k)}|n+k,-k\rangle + i\sqrt{n}\alpha|n\rangle \tag{1.103}$$

and the $(-1)^{\mathcal{N}}$ operator like

$$(-1)^{\mathcal{N}}L_{-n}|\alpha\rangle = L_{-n}|\alpha\rangle - i2\sqrt{n}\alpha|n\rangle \tag{1.104}$$

so that inserting it in the calculation of the norms:

$$\langle\alpha|L_n(-1)^{\mathcal{N}}L_{-n}|\alpha\rangle = \langle\alpha|L_nL_{-n}|\alpha\rangle - 4nh_\alpha \tag{1.105}$$

with

$$h_\alpha = \frac{\alpha^2}{2}. \tag{1.106}$$

We notice that to have the usual normalization

$$\langle\alpha|L_nL_{-n}|\alpha\rangle = 2nh_\alpha + \frac{1}{12}n(n^2-1) \tag{1.107}$$

we should multiply the first coefficien of the operator expansion L_n e L_{-n} for a $\sqrt{2}$ factor. In these calculation it is used the sum

$$\sum_{0<k<n} k(n-k) = \frac{n(n^2-1)}{6}. \tag{1.108}$$

The $(-1)^{\mathcal{N}}$ operator, in this case, does not alter the norms of the descendants of the ground state, but just the norms of the descendants of the other primaries. In any case, the free boson theory is unitary, and its introduction is not required.

Coulomb gas An explicit realization of non-unitarity scheme can be found in the Coulomb gas formalis, which conisits in the lagrangian formulation

$$\mathcal{L}_{\alpha_0} = \frac{1}{8\pi}\{\partial_\mu\varphi\partial^\mu\varphi - i\sqrt{2}\alpha_0\partial\bar{\partial}\delta(z-\infty)\delta(\bar{z}-\infty)\varphi\} \tag{1.109}$$

which is recongnized as a conformal theory with central charge

$$c = 1 - 24\alpha_0^2. \tag{1.110}$$

The peculiarity of this formulation consists in having the lagrangian always non-hermitian, for both unitary and non-unitary models. The energy-momentum tensor is

$$T(z) = -\frac{1}{2} : \partial\varphi\partial\varphi : +i\sqrt{2}\alpha_0\partial^2\varphi \tag{1.111}$$

which is non-hermitian. We remember that in this model

$$a_0|\alpha,\alpha_0\rangle = \sqrt{2}\alpha|\alpha,\alpha_0\rangle \tag{1.112}$$

and Virasoro generators are

$$\begin{aligned} L_n^{(\alpha_0)} &= L_n + i\sqrt{2n}\alpha_0(n+1)\tilde{a}_n \\ L_{-n}^{(\alpha_0)} &= L_{-n} + i\sqrt{2n}\alpha_0(n-1)\tilde{a}_n^\dagger \\ L_0^{(\alpha_0)} &= \sum_{k=1}^{\infty} a_{-k}a_k + \frac{1}{2}a_0^2 - \sqrt{2}\alpha_0 a_0 \\ L_0|\alpha,\alpha_0\rangle &= (\alpha^2 - 2\alpha\alpha_0)|\alpha,\alpha_0\rangle = h_\alpha|\alpha,\alpha_0\rangle. \end{aligned} \tag{1.113}$$

A dual Fock space structure F^*_{α,α_0} is constructed by

$$\langle x|Ay\rangle = \langle xA^t|y\rangle, \tag{1.114}$$

obtained through the transposition rules

$$a_n^t = -a_{-n} \qquad a_0^t = 2\sqrt{2}\alpha_0 - a_0. \tag{1.115}$$

and normalization

$$\langle\alpha,\alpha_0|\alpha,\alpha_0\rangle = 1. \tag{1.116}$$

Rewriting the whole expression for Virasoro operator for clarity

$$L_{-n}^{(\alpha_0)} = -\frac{1}{2}\sum_{-n<k<0}\sqrt{-k(n+k)}\tilde{a}_{n+k}^\dagger\tilde{a}_{-k}^\dagger + i\sqrt{2n}\alpha_0(n-1)\tilde{a}_n^\dagger + i\sqrt{n}\tilde{a}_n^\dagger a_0$$

$$L_n^{(\alpha_0)} = -\frac{1}{2}\sum_{0<k<n}\sqrt{k(n-k)}\tilde{a}_k\tilde{a}_{n-k} + i\sqrt{2n}\alpha_0(n+1)\tilde{a}_n - i\sqrt{n}\tilde{a}_n a_0 \qquad (1.117)$$

let us see again that the $(-1)^{\mathcal{N}}$ can act

$$(-1)^{\mathcal{N}} L_{-n}^{(\alpha_0)}|\alpha,\alpha_0\rangle = L_{-n}^{(\alpha_0)}|\alpha,\alpha_0\rangle - i2\sqrt{2n}\alpha_0(n-1)|n\rangle - i2\sqrt{n}\alpha|n\rangle. \qquad (1.118)$$

Denoting the operator as

$$(-1)^{\mathcal{N}} = \gamma \qquad (1.119)$$

and we thus find the interesting relation

$$\gamma L_{-n}^{\dagger(\alpha_0)}\gamma = L_n^{(\alpha_0)}. \qquad (1.120)$$

Doing the calculatinos we obtain that the usual norm gives

$$\langle\alpha,\alpha_0|L_n^{(\alpha_0)}L_{-n}^{(\alpha_0)}|\alpha,\alpha_0\rangle = 2nh + \frac{n(n^2-1)}{12} - 2\alpha_0^2 n(n^2-1) = \qquad (1.121)$$

$$2nh_\alpha + \frac{1}{12}c_0 n(n^2-1) \qquad (1.122)$$

as expected for the general case with

$$c_0 = 1 - 24\alpha_0^2 \qquad (1.123)$$

while the corrective term gives

$$-4nh_\alpha + 4n\alpha_0^2(n^2-1) = -4nh_\alpha - 4n(n^2-1)h_{\alpha_0}. \qquad (1.124)$$

The new norm is then

$$\langle\alpha,\alpha_0|L_n^{(\alpha_0)}\gamma L_{-n}^{(\alpha_0)}|\alpha,\alpha_0\rangle = -2nh_\alpha + \frac{n(n^2-1)}{12} + 2\alpha_0^2 n(n^2-1) \qquad (1.125)$$

and we see that can be regardes as coming from a theory possessing a central charge

$$c_0' = 1 + 24\alpha_0^2. \qquad (1.126)$$

At this point we could hope to find also some metric operator η such that

$$\langle h|T(z)\eta T(0)|h\rangle = \frac{c_{eff}}{2z^4} \qquad (1.127)$$

for reasons that will be clarified in the following chapters.

1.3 Integrability

A quantum theory is said to be integrable if an infinite numbers of conserved charges $\mathcal{Q}_s$ are present, $[\mathcal{Q}_s, H] = 0$ and in reciprocal involution, i.e.

$$[\mathcal{Q}_s, \mathcal{Q}_{s'}] = 0. \tag{1.128}$$

where s is a spin index. When this happens the theory is said *exactly solvable*, which means that the exact mass spectrum and S-matrix can be calculated and a procedure can be given for the exact computation of excited states and correlation functions [14]. In two dimensional QFT these charges are given by

$$\mathcal{Q}_s = \oint T_{s+1} dz + \Theta_{s-1} d\bar{z}. \tag{1.129}$$

As usual, each charge is associated to the zeroth component of some conserved current J^μ_s:

$$J^0_s = T_{s+1} + \Theta_{s-1}, \tag{1.130}$$

$$J^1_s = T_{s+1} - \Theta_{s-1}. \tag{1.131}$$

such that $\partial_\mu J^\mu_s = 0$. In the case of CFT, where $\Theta = 0$ the conserved charges are the integration of the various descendant of some primary field; usually the conformal family of the identity $[I]$ is considered

$$\mathcal{Q}_s = \oint T_{s+1} dz \tag{1.132}$$

where

$$T_{s+1} = \sum_k a_k T^{(k)}_{s+1} \tag{1.133}$$

with

$$T^{(k)}_{s+1} = L_{-n_1} ... L_{-n_k} I, \qquad \sum_i n_i = s + 1 \tag{1.134}$$

and since here the holomorphic and antiholomorphi sector are factorized, the conservation equation

$$\bar{\partial} T_{s+1}(z) = 0 \tag{1.135}$$

is satisfied.

Note on CFT particles It has been said that in $(1+1)$ dimensions there is no free massless scalar field theory [15]. This statement is clarified in note 10 of the same paper: even considering a free massless theory, in which the expected dispersion relation is $p_\mu p^\mu = 0$, say the massless fermion

$$\mathcal{L} = \frac{1}{2}\bar{\psi}(i\gamma^\mu \partial_\mu)\psi \tag{1.136}$$

this can be shown to contain particles. In fact, taking the usual definition of particle as a normalized eigenstate of the Casimir operator $p_\mu p^\mu$

$$p_\mu p^\mu |\text{particle}\rangle = m^2 |\text{particle}\rangle \tag{1.137}$$

it is found, for example, that the scalar state

$$: \psi\bar{\psi} : (0)|0\rangle = | : \psi\bar{\psi} :\rangle \tag{1.138}$$

is in fact an eigenstate of the same operator

$$p_\mu p^\mu | : \psi\bar{\psi} :\rangle = M^2 | : \psi\bar{\psi} :\rangle. \tag{1.139}$$

where M can be seen as some energy scale. We also note how this picture reconciles with integrability: it can be found that $\mathcal{Q}_1$ and $\mathcal{Q}_{-1}$ coincide with the light-cone components of the momentum

$$\mathcal{Q}_1 = P = P^{(0)} + P^{(1)} \tag{1.140}$$

$$\mathcal{Q}_{-1} = \bar{P} = P^{(0)} - P^{(1)} \tag{1.141}$$

thus it is

$$P\bar{P} = \mathcal{Q}_1 \mathcal{Q}_{-1}. \tag{1.142}$$

These excitations have anyway peculiar behaviour, from their two-points correlators

$$\langle : \psi\bar{\psi} : (x) : \psi\bar{\psi} : (0)\rangle \sim \frac{1}{x^\Delta} \tag{1.143}$$

which is divergent in the origin instead of presenting a delta-function behaviour. This also means that these particles have null probabilty to be produced out from the vacuum.

$$\langle : \psi\bar{\psi} : (x)\rangle = 0. \tag{1.144}$$

The emerging picture is somehow that of massless excitatioins, traveling at the speed of light, but with some definite energy, being always interacting at every scale, impossible to produce or annihilate; virtual particles cannot come into being and real particles cannot go into virtual being. Moreover, as it could be reconstructed by the Coulomb gas formalism, they could be thought as composed by more elementary excitations, what we could call bosonic photons. Particles then, in CFT, can be defined as eigenstates of these infinite charges [16].

$$\mathcal{Q}_s|\text{particles}\rangle = \omega_s^{(a)}|\text{particles}\rangle \tag{1.145}$$

where a is the particle type, and to find their masses M_a would be a main goal of S-matrix theory. But, since in CFT, due to conformal symmetry, it is not possible to define asymptotic free states, the whole S-matrix theory seems not to apply there. How this obstacle is being circumvented is explained in the following section.

1.4 S-matrix theory

A complementary view on $(1+1)$ dimensional quantum field theory is given by S-matrix theory. The concepts about integrability, introduced above, are deeply related to this topic, and in particular allows one to know the S-matrix in great details, as explained in [16]. As is known, in $(1+1)$ dimensions the dispersion relation $E^2 - p^2 = m^2$ is solved for the components as

$$p^{(0)} = m\,\cosh(\theta) \qquad p^{(1)} = m\,\sinh(\theta) \tag{1.146}$$

or alternatively, using light-cone componets

$$p = p^{(0)} + p^{(1)} = m\,e^{\theta} \qquad \bar{p} = p^{(0)} - p^{(1)} = m\,e^{-\theta}. \tag{1.147}$$

In these formulas θ is called rapidity of the considered particles, and a specific operator $A_a(\theta)$, when applied to the vacuum, has the role to create a particle of the a kind, with rapidity θ:

$$A_a(\theta)|0\rangle = |A_a(\theta)\rangle. \tag{1.148}$$

These kind of states are normalized as follows

$$\langle A_b(\theta_b)|A_a(\theta_a)\rangle = 2\pi\delta_{ab}\delta(\theta_b - \theta_a). \tag{1.149}$$

When applying one of the conserved charges to a particle state

$$\mathcal{Q}_s|A_a(\theta)\rangle = \omega_s^{(a)}|A_a(\theta)\rangle \tag{1.150}$$

which can be further expressed as

$$\omega_s^{(a)} = \chi_s^{(s)} e^{s\theta} \tag{1.151}$$

and $\chi_s^{(s)}$ is called the eigenvalue of the charge. In the considered case of $(1+1)$ dimensions there are some constraints on the physics of the quantum fields and particles, that concerns the S-matrix as well. In particular particle production is absent, the momenta are unchanged in the final state compared with the initial one, and each each $n \to n$ S-matrix is factorizable into a product of $2 \to 2$ S-matrices. Thus the physics of such systems is encoded in the two-particles S-matrix, which is the key element of the theory, and to find it becomes the main goal of the theory. The two-particles S-matrix is also linked to the so-called Zamolodchikov algebra,

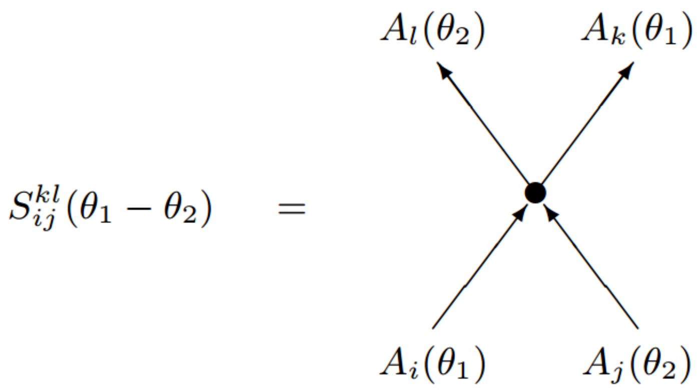

Figure 1.1: Graphical representation of a two-particles scattering associated to the two-particles S-matrix.

expressed by the relation

$$|A_i(\theta_1)A_j(\theta_2)\rangle_{in} = S_{ij}^{kl}(\theta_1 - \theta_2)|A_k(\theta_1)A_l(\theta_2)\rangle_{out} \tag{1.152}$$

The matrix is non-zero only when $m_i = m_k, m_j = m_l$ Moreover the two-particles S-matrix posseses a number of properties:

$$\mathcal{P} - \text{invariance} : S_{ij}^{kl}(\theta) = S_{ji}^{lk}(\theta) \tag{1.153}$$

$$\mathcal{C} - \text{invariance} : S_{ji}^{lk}(\theta) = S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}(\theta) \tag{1.154}$$

$$\mathcal{T} - \text{invariance} : S_{ij}^{kl}(\theta) = S_{lk}^{ji}(\theta) \tag{1.155}$$

and as a consequence of integrability it obeys the Yang-Baxter equation

$$S_{ij}^{ab}(\theta_{12})S_{bk}^{cl}(\theta_{13})S_{ac}^{nm}(\theta_{23}) = S_{jk}^{ab}(\theta_{23})S_{ia}^{nc}(\theta_{13})S_{cb}^{ml}(\theta_{12}). \tag{1.156}$$

And the so-called crossing invariance condition

$$S_{ij}^{kl}(\theta) = S_{i\bar{l}}^{k\bar{j}}(i\pi - \theta). \tag{1.157}$$

But maybe the relations that are more critical in this context are the unitarity conditions. There is a delicate issue here about distinguishing among close but different concepts, such as probability conservation, unitarity/hermiticity and reality of the spectrum. In particular the common hermiticity relation is written as

$$(S_{ij}^{kl}(\theta))^* = S_{lk}^{ji}(-\theta^*) \tag{1.158}$$

while the relation

$$\sum_{n,m} S_{ij}^{nm}(\theta) S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l \tag{1.159}$$

which in compact notation can be writtes as

$$S(\theta)S(-\theta) = 1. \tag{1.160}$$

has been called R-matrix unitarity, since is related to the corresponding unitarity of the quantum group R-matrix. These two concepts together imply the so-called *two-particle unitarity*, encoded in the equation

$$\sum_{k,l} S_{ij}^{kl}(\theta)(S_{mn}^{kl}(\theta))^* = \delta_{im}\delta_{jn}. \tag{1.161}$$

In integrable theory, where only two-particle scattering is possible, this last becomes

$$S(\theta)S^\dagger(\theta) = 1 \tag{1.162}$$

which encodes the normalization of probability that some initial state $|i\rangle$ evolves as a final state $|f\rangle$ in any possible way. In fact, in any non-unitary theory a different relation is expected to hold, that could be called quasi-hemiticity relation. The matrix element $S_{fi} = \langle f|S|i\rangle$ can be expanded as follows. Taking the initial state as normalizable, it can be expanded in linear superposition of basis vectors

$$|i\rangle = \sum_n a_n |n\rangle, \qquad \sum_n |a_n|^2 = 1; \tag{1.163}$$

where has to be remembered that in quasi-hermitian case, the left basis is

$$\langle \tilde{n}|. \tag{1.164}$$

The total probability that the state evolves as a final state in any basis vectors is

$$1 = \sum_n |\langle \tilde{n}|S|i\rangle|^2 = \sum_n \langle \tilde{i}|S^\ddagger|n\rangle\langle \tilde{n}|S|i\rangle \tag{1.165}$$

which means, in operatorial form

$$S^{\ddagger}(\theta)S(\theta) = 1 \tag{1.166}$$

In the context of CFT the analysis of the S-matrix has been constructed for many perturbed CFTs, even non-unitary ones; this wil be commented later. Every perturbation gives a different particle content and scattering theory. In general the perturbed theory could consist of a massless phase leading to a new fixed point, if the correlation length is infinite $\xi = \infty$ and if the correlation function $G(r) = \langle \varphi(r)\varphi(0) \rangle$ has the behaviour

$$G(r) = \begin{cases} r^{-2x^{(1)}} & r \to 0 \\ r^{-2x^{(2)}} & r \to \infty \end{cases} \tag{1.167}$$

or it might constitute a massive phase, and in this case a mass can be associated to the correlation length $\xi = m^{-1}$ and

$$G(r) = \begin{cases} r^{-2x} & r \to 0 \\ e^{-mr} & r \to \infty \end{cases} \tag{1.168}$$

For a CFT, the S-matrix has been thought to be absent, since it is not possible to construct truly asymptotic states. Some ideas about CFT S-matrix have been anyway proposed, as in [17]. In this cited paper, a massless flow is considered from some UV-CFT to another IR-CFT, guided by some perturbation. Along these massless flow the theory is massless but not scale-invariant, so there are anyway mass scales M, excitations can be classified in right-movers and left-movers and energy of the movers can be parametrized as

$$E = p = \frac{M}{2}\, e^{\theta} \qquad \text{for right-movers} \tag{1.169}$$

$$E = -p = \frac{M}{2}\, e^{-\theta} \qquad \text{for left-movers.} \tag{1.170}$$

and their limit values link to the fixed points

$$M \to 0 \qquad \text{UV fixed point} \tag{1.171}$$

$$M \to \infty \qquad \text{IR fixed point} \tag{1.172}$$

So the matrix elements can be divided in elements relative to particles moving in the same direction S_{RR}, S_{LL}, and particles moving in reative opposite direction S_{RL}, S_{LR}. What is found is that the first ones depend just on rapidity differences, while the second ones also on the mass-scale

$$S_{RR}(\theta),\ S_{LL}(\theta) \qquad S_{RL}(\theta, M)\ S_{LR}(\theta, M) \tag{1.173}$$

so that S_{RR} and S_{LL} depends only on the properties of the infrared fixed point, and can in this sense be formally considered the S-matrix of the conformal field theory, eve if, from the intuitive point of view, no scattering happens at all. So that massless S-matrix can be viewd as describing the infrared theory perturbed by some *irrelevant* field. As a technical note due to Zamolodchikov we add the modification of the crossing symmetry for these S-matrices: in this context it becomes

$$S_{ij}^{kl}(\theta) = S_{i\bar{l}}^{k\bar{j}}(i\pi + \theta). \tag{1.174}$$

1.5 Integrability out of the critical point

To tackle the problem of understanding the RG flows among minimal models, we collect first here some knowledge about integrability away from critical point. If the RG flow bring from some UV-CFT to some other IR-CFT, both being integrable, the perturbation must be of the kind which does not destroy integrability. The analysis of this process was conducted by A.B. Zamolodchikov, and is called the *counting argument*, which we summarize here, on the line of [16]. Consider a conformal model $M_{p,q}$ deformed by a relevant primary scalar field $\Phi_{lk} = \phi_{lk}(z)\phi_{lk}(\bar{z})$, with anomalous dimension $x = 2\Delta < 2$. The perturbed action is

$$\mathcal{A} = \mathcal{A}_0 + \lambda \int \Phi_{lk}(z, \bar{z}) d^2 z. \tag{1.175}$$

If $J_{s+1}(z)$ is a conserved current of the confrormal model $M_{p,q}$, i.e.

$$\bar{\partial} J_{s+1}(z) = 0 \tag{1.176}$$

of spin $s+1$, the question is, is there any conserved current after the perturbation? The Ward identity of the current $J_{s+1}(z, \bar{z})$ in the perturbed systems can be expressed in terms of the conformal Ward identity:

$$\langle J_{s+1}(z, \bar{z})...\rangle = \langle J_{s+1}(z)...\rangle_0 + \lambda \int dw d\bar{w}\ \langle J_{s+1}(z)\Phi_{lk}(w, \bar{w})...\rangle_0 + \mathcal{O}(\lambda^2) \tag{1.177}$$

which, together with the OPE

$$J_{s+1}(z)\Phi_{lk}(w, \bar{w}) = \sum_{n=2}^{m} \frac{C_{lk}^{(n)}}{(z-w)^n} \Phi_{lk}^{(n)}(w, \bar{w}) + \frac{1}{z-w} B_{lk}(w, \bar{w}) + ... \tag{1.178}$$

gives the equation

$$\bar{\partial} J_{s+1}(z, \bar{z}) = \lambda(B_{lk}(z, \bar{z}) - C_{lk}^{(2)} \partial \Phi_{lk}^{(2)}). \tag{1.179}$$

In these formulas $\Phi_{lk}^{(n)}$ and B_{lk} are descendants of the perturbing field. The existence of a conserved current away from equilibrium depends then on the possibility to write the r.h.s. of the last equation as a total derivative. A.B. Zamolodchikov discussed this possibility in the general case. Let $\hat{\Lambda}_{s+1} = \Lambda_{s+1}/L_1\Lambda_s$ be the space of quasi-primary descendant fields of the identity operator and $\hat{\Phi}_s = \Phi_s/L_{-1}\Phi_{s-1}$ the quotient space at level s of the perturbing field. The linear map

$$\bar{\partial}\ :\ \hat{T}_{s+1} \longrightarrow \lambda\hat{\Phi}_s$$

has non-zero kernel when

$$\mathrm{dim}\hat{T}_{s+1} > \mathrm{dim}\hat{\Phi}_s. \tag{1.180}$$

The dimensions of such spaces are obtainable through the characters, using the following formulas

$$\sum_{s=0}^{\infty} q^s \mathrm{dim}\hat{T}_{s+1} = (1-q)\tilde{\chi}_{1,1}(q) + q$$

$$\sum_{s=0}^{\infty} q^{s+\Delta_{kl}} \mathrm{dim}(\hat{\Phi_{k,l}})_{s+1} = (1-q)\tilde{\chi}_{k,l}(q) \tag{1.181}$$

If this condition is fulfilled, then there are necessarily some fields $T_{s+1}(z,\bar{z}) \in \hat{T}_{s+1}$ and $\Phi_{s-1}(z,\bar{z}) \in \hat{\Phi}_{s-1}$ such that

$$\partial T_{s+1}(z,\bar{z}) = \lambda\bar{\partial}\Phi_{s-1}(z,\bar{z}) \tag{1.182}$$

i.e. there is a conserved charge

$$\mathcal{Q}_s(z,\bar{z}) = \int T_{s+1}dz + \lambda\Phi_{s-1}d\bar{z} \tag{1.183}$$

which, since we are in the context of renormalizable theory, is also related to the trace of the energy-momentum tensor. The presence of conserved charges happens also to be facilitated by the existence of null-vectors, whose structure we might summarize here.

Constraints from null vectors As it is known, every level N of Virasoro modules hosts $P(N)$ states, where $P(N)$ is the partition of N. As the Hardy-Ramanujan approximate formula tells,

$$\lim_{N\to\infty} P(N) \approx \frac{e^{[\pi\sqrt{\frac{2N}{3}}]}}{4\sqrt{3}N} \tag{1.184}$$

this number grows very fast. Among these states some null states, say $|\chi\rangle$, might hide i.e. states with zero norm

$$\langle\chi|\chi\rangle = 0. \tag{1.185}$$

It is also known that if the identity is associated to the field $\phi_{r,s}$ then the level $N_{r,s} = rs$ hosts a null vector, say $|\chi_{r,s}\rangle$. Because of the state-field correspondence, typical of CFT, to every null-vecto will be associated a null field, that is a field $\chi(z, \bar{z}) = 0$. It will have a chiral structure like

$$\chi(z) = \sum_k b_k T_n^{(k)}(z) = \tilde{T}_n(z) \tag{1.186}$$

and its O.P.E. with any other vanish.

$$\tilde{T}_n(z)\Phi_{r,s}(0) = 0. \tag{1.187}$$

Following [19] we will call it $T_{(rs)} \sim 0$. When this happens the theory possesses some constraints, related to its null vectors

$$T(z)_{rs}\phi(0)_{p,q} = 0. \tag{1.188}$$

For example with field $\phi_{1,2}$ this gives

$$(L_2 - \frac{3}{4\Delta_{1,2} + 2} L_1^2)\phi_{1,2} = 0. \tag{1.189}$$

This set of constraint allows the expression of higher descendants of fields in terms of the Ginzburg-Landau (GL) fields and its derivatives. To better explain this mechanics, and the interplay among conserved charges and null vectors we shall consider, in the next chapter, the Lee-Yang model.

Chapter 2

GL-minimal models correspondence

2.1 Landscape of minimal models and flows between them

Minimal models are conformal theories denoted by the symbols $\mathcal{M}_{p,q}$, where p and q are relative prime positive integers, with $p < q$. It is interesting to represent them in picture, where the numbers represent their position in the plane $\mathbf{N} \times \mathbf{N}$: In studying the graphics we notice an interesting symmetric pattern, that might help to clarify the structure of the whole ensamble of models and, maybe, the structure of renormalization group flows among them. Leaving asides the diagonal of unitary minimal models, the first column $\{2, 4+n\}$, with $n = 1, 2, ...$ presents the same structure than the first diagonal $\{2+n, 4+n\}$. The same seems to be true for the second column $\{3, 5+n\}$ and the second diagonal $\{3+n, 5+n\}$, etc. Another observation, that encodes the firs one is that, always leaving aside the unitary diagonal, each raw has a central symmetry. This could be expressed in functional form, with the aid of the so-called Smarandache function, found in the mathematical literature [20]. The Smarandache Coprime function $C_k(n_1, ..., n_k)$ is defined as follows

$$C_k(n_1, ..., n_k) = \begin{cases} 0 & \text{if } n_1, ..., n_k \text{ are coprime} \\ 1 & \text{if they are not} \end{cases} \tag{2.1}$$

and the symmetry observed on the table could be written in terms of this function, and in the considered domain, as

$$C_2(p, q) = C_2(q - p, q). \tag{2.2}$$

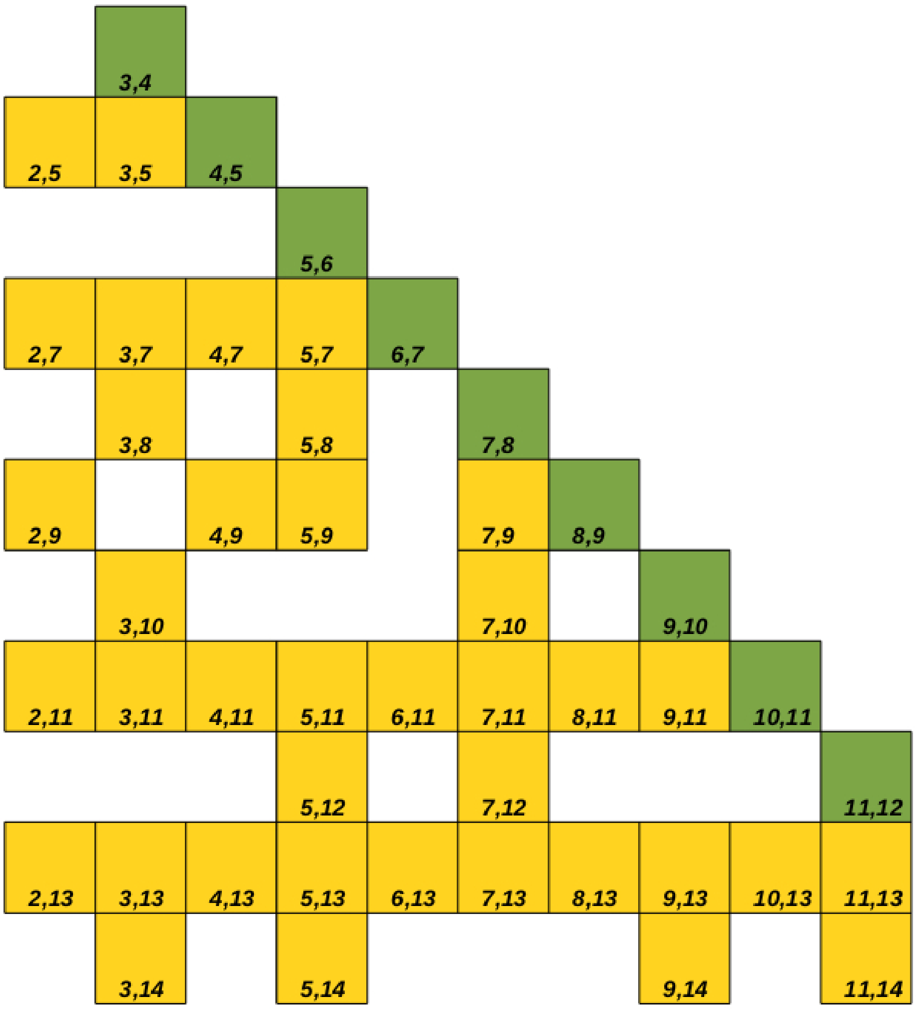

Figure 2.1: Representation of minimal models on the $\mathbf{N} \times \mathbf{N}$ plane. Dark color stays for unitary models.

Minimal models exhaust all universality classes with a finite number of primary scaling fields. These theories have central charge

$$c = 1 - 6\frac{(q-p)^2}{qp}. \tag{2.3}$$

All scaling fields belong to degenerate representations of the Virasoro algebra; their correlation functions are bilinear forms of generalized hypergeometric functions. We shall here restrict the analysis to *diagonal* theories, which have chiral primary fields

$$\Phi_{(r,s)}(z,\bar{z}) = \phi_{(r,s)}(z) \otimes \phi_{(r,s)}(\bar{z}) \tag{2.4}$$

with

$$1 \leq r \leq p-1 \qquad 1 \leq s \leq q-1.$$

Each of this models has its own field content, i.e. a certain number of primary fields with their conformal dimension. When considering the perturbation of a particular minimal model with some relavant operator (with $\Delta < 1$) and the renormalization group acts, two possibilities are at sight, as already explained. The first is that the resulting theory is a massive one, meaning that some scale appears in the theory, the second is that the RG flow brings to a new fixed point, i.e. a new minimal model. Understanding the structure of this flows is at the present moment a major open problem of conformal field theory. Insofar some results have been established and proven, new ones have been conjectured and left as lines of research. The first of this RG flows was suggested in the paper [22] and regards unitary minimal models; for them it was showed that the flow

$$\mathcal{M}_{p+1} + \lambda\Phi_{(1,3)} \to \mathcal{M}_p, \tag{2.5}$$

with $\lambda < 0$ exists. Similar cascades, which preserve the *nonunitarity index* $n = q-p$ have been discovered also for other non-unitary models, which generalize the previews result. For example in [21] the

$$\mathcal{M}_{p,q} + \lambda\Phi_{(1,3)} \to \mathcal{M}_{p-n,q-n} \tag{2.6}$$

series has been proven, at least for large values of $m = p/n$, and extended later for any value of m. We observe that graphically the $\Phi_{(1,3)}$ flow consists of diagonal lines, with n-lengthy jumps. The scenario of flows is anyway much reacher, and the whole phase space landscape has still to be scouted, as stressed, for example, in [21]. The flow

$$\tilde{\mathcal{M}}_p + \lambda\psi \to \tilde{\mathcal{M}}_{p-1} \tag{2.7}$$

has been introduced, for example in [24], where

$$\tilde{\mathcal{M}}_p = \begin{cases} \mathcal{M}_{\frac{p+1}{2},p} & \text{for p odd} \\ \mathcal{M}_{\frac{p}{2},p+1} & \text{for p even} \end{cases} \tag{2.8}$$

and

$$\psi = \begin{cases} \phi_{2,1} & \text{for p odd} \\ \phi_{1,5} & \text{for p even.} \end{cases} \tag{2.9}$$

In the same paper some comments have been written about the parity/oddness of the perturbing field, and the nature of the associated perturbing parameter. In particular the authors reported that when the perturbation is *even* massless and massive phases are related by the transformation $\lambda \longleftrightarrow -\lambda$, while when the perturbation is *odd*, only one of the two possible behaviours was thought to be allowed. Anyway this holds when only the $\lambda \in \mathbf{R}$ was considered; while allowing the perturbing parameter to take complex values, $\lambda \in \mathbf{C}$, the possibility of having two phases is restored. So, when the perturbation is odd, the coupling tranformation to consider, to change the phase of the system, would rather be $\lambda \longleftrightarrow i\lambda$. To resume the situation:

$$\mathcal{M} + \lambda\phi_{even} \rightarrow \qquad \text{to change behaviour : } \lambda \longleftrightarrow -\lambda \tag{2.10}$$

$$\mathcal{M} + \lambda\phi_{odd} \rightarrow \qquad \text{to change behaviour : } \lambda \longleftrightarrow i\lambda \tag{2.11}$$

The possibility to consider complex coupling, and having anyway sensible theories due to pseudo-hermiticity, enlarge enormously the possibilities of RG flows. We quote from [21]:

> Under the crossovers considered here, which preserve the nonunitarity index n, the central charge decreases. This does not rule out crossovers between different sequences that may violate the c-theorem. In fact, the mean field analysis suggests that the usual Ising fixed point is linked to the Yang-Lee fixed point through a crossover induced by the imaginary magnetic field and the reduced temperature in a fine-tuned linear combination. More generally, we should expect nonunitary minimal universailty classes to appear in the phase diagram of the Landau-Ginzburg model when the thermodynamic parameters λ_j take imaginary values. All that hints at an extremally reach scenario yet to be discovered.

These results have conducted to the hypothesis of a multicritical generalization of the Lee-Yang model, with some Lee-Yang like singularities related to multicritical Ising unitary $\mathcal{M}_{p,q}$,but this is not straightforward and needs further investigations. Minimal models have effective central charge $c_{eff} = c - 24\Delta_0$

$$c_{eff} = 1 - 6\frac{1}{pq}. \tag{2.12}$$

where Δ_0 is the lowest dimension of the theory.

2.2 Ginzburg-Landau representation of unitary minimal models

Insofar minimal models have been described as abstract entities. In statistical mechanics the most simple of them are associated to known statistical systems, as the Ising model, the multicritical Ising models, the Lee-Yang model. Since in statistical physics a description in terms of order parameters is often adopted, also these minimal models are associated to order parameters. In particular, Ginzburg-Landau description lead to some bosonic field theory, to describe the mean field behaviour of the system, but it can also be thought of as a quantum field theory on its own rights. The explicit construction of Ginzburg Landau lagrangians starting from the minimal model structure was conducted in [22], by A.B. Zamolodchikov. He interpreted the subset of *unitary* minimal models $\mathcal{M}_{p,p+1} \equiv \mathcal{M}_p$ as the multicritical points of Ginzburg-Landau models of a single scalar field with Z_2-symmetric lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \phi + \sum_{k=1}^{N} g_k \varphi^{2k} \tag{2.13}$$

where the field φ can be regarded as the above mentioned order parameter of some underlying statistical theory. At the $(p-1)$-critical point, the potential can be described by

$$V(\varphi) = g_{p-1}\varphi^{2(p-1)}. \tag{2.14}$$

as discussed again in [21]. Here the vanishing of all other parameteres describes a critical surface

$$\mathcal{C}_p : g_1 = g_2 = ...g_{2p-3} = 0 \tag{2.15}$$

which is nested into all manifolds of lower criticality:

$$...\mathcal{C}_p \subset \mathcal{C}_{p-1} \subset ...\mathcal{C}_3. \tag{2.16}$$

The explicit relation among GL_{p-1} models and the corresponding minimal model $\mathcal{M}_p$ is encoded in the respective operator algebras, as it is expained in [17] and it can be pictured in the Kac table, where operators are classified according to their anomalous dimensionality. A typical Kac table is shown here. The identification of the two theories is complete if the operator content of the firs correspond to the operator content of the latter. The metching can be seen in the following constructive way: let us start with the identification

$$\varphi \sim \Phi_{(2,2)} \tag{2.17}$$

then take its OPE expansion

$$\varphi(x) \times \varphi(0) = |x|^{d_2 - 2d_1} \Phi_2 + ... \text{ less singular terms} \tag{2.18}$$

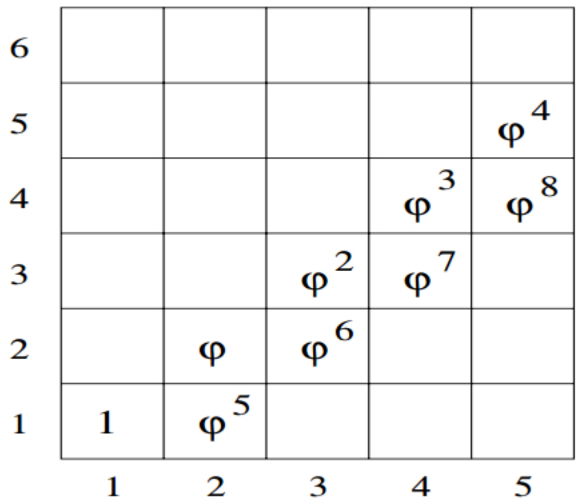

Figure 2.2: The Kac table for the unitary model M_6, where just relevant fields are represented (once).

A successive identification is

$$: \varphi^2 : \sim \Phi_2 \equiv \Phi_{(3,3)}. \tag{2.19}$$

Iteratively expand

$$\varphi(x) \times : \varphi^2(0) : \tag{2.20}$$

and so on with the identifications. The process can continue until it ends up when the equation of motion

$$\partial\bar{\partial}\varphi = g_{2p-3} : \varphi^{2p-3} : \tag{2.21}$$

is reached. In these cases the central charge has value:

$$c_p = 1 - \frac{6}{p(p+1)} \tag{2.22}$$

2.3 Towards a GL representation of non-unitary minimal models

2.3.1 The Lee-Yang model: the first of a series?

The scaling Lee-Yang model has Ginzburg-Landau hamiltonian

$$H = \int dx[\frac{1}{2}(\partial\varphi)^2 + i(h - h_c)\varphi + ig\varphi^3] \tag{2.23}$$

and its S-matrix is

$$S_{LY} = \frac{\tanh\frac{1}{2}(\theta + i\frac{2\pi}{3})}{\tanh\frac{1}{2}(\theta - i\frac{2\pi}{3})}. \tag{2.24}$$

As it has been already said this theory can be regarded as a pseudo-hermitian $\mathcal{PT}$-symmetric quantum field theory, and as such a pseudo-hermiticity relation can be recovered through the aid of a $\mathcal{C}$-operator, defined in [9] to have the following action

$$\mathcal{C}\varphi\mathcal{C} = -\varphi, \tag{2.25}$$

so that the required relation

$$\mathcal{C}H^\dagger\mathcal{C} = H. \tag{2.26}$$

is respected, together with

$$\mathcal{C}^2 = 1. \tag{2.27}$$

Moreover, Cardy and Mussardo wrote that the N-particles states

$$|N\rangle = \lim_{t_1,\ldots,t_N \to -\infty} \varphi(t_1, x_1)\ldots\varphi(t_N, x_N)|0\rangle \tag{2.28}$$

are eigenvector of such an operator, with eigenvalue $(-1)^N$:

$$\mathcal{C}|N\rangle = (-1)^N|N\rangle. \tag{2.29}$$

They also noticed that the S-matrix, while obeying, by construction, the so-called unitarity relations

$$S(\theta)S(-\theta) = 1, \tag{2.30}$$

$$SS^\dagger = 1. \tag{2.31}$$

presents a residue at his pole with opposite sign compared to what expected in a unitary theory. In fact the LSZ reduction formula would give, for a unitary theory

$$\mathcal{S} \sim \frac{-i\lambda^2}{s - m_a^2}. \tag{2.32}$$

As it has been suggested the introduction of the $\mathcal{C}$-operator should explain the difference in the residue sign. In fact, considering the amplitude for a given process

$$\langle f|i\rangle \tag{2.33}$$

it can be calculated through matrix elements

$$\langle f|1|i\rangle = \langle f|SS^\dagger|i\rangle = \sum_n \langle f|S|n\rangle\langle n|\mathcal{C}S^\dagger|i\rangle = \sum_n (-1)^n \langle f|S|n\rangle\langle n|S^\dagger|i\rangle. \tag{2.34}$$

When $h = h_c$ we fall in the simplest non-unitary minimal model, and the simpler minimal model ever, that is the model $\mathcal{M}_{2,5}$, which posseses a GL representation as well. Its hamiltonian is thought to be

$$H = \int dx[\frac{1}{2}(\partial\varphi)^2 + ig\varphi^3]. \tag{2.35}$$

The theory possesses only one primary field, leaving the identity operator aside, and it is $\varphi = \Phi_{1,2}$ itself, having anomalous dimension $\Delta = -\frac{1}{5}$. Applying the fusion rules and field identification, as for the unitary case, we are directly led to the equaiton of motion

$$L_{-1}\bar{L}_{-1}\varphi \ \sim :\varphi^2: \tag{2.36}$$

and since its identity operator coincides also with the field $\Phi_{1,4}$, the existence of a null vector at level 4 is predicted, as explained in [19], and the associated null operator T_4. Applying it to φ

$$T_4\varphi = 0 \tag{2.37}$$

a series of constraint is obtained, the first of them being

$$L_{-2}\varphi = \frac{3}{4\Delta_{1,2} + 2} L_{-1}^2\varphi \tag{2.38}$$

which implies

$$L_{-2}\bar{L}_{-2}\varphi \ \propto \ L_{-1}^2\bar{L}_{-1}^2\varphi = L_1\bar{L}_1 :\varphi^2: . \tag{2.39}$$

As mentioned above, in CFT instead of massive asymptotic states (particles), right (+) and left (-) movers are introduced so that four matrix elements are needed: $S_{++}, S_{--}, S_{+-}, S_{-+}$. Following [24] and requiring parity symmetry it can be stated that $S_{++} = S_{--}$, and in analogy with the perturbed case, also $S_{+-} = S_{-+}$. The elements are given by

$$S_{++} = S_{--} = S_{LY} \tag{2.40}$$

$$S_{+-} = S_{-+} = 1 \tag{2.41}$$

A possible idea, even though not proven, is that the unitary S matrix so far constructed could be indeed the hermitian counterpart of some non-hermitian one, the relation between them being always encoded in the metric operator η. Always according to [24] the proposed S-matrix can also be regarded as describing the massless perturbation

$$M_{3,5} + \Phi_{2,1}. \tag{2.42}$$

So, in next paragraphs of this theis we have explored the possibility of a GL descriprion of the UV model, namely $M_{3,5}$. The critical Yang-Lee model can be also formulated along with the Coulomb gas formalism. In this context the role of the fundamental field φ is taken by the vertex operator V_α. This lead us to imagine that

$$\mathcal{C}V_\alpha(0)\mathcal{C} = -V_\alpha(0) \tag{2.43}$$

where the $|N\rangle$ state is now

$$|N\rangle = \lim_{|z|_1 \ldots, |z|_N \to 0} V_\alpha(z_1, z_1) \ldots V_\alpha(z_N, \bar{z}_N)|0\rangle.$$

2.3.2 $M_{2,7}$ conjecture

Following similar lines of reasoning we have conjectured that the non-unitary minimal model $M_{2,7}$ corresponds to a Ginzburg-Landau theory of the kind

$$V(\varphi) \sim \varphi^4 \tag{2.44}$$

with possibly complex coupling. Starting from the identification of the fundamental field $\varphi(x) = \Phi_{1,3}(x)$, i.e. the one with lowest dimension $\Delta_{1,3} = -\frac{3}{7}$, the other renormalized fields can be built through the above mentioned procedure, taken from works by Zamolodchikov [22] and Keke Li [23] and suitably modified to adapt to the non-unitary case. Given two operators A and B already identified

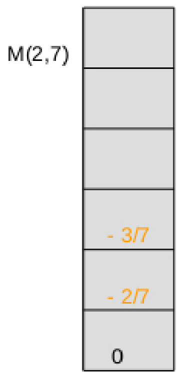

Figure 2.3: The Kac table of the model $M_{2,7}$.

with scalar operators of the conformal theory, their operator product is composed, ignoring already identified terms, keeping the less regular one, to be identified with a new operator C to suitably renormalize. For example, in the $M_{2,7}$ case, the fusion rules are

$$\Phi_{1,3} \times \Phi_{1,3} = [\Phi_{1,1}] + [\Phi_{1,2}] + [\Phi_{1,3}] \tag{2.45}$$

$$\Phi_{1,3} \times \Phi_{1,2} = [\Phi_{1,2}] + [\Phi_{1,3}] \tag{2.46}$$

so that the first operator expression is

$$\lim_{x \to 0} \varphi(x)\varphi(0) = a|x|^{-2\Delta_{1,3}}[1](0) + a'|x|^{\Delta_{1,2}-2\Delta_{1,3}}[\Phi_{1,2}](0) + a''|x|^{-\Delta_{1,3}}[\Phi_{1,3}](0) \tag{2.47}$$

and the identification can be written as

$$: \varphi^2 : (0) =: \lim_{x \to 0} |x|^{-\Delta_{1,2}+2\Delta_{1,3}}\varphi(x)\varphi(0) := a'\Phi_{1,2}(0). \tag{2.48}$$

Moving further with the next one:

$$\lim_{x \to 0} \varphi(x) : \varphi^2(0) := a'\Phi_{1,3}(x)\Phi_{1,2}(0) = a'b|x|^{-\Delta_{1,3}}[\Phi_{1,2}](0) + b'|x|^{-\Delta_{1,2}}\Phi_{1,3}(0) +$$

$$\tilde{b}'|x|^{-\Delta_{1,2}+2}L_{-1}\bar{L}_{-1}\Phi_{1,3}(0) + ... \tag{2.49}$$

the equation of motion can be obtained

$$\tilde{b}'\partial\bar{\partial}\varphi(0) =: \lim_{x \to 0} |x|^{\Delta_{1,2}-1}\varphi(x) : \varphi^2(0) ::=: \varphi^3 : (0) \tag{2.50}$$

$$\partial\bar{\partial}\varphi(0) = \frac{1}{\tilde{b}'} : \varphi^3 : (0). \tag{2.51}$$

At this point, to determine the potential, it is necessary to know the value of the structure constant which appears in the equation of motion. Before to do this, let us try first to extend the conjecture to the whole series $M_{2,q}$.

2.3.3 $M_{2,q}$ conjecture

The Kac table of this series consists of a single column, with $(q-1)/2$ different fields, all of them having negative dimension. We are showing in figure 2.3 the first tables of the series. The field $\Phi_{1,1}$ is always the identity, and the field $\Phi_{1,\frac{q-1}{2}}$ is always φ. So that, in the models $M_{2,7}$ and $M_{2,9}$ the equation of motion of the fundamental field comes always from the O.P.E. of φ itself and the field $\Phi_{1,2}$ which should then correspond to the highest power in the potential, namely $\varphi^{\frac{q+1}{2}}$. Thus, to calculate the structure constant which appears in the potential we need to consider the following fusion

$$\Phi_{1,\frac{q-1}{2}} \times \Phi_{1,2} \tag{2.52}$$

and find the coefficient relative to the first descendant of the fundamental field. It can be written as

$$\beta\beta' C^{1,\frac{q-1}{2}}_{1,\frac{q-1}{2};1,2} \equiv \beta\beta' C^{1,s_q+1}_{1,s_q;1,2} \tag{2.53}$$

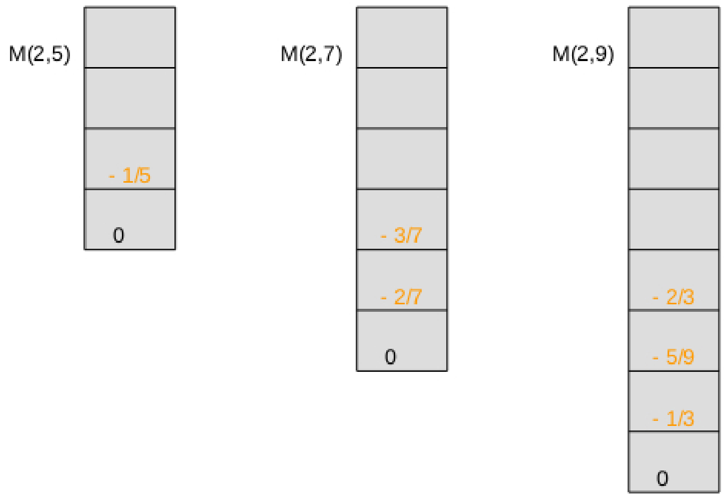

Figure 2.4: Kac tables for first of the non-unitary models of the series $M_{2,q}$.

where we remember that, in these models $\Phi_{1,\frac{q-1}{2}} = \Phi_{1,\frac{q+1}{2}}$. This fact allows for the use of the general formula [16]

$$(C^{1,s_q+1}_{1,s_q;1,2})^2 = \frac{\Gamma(2-2\rho)\Gamma(1-s_q\rho)\Gamma(\rho)\Gamma(-1+\rho+s_q\rho)}{\Gamma(-1-2\rho)\Gamma(s_q\rho)\Gamma(1-\rho)\Gamma(2-\rho-s_q\rho)} \tag{2.54}$$

where

$$\rho = \alpha_-^2(q) = \frac{q^2-q+2-(q-2)\sqrt{q^2-2q+4}}{q} \tag{2.55}$$

since

$$\alpha_- = \alpha_0 - \sqrt{\alpha_0^2+1}. \tag{2.56}$$

Comparing the central charge

$$c_{\alpha_0} = 1 - 24\alpha_0^2 \tag{2.57}$$

with

$$c = 1 - \frac{6(q-2)^2}{2q} \tag{2.58}$$

it is found

$$\alpha_0^2 = \frac{(q-2)^2}{2q}. \tag{2.59}$$

Actually, more than the exact value of the coefficient, what is important is its sign and nature (if it is real or complex). What is left to calculate are the signs of the

β coefficients, which are anyway real numbers. Let us proceed as in [19]. Take the O.P.E.

$$\Phi_{1,2}(z,\bar{z})\Phi_{1,\frac{q-1}{2}}(0,0)=\sum_{(k,\bar{k})}\sum_{p}C^{p\{k,\bar{k}\}}_{1,2;1,\frac{q-1}{2}}z^{\Delta_p-\Delta_1-\Delta_2+k}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_1-\bar{\Delta}_2+\bar{k}}\Phi_p^{\{k,\bar{k}\}}(0,0) \tag{2.60}$$

and apply it to the conformal state -which in non-unitary cases is not the ground state- and write

$$\Phi_{1,2}(z,\bar{z})|\Delta_0\bar{\Delta}_0\rangle=\sum_{(k,\bar{k})}C^{p}_{1,2;1,\frac{q-1}{2}}z^{\Delta_p-\Delta_1-\Delta_2}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_1-\bar{\Delta}_2}\psi(z)\bar{\psi}(\bar{z})|\Delta_p\bar{\Delta}_p\rangle \tag{2.61}$$

where it has been defined

$$\psi(z)=\sum_{(k,\bar{k})}z^k\beta^{p\{k,\bar{k}\}}_{1,2;1,\frac{q-1}{2}}L_{-k}...L_{-k_N} \tag{2.62}$$

so that the equation can be also be written as

$$\Phi_{1,2}(z,\bar{z})|\Delta_0\bar{\Delta}_0\rangle=C^{p}_{1,2;1,\frac{q-1}{2}}z^{\Delta_p-\Delta_1-\Delta_2}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_1-\bar{\Delta}_2}|z,\Delta_p\rangle|\bar{z},\bar{\Delta}_p\rangle. \tag{2.63}$$

where an expansion in levels can be made

$$|z,\Delta_p\rangle=\sum_{N=0}^{\infty}z^N|N,\Delta_p\rangle \tag{2.64}$$

Remember that in this particular case

$$\Delta_{1,\frac{q-1}{2}}=\Delta_{1,\frac{q+1}{2}}. \tag{2.65}$$

Applying the operator L_n to the left side:

$$L_n\Phi_{1,2}(z,\bar{z})|\Delta_0\bar{\Delta}_0\rangle=[L_n,\Phi_{1,2}(z,\bar{z})]|\Delta_0\bar{\Delta}_0\rangle= \tag{2.66}$$

$$(\Delta_{1,2}(n+1)z^n+z^{n+1}\partial)\Phi_{1,2}(z,\bar{z})|\Delta_0\bar{\Delta}_0\rangle \tag{2.67}$$

and that should give

$$\beta^{p\{1\}}_{1,2;1,\frac{q-1}{2}}=\frac{1}{2} \tag{2.68}$$

so that we can conclude that to know the sign of the constant associated to the equation of motion it is enough to consider just the algebra of primaries. Thus, the model $M_{2,7}$ $M_{2,9}$ can be thought of as having GL potential

$$V(\varphi)\sim-\lambda_4\varphi^4 \tag{2.69}$$

$$V(\varphi)\sim i\lambda_5\varphi^5. \tag{2.70}$$

For some models in the same series with higher q, this analysis is modified, since the algebra of the fundamental field closes leaving aside some elements of the Kac table. For example, in the model $M_{2,11}$ the field $\Phi_{1,2}$ is not touched by the fusion rules related to the fundamental $\Phi_{1,5}$, so it might generate some constraint to be associated to the lagrangian. Calling for example σ the field left aside, we could pair a φ^4 GL potential with a $\sigma^2 = \varphi^3$ constraint. A similar situation is found for the model $M_{2,13}$ where $V(\varphi) \sim \varphi^6$ and $\sigma^2 = \varphi^4$. If there is some other dynamical aspect to consider it is not clear to the author at the present state of the art.

2.3.4 The serie $M_{2,q}$ and the restricted sG theory

In paper [8] the series $M_{2,q}$, which was parametrized as $M_{2,2n+3}$, was related to the sG theory in the following way. The sG theory

$$\mathcal{L}_{SG} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{m^2}{\gamma}(\cos\sqrt{\gamma}\varphi - 1) \tag{2.71}$$

is known to contain different kind of particles, namely solitons and breathers. For the values of the parameter

$$\xi = \frac{\pi\gamma}{8\pi - \gamma} = \frac{2\pi}{2n+1} \tag{2.72}$$

the possibility to restrict the theory to the soliton-free sector was proven by Smirnov, through a projection operator P (not to be confused with the parity operator), to obtain the equivalence with the perturbed minimal model $M_{2,2n+3} + \Phi_{1,3}$. Since for the sG theory $T_{\mu\nu}$ is non-local,

$$[T^{\mu\nu}(x), T^{\mu'\nu'}(0)] \neq 0 \qquad x_\mu x^\mu < 0 \tag{2.73}$$

a restricted energy-momentum tensor was proposed, through the construction of the equivalent tensor

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + i2^{-5/2}\frac{\sqrt{\gamma}}{\xi}\epsilon^{\mu\mu'}\epsilon^{\nu\nu'}\partial_{\mu'}\partial_{\nu'}\varphi \tag{2.74}$$

and the above mentioned projection:

$$\tilde{\mathcal{T}}_{\mu\nu} = P\tilde{T}^{\mu\nu}P. \tag{2.75}$$

The new energy-momentum tensor is now local

$$[\mathcal{T}^{\mu\nu}(x), \mathcal{T}^{\mu'\nu'}(0)] = 0 \qquad x_\mu x^\mu < 0. \tag{2.76}$$

and has ultra-violet central charge

$$c_n = 1 - \frac{6(2n+1)^2}{2(2n+3)} \tag{2.77}$$

but it is not hermitian any more, while its trace is

$$\tilde{\mathcal{T}}_{\mu\mu} = \frac{M_1^2}{4\sin\xi} P e^{i\sqrt{\gamma}\varphi} P \tag{2.78}$$

and has dimensions

$$(\frac{1-2n}{2n+3}, \frac{1-2n}{2n+3}),$$

which prove the statement. To show how close Smirnov reasoning was with the mentioned results about pseudo-hermitian systems, we quote from his article [8]:

> We have an amusing situation when the energy-momentum tensor is not self-adjoint and can not be made self-adjoint without loss of locality. At the same time energy and momenutm are self-adjoint and energy is positive. The situtation undermines the usual views, in particular it demonstrates that the definition of positivity accepted in CFT, which is nothing other than the assumption of self-adjointness of the energy-momentum tensor, does not necessarily correspond to the principal ideas of positivity of energy and unitarity of the S-matrix.

In fact, the soliton free sector, has m_1-breather-m_2-breather S-matrix

$$S_{ab}(\theta) = \coth\frac{1}{2}(\theta - \frac{1}{2}i\xi(m_1+m_2))\tanh\frac{1}{2}(\theta + \frac{1}{2}i\xi(m_1+m_2))$$

$$\coth\frac{1}{2}(\theta - \frac{1}{2}i\xi(|m_1-m_2|)\tanh\frac{1}{2}(\theta + \frac{1}{2}i\xi|m_1-m_2|)$$

$$\prod_{j=1}^{\min(m_1,m_2)-1} \tanh^2\frac{1}{2}(\theta + \frac{1}{2}i\xi(|m_1-m_2|+2j) \tag{2.79}$$

which, for $m_1 = m_2 = 1$ gives S_{YL}. Moreover, since the so-restricted theory has been proven to be equivalent to $M_{2,2n+3} + \Phi_{1,3}$ these S-matrix could give interesting informations about the lagrangian structure of these models, confirm or confute our conjectures.

2.3.5 Towards a GL-representation of any non-unitary minimal model

When we try to analize in the same way other minimal models we encounter wider difficulties to reproduce the structure of familiar lagrangians. In these cases we attempt to find in the fusion rules constrained lagrangian systems, as it was begun in the previews section. They might have the general structure

$$\begin{array}{c} \mathcal{L} = \frac{1}{2}(\partial\sigma)^2 + V(\sigma,\varphi) \\ f(\varphi,\sigma) = 0 \end{array} \tag{2.80}$$

For example the $M_{3,5}$ model would consist, according to its fusion rules, in the still simple

$$\begin{array}{c} \mathcal{L}_{3,5} = \frac{1}{2}(\partial\sigma)^2 + g\sigma^3 \\ \sigma \ \sim : \varphi^2 : . \end{array} \tag{2.81}$$

This should also reconcile with the predicted flow $M_{3,5} + \Phi_{2,1} \to M_{2,5}$. This could be heuristically explained in this picture, since the field $\Phi_{2,1}$ coincides with $: \varphi^3 :$, that would lead, at IR point, to the required model. Along the flow, then, should be admitted that the constraint unties, and φ and σ come to coincide. So it seems that a different field, $\sigma(z,\bar{z})$, respect to the lowest dimension one $\varphi(z,\bar{z})$, is somehow the lagrangian field. Even for the next simpler models, anyway, the algebra of fields entwines in complex structures, which we have not been able to understand clearly. From the analysis of the Kac tables of these first models, anyway, some recurrence appear and hopefully will help in the task. To give some example of these recurrence we show the first tables of the non-unitary series $M_{3+n,5+n}$, where n is the non-unitary index, in this case equal to 3. A flow between them is driven by the perturbing field $\Phi_{1,3}$.

Another group of model which we think might present some common structure is the $M_{3,5+3k}$ series; again we show the structure of their Kac tables:

M(3,5)

¾	
1/ 5	
- 1/20	
0	

M(5,7)

3 3/ 4	1 4/5		
2 2/ 7	117/140		
1 5/28	8/35		
3/ 7	- 3/140		
1/28	3/35		
0	11/20		

M(7,9)

8 3/ 4	5 5/ 7	3 9/28			
6 1/ 3	3 67/84	1 19/21			
4 11/36	2 16/63	221/252			
2 2/ 3	1 11/84	5/21			
1 5/12	8/21	- 1/84			
5/ 9	5/252	8/63			
1/12	1/21	55/84			
0	13/28	1 4/ 7			

Figure 2.5: The Kac table for some non-unitary models of the series $M_{3+n,5+n}$, where n is the non-unitarity index. Color grey means fields are relevant, color orange is associated to negative dimensions.

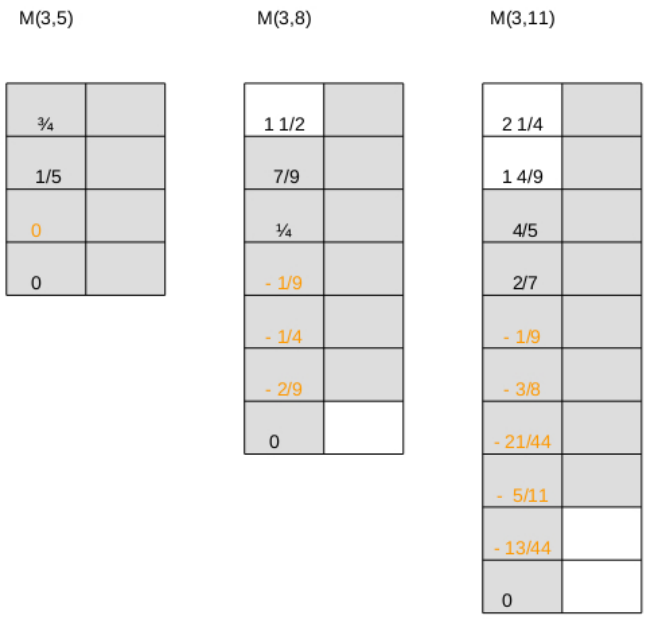

Figure 2.6: The Kac table for some non-unitary models of the series $M_{3,5+3k}$, where $k = 1, 2....$ Color grey means fields are relevant, color orangeis associated to negative dimensions.

Chapter 3

Towards a c_{eff}-theorem

Conformal field theories are associated to fixed points of the renormalization group (RG), which are conneced to the critical behaviour of statistical systems. As emphasized in [25], a full analysis should include, together with the construction of the conformally invariant field theory, the calculation of the corresponding universality class, i.e., in essence, the description of the structure of the RG in a certain neighbourhood of the fixed point, and this is tackled using perturbation theory. One of the main results in (1+1) dimensional RG theory is the so-called c-theorem, demonstrated by A.B. Zamolodchikov. The physical content of this theorem is very deep, stating that along the RG flow there is information loss, which determines it as something similar to irreversible processes. The c-theorem was proven for unitary (1+1) dimensional quantum field theory; we wonder if a similar result could be proven also for pseudo-hermitian (1+1) quantum field theory, whose RG structures have only started to emerge.

3.1 Zamolodchikov's c-theorem

To tell about the c-theorem we quote directly the abstract from the original paper [26]:

> There exists a function $c(g)$ of the coupling constant g in a 2D renormalizable field theory which decreases monotonically under the influence of a renormalization-group transformation. This function has constant values only at fixed points, where c is the same as the central charge of a Virasoro algebra of the corresponding conformal field theory.

Let us report some note on the passages of that dense demonstration. A general action

$$S = \int_{\Lambda} dx^2 \; \mathcal{L}(x; g) \tag{3.1}$$

is characterized by dimensionless parameters g and a certain cut-off Λ. The Renormalization Group acts as a scale transformation on the coordinates

$$x^\mu \to x^\mu e^s \sim x^\mu(1+s) \tag{3.2}$$

and as a redefinition of the cut-off:

$$\Lambda \to \Lambda e^{-s} \sim \Lambda(1-s) \tag{3.3}$$

The combination of the two leads to a transformed action which can be seen as the same action with modified (running) parameters

$$S = \int_\Lambda dx^2 \; \mathcal{L}(x; g(s)) \tag{3.4}$$

where the running is ruled by the so-called beta functions

$$\frac{dg}{ds} = \beta(g). \tag{3.5}$$

The behaviour of n-point correlation functions under the action of the RG is ruled by the Callan-Symanzik equation

$$\sum_{i=1}^{n} \langle (\frac{1}{2} x_i^\mu \frac{\partial}{\partial x_i^\mu} + \hat{\gamma}^{(i)}(g)) A_1(x_1)...A_n(x_n) \rangle - \sum_a \beta^a(g) \frac{\partial}{\partial g_a} \langle A_1(x_1)...A_n(x_n) \rangle = 0 \tag{3.6}$$

where the operator of anomalous dimensions $\hat{\gamma}(g)$ is defined as

$$\hat{\gamma}(g)) = \hat{D} + \beta^a \frac{\partial}{\partial g_a} \tag{3.7}$$

the $\hat{D}$ operator implementing the internal scale transformation of the fields. In the context of (1+1) quantum field theory complex variables can be introduced

$$z = x_1 + ix_2 \qquad \bar{z} = x_1 - ix_2 \tag{3.8}$$

together with the notation $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$, $\Theta = T_{z\bar{z}} = T_{\bar{z}z} = T^\mu_\mu$ to describe the components of the energy-momentum tensor and its trace. Zamolodchikov considered the two-points functions

$$\langle T(z,\bar{z}) T(0,0) \rangle = \frac{F(z\bar{z})}{z^4} \tag{3.9}$$

$$\langle \Theta(z,\bar{z}) T(0,0) \rangle = \frac{H(z\bar{z})}{z^3 \bar{z}} \tag{3.10}$$

$$\langle \Theta(z, \bar{z})\Theta(0, 0)\rangle = \frac{G(z\bar{z})}{z^2\bar{z}^2} \tag{3.11}$$

which at the conformal fixed point are equal to zero, except the first one, which has value

$$\langle T(z, \bar{z})T(0, 0)\rangle_{CFT} = \frac{c}{2z^2}. \tag{3.12}$$

Assuming translational and rotational symmetries means local conservation of the energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.13}$$

which in complex coordinates reads

$$\bar{\partial}T + \frac{1}{4}\partial\Theta = 0. \tag{3.14}$$

The conservation equation can be joined in OPE to $T(0, 0)$ giving

$$\langle \bar{\partial}T(z, \bar{z})T(0, 0) + \frac{1}{4}\partial\Theta(z, \bar{z})T(0, 0)\rangle = 0, \tag{3.15}$$

where the symbol

$$R = |z| \tag{3.16}$$

is introduced. The above expressions for the correlators allowed Zamolodchikov to obtain a quantity c(R, g) whose behaviour follows

$$\dot{c} = \frac{\partial c}{\partial R}R = -\frac{3}{4}G. \tag{3.17}$$

Then, fixing R, the RG equation give

$$\beta^i \frac{\partial}{\partial g^i}c(g) = -12G_{ij}(g)\beta^i(g)\beta^j(g) \tag{3.18}$$

where the symmetric matrix G_{ij} is positive-definite, and this proves the theorem. This point is central for the purposes of this thesis, because this last assumption coincides with the concept of unitarity. How this assumption of positivity

$$\langle \Theta(z, \bar{z})\Theta(0, 0)\rangle \geq 0 \tag{3.19}$$

is related to unitarity can be seen by the following: in the axiomatics of Constructive Quantum Field Theory the state

$$|\theta\rangle = \int dz\ f(z)\Theta(z)|0\rangle \tag{3.20}$$

can be defined (here only one holomorphic component is expressed), where the test functions $f \in \mathcal{C}^\infty_\gamma$ are introduced, where γ is a space-time compact support, so that its norm is

$$\langle\theta|\theta\rangle = \int\int dzdw\ f(z)\bar{f}(w)\ \langle 0|\Theta^\dagger(z)\Theta(w)|0\rangle. \tag{3.21}$$

The two point correlators that appear are the so called Wightman-Schwinger functions

$$\langle 0|\Theta(z)\Theta(w)|0\rangle = W_\Theta(z,w) \tag{3.22}$$

The condition of unitarity requires this norm to be positive, for any test function. Considering that

$$\Theta^\dagger = \Theta \tag{3.23}$$

we recover Zamolodchikov positivity condition. For the same reason, since in renormalizable theories

$$\Theta = \beta^i(g)\Phi_i \tag{3.24}$$

the matrix

$$G_{ij}(z,\bar{z}) = \langle\Phi_i(z,\bar{z})\Phi_j(0,0)\rangle \tag{3.25}$$

is positive definite. In non-unitary CFTs the c-theorem can be violated, as many case examples show. Since in this thesis non-unitary CFTs are considered, and thought as $\mathcal{PT}$-symmetric models, we think that some extentions of the c-theorem to non-unitary cases could rely on pseudo-hermiticity concepts. The validity of a so-called c_{eff}-theorem is in the air since a long time, but its proof is still missing. A possible line of demonstrations would consider the fact that in non-unitary CFT the ground state is not the conformal vacuum, but a different state $|\phi(0)\rangle$ where ϕ is the field with lowest conformal dimension, and the fact that the need to modify the norms in a $\mathcal{PT}$-symmetric way. So, Zamolodchikov arguments could maybe adapt to correlators like

$$\langle\phi(\infty)|T(z)\eta T(0)|\phi(0)\rangle. \tag{3.26}$$

which could be regarded as η-Wightman functions $W^{(\eta)}_{T\phi}(z,0)$ where elements like $T(z)\phi(0)$ could maybe be thought of as single operators. Still without a proof, we hope to obtain some more clue, studying another c-theorem, composed in recent years by a couple of physicists: Casini and Huerta.

3.2 Casini and Huerta c-theorem

In 2006 Casini and Huerta proposed another demonstration of the c-theorem [27], based on general properties of entanglement entropy, in particular on the strong sub-additivity (SSA) property. Before entering in the details of the proof we recollect a brief introduction to entanglement entropy itself.

Brief history of entanglement Entanglement can be considered the key element of quantum systems. Since the idea of its presence was in germ conceived and shared by Schrödinger it has been object to passionate research. Even if the word *entanglement* was coined by him in 1935 after the publication of the EPR article, the sense of its over-spatial logical structure was already expressible; he did it in one paper, in small parenthesis [28]:

> A disrupted particle is still to be included in the system, until the separation is also *logically* - by decomposition of configuration space-completed.

In March 1935 Einstein, Podolsky and Rosen expressed the entangled state as a trial to completeness of quantum mechanics [29]. The EPR state corresponds to a couple of particle, having had some kind of interaction, with null total momentum and fixed reciprocal distance:

$$\Psi(x_1, x_2) = \int_{\infty}^{\infty} dp \; e^{(\frac{2\pi i}{h})(x_1 - x_2 + x_0)\cdot p} \tag{3.27}$$

In October, that same year, as mentioned, Schrödinger generated the word:

> When two systems, of which we know the states by their respective representatives, enter in temporary physical interactions due to known forces between them, and then after a time of mutual influence the system separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical line of thought. By the interactions the two representatives (or ψ functions) have become entangled.

One important attempt to interpret the phenomenon was given by Bohm, in 1952 [30]. He actually suggested a hidden variable theory, where observables have hidden but precise values, and their quantum behaviour is generated by the so called quantum potential. In the case of a many body system, the quantum potential would have the expression

$$U(x_1, x_2, ..., x_n) = -\frac{\hbar}{2mR} \sum_{s=1}^{n} \nabla_s^2 R(x_1, x_2, ..., x_n) \tag{3.28}$$

were R is the squared modulo of the wave function written in polar form :

$$\psi = R(x_1, x_2, ..., x_n) e^{iS(x_1, x_2, ..., x_n)} \tag{3.29}$$

and it would be the physical entity responsible for the non-local correlations typical of entanglement [31]:

> In our suggested new interpretation of the quantum theory [...] the ”quantum-mechanical” forces may be said to transmit uncontrollable disturbances instantaneously from one particle to another through the medium of the ψ-field.

The truly non-local nature of the phenomenon became accessible after Bell's paper, 1964 [32], where he constructed a statistical observable $\mathcal{B}$ to associate to bipartite systems, and found some expected constraints on it. For hidden local variables theories (HLV), it is expected to respect the inequality

$$\mathcal{B}_{HLV} \leq 2 \tag{3.30}$$

while in the case of quantum mechanics (QM) he prooved that for entangled states a base can alway be found in which the same inequality is violated, meaning that it can be found, and in principle measured

$$\hat{\mathcal{B}}_{QM} \geq 2 \tag{3.31}$$

probing the non-local nature of entanglement correlations. The first measure of such a violation was conduced in 1972, by Clauser. If it is now accepted that entanglement involves non-local correlations, it is also known that no contradiction arise between quantum mechanics and special relativity, because no signal can be transmitted through entangled states. This kind of non-locality [33]

> (which perhaps is *sui generis* and appropriately named “passion at a distance”)

is then compatible with special relativity. Some authors have pose this theoretical situation as *peaceful coexistence* [34]:

> Indeed, it may be appropriate to introduce a notation which is familiar in a political context in order to summarize the relation between quantum mechanics (QM) and special relativity theory (SR):
>
> **QM ☮ SR.**

Some others move forwards and are working to insert non-locality, together with causality, as fundational principles of quantum mechanics. Research on foundations is of course still open [35]:

> what is the minimal set of physical principles - “*nonlocality* plus *no signalling* plus *something else simple and fundamental*” as Shimony put it- from which we may derive quantum mechanics?

Nowadays, entanglement is largely studied in many-body quantum systems, quantum field theories and quantum statistical mechanics. Moreover interesting perspectives are suggesting entanglement as a fundamental key to unlock deeper understanding of black holes and eventually quantum gravity.[36] It is interesting to observe the evolution of the concept: from a spooky misbelieved particularity of quantum theory, to widespred physical condition of the elements of the Universe. Actually, even if the very nature of this aspect of nature is not at all less conterintuitive than it was at the beginning, we could safely affirm that virtually *everything is entangled* [37].

Information theory has offered an essential axiomatization of properties that a well defined entanglement measure is required to fulfill. Axioms are the following:

- Normalization: $E(\rho) = 0$ for separable states $\rho = \sum_j c_j \; \omega_j \otimes \sigma_j$
- Non-growing under local operations and classical comunications (LOCC) transformations group: $E(\Theta\rho) \leq E(\rho)$; in particular the LO group is composed by unitary transformations, under which: $E(U^\dagger \rho U) = E(\rho)$
- Convexity: $E(\lambda\rho_1 + [1-\lambda]\rho_2) \leq \lambda E(\rho_1) + [1-\lambda]E(\rho_2), \qquad 0 < \lambda < 1$
- Subadditivity: $E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma)$ for every couple of pure states ρ and σ

Entanglement Entropy Various entanglement measures have been proposed in the literature. Restricting to pure states, entanglement entropy has been adopted as an adecuate one, and a family of functions are used for the purpose in the case of bi-partition

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B, \tag{3.32}$$

the Rnyi entanglement entropies (EEs) $S_n, \; n \leq 1$

$$S_n(A) = \frac{1}{1-n} \log \mathrm{Tr}_A(\rho_A^n) \tag{3.33}$$

where the special case $S = S_1$ is the von Neumann EE

$$S(A) = \mathrm{Tr}_A \, \rho_A \log \rho_A \tag{3.34}$$

If ρ_A is the density matrix of the vacuum state of a quantum field theory, the EE is also called *geometric entropy* (GE). When applaying EE-GE calculations to those systems related to conformal theories, an interesting connection emerges among entanglement entropy and the central charge of the associated conformal theory. As explained in details by Casini and Huerta in [38], the (SSA) property can be

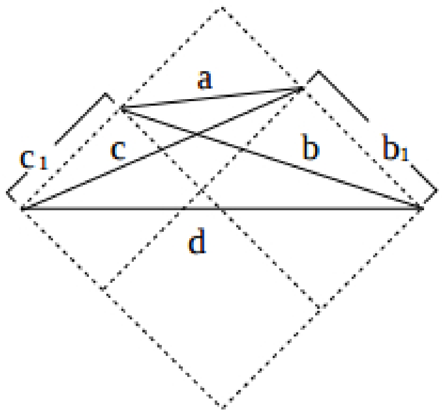

Figure 3.1: Space-time surfaces and their causal domains.

applied to regions of $1+1$ Minkowki space-time, as shown in figure: where time is the vertical axis, the horizontal axis is the spatial coordinate x and the null lines are drawn at 45°; b and c are two intersecting Cauchy surfaces, a is another surface having cuasal domain equivalent to the intersection of causal domains of a and b, and d is a fourth surface, whose domain is equivalent to the union of causal domains of b and c. We could express relations among causal domains with the notation:

$$\mathcal{D}_a = \mathcal{D}_b \cap \mathcal{D}_c \qquad\qquad \mathcal{D}_d = \mathcal{D}_b \cup \mathcal{D}_c \tag{3.35}$$

They appliy SSA to the non time-like surface

$$c_1 \cup a \cup b_1 \tag{3.36}$$

and read

$$S(c_1 \cup a) + S(a \cup b_1) \geq S(a) + S(c_1 \cup a \cup b_1) \tag{3.37}$$

followed by the identification

$$S(c_1 \cup a) = S(c) \tag{3.38}$$

$$S(a \cup b_1) = S(b) \tag{3.39}$$

$$S(c_1 \cup a \cup b_1) = S(d) \tag{3.40}$$

as a consequence of the unitarity of the causal evolution. They finally arrive at

$$S(b) + S(c) \leq S(a) + S(d) \tag{3.41}$$

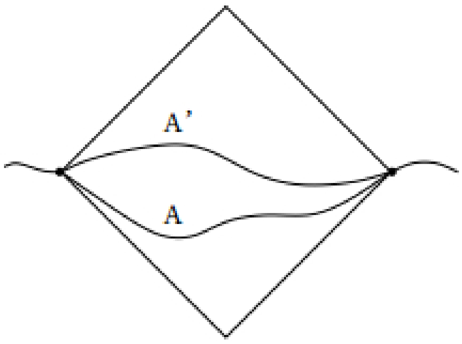

Figure 3.2: Equivalent space-time surfaces respect to entanglement entropy.

In general, causal structure and unitarity imply that this entropy must be the same for different spatial sets having the same causal domain of dependence, in the picture A and A'.

$$S(A) = S(A') \tag{3.42}$$

To explicit this we recall that

$$S(A) \equiv S(\rho_A) \tag{3.43}$$

and because under unitary evolution

$$S(U^\dagger \rho U) = S(\rho); \tag{3.44}$$

we can then recostruct the following relations, introducing a parameter τ along the space-time surface

$$\rho_A = \int_{\tau_i}^{\tau_f} d\tau \rho_{x(\tau)}(t(\tau)) = \int_{\tau_i}^{\tau_f} d\tau\, U^\dagger(t(\tau))\rho_{x(\tau)}(0)\, U(t(\tau)) \tag{3.45}$$

and applying the convexity property to the continuum case:

$$S(A) = \int_{\tau_i}^{\tau_f} d\tau\, S(\rho_{x(\tau)}(t[\tau])) = \int_{\tau_i}^{\tau_f} d\tau\, S(U^\dagger(t[\tau])\rho_{x(\tau)}(0)\, U(t[\tau])) =$$

$$S(|x(\tau_f) - x(\tau_i)|) = S(A'), \tag{3.46}$$

where it can be seen that the precise space-time trajectory followed is not relevant in terms of entropy. The relativistic geometry then gives the relation

$$ad = bc \tag{3.47}$$

which can be rewritten as

$$\frac{c}{a} = \frac{d}{b} = \lambda \leq 1 \tag{3.48}$$

or

$$c = \lambda a \qquad\qquad d = \lambda b \tag{3.49}$$

so that, inserting these latters in the SSA inequality we find that the difference of the entropies $S(b) - S(a)$ is non-increasing under dilatations:

$$S(b) - S(a) \geq S(\lambda b) - S(\lambda a) \tag{3.50}$$

An entropic c-function depending on a single parameter (and containing all the information in $S(b) - S(a)$) can be defined by

$$c(r) = r\frac{dS(r)}{dr} \tag{3.51}$$

which is dimensionless, universal, positive and satisfies

$$c'(r) = r\,S''(r) + S'(r) \leq 0. \tag{3.52}$$

At the conformal points the entropy can be explicitly evaluated with very interesting methods [39] (that we do not report here) and is given by

$$S(r) = \frac{c}{3}\log(r/\epsilon) \tag{3.53}$$

where ϵ is a cutoff scale. Out of the critical point, it has been also showed that the entanglement entropy accords itself with the natural scales of the theory, i.e. the correlation length ξ:

$$S(r) = \frac{c}{3}\log(\xi/\epsilon). \tag{3.54}$$

In thinking about non-unitary CFTs, some authors are trying to formalize similar results [40] for that class of systems. In particular, operating in similar ways than in [39], they have argued that the entanglement entropy behaves here according to

$$S(r) = \frac{c_{eff}}{3}\log(r/\epsilon). \tag{3.55}$$

suggesting that, in general, it is this effective central charge, rather than the central charge, to provide informations about the nature of the critical point. For effctive central charge it is meant the quantity $c_{eff} = c - 24\Delta_0$, built with the proper central charge and the lowest dimension Δ_0. More on this topic has been written, for example, in [41], even if the main obstacle in generalizing out of the critical point these results, as wished is the missing presence of a proper c_{eff}-theorem.

3.3 Conjecture about a c_{eff}-theorem

To apply Casini and Huerta reasoning to proove a possible c_{eff}-theorem it seems that the CFT $\mathcal{PT}$-symmetric structure should be understood in details. We collect here just some intuition, which cannot be regarded yet as a rigorous proof. The main entity to consider to obtain the entanglement entropy is the vacuum density matrix of a (non-unitary) CFT

$$\varrho = |\Phi\rangle\langle\tilde{\Phi}| \tag{3.56}$$

The main point that this thesis suggests, taking various results from the literature, is to consider CFT as a $\mathcal{PT}$-symmetric QFT. As discussed in [40], regarding it as such, unbroken $\mathcal{PT}$-symmetry for the ground state holds:

$$\mathcal{PT}|\Phi(0)\rangle = |\Phi(\infty)\rangle. \tag{3.57}$$

Since the ground state, in non-unitary CFT, is associated to the field with minimum (negative) anomalous dimension Φ, in diagonal models it can be chirally factorized:

$$\Phi(z,\bar{z}) = \phi(z)\bar{\phi}(\bar{z}) \tag{3.58}$$

within radial quantization centered on the field, with coordinate

$$z = e^{ix+\tau}. \tag{3.59}$$

The parity transformation, which sends x in $-x$ consists in

$$\mathcal{P}: \quad z \longmapsto \bar{z} \tag{3.60}$$

exchanges the chiral component of the field between themselves. Since in general the time-reversal operator exchanges left and right states

$$\mathcal{T}|\Psi_R\rangle = |\Psi_L\rangle \qquad\qquad \mathcal{T}|\Psi_L\rangle = |\Psi_R\rangle \tag{3.61}$$

then it can be confirmed that

$$\mathcal{PT}|\Phi\rangle = |\Phi\rangle. \tag{3.62}$$

Considering also a $\mathcal{C}$-operator whose action is like the one proposed by Cardy and Mussardo, with respect to the fundamental field

$$\mathcal{C}|\Phi\rangle = (-1)|\Phi\rangle \tag{3.63}$$

we can conclude that

$$\varrho = -|\Phi\rangle\langle\Phi|. \tag{3.64}$$

This could mean that the calculation of entanglement entropy for non-unitary CFT can be done essentially as always, since its values are unaffected by the presence of metric operators

$$\varrho_A = \mathrm{Tr}_{\bar{A}}\varrho = -\sum_{\bar{A}} \langle \bar{A}\eta|\Phi\rangle\langle\Phi|\bar{A}\rangle = \sum_{\bar{A}} \langle \bar{A}|\Phi\rangle\langle\Phi|\bar{A}\rangle \tag{3.65}$$

and so is $S(A)$. This would in fact justify the results already obtained by some authors, conducted without considering the problems related to the metric. The simple action of pseudo-hermitian symmetry for the ground state at conformal point allows to formulate some heuristic conjecture about the possibilities of a c_{eff}-theorem. When perturbed with some operator, say Φ_{kl} with a coupling λ in the case that the $\mathcal{PT}$-symmetry is unbroke, and

$$[\mathcal{PT}, (\lambda\Phi_{kl})] = 0 \tag{3.66}$$

the original $\mathcal{CPT}$ inner product can be inherited by the perturbed action. Since the evolution operator can be still made unitary through the aid of the metric it could be true that the whole argument of Casini and Huerta applies, and prooves the theorem.

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