ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

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Applications of elliptic functions to solve differential equations

Tesi di Laurea in Analisi Matematica

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Introduction

The integrals $\int R(t, \sqrt{p(t)})$, where R is a rational function and $p(t)$ is a polynomial of third and fourth degree without multiple roots, are called *elliptic integrals*, because they first occur in the formula for the arc length of the ellipse. The functions obtained by inverting elliptic integrals are called *elliptic functions*, and the curves that require elliptic functions for their parametrization are called *elliptic curves*.

Elliptic integrals arise in many important problems of geometry and mechanics. Indeed the arc of length of an ellipse represents the first approximation of the orbit's lenght of a planet around the sun.

In this thesis we shall consider other relations between mechanics and elliptic integrals. First, we shall compute the action-angle variables of one-degree of freedom Hamiltonian systems with a cubic or quadratic potential using elliptic integrals. Then we shall redo the computations of Moser [11] for the integrable systems of the geodesics on an ellipsoid and of the Neumann system on a sphere in $\mathbb{R}^n$, since in the case n=2 they can be solved by elliptic integrals.

The theory of elliptic functions and elliptic integrals has a long history. It began with the discovery of a remarkable property of the arc of the *leminiscate of Bernouilli*, made in 1718 by the Italian count Fagnano.

The leminiscate is the locus of a point $p = (x, y)$ in the plane so that the product of its distances from two fixed points, called the *foci*, has constant value c^2.

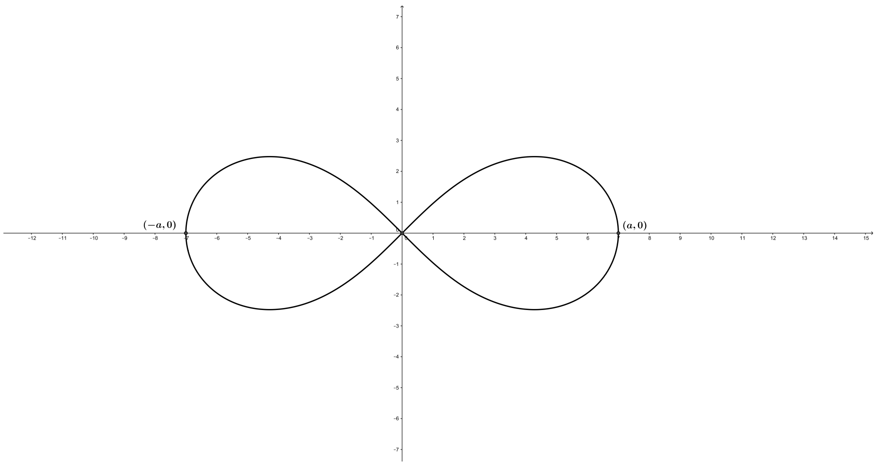


Figure 1: Leminiscate of Bernouilli

We choose the foci as $(\pm a, 0)$ with $c = a$, so that the leminiscate passes throught the origin and is symmetric with respect to the two coordinate axes. Let $a = 1/\sqrt{2}$ and r be the distance between the point p and the origin. Since the curve is symmetric with respect to both coordinate axes, we can restrict the curve to the first quadrant.

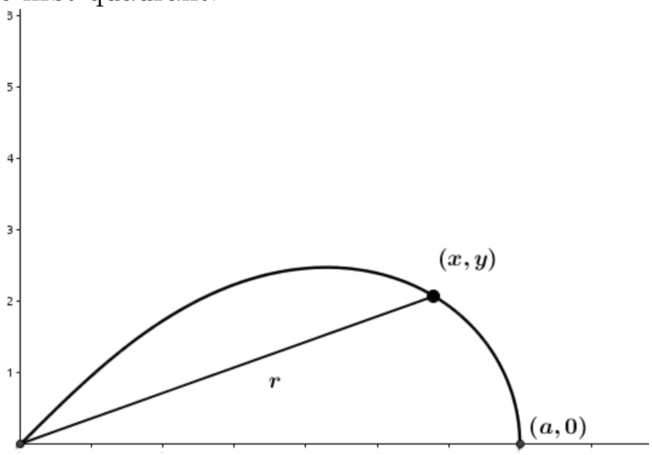


Figure 2: Arc of a leminiscate

The arc length s in function of r satisfies

$$\frac{ds}{dr} = \frac{1}{\sqrt{1 - r^4}},$$

so that the lenght of the leminiscate, restricted to the first quadrant, is:

$$s = \int_0^1 \frac{dr}{\sqrt{1 - r^4}}.$$

Fagnano discovered how to double and halve a leminiscate arc given by its end point using ruler and compass alone.
In 1751, Euler started his investigations closely related to the work of Fagnano, which led him to the discovery of the addition theorem for elliptic integrals. He knew that

$$\int_0^u \frac{dt}{\sqrt{1-t^2}} + \int_0^v \frac{dt}{\sqrt{1-t^2}} = \int_0^r \frac{dt}{\sqrt{1-t^2}}.$$

from the trigonometric addition theorem

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

by substituting

$$\begin{cases} u = \sin(x), \\ v = \sin(y), \\ r = \sin(x+y) = u(\sqrt{1-v^2}) + v(\sqrt{1-u^2}). \end{cases}$$

In the case of the leminiscate, he replaced the expression $(1-u^4)$ under the radical sign by the polynomial $P(u) = 1 + au^2 - u^4$, where a is an arbitrary constant, and thus proved the addition theorem

$$\begin{cases} r = \frac{u\sqrt{P(v)}+v\sqrt{P(u)}}{1+u^2v^2} \quad \text{and} \\ \int_0^u \frac{dt}{\sqrt{P(t)}} + \int_0^v \frac{dt}{\sqrt{P(t)}} = \int_0^r \frac{dt}{\sqrt{P(t)}}. \end{cases}$$

Finally Euler discovered the invariance of the integral under the fractional linear transformation

$$u = \frac{\alpha w + \beta}{\gamma w + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0.$$

Indeed, by applying it at the variable u of a generic polynomial $P(u)$ of fourth degree, he found that

$$H(w) = (\gamma w + \delta)^4 P(u)$$

is again a polynomial of fourth degree in w. In particular, with a suitable choice of the values of a, α, β, γ and δ, $H(w)$ can take the form

$$H(w) = 1 + aw^2 - w^4.$$

Since

$$du = \frac{\alpha\delta - \beta\gamma}{(\gamma w - \delta)^2} dw,$$

it follows that

$$(\alpha\delta - \beta\gamma)^{-1}\frac{du}{\sqrt{P(u)}} = \frac{dw}{\sqrt{H(w)}}.$$

This approach yields the general form of the Euler addition theorem.

The idea of inverting elliptic integrals to obtain elliptic functions is due to Gauss, Abel and Jacobi.
Gauss first considered inverting an elliptic integral in 1796, in the case of

$$\int \frac{dt}{\sqrt{1-t^3}}.$$

The following year he inverted the leminiscatic integral

$$u = \int_0^x \frac{dt}{\sqrt{1-t^4}},$$

defining the leminiscatic sine function $x = sl(u)$, and he found $sl(u)$ is doubly periodic.
Thus Gauss discovered the double periodicity property of the elliptic functions, but refrained from publishing any results. Beetween 1827 and 1829, this property was discovered (and released) by Abel.

This dissertation consists of five different chapters.
The first chapter introduces the necessary definitions concerning differential geometry, symplectic geometry and complex analysis, which should be useful to the reader to gain a better understanding of the following chapters.
In the second chapter we write some of the theory of the elliptic functions, which are doubly-periodic meromorphic functions on $\mathbb{C}$. We consider the fundamental parallelogram of the periods, and use it to characterize the the other properties of the elliptic functions. We construct elliptic functions of order $N \geq 3$, using the convergence of certain infinity sums over a lattice and the infinite products of meromorphic functions. Thanks to them we define the σ and the ζ Weierstrass functions. We define the $\wp$ Weierstrass function from the ζ function, and show that it is an elliptic function of second order. The $\wp$ Weierstrass function solves the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

We then use the property of the σ, ζ, $\wp$ Weierstrass functions to characterize some elliptic integrals. Moreover, from the differential equation we can parametrize elliptic curves and describe some of their properties.

The third chapter covers Lagrangian mechanics which describes the motion of a mechanical system by solving the Lagragian equations of motion that is obtained from the calculus of variations.
Throughout the fourth chapter we obtain the Hamiltonian function by applying the Legendre transformation to the Lagrangian function. From this we obtain the Hamiltonian system of equations. We then define the Hamiltonian systems as a triple (M, ω, H) where M is an even-dimensional manifold, ω a symplectic structure and $H \in C^{\infty}(M, \mathbb{R})$ the Hamiltonian function. With the Arnold-Liouville theorem, we see the condition for which a Hamiltonian system is integrable by quadrature. The chapter is concluded by with two basic examples of one degree of freedom Hamiltonian system that involve the elliptic integrals. In particular is explained how transform the elliptic integrals of the first kind with cubic and quartic polynomials into the Weierstrass form.
In the last chapter the integrable systems of the geodesic on an ellipsoid and the Neumann system on a sphere are considered. These two systems are linked by the Gauss map. Here, Moser's computations [11] are redone and it is shown that in the case $n = 2$ the integral associated to the quadrature becomes elliptic.
Certain calculations of the two examples shown in chapter four and certain proofs in the last chapter were made by using Matlab programs. Finally, the codes are listed in the appendix.

Chapter 1

Mathematical Preliminares

1.1 Differentiable manifold

Definition 1.1.1. An *n-dimensional smooth manifold* is a topological space M together with a countable collection of open sets $\{U_\alpha\}$ called the *coordinate charts* such that

- $\bigcup\limits_{\alpha} U_\alpha = M$;
- let $V_\alpha \subseteq \mathbb{R}^n$ be an open set. There exist one-to-one homeomorphisms $\phi_\alpha : U_\alpha \longrightarrow V_\alpha$, called the *coordinate functions*, such that for any pair of overlapping coordinate charts the maps

  $$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

  are smooth (i.e. infinitely differentiable) functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.

The pair (U_α, ϕ_α) is called *chart*, while the whole family $\big\{(U_\alpha, \phi_\alpha)\big\}$ is called *atlas.*

Definition 1.1.2. Let $f : M \longrightarrow N$ be a map from an m-dimensional manifold to an n-dimensional manifold. f is said *differentiable* at $p \in M$ if there are (U, ϕ) and (V, ψ) charts in M and N respectively, with $p \in U$ and $f(p) \in V$, such that $f(U) \subset V$ and

$$\psi \circ f \circ \phi^{-1} : \phi(U) \longrightarrow \psi(V)$$

is C^∞ with respect to each $\phi(p)$. Differentiable maps are also said to be *smooth*. f is a *diffeomorphism* if $\psi \circ f \circ \phi^{-1}$ is invertible and the inverse is C^∞.

Definition 1.1.3. Let $\phi : (-\epsilon, \epsilon) \longrightarrow M$ be a differentiable curve, $\epsilon > 0$, such that $\phi(0) = \mathbf{x}$. The *tangent vector* to M at $\mathbf{x}$ is a velocity vector of curves on M passing thought $\mathbf{x}$:

$$\dot{\mathbf{x}} = \lim_{t \longrightarrow 0} \frac{\phi(t) - \phi(0)}{t}.$$

Definition 1.1.4. Two curves $\phi(t)$ and $\psi(t)$ are *equivalent* if $\phi(0) = \psi(0) = \mathbf{x}$ and $\lim\limits_{t \longrightarrow 0}(\phi(t) - \psi(t))/t = 0$ in some chart.

It's easy to check that the set of tangent vectors is closed under scalar multiplication and addition.

Definition 1.1.5. Let M be a differentiable manifold of dimension n, and let $\mathbf{x}$ be a point of M. The set of all tangent vectors to M at $\mathbf{x}$ is a vector space of dimension n called the *tangent space* to M at $\mathbf{x}$, and is denoted by $T_x M$.

Definition 1.1.6. Let U be a chart of an atlas for M with coordinates $q_1, \dots, q_n$. Then the components of the tangent vector to the curve $\mathbf{q} = \phi(t)$ are the numbers $\xi_1, \dots, \xi_n$, where $\xi_i = (d\phi_i/dt)\big|_{t=0}$.

Definition 1.1.7. The union of the tangent spaces to M at the various points, $\bigcup\limits_{x \in M} T_x M$, has a natural differentiable manifold structure, the dimension of which is twice the dimension of M. This manifold is called the *tangent bundle* of M and is denoted by TM.

A point $(x, \xi) \in TM$ is a point $q \in M$ and a vector $\xi \in T_x M$, tangent to M at $\mathbf{x}$. Let $q_1, \dots, q_n$ be local coordinates on M, and $\xi_1, \dots \xi_n$ be the components of tangent vector in this coordinate system. Then the $2n$ numbers $(q_1, \dots, q_n, \xi_1, \dots \xi_n)$ give the local coordinate system on TM. One sometimes writes dq_i for ξ_i.

1.2 Differential forms

1.2.1 Exterior form

Definition 1.2.1. A f*orm of degree* 1 (or a 1-*form*) on $\mathbb{R}^n$ is a linear function $\omega : \mathbb{R}^n \longrightarrow \mathbb{R}$, i.e., $\forall \lambda_1, \lambda_2 \in \mathbb{R}$ and $\forall \xi_1, \xi_2 \in \mathbb{R}^n$,

$$\omega(\lambda_1 \xi_1 + \lambda_2 \xi_2) = \lambda_1 \omega(\xi_1) + \lambda_2 \omega(\xi_2),$$

The space of 1-forms on $\mathbb{R}^n$ is itself n-dimensional, and is also called the dual space $(\mathbb{R}^n)^*$.
Suppose that we have choosen a linear coordinate system $x_1, \dots, x_n$ on $\mathbb{R}^n$. Each coordinate x_i is itself a 1-form. These 1-forms are linearly independent. Therefore, every 1-form ω takes the form

$$\omega = a_1 dx_1 + \cdots + a_n dx_n, \qquad a_i \in \mathbb{R}.$$

The value of ω on a vector ξ is equal to

$$\omega(\xi) = a_1 x_1(\xi) + \cdots + a_n x_n(\xi), \qquad a_i \in \mathbb{R}.$$

where $x_1(\xi), \dots, x_n(\xi)$ are the components of ξ in the choosen coordinate system.

Definition 1.2.2. An *exterior form of degree 2* (or a *2-form*) is a function of pairs of vectors $\omega^2 : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, which is bilinear and skew-symmetric:

$$\begin{gathered} \omega^2(\lambda_1 \xi_1 + \lambda_2 \xi_2, \xi_3) = \lambda_1 \omega^2(\xi_1, \xi_3) + \lambda_2 \omega^2(\xi_2, \xi_3) \\ \omega^2(\xi_1, \xi_2) = -\omega^2(\xi_2, \xi_1) \\ \forall \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \xi_1, \xi_2, \xi_3 \in \mathbb{R}^n. \end{gathered}$$

The set of all 2-forms on $\mathbb{R}^n$ becomes a real vector space with the addition and the multiplication by scalars.

Definition 1.2.3. A 2-form is *non-degenerate* if

$$\omega^2(\xi, \eta) = 0, \quad \forall \eta \in \mathbb{R}^n \Longrightarrow \xi = 0$$

1.2.2 Exterior product

Let ξ be a vector in $\mathbb{R}^n$. Given two 1-forms ω_1 and ω_2, we can define a mapping of $\mathbb{R}^n$ to the plane $\mathbb{R} \times \mathbb{R}$ by associating to ξ the vector $\omega(\xi)$ with components $\omega_1(\xi)$ and $\omega_2(\xi)$ in the plane with coordinates ω_1, ω_2.

Definition 1.2.4. The value of *exterior product* $\omega_1 \wedge \omega_2$ on the pair of vectors $\xi_1, \xi_2 \in \mathbb{R}^n$ is the oriented area of the image of the parallelogram with sides $\omega(\xi_1)$ and $\omega(\xi_2)$ on the ω_1, ω_2 - plane:

$$(\omega_1 \wedge \omega_2)(\xi_1, \xi_2) = \begin{vmatrix} \omega_1(\xi_1) & \omega_2(\xi_1) \\ \omega_1(\xi_2) & \omega_2(\xi_2) \end{vmatrix}.$$

$\omega_1 \wedge \omega_2$ is a 2-form, so is bilinear and skew symmetic:

$$\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$$
$$(\lambda_1\omega_1 + \lambda_2\omega_2) \wedge \omega_3 = \lambda_1\omega_1 \wedge \omega_3 + \lambda_2\omega_2 \wedge \omega_3.$$

Follows

$$\omega_i \wedge \omega_i = 0.$$

1.2.3 Differential forms

We give here the definition of differential forms on differentiable manifolds.

Definition 1.2.5. A *differential form of degree 1* (or a *1-form*) on a manifold M is a smooth map

$$\omega : TM \longrightarrow \mathbb{R}$$

of the tangent boundle of M to the line, linear on each tangent space T_xM.

Let $M = \mathbb{R}^n$ be with coordinates $x_1, \dots, x_n$. Let $\xi \in T_x\mathbb{R}^n$ be a vector,

$$\xi = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i},$$

with

$$dx_i(\xi) = a_i(\xi), \qquad dx_i(\frac{\partial}{\partial x_j}) = \delta_{i,j} \quad i = 1, \dots, n.$$

The n 1-forms $dx_1, \dots, dx_n$ on T_xM are linearly independent and form a basis for the n-dimensional space of 1-forms on T_xM. So we can say:

Proposition 1.2.1. *Let ω be a differential 1-form on the space M with a given coordinate system $x_1, \ldots x_n$. Then ω can be written uniquely in the form*

$$\omega = a_1(x)dx_1 + \cdots + a_n(x)dx_n$$

where the coefficients $a_i(x)$ are smooth functions.

Definition 1.2.6. Let M be an n-dimensional differentiable manifold. A 1-form on the tangent space to M at point x is called *cotangent vector* to M at x. The set of all cotangent vectors to M at x forms an n-dimensional vector space, dual to the tangent space T_xM. We will denote this vector space of *cotangent vectors* by T_x^*M and call it the *cotangent space* to M at x.
The union of the cotangent spaces to the manifold at all of its points is called the *cotangent boundle* of M and is denoted by T^*M. The cotangent bundle has a natural structure of a differentiable manifold of dimension $2n$.

Definition 1.2.7. A *differentiable k-form* ω^k at point x of a manifold M is an exterior k-form on the tangent space T_xM to M at x, i.e. a k-linear skew-symmetric function of k vectors $\xi_1, \ldots \xi_k$ tangent to M at x.

Theorem 1.2.2. *Every differential k-form on the space M with a given co-ordinate system $x_1, \ldots x_n$ can be written uniquely in the form*

$$\omega^k = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $a_{i_1, \ldots, i_k}(x)$ are smooth functions on $\mathbb{R}^n$.

Definition 1.2.8. We define the *exterior derivative* of the k-form

$$\omega^k = \sum_{i=1}^{n} a_{i1,\ldots,ik} dx_{i1} \wedge \cdots \wedge dx_{ik}$$

be the $(k+1)$-form

$$d\omega^k = \sum_{i=1}^{n} da_{i1,\ldots,ik} \wedge dx_{i1} \wedge \cdots \wedge dx_{ik}.$$

So, if ω is a 1-form, ,

$$\omega = a_1 dx_1 + \cdots + a_n dx_n$$

its exterior derivative is

$$d\omega = da_1 \wedge dx_1 + \cdots + da_n \wedge dx_n.$$

Theorem 1.2.3 (Stoke's Theorem)**.** *Let M be a n-dimensional orientable manifold, i.e it can be given an orientation, and ω a k-form.*
Then the integral of a differential form ω over the boundary of M is equal to the integral of its exterior derivative $d\omega$ over M, so

$$\int_{\partial M} \omega = \int_M d\omega.$$

Definition 1.2.9. A differential form ω on a manifold M is *closed* if its exterior derivative is zero: $d\omega = 0$.

1.3 Symplectic geometry

Symplectic vector spaces

Definition 1.3.1. A symplectic linear structure on $\mathbb{R}^{2n}$ is a non-degenerate bilinear skew-symmetric 2-form given in $\mathbb{R}^{2n}$. This form is called the *skew scalar product* and is denoted by $[\xi, \eta] = -[\eta, \xi]$.
The pair $(\mathbb{R}^{2n}, [,])$ is called the *symplectic vector space.*

Definition 1.3.2. Let $(p_1, \dots, p_n, q_1, \dots q_n)$ be coordinate functions on $\mathbb{R}^{2n}$, and ω^2 be the form

$$\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

This form is nondegenerate and skew-symmetric, it can be taken for a skew-scalar product: $[\xi, \eta] = \omega^2(\xi, \eta)$. In this way $\mathbb{R}^{2n} = (\mathbf{p}, \mathbf{q})$ receives a symplectic structure and $(\mathbb{R}^{2n}, \omega^2)$ is called the *standard symplectic structure.*

Definition 1.3.3. A *symplectic basis* is a set of $2n$ vectors $\mathbf{e}_{p_i}, \mathbf{e}_{q_i}$, $i = 1, \dots, n$, whose scalar products have the following structure:

$$[\mathbf{e}_{p_i}, \mathbf{e}_{p_j}] = [\mathbf{e}_{q_i}, \mathbf{e}_{q_j}] = 0, \quad [\mathbf{e}_{p_i}, \mathbf{e}_{q_j}] = \delta_{ij} \quad \forall i, j = 1, \dots n,$$

where δ_{ij} is the *Kronecker delta function.*

If we take the vectors of a symplectic basis as a coordinate unit vectors, we obtain a coordinate system p_i, q_i in which $[\quad , \quad]$ takes the standard form $dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$. Such a coordinate system is called *symplectic.*

Definition 1.3.4. A linear transformation $S : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ of the symplectic space $\mathbb{R}^{2n}$ to itself is called *symplectic* if, $\forall \xi = \sum_{i=1}^{n} \left[\alpha_i(\mathbf{p},\mathbf{q})e_{p_i} + \beta_i(\mathbf{p},\mathbf{q})e_{q_i}\right]$ and $\forall \eta = \sum_{i=1}^{n} \left[\gamma_i e_{p_i} + \delta_i e_{q_i}\right]$, it preserves the skew-scalar product:

$$[S\xi, S\eta] = [\xi, \eta] = \alpha_i\delta_j - \beta_i\gamma_j, \qquad \forall \xi, \eta \in \mathbb{R}^{2n}.$$

It follows that a transformation $S : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ of the standard symplectic space $(\mathbf{p}, \mathbf{q})$ is symplectic if and only if it is linear and canonical, i.e. preserves the differential 2-form

$$\omega^2 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

In this coordinate system, the transformation is given by a $2n \times 2n$ matrix S.

Theorem 1.3.1. *A transformation is symplectic if and only if its matrix S in the symplectic coordinate system $(\boldsymbol{p}, \boldsymbol{q})$ satisfies the relation*

$$S'IS = I,$$

where

$$I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

S' is the transpose of S and E is the $n \times n$ identity matrix.

Symplectic atlas

An atlas of a manifold M^{2n} is called symplectic if the standard symplectic structure $\omega^2 = d\mathbf{p} \wedge d\mathbf{q}$ is introduced into the coordinate space $\mathbb{R}^{2n} = (\mathbf{p}, \mathbf{q})$, and the transfer from one chart to another is realized by canonical (i.e. ω^2 preserving) transformation $\phi_\beta \circ \phi_\alpha^{-1}$.

Theorem 1.3.2 (Darboux's theorem)**.** *Let ω^2 be a closed nondegenerate differential 2-form in a neighborhood of a point $\boldsymbol{x}$ in the space $\mathbb{R}^{2n}$. Then in some neighborhood of $\boldsymbol{x}$ one can choose a coordinate system $(p_1, \ldots, p_n, q_1, \ldots q_n)$ such that the form has the standard form:*

$$\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

1.4 Complex analysis review

The Riemann Sphere

In order to discuss meromorphic functions, we use the extended complex plane

$$\Sigma = \mathbb{C} \cup \{\infty\}$$

where ∞ is an extra point called the *point at infinity.* Σ may be regarded as being a sphere. Indeed, consider the 2-sphere

$$S^2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\right\}$$

in $\mathbb{R}^3$, and identify the complex plane $\mathbb{C}$ with the plane $x_3 = 0$ by identifying $z = x + iy$ $(x, y \in \mathbb{R})$ with $(x, y, 0) \forall z \in \mathbb{C}$. If $N = (0, 0, 1)$ is the north pole of S^2, then the stereographic projection from N to the plane $(x, y, 0)$ gives a bijective map

$$\begin{aligned} \pi : S^2 \setminus \{N\} &\longrightarrow \mathbb{C} \\ Q &\longmapsto A, \end{aligned}$$

where $A \in \mathbb{C}$, $Q \in S^2 \setminus \{N\}$, and A, Q, N are collinear.
π is an homorphism between $S^2 \setminus \{N\}$ and $\mathbb{C}$ (see [6]), so we can extend $\pi : S^2 \setminus \{N\}$ to a bijection $\pi : S^2 \longrightarrow \Sigma$ by defining $\pi(N)$ to be ∞, and use it to transfer algebraic and topological properties from Σ to S^2 and vice-versa. Since Σ has the same topological properties as the sphere S^2, Σ is often referred as the *Riemann sphere.*

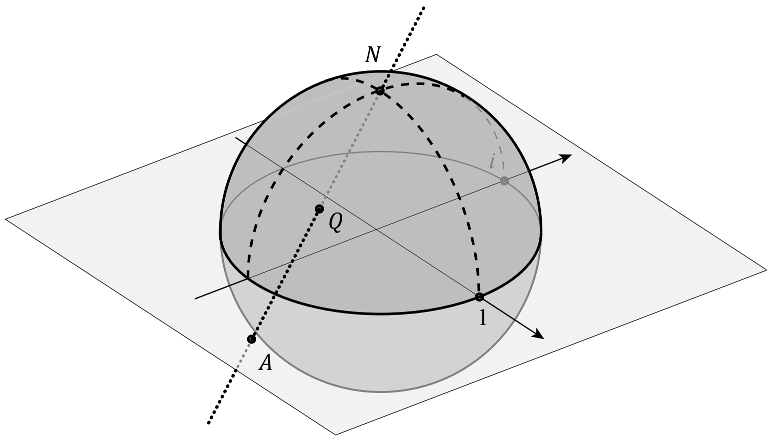


Figure 1.1: Riemann sphere

Analytic and meromorphic functions

Let $U \subset \mathbb{C}$ be an open set (i.e. $\forall z_o \in U, \exists r > 0$ s.t. $\{|z - z_0| < r\} \subset U$).

Definition 1.4.1. A function f is said to be *holomorphic* on U if f is complex differentiable on each point of U. i.e. at $z_0 \in \mathbb{C}$, the limit

$$\lim_{h \longrightarrow o} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in $\mathbb{C}$.

Theorem 1.4.1. *Let $f : U \longrightarrow \mathbb{C}$, and write f=u+iv, where u, v are real valued functions.*

- *if $f \in C^1(U)$ satisfies the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

  then f is holomorphic in U

- *If f is given by a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges on an open disc $B(z_0, r)$ of centre z_0 and radious of convergence $r > 0$, then f is holomorphic on $B(z_0, r)$.

Definition 1.4.2. A function $f : U \longrightarrow \mathbb{C}$ is said to be *analytic* on U if for each point $z_0 \in U$ there exists an open disc $B(z_0, r) \subseteq U$ in which f can be written as the sum of a power series centrered at z_0, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{for } z \in B(z_0, r).$$

Proposition 1.4.2. *f analytic on $U \Longrightarrow f$ holomorphic on U.*

Definition 1.4.3. Let $0 \leq r_1 < r$ and $z_0 \in \mathbb{C}$. Let $A = \{z \in \mathbb{C} \, | r_1 < |z - z_0| < r_2\}$ an annulus. $f(z)$ is called *Laurent series* is has the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \tag{1.1}$$

where the coefficients a_n, b_n for $n \in \mathbb{Z}$ are complex numbers, and both the series converge absolutely on A and uniformly in sets of the form $B_{\rho_1,\rho_2} = \{z|\rho_1 \leq |z - z_0| \leq \rho_2\}$, where $r_1 < \rho_1 < \rho_2 \leq r_1$.

Let's see the special case of the Laurent series when $r_1 = 0$:

Definition 1.4.4. If f is analitic on $\{z \in \mathbb{C}|0 < z - z_0| < r_2\}$, which is deleted r_2 neighborhood of z_0, we say that z_0 is an *isolated singularit*y.

- If z_0 is an isolated singularity of f ad if all but a finite number of the b_n in 1.1 are zero, then z_0 is called a *pole* of f. If k is the highest integer such that $b_k \neq 0$, z_0 is called a *pole of order k*. If z_0 is a first-order pole, we also say it is a *simple pole*. The Laurent series has the form

$$\frac{b_k}{(z - z_0)^k} + \dots \frac{b_1}{(z - z_0)} + a_0 + a_1(z - z_0) + \dots$$

  The part

$$\frac{b_k}{(z - z_0)^k} + \dots \frac{b_1}{(z - z_0)}$$

  is called *principal part* of f at z_0.

- if an infinite number of b_k in 1.1 are nonzero, then z_0 is called an *essential singularity*.

- We call b_1 in 1.1 the *residue* of f at z_0, and we denote it with $\underset{z=z_0}{Res}(f(z))$.
- If all the b_k in 1.1 are zero, we say that z_0 is a *removable singularity*, then
$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$
i.e. is a convergent power series.

Definition 1.4.5. Let f be an analytic function and $F(z) = f(1/z)$. Then we say that:

- f has a *pole of order* k at ∞ if F has a pole of order k at 0;
- f has a *zero of order* k at ∞ if F has a zero of order k at 0;
- we define $\underset{w=\infty}{Res}(f(w)) = -\underset{z=0}{Res}(\frac{1}{z^2}F(z))$, where $w = \frac{1}{z}$.

Definition 1.4.6. A function is said to be *meromorphic* in A if it is analytic on A, except for poles in A.

Theorem 1.4.3 (Residue theorem)**.** *Let f be analytic on a region $A \setminus \{z_0\}$ and have an isolated singularity at z_0. If γ is any circle around z_0 in A whose interior, except for the point z_0, lies in A, then*
$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \underset{z=z_0}{Res}(f(z)).$$

Proposition 1.4.4. *Let γ be a simple closed curve in $\mathbb{C}$. Let f be analytic along γ and have only finitely many singularities outside γ at the point $z_1, \dots, z_n$. Then*
$$\frac{1}{2\pi i}\int_\gamma f = \sum_{i=1}^{n} \underset{z=z_i}{Res}(f(z)) + \underset{z=\infty}{Res}(f(z)).$$

Theorem 1.4.5 (Cauchy's Residues Theorem)**.** *Suppose f and g are analytic in a neirborhood of z_0 with zeros there of order n and k respectively.*
Let $h(z) = \dfrac{f(z)}{g(z)}$. Then

- *if $k > n$, then h has a pole of order $k-n$ at z_0;*
- *if $k = n$, then h has a removable singularity with nonzero limit at z_0;*
- *if $k < n$, then h has a removable singularity at z_0, and setting $h(z_0) = 0$ produces an analytic function with a zero of order $n-k$ at z_0.*

Chapter 2

Elliptic functions

2.1 Periodic functions

Definition 2.1.1. Let f be a function defined on the complex plane $\mathbb{C}$. Then a complex number ω is called a *period* of f if

$$f(z+\omega) = f(z)$$

$\forall z \in \mathbb{C}$, and f is called *periodic* if it has period $\omega \neq 0$.

The set Ω_f of periods of a function f has two important properties: one algebraic, valid for all f, and one topological, valid for a non-constant meromorphic functions f.

Theorem 2.1.1. *Let Ω_f be the set of periods of a function f defined on $\mathbb{C}$; then Ω_f is a subgroup of the additive group $\mathbb{C}$.*

Theorem 2.1.2. *Let Ω_f be the set of periods of a non-constant meromorphic function f defined on $\mathbb{C}$; then Ω_f is a discrete subset of $\mathbb{C}$.*

We now show that there are three types of discrete subgroups of $\mathbb{C}$, isomorphic to $0, \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ respectively.

Theorem 2.1.3. *Let Ω be a discrete subgroup of $\mathbb{C}$. Then one of the followings holds:*

i. $\Omega = \{0\}$;

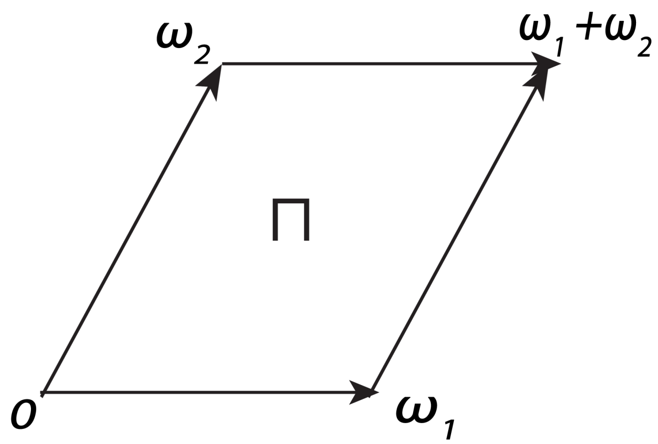
ii. $\Omega = \{n\omega_1 | n \in \mathbb{Z}\}$ *for some fixed* $\omega_1 \in \mathbb{C} \setminus \{0\}$, *and so* Ω *is isomorphic to* $\mathbb{Z}$;

iii. $\Omega = \{m\omega_1 + n\omega_2 | n, m \in \mathbb{Z}\}$ *for some fixed* $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, *where* ω_1 *and* ω_2 *are linearly independent over* $\mathbb{R}$, *i.e* $Im\left(\frac{\omega_2}{\omega_1}\right) \neq 0$. *In this case,* Ω *is isomorphic to* $\mathbb{Z} \times \mathbb{Z}$.

Definition 2.1.2. If a function f has is set Ω_f of periods of type (ii), then f is *simply periodic*; if Ω_f is of type (iii), then f is *doubly periodic.*
Groups Ω of type (iii) are called *lattices*, and a pair $\{\omega_1, \omega_2\}$ such that

$$\Omega = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$$

is called *basis* for the lattice.
The parallelogram $\sqcap$ with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$ is called *fundamental parallelogram* for Ω .



If $\sqcap_1$ is another fundamental parallelogram for Ω , then $\sqcap$ and $\sqcap_1$ have the same area. Indeed

Theorem 2.1.4. *Let* Ω *be a lattice with basis* $\{\omega_1, \omega_2\}$. *Then* $\{\omega_1', \omega_2'\}$ *is a basis for* $\Omega \iff \exists\ a, b, c, d \in \mathbb{Z}$ *such that*

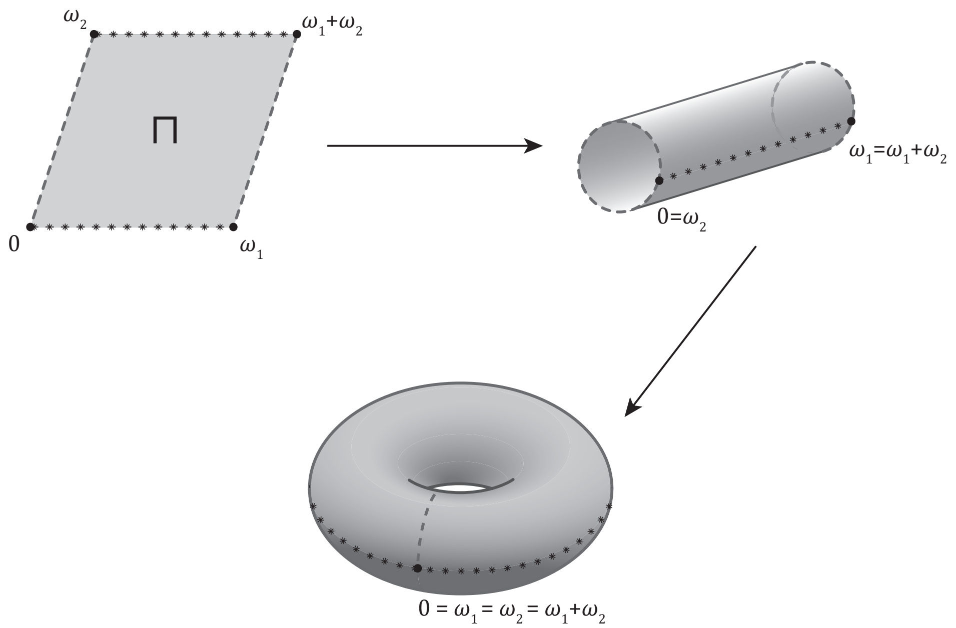
$$\omega_1' = a\omega_1 + b\omega_2$$
$$\omega_2' = c\omega_1 + d\omega_2.$$

and $ad - bc = \pm 1$.

Given a lattice Ω , we define $z_1, z_2 \in \mathbb{C}$ to be *congruent* mod Ω , written $z_1 \sim z_2$, if $z_1 - z_2 \in \Omega$, i.e.

$$z_1 \equiv z_2 (mod\ \Omega) \iff z_2 = z_1 + \omega, \quad \text{for some } \omega \in \Omega.$$

Congruence mod Ω is easily to be an equivalence relation on $\mathbb{C}$.
We can identify doubly periodic functions with functions on a quotient space $\mathbb{C}/\Omega$, and this is identified with the torus obtained by gluing together both opposite pairs of edges of the fundamental parallelogram $\sqcap$.



2.2 General properties of elliptic functions

Definition 2.2.1. A meromorphic function $f : \mathbb{C} \longrightarrow \Sigma$ is *elliptic* with respect to a lattice $\Omega \subseteq \mathbb{C}$ if f is doubly periodic with respect to Ω , i.e. if

$$f(z + \omega) = f(z) \qquad \forall z \in \mathbb{C}, \omega \in \Omega,$$

so that each $\omega \in \Omega$ is a period of f.
If f is elliptic with respect to Ω , then we may regard f as a function $f : T \longrightarrow \Sigma$, where T is the torus $T = \mathbb{C}/\Omega$.

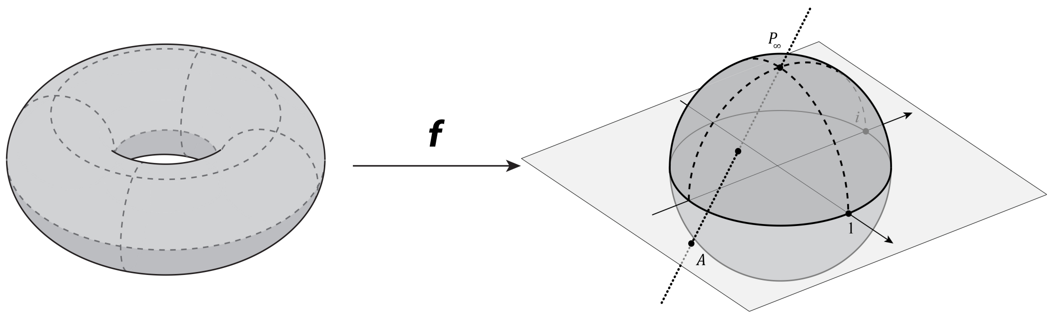


Figure 2.1: $f : T \longrightarrow \Sigma$

Definition 2.2.2. The *order* of the elliptic function $f(z)$, written $N = N(f)$ is the number of poles counted with multiplicity of $f(z)$ mod Ω , i.e N is the sum of the orders of the poles of $f(z)$ in the interior of $\sqcap$.

Theorem 2.2.1. *An elliptic function f has order $N = 0 \iff f$ is constant.*

Proof. ($\iff$) Let $\sqcap$ be a (closed) period parallelogram for the elliptic function $f(z)$. Suppose $f(z)$ has order $N = 0$, i.e. it has no poles and so f is holomorphic on $\mathbb{C}$. In particular it is continuous and $f(\Omega) = f(\sqcap)$. Since the continuous image of a closed and bounded set is bounded, $f(\sqcap)$ is a bounded subset of $\mathbb{C}$. So $\exists M > 0$ such that $\forall z \in \sqcap$, $\left|f(z)\right| \leq M$. $\forall z \in \mathbb{C}, \exists\ m, n \in \mathbb{Z}$ such that $z_1 = z - m\omega_1 - n\omega_2 \in \sqcap$ and $f(z) = f(z_1) < M$. Then f is bounded and holomorphic on $\mathbb{C}$, and, for the Liouville theorem, f is constant. $\square$

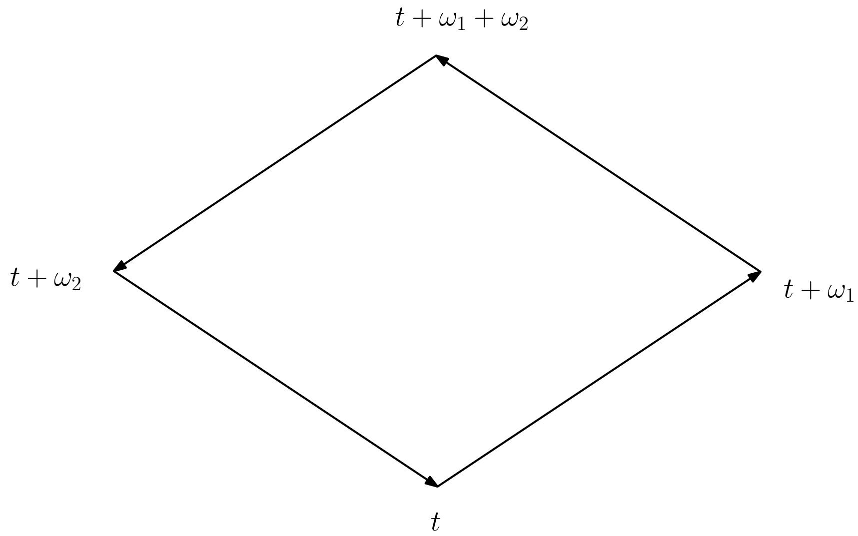
Theorem 2.2.2. *Let $f(z)$ be elliptic, let $\sqcap$ be a period parallelogram for $f(z)$ with no poles of $f(z)$ on $\partial\sqcap$. Let $b_1, \ldots, b_s$ be the poles of $f(z)$ in $\sqcap$. Then the sum of the residues at poles of $f(z)$ inside $\sqcap$ is zero, i.e.*

$$\sum_{i=1}^{s} \underset{z=b_i}{Res}(f(z)) = 0.$$

Proof. We use the Cauchy's Residue Theorem

$$\sum_{i=1}^{s} \underset{z=b_i}{Res}(f(z)) = \frac{1}{2\pi i} \oint_{\partial\sqcap} f(z)dz.$$

So



$$\frac{1}{2\pi i}\Big(\int_{x=0}^{1} f(t+\omega_1 x)\omega_1 dx + \int_{x=0}^{1} f(t+\omega_1+\omega_2 x)\omega_2 dx$$
$$+\int_{x=0}^{1} f(t+\omega_1+\omega_2-\omega_1 x)(-\omega_1)dx + \int_{x=0}^{1} f(t+\omega_2-\omega_2 x)(-\omega_2)dx\Big).$$

By substitution $y = 1-x$ in the last two integrals, and since ω_1 and ω_2 are periods, we have

$$\frac{1}{2\pi i}\Big(\int_{x=0}^{1} f(t+\omega_1 x)\omega_1 dx + \int_{x=0}^{1} f(t+\omega_2 x)\omega_2 dx$$
$$-\int_{y=0}^{1} f(t+\omega_1 y)\omega_1 dy - \int_{y=0}^{1} f(t+\omega_2 y)\omega_2 dy\Big) = 0.$$

□

Corollary 2.2.3. *There are no elliptic function of order $N = 1$.*

Proof. Suppose f is elliptic of order $N = 1$. So we have a pole at $z = b$, and the Laurent series has principal part $\frac{a_{-1}}{z-b}$, with residue $\underset{z=b}{Res}(f(z)) = a_{-1} \neq 0$. But by Theorem 2.2.2

$$0 = \sum \underset{z=b}{Res}(f(z)) = a_{-1}$$

and we have the contraddiction.

□

Theorem 2.2.4. *Let $f(z)$ be a non constant elliptic function of order N. Then the number of zeros of $f(z)$, mod Ω, counted with multiplicity, equals the number of the poles of $f(z)$, mod Ω, counted with multiplicity.*

Let $\sqcap$ be a period parallelogram with no zeros and no poles of $f(z)$ on $\partial\sqcap$. Let $z = a_1, \ldots, a_r$ be the zeros of $f(z)$ inside $\sqcap$ with orders $k_1, \ldots, k_r$ respectively. Let $z = b_1, \ldots, b_s$ be the poles of $f(z)$ inside $\sqcap$ with orders $l_1, \ldots, l_s$ respectively. Then

$$\sum_{i=1}^{r} k_i = \sum_{j=1}^{s} l_j = N.$$

Proof. $\frac{f'(z)}{f(z)}$ is meromorphic, and since $\partial\sqcap$ contains no poles or zeros of f, f'/f is analitic on $\partial\sqcap$. Since f'/f is elliptic, from the previous theorem, we have

$$\oint_{\partial\sqcap} \frac{f'(z)}{f(z)} = 0 \Longrightarrow \sum Res(\frac{f'}{f}) = 0.$$

Now, f'/f has poles at poles and zeros of f and nowhere else. Suppose f has zeros at $z = a$ with multiplicity k. So

$$f(z) = (z-a)^k g(z)$$

where g is analytic and $g(a) \neq 0$, and

$$f'(z) = k(z-a)^{k-1} g(z) + (z-a)^k g'(z) \qquad \text{near } z = a.$$

Then

$$\frac{f'(z)}{f(z)} = \frac{k(z-a)^{k-1} g(z) + (z-a)^k g'(z)}{(z-a)^k g(z)} = \frac{k}{(z-a)} + \frac{g'(z)}{g(z)},$$

so that f'/f has residue k at $z = a$.
Let's do the same for f with a pole $z = a$ of multiplicity k.

$$f(z) = (z-a)^{-k} g(z) \Longrightarrow f'(z) = -k(z-a)^{-k-1} g(z) + (z-a)^{-k} g'(z) \quad \text{near } z = a$$

Then we have

$$\frac{f'(z)}{f(z)} = \frac{-k}{(z-a)} + \frac{g'(z)}{g(z)}.$$

So that f'/f has residue $-k$ at $z = a$. Since the sum of the residues in zero, the number of zeros must be equal to the number of poles counting with multiplicity. So $f(z) = 0$ has N solution, as required. $\square$

Corollary 2.2.5. *If f has order N then f takes each value $c \in \Sigma$ exactly N times.*

Theorem 2.2.6. *The sum of the places where $f(z)$ has zeros (counted with multiplicity) equals the sum of the places where $f(z)$ has poles (counted with multiplicity), i.e*

$$\sum_{i=1}^{r} k_i a_i - \sum_{j=1}^{s} l_j b_j \equiv 0 \ mod \ \Omega.$$

Proof. First we prove that

$$\sum_{i=1}^{r} k_i a_i - \sum_{j=1}^{s} l_j b_j = \frac{1}{2\pi i} \oint_{\partial \sqcap} z \frac{f'(z)}{f(z)}.$$

Since the poles of zf'/f are at the zeros and poles of f, and if f has a zero of multiplicity k at $z = a$, then

$$\begin{aligned} f(z) &= (z-a)^k g(z) \\ f'(z) &= k(z-a)^{k-1} g(z) + (z-a)^k g'(z) \end{aligned}$$

near $z = a$ and with g analytic, $g(z) \neq 0$.

$$z \frac{f'(z)}{f(z)} = \frac{zk(z-a)^{k-1} g(z) + z(z-a)^k g'(z)}{(z-a)^k g(z)} = \frac{zk}{(z-a)} + \frac{zg'(z)}{g(z)}$$

so $Res \left(z \frac{f'(z)}{f(z)}, a \right) = ka.$
Suppose $z = b$ is a pole of $f(z)$ of order l, so

$$f(z) = (z-b)^{-l} g(z), \qquad \text{near } z = b, g(b) \neq 0, g \text{ analytic}$$

and with the same calculation $Res \left(z \frac{f'(z)}{f(z)}, b \right) = -lb.$
By the Residue Theorem

$$\sum_{i=1}^{r} k_i a_i - \sum_{j=1}^{s} l_j b_j = \frac{1}{2\pi i} \oint_{\partial \sqcap} z \frac{f'(z)}{f(z)}.$$

Let's calculate $\frac{1}{2\pi i}\oint_{\partial\sqcap} z\frac{f'(z)}{f(z)}$ in the same way we did in Theorem 2.2.2

$$\frac{1}{2\pi i}\oint_{\partial\sqcap} z\frac{f'(z)}{f(z)} =$$

$$\frac{1}{2\pi i}\Big(\int_{x=0}^{1}(t+\omega_1 x)\frac{f'(t+\omega_1 x)}{f(t+\omega_1 x)}\omega_1 dx + \int_{x=0}^{1}(t+\omega_1+\omega_2 x)\frac{f'(t+\omega_1+\omega_2 x)}{f(t+\omega_1+\omega_2 x)}\omega_2 dx$$

$$-\int_{x=0}^{1}(t+\omega_1 x+\omega_2)\frac{f'(t+\omega_1 x+\omega_2)}{f(t+\omega_1 x+\omega_2)}\omega_1 dx - \int_{x=0}^{1}(t+\omega_2 x)\frac{f'(t+\omega_2 x)}{f(t+\omega_2 x)}\omega_2 dx\Big)$$

$$= \frac{1}{2\pi i}\left\{-\omega_2\int_{x=0}^{1}\frac{f'(t+\omega_1 x)}{f(t+\omega_1 x)}\omega_1 dx + \omega_1\int_{x=0}^{1}\frac{f'(t+\omega_2 x)}{f(t+\omega_2 x)}\omega_2 dx\right\}$$

$$= \frac{1}{2\pi i}\left\{-\omega_2\left[log(f(t+\omega_1 x))\right]_0^1 + \omega_1\left[log(f(t+\omega_2 x))\right]_0^1\right\}$$

$$= \frac{1}{2\pi i}\{-\omega_2 2\pi i m + \omega_1 2\pi i n\} = -\omega_2 m + \omega_1 n, \qquad \text{for some } n, m \in \mathbb{Z}$$

which is an element of Ω , as required.

□

Theorem 2.2.7. *Let f and g be elliptic functions with respect to Ω , with zeros and poles of the same orders at the same points of $\mathbb{C}$. Then f and g differ by a (non zero) multiplicative constant, i.e.*

$$\exists c \in \mathbb{C},\ c \neq 0,\ \textit{such that } g(z) = cf(z).$$

Proof. Consider

$$h(z) = \frac{g(z)}{f)(z)}.$$

This is clearly elliptic. The only possible zeros and poles of $h(z)$ are at zeros and poles of $f(z)$ and $g(z)$. Suppose z_0 is a zero (respectively pole) of order k (respect. $-k$, $k > 0$) of f and hence of g.

$$f(z) = (z - z_0)^k f_0(z)$$
$$g(z) = (z - z_0)^k g_0(z)$$

where $f_0(z)$ and $g_0(z)$ are holomorphic near $z = z_0$ and $g_0(z_0), f_0(z_0) \neq 0$

$$h(z) = \frac{(z-z_0)^k f_0(z)}{(z-z_0)^k g_0(z)} = \frac{g_0(z)}{f_0(z)}.$$

So z_0 is a removable singularity of $h(z)$ and $h(z)$ is elliptic function without poles. By Theorem 2.2.1 $h(z)$ is a constant c.

$$c = \frac{g(z)}{f(z)} \Longrightarrow g(z) = cf(z).$$

□

Theorem 2.2.8. *Let f and g be elliptic functions with respect to Ω , with poles at the same points in $\mathbb{C}$ and with the same principal parts at these points, Then $f(z)$ and $g(z)$ differ by an additive constant, i.e.*

$$\exists\ c \in \mathbb{C} \text{ such that } g(z) = f(z) + c$$

Proof. Let

$$h(z) = g(z) - f(z).$$

$h(z)$ is elliptic and has no poles since $f(z)$ and $g(z)$ have the same poles. So by Theorem 2.2.1 $h(z)$ is a constant c.

$$c = g(z) - f(z) \Longrightarrow g(z) = f(z) + c.$$

□

2.3 Construction of elliptic functions of order $N \geq 3$ with prescribed periods

In this section we construct elliptic functions of order $N \geq 3$ woth prescribed periods using summation over the lattice Ω . To clarify the meaning of summation over Ω we must first describe a particular ordering of Ω . Let's define

$$\Omega_0 = \{0\}$$
$$\Omega_r = \Big\{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z} \text{ and } max \left\{|m|, |n|\right\} = r, \Big\}$$
$$\sqcap_0 = \{0\}$$
$$\sqcap_r = \Big\{m\omega_1 + n\omega_2 | m, n \in \mathbb{R} \text{ and } max \left\{|m|, |n|\right\} = r, \Big\}$$

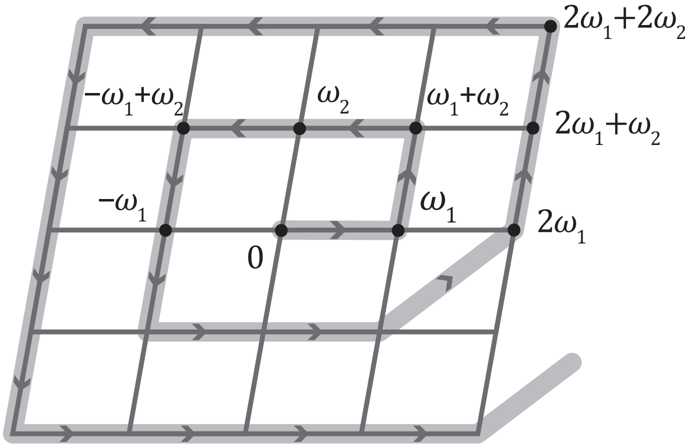


Figure 2.2: Summation over Ω

We have $\sqcap_r \supseteq \Omega_r \forall r$, Ω is a disjoint union $\Omega = \{0\} \cup \Omega_1 \cup \cdots \cup \Omega_r \cup \ldots$, and for each $r \geq 1$ we have

$$|\Omega_r| = 8r. \tag{2.1}$$

We can order the elements of Ω by starting at 0 and then listing the elements of $\Omega_1, \Omega_2, \ldots$ in turn, rotating around each Ω_r, in the order $r\omega_1, r\omega_1 + \omega_2, \ldots, r\omega_1 - r\omega_2$ (*spiralling order*).

If we denote this ordering by $\omega^{(0)}, \omega^{(1)}, \omega^{(2)}, \ldots$, then $\omega^{(0)} = 0, \omega^{(1)} = \omega_1, \omega^{(2)} = \omega_1 + \omega_2$ and so on.

So we define the *sum over a lattice*

$$\sum_{\omega \in \Omega} = \sum_{r=0}^{\infty} \sum_{\omega \in \Omega_r}.$$

Notation: $\sum_{\omega \in \Omega}'$ means omit the term $\omega = 0$, i.e.

$$\sum_{\omega \in \Omega}' = \sum_{r=1}^{\infty} \sum_{\omega \in \Omega_r}. \tag{2.2}$$

Proposition 2.3.1. *If $s \in \mathbb{R}$, then $\sum_{\omega \in \Omega}' |\omega|^{-s}$ converges if and only if $s > 2$.*

Proof. Let D and d be the greatest and the least distances of a point of the parallelogram $\sqcap_1$ from 0. Since $\sqcap_r = \{rz | z \in \sqcap_1\}$, the greatest and the least distances of a point in $\sqcap_r$ from 0 are rD and rd respectively. In particular, if $\omega \in \Omega_r$, then

$$rd \leq |\omega| \leq rD$$

so, by (2.1) and (2.2) we have

$$r^{-s}min\left\{D^{-s}, d^{-s}\right\} \leq |\omega|^{-s} \leq r^{-s}max\left\{D^{-s}, d^{-s}\right\} \Longrightarrow$$
$$8rr^{-s}min\left\{D^{-s}, d^{-s}\right\} \leq \sum_{\omega\in\Omega} |\omega|^{-s} \leq 8rr^{-s}max\left\{D^{-s}, d^{-s}\right\} \Longrightarrow$$
$$\sum_{r=1}^{\infty} 8rr^{-s}min\left\{D^{-s}, d^{-s}\right\} \leq \sum_{\omega\in\Omega_r}{}' |\omega|^{-s} \leq \sum_{r=1}^{\infty} 8rr^{-s}max\left\{D^{-s}, d^{-s}\right\} \Longrightarrow$$
$$8min\left\{D^{-s}, d^{-s}\right\} \sum_{r=1}^{\infty} r^{1-s} \leq \sum_{\omega\in\Omega}{}' |\omega|^{-s} \leq 8max\left\{D^{-s}, d^{-s}\right\} \sum_{r=1}^{\infty} r^{1-s}.$$

From calculus, we know that

$$\sum_{\omega\in\Omega}{}' |\omega|^{-s} \quad \text{converges} \Longleftrightarrow \sum_{r=1}^{\infty} r^{1-s} \quad \text{converges}$$

i.e. if and only if $1 - s < -1 \Longrightarrow s > 2$. □

To prove the existence of meromorphic function with poles of order $N \geq 3$ we shall use Weierstrass M-test, and so the convergence.

Definition 2.3.1. Let (u_n) be a sequence of functions $u_n : E \longrightarrow \mathbb{C}$, defined on some set E. We say that u_n *converges uniformly* to a function $u : E \longrightarrow \mathbb{C}$ if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|u_n(z) - u(z)| < \epsilon \; \forall n > n_0$ and $\forall z \in E$.

Definition 2.3.2. Let R be a region in $\mathbb{C}$, nd let (u_n) be a sequence of functions $u_n : R \longrightarrow \mathbb{C}$; then (u_n) *converges uniforly on all compact subsets* of R if, for each compact $K \subseteq R$, the sequence of restrictions $(u_n\big|_K)$ converges uniformly in K.

Theorem 2.3.2. *Let (u_n) be a sequence if analytic functions on a region $R \subseteq \mathbb{C}$, uniformly convergent to a function u on all compact subsets of R. Then u is analytic on R, and the sequence of derivatives (u_n') converges uniformly to u' on all compact subsets of R.*

We can extend this result from sequences to series: we say that $\sum_{n=0}^{\infty} u_n(z)$ converges uniformly to $u(z)$ on a set E if the sequence of partial sums $\sum_{n=0}^{m} u_n(z)$ converges uniformly to $u(z)$ on E as $m \longrightarrow \infty$.

Corollary 2.3.3. *Let (u_n) be a sequence of analytic functions on a region $R \subseteq \mathbb{C}$; if $\sum_{n=0}^{\infty} u_n(z)$ is uniformly convergent to $u(z)$ on all compact subsets of R, then $u(z)$ is analytic on R and $\sum_{n=0}^{\infty} u_n'(z)$ is uniformly convergent to $u'(z)$ on all compact subsets of R.*

Theorem 2.3.4 (Weierstrass M-test). *let (u_n) be a sequence of functions $u_n : E \longrightarrow \mathbb{C}$, defined on some set E, such that*

- *for each $n \in \mathbb{N}$ there exists $M_n \in \mathbb{R}$ satisfying $|u_n(z)| \leq M_n$ for all $z \in E$,*
- *$\sum_{n=0}^{\infty} M_n$ converges.*

Then $\sum_{n=0}^{\infty} u_n(z)$ converges uniformly on E, and converges absolutely for each $z \in E$.

Suppose that (u_n) is a sequence of meromorphic functions on a region R, and for each compact subset $K \subseteq R$ there exists $N_k \in \mathbb{N}$ such that:

- $u_n(z)$ has no poles (and so it is analytic) in K for $n > N_k$,
- $\sum_{n>N_k} u_n(z)$ is uniformly convergent on K.

Then we say that $\sum u_n(z)$ converges uniformly on all compact subsets of R. Since $\sum_{n \leq N_K} u_n(z)$ is meromorphic on the interior $\mathring{K}$ of K (being a sum of finitely many meromorphic function), and since $\sum_{n>N_K} u_n(z)$ is analytic on $\mathring{K}$, the function

$$\sum_{n=0}^{\infty} u_n(z) = \sum_{n \leq N_K} u_n(z) + \sum_{n>N_K} u_n(z)$$

is meromorphic on $\mathring{K}$, its poles being included among the poles of the functions $u_n(z)$ for $n \leq N_K$. Since each point $z \in R$ has a neighbourhood with compact closure $K \subseteq R$, $\sum u_n(z)$ is meromorphic on R.

Theorem 2.3.5. *Let $\sum u_n(z)$ be a series of meromorphic functions on a region $R \subseteq \mathbb{C}$, uniformly convergent to $u(z)$ on all compact subsets of R. Then $u(z)$ is meromorphic on R, and the series $\sum u_n'(z)$ of the derivatives converges uniformly to $u'(z)$ on all compact subsets of R.*

Theorem 2.3.6. *For each integer $N \geq 3$, the function*

$$F_N(z) = \sum_{\omega \in \Omega} (z - \omega)^{-N}$$

is elliptic of order N with respect to Ω .

Proof. Let's prove first that $F(z)$ is a meromorphic function. Let K be a compact set in $\mathbb{C} \setminus \Omega$ (i.e closed and bounded).
Let's suppose that $|z| < R \quad \forall z \in K$. There are only finitely many $\omega \in \Omega$ such that $|\omega| < 2R$.
Let $\Phi = \{\omega \in \Omega | |\omega| > 2R\}$ $\forall z \in K$ and $\omega \in \Phi$. Then

$$|z| < R < \frac{|\omega|}{2} \text{ and}$$
$$|z - \omega| \geq |\omega| - |z| \geq \frac{1}{2}|\omega|.$$

Thus

$$|z - \omega|^{-N} \leq 2^N |\omega|^{-N}.$$

For $z \in K$

$$\sum_{\omega \in \Omega} |z - \omega|^{-N} \leq 2^N | \sum_{\omega \in \Omega} \omega|^{-N} \quad \text{which converges.}$$

So, by the *Weierstrass M-test*, $z \in K$

$$\sum_{\omega \in \Omega} |z - \omega|^{-N} \qquad \text{converges absolutely and uniformly in } K.$$

Since each term $(z - \omega)^N$ is analytic on K, Corollary 2.3.3 implies that $F_N(z)$ is analytic on $\mathbb{C} \setminus \Omega$. In the same way, using Theorem 2.3.5, we can show that $F_N(z)$ is meromorphic at each $\omega \in \Omega$. This means $F_N(z)$ is meromorphic and only has poles when $(z - \omega)^{-N}$ has poles for some $\omega \in \Omega$, i.e a pole of order N at each lattice point and no others.
Now we show that F_N is doubly periodic.
Let $\omega_0 \in \Omega$.

$$\begin{aligned} F_N(z + \omega_0) &= \sum_{\omega \in \Omega} \big((z + \omega_0) - \omega\big)^{-N} = \sum_{\omega \in \Omega} \big(z - (\omega_0 - \omega)\big)^{-N} \\ &= \sum_{\omega' \in \Omega} (z - \omega')^{-N} = F_N(z), \qquad \text{where } \omega' = (\omega - \omega_0) \in \Omega. \end{aligned}$$

So $F_N(z)$ is doubly periodic with period lattice Ω . Hence it is elliptic, with single pole of order $N \mod \Omega$. Thus F_N has order N. $\square$

Note we know that there exist elliptic functions of all orders $N \geq 3$, and none of order $N = 1$.
In the next section, we will discute about the Wieierstrass elliptic function, that is an example of elliptic function of order $N = 2$.

2.4 Weierstrass functions

Let $\Omega = \Omega(\omega_1, \omega_2)$ be a lattice with basis $\{\omega_1, \omega_2\}$, and let $\sqcap$ be a fundamental parallelogram for Ω with no elements of Ω on $\partial\sqcap$. By corollary 2.2.3 we know that f cannot have just one simple pole in $\sqcap$, so the simplest non-constant elliptic function has order 2, with either two simple poles or else a single pole of order 2 in $\sqcap$. In this section we shall introduce the Weierstrass function $\wp(z)$ which is elliptic of order 2 in $\sqcap$. We shall derive $\wp(z)$ from the Weierstrass sigma-function $\sigma(z)$.

Definition 2.4.1. The Weierstrass *sigma-function* $\sigma(z)$ is defined by the following infinite product

$$\sigma(z) = z \prod_{\omega \in \Omega}{}' g(\omega, z)$$

where

$$g(\omega, z) = \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right).$$

$\sigma(z)$ is holomorphic, is odd ($\sigma(-z) = -\sigma(z)$), and has simple zero at each $\omega \in \Omega$ and nowhere else.

Definition 2.4.2. The Weierstrass *zeta-function* $\zeta(z)$ is defined by

$$\begin{aligned} \zeta(z) &= \frac{\sigma'(z)}{\sigma(z)} \\ &= \frac{d}{dz} Log\left(\sigma(z)\right) \\ &= \frac{1}{z} + \sum_{\omega \in \Omega}{}' \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right). \end{aligned} \tag{2.3}$$

Since $\sigma(z)$ is an odd function, $\zeta(z)$ is also odd. It has simple poles at the lattice-points, and is analytic on $\mathbb{C} \setminus \Omega$. As a series of meromorphic functions, $\zeta(z)$ converges uniformly on compact subsets of $\mathbb{C}$ and so we may differentiate term-by-term to obtain a meromorphic function $\zeta'(z)$.

Definition 2.4.3. The Weierstrass $\wp$*-function* is defined by $\wp(z) = -\zeta'(z)$, then we have

$$\wp = \frac{1}{z^2} + \sum_{\omega\in\Omega}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{2.4}$$

Theorem 2.4.1. *The Weierstrass $\wp$-function has the following properties:*

1. $\wp'(z) = -2F_3(z) = -2\sum_{\omega\in\Omega}(z-\omega)^{-3}$ *;*

2. $\wp(z)$ *is even ;*

3. $\wp(z)$ *is an elliptic function with period lattice* Ω *;*

4. $\wp(z)$ *has order 2, with double pole at each* $\omega \in \Omega$ *;*

Proof. 1. $\wp(z)$ converges uniformly on compact sets

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\omega\in\Omega}{}' \left(\frac{-2}{(z-\omega)^3} \right) = \sum_{\omega\in\Omega} \frac{1}{(z-\omega)^3} = -2F_3(z)$$

2. Since $\wp(z) = -\zeta'(z)$ and $\zeta'(z)$ is even, than $\wp(z)$ is even.

3. We have seen that $\wp(z)$ is meromorphic, so it is sufficient to prove the doubly periodicity.
Since $\wp'(z)$ is elliptic, we have $\forall \omega \in \Omega$

$$\wp'(z+\omega) = \wp'(z).$$

So, integrating,

$$\wp(z+\omega) = \wp(z) + c, \quad \forall z \in \mathbb{C}$$

where c is a constant.
Putting $z = -\omega/2$ we have $c = \wp(\omega/2) - \wp(-\omega/2) = 0$ since $\wp(z)$ is even.
Thus $\wp(z+\omega) = \wp(z)$ $\forall$ $z \in \mathbb{C}$ and $\omega \in \Omega$. So $\wp(z)$ is elliptic.

4. Clearly each point of Ω is a pole of order 2 and $\wp(z)$has not other poles.
So Ω is exactly the period lattice and $\wp(z)$ has order 2.

□

2.5 The Addition Theorems for the Weierstrass functions

Definition 2.5.1. A meromorphic function $f(z) : \mathbb{C} \longrightarrow \mathbb{C} \cup \left\{ \infty = \widehat{\mathbb{C}} \right\}$ is said to have the addition formula if there exixts a rational function in three variables $R : \widehat{\mathbb{C}}^3 \longrightarrow \mathbb{C}$ so that

$$R(f(n), f(v), f(u+v)) = 0 \quad \forall u, v \in \mathbb{C}.$$

For example:

- $f(z) = \dfrac{z}{z+1}$

$$\frac{f(u)}{1 - f(u)} + \frac{f(v)}{1 - f(u)} = \frac{f(u+v)}{1 - f(u+v)};$$

- $f(z) = e^z$

$$e^{u+v} = e^u e^v;$$

- $f(z) = tan(z)$

$$f(u+v) = \frac{f(u) + f(v)}{1 - f(u)f(v)}.$$

Theorem 2.5.1 (Weierstrass Theorem)**.** *A meromorphic function has the addition formula* $\Longleftrightarrow$ *it is:*

- *rational*
- *simply periodic or*
- *doubly periodic.*

Also the Weierstrass functions ζ and $\wp$ are provided with the addition formula, and are (see [2])

$$\wp(u+v) + \wp(u) + \wp(v) = \frac{1}{4} \left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2 \tag{2.5}$$

$$\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2} \left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right], \tag{2.6}$$

where 2.6 comes from 2.5 since

$$\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} = 2\zeta(u+v) - 2\zeta(u) - \zeta(v).$$

2.6 The differential equation for $\wp(z)$

In this section we derive an important equation connecting $\wp(z)$and $\wp'(z)$, obtained from the Laurent series near $\wp(z) = 0$. We start by finding the Laurent series for

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\in\Omega}{}' \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right). \tag{2.7}$$

Let $m = min\left\{|\omega| \mid \omega \in \Omega \setminus \{0\}\right\}$, and let $D = \left\{z \in \mathbb{C} | |z| < m\right\}$, the largest open disc centered at 0 and containing no other lattice-points. Since

$$\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} = \frac{z^2}{\omega^2(z-\omega)},$$

we see, by comparison with $\sum' |\omega|^{-3}$, that $\sum' \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$ is absolutely convergent for each $z \in \mathbb{C} \setminus \Omega$. Moreover, for each $\omega \in \Omega \setminus \{0\}$, the binomial series

$$\frac{1}{z-\omega} = -\frac{1}{\omega} \left(\frac{1}{1-\frac{z}{\omega}} \right) = -\frac{1}{\omega} \sum_{j=0}^{\infty} \left(\frac{z}{\omega} \right)^j$$

is absolutely convergent for $z \in D$, so we may substitute this in 2.7 to obtain

$$\begin{aligned} \zeta(z) &= \frac{1}{z} + \sum_{\omega\in\Omega}{}' \left(-\frac{1}{\omega} \sum_{j=0}^{\infty} \left(\frac{z}{\omega} \right)^j + \frac{1}{\omega} + \frac{z}{\omega^2} \right) \\ &= \frac{1}{z} + \sum_{\omega\in\Omega}{}' \left(-\frac{1}{\omega} \sum_{j=2}^{\infty} \left(\frac{z}{\omega} \right)^j \right) \\ &= \frac{1}{z} + \sum_{j=2}^{\infty} z^j \sum_{\omega\in\Omega}{}' \left(-\frac{1}{\omega^{j+1}} \right). \end{aligned}$$

Define

$$G_k = G_k(\Omega) = \sum_{\omega\in\Omega}{}' \frac{1}{\omega^k}, \quad k \in \mathbb{Z}, k \geq 3$$

called the *Eisenstein series* for Ω , absolutely convergent for $k \geq 3$, and $G_k = 0$ for k odd.

So the *Laurent series for* $\zeta(z)$ becomes

$$\zeta(z) = \frac{1}{z} - \sum_{n=2}^{\infty} z^{2n-1} G_{2n}$$

and hence

$$\wp(z) = -\zeta'(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)z^{2n-2}G_{2n}; \tag{2.8}$$

this is the *Laurent series for* $\wp(z)$, valid for $z \in D$. From this we obtain

$$\wp'(z) = \frac{-2}{z^3} + 6G_4 z + 20G_6 z^3 + \dots,$$

and so

$$\begin{aligned}\wp'(z)^2 &= \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + z^2\phi_1(z)\\ 4\wp(z)^3 &= \frac{4}{z^6} + \frac{36G_4}{z^2} + 60G_6 + z^2\phi_2(z)\\ 60G_4\wp(z) &= \frac{60G_4}{z^2} + z^2\phi(z)\end{aligned}$$

where ϕ_1, ϕ_2, ϕ_3 are power series convergent in D. These last three equations give

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6 = z^2\phi(z)$$

where $\phi(z) = \phi_1 - \phi_2(z) + \phi_3(z)$ is a power series convergent in D. As $\wp$ and $\wp'$ are elliptic with respect to Ω , the function

$$f(z) = \wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6$$

is also elliptic. Since $f(z) = z^2\phi(z)$ in D, with $\phi(z)$ analytic, f vanishes at 0 and hence at all $\omega \in \Omega$. However, by its construction f can have poles only where $\wp(z)$or $\wp'$ have poles, that is, at the lattice-points. Therefore f has no poles and so, by Theorem 2.2.1 , $f(z)$ is a constant, which must be zero since $f(0) = 0$. Thus we have proved the following

Theorem 2.6.1.

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6. \tag{2.9}$$

The equation 2.9 is the *differential equation for* $\wp(z)$. If we denote with

$$\begin{aligned}g_2 &= 60G_4 = 60\sum{}'\omega^{-4}\\ g_3 &= 140G_6 = 140\sum{}'\omega^{-6}\end{aligned}$$

we can write 2.9 like

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{2.10}$$

2.7 Elliptic integrals

Definition 2.7.1. An integral of the form

$$\int R(z,w)dz, \tag{2.11}$$

where $R(z,w)$ is a rational function of its argument and

$$w^2 = a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4$$

is a polynomial of third or fourth degree in z without multiple roots, is called *elliptic integral.*

We saw the function $x = \wp(u)$ satisfies the differential equation

$$\left(\frac{dx}{du}\right)^2 = 4x^3 - g_2x - g_3 = 4(x-a)(x-b)(x-c).$$

Then

$$du = \frac{dx}{2\sqrt{(x-a)(x-b)(x-c)}} \tag{2.12}$$

$$u(x) = \int_{\wp}^{\infty} \frac{dx}{2\sqrt{(x-a)(x-b)(x-c)}}. \tag{2.13}$$

The integral 2.13 is called an *elliptic integral of the first kind.*
With the help of a suitable linear fractional transformation it is possible to reduce the radical in the integral 2.11 to the form $\sqrt{4x^3 - g_2x - g_3}$. This linear fractional transformation has the form

$$z = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad (\alpha\delta - \beta\gamma) = 1.$$

Then

$$\sqrt{a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4} = \frac{\sqrt{4x^3 - g_2x - g_3}}{(\gamma x + \delta)^2},$$

and

$$\int R(z, \sqrt{a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4})dz =$$

$$= \int R\left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{\sqrt{4x^3 - g_2x - g_3}}{(\gamma x + \delta)^2}\right)\frac{dx}{(\gamma x + \delta)^2}.$$

In particular

$$\int \frac{dz}{\sqrt{a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4}} = \int \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}. \tag{2.14}$$

So, instead of 2.11 we can consider 2.14.

If we introduce the function $\wp$ corresponding to the invariant g_2 and g_3, and let $x = \wp(u)$, then an elliptic integral

$$\int R(x, \sqrt{4x^3 - g_2 x - g_3})dx = \int R(\wp(u), +\wp'(u)\wp'(u)du = \int R_1(\wp(u), \wp'(u)du, \tag{2.15}$$

i.e- we arrive at the integral of an elliptic function.

Expanding the elliptic function $R_1(\wp, \wp')$ into partial fractions, and using the addition theorems for the function ζ and $\wp$ it may be prove (see [2] for calculation)

$$\int R_1(\wp, \wp')du = Cu + \sum_{i=1}^{n} A_i \log \sigma(u - a_i) + A\zeta(u) + R^*(\wp, \wp').$$

where A is a constant and A_i is the residue of our elliptic function with respect to the pole a_i. Since $\sum_{i=1}^{n} A_i = 0$, this formula can be rewritten in the form

$$\int R_1(\wp, \wp')du = Cu + \sum_{i=1}^{n} A_i \log \frac{\sigma(u - a_i)}{\sigma(u)} + A\zeta(u) + R^*(\wp, \wp'). \tag{2.16}$$

$R^*(\wp, \wp')$ is an elliptic integral, but the other three terms on the right-hand side are not.

Let us pass from the variable u to $z = \wp(u)$. Then the last term on the RHS of 2.16 is written in the form R*(z,w), where $w = 4z^3 - g_2 z - g_3$, and this is the algebraic part of the integral 2.14. The remaining part is constructed with the functions

$$u = \int \frac{dz}{w}, \tag{2.17}$$

$$\zeta(u) = -\int \wp(u)du = -\int \frac{zdz}{w}, \tag{2.18}$$

and

$$\ln \frac{\sigma(u-a)}{\sigma(u)} + u\zeta(a) = \frac{1}{2} \int \frac{\wp'(u) + \wp'(a)}{\wp(u) - \wp(a)} du = \tag{2.19}$$

$$\frac{1}{2} \int \frac{w + w_0}{z - z_0} \frac{dz}{w} \tag{2.20}$$

where we replaced $z = \wp(u)$, $w = \wp'(u)$ and so $\wp(a) = z_0$ and $\wp'(a) = w_0$.
The integral 2.17 an elliptic integral of the first kind, 2.18 is called *elliptic integral of the second kind*, with a pole of order two, and 2.19 is called *elliptic integral of the third kind*, and has two simple poles.
Thus, every elliptic integral is the result of adding elliptic integrals of three kinds and a certain function of z and w.

2.8 Real elliptic curves

We have seen that $\wp$ satisfies a differential equation $(\wp')^2 = p(\wp)$, where $p(x)$ is the cubic polinomial $4x^3 - g_2x - g_3$, so every point $t \in \mathbb{C}/\Omega$ determines a point $(\wp(t), \wp'(t))$ on the elliptic curve

$$E = \left\{(x, y) \in \Sigma \times \Sigma | y^2 = p(x)\right\}. \tag{2.21}$$

We can think of E as the graph of the equation $y^2 = p(x)$, for $x, y \in \Sigma$. As a subset of $\Sigma \times \Sigma$, E has a natural topology and is homemomorphic to a torus.
Let's concentrate on the real points of E, those for which $x, y \in \mathbb{R}$, under the assumption thet the coefficients g_2, g_3 of $p(x)$ are real. We define the real elliptic curve E_R to be $\left\{(x, y) \in \mathbb{R} | y^2 = p(x)\right\}$, the graph of $y^2 = p(x)$ as an equation between real variables. Clearly E_R is symmetric about the x-axis of $\mathbb{R}^2$.
First we examine the condictions under which the coefficients g_2, g_3 are real. We define a meromorphic function $f : \mathbb{C} \longrightarrow \Sigma$ to be real if $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$. We define a lattice Ω to be real if $\overline{\Omega} = \Omega$ (where $\overline{\Omega}$ denotes $\left\{\overline{\omega} | \omega \in \Omega\right\}$).

Theorem 2.8.1. *The following conditions are equivalent:*

1. g_2, $g_3 \in \mathbb{R}$;
2. $G_k \in \mathbb{R}$, $\forall k \geq 3$;
3. $\wp$ *is a real function;*
4. Ω *is a real lattice.*

Proof. $1 \Longrightarrow 2$

Differentiating $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, and then dividing by $2\wp'$ (which is not identically zero) we have

$$\wp'' = 6\wp^2 - \frac{g_2}{2}. \tag{2.22}$$

Now by 2.8, $\wp(z)$ has Laurent expansion

$$\wp(z) = z^{-2} + \sum_{n=1}^{\infty} a_n z^{2n},$$

valid near $z = 0$, where

$$a_n = (2n+1)G_{2n+2} = (2n+1)\sum_{\omega}{}' \omega^{-2n-2}.$$

The coefficient of z^{2n} in the expansion of $\wp''(z)$ is therefore $(2n+2)(2n+1)a_{2n+1}$, while the coefficient of z^{2n} in $\wp(z)^2$ is $2a_{n+1} + \sum_{r+s=n} a_r a_s$. Equating the coefficients in 2.22 we have, for each $n \geq 1$,

$$(2n+2)(2n+1)a_{n+1} = 12a_{n+1} + 6\sum_{r+s=n} a_r a_s$$

and hence

$$(2n+5)(n-1)an+1 = 3\sum_{r+s=n} a_r a_s.$$

So, for $n \geq 2$ we have

$$a_{n+1} = \frac{3}{(2n+5)(n-1)}\sum_{r+s=n} a_r a_s.$$

By induction on n, we see that each coefficient a_n is a polynomial in a_1 and a_2, with rational coefficients. Using $a_n = (2n+1)G_{2n+2}$, $g_2 = 60G_4$ and $g_3 = 140G_6$, we see that each G_k (k even,$k \geq 4$), is a polynomial in g_2 and g_3 with rational coefficients, so if g_2 and g_3 are real then so is G_k; since $G_k = 0$ for all odd k, (2) is proved.

$2 \Longrightarrow 3$

If $G_k \in \mathbb{R}\forall k \geq 3$, then the coefficients of the Laurent series for $\wp(z)$are real, so $\overline{\wp(\overline{z})} = \wp(z)$ near $z = 0$. Now $\wp(z)$and $\overline{\wp(\overline{z})}$ are meromorphic functions, identically equal on a neighbourhood of 0, so they are identically equal on $\mathbb{C}$. Thus $\wp$ is a real function.

$3 \Longrightarrow 4$
Let $\omega \in \Omega$. Then $\wp(z+\overline{\omega}) = \overline{\wp(\overline{z}+\omega)} = \overline{\wp(\overline{z})}$ since $\wp$ is real and has ω as a period. Thus $\overline{\omega} \in \Omega$, so $\overline{\Omega} \subseteq \Omega$. Taking complex conjugates, we have $\Omega = \overline{\overline{\Omega}} \subseteq \overline{\Omega}$, so $\Omega = \overline{\Omega}$ and Ω is real.
$4 \Longrightarrow 1$
This follow immediately from $g_2 = 60\sum'_{\omega}\omega^{-4}$ and $g_3 = 140\sum'_{\omega}\omega^{-6}$. $\square$

Definition 2.8.1. We say that Ω is *real rectangular* if $\Omega = \Omega\{\omega_1, \omega_2\}$, where ω_1 is real and ω_2 is purely imaginary.
We say that Ω is *real rhombic* if $\Omega = \Omega\{\omega_1, \omega_2\}$, where $\overline{\omega}_1 = \omega_2$.
The fundamental parallelogram with vertices $0, \omega_1, \omega_2$ and $\omega_3 = \omega_1 + \omega_2$ is rectangular or rhombic respectively.

Theorem 2.8.2. *A lattice Ω is real if and only if it is real rectangular or real rhombic*

Proof. $\Longleftarrow$ If $\Omega = \Omega(\omega_1, \omega_2)$ is real rectangular, with $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\mathbb{R}$, then $\overline{\Omega} = \Omega(\overline{\omega}_1, \overline{\omega}_2) = \Omega(\omega_1, -\omega_2) = \Omega(\omega_1, \omega_2) = \Omega$, so Ω is real. A similar argument applies to real rhombic lattices.
$\Longrightarrow$ Suppose that Ω is real. If $\omega \in \Omega$, then $\omega + \overline{\omega}$, $\omega - \overline{\omega} \in \Omega$, so Ω contains both real and purely imaginary elements, and these form discrete subgroups $\Omega \cap \mathbb{R} = \lambda\mathbb{Z}$ and $\Omega \cap i\mathbb{R} = \mu i\mathbb{Z}$ for certain $\lambda, \mu \in \mathbb{R}$, $\lambda.\mu > 0$. Clearly $\Omega \supseteq \lambda\mathbb{Z} + \mu i\mathbb{Z}$, and if we have equality, then there exists $\omega \in \Omega \setminus (\lambda\mathbb{Z} + \mu i\mathbb{Z})$. By addind a suitable element of $\lambda\mathbb{Z} + \mu i\mathbb{Z}$ to ω we may assume that $0 \leq Re(\omega) < \mu$. Now

$$2\omega = (\omega + \overline{\omega}) + (\omega - \overline{\omega}),$$

with $\omega + \overline{\omega} \in \Omega \cap \mathbb{R} = \lambda\mathbb{Z}$ and $\omega - \overline{\omega} \in \Omega \cap i\mathbb{R} = \mu i\mathbb{Z}$, so we have

$$2\omega = m\lambda + n\mu i$$

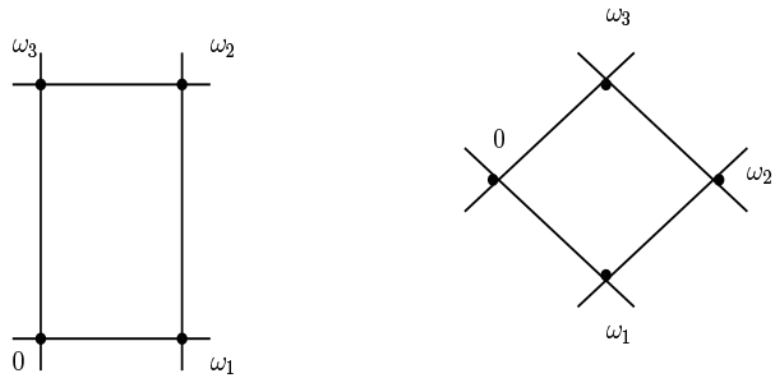
for integers m, n, and the condictions on $Re(\omega)$ and $Im(\omega)$ force m and n to take value 0 or 1. Since ω is neither real or purely imaginary, we must have $m = n = 1$, and so $\omega = \frac{1}{2}(\lambda + \mu i)$. Thus every element of $\Omega \setminus (\lambda\mathbb{Z} + \mu i\mathbb{Z})$ has the form

$$\frac{1}{2}(\lambda + \mu i) + a\lambda + b\mu i = (a+b+1)\left(\frac{\lambda + \mu i}{2}\right) + (a-b)\left(\frac{\lambda - \mu i}{2}\right),$$

for integers a, b, while every elememt of $\lambda\mathbb{Z} + \mu i\mathbb{Z}$ has the form

$$a\lambda + b\mu i = (a+b)\left(\frac{\lambda+\mu i}{2}\right) + (a-b)\left(\frac{\lambda-\mu i}{2}\right).$$

Thus $\Omega = \Omega(\frac{1}{2}(\lambda+\mu i), \frac{1}{2}(\lambda-\mu i))$, which is a real rhombic. □



Ω real rectangular Ω real rhombic

2.9 The discriminant of a cubic polynomial

In this section we give a necessary and sufficient condiction for p to have distinct roots. We saw that Weierstrass'elliptic function $\wp$ satisfies a differential equation $\wp' = \sqrt{p(\wp)}$, where p is a cubic polynomial on the form

$$p(z) = 4z^3 - g_2 z - g_3, \quad (g_2, g_3 \in \mathbb{C}). \tag{2.23}$$

Any polynomial in this form is said to be in the *Weierstrass normal form.* By means of a substitution $\theta : z \mapsto az + b (a, b \in \mathbb{C}, a \neq 0)$, any cubic polynomial may be brought into this form; now $\theta : \mathbb{C} \longrightarrow \mathbb{C}$ is a bijection, preserving multiplicities of roots, so without loss of generality we can restrict our attenction to cubic polynomials p in the Weierstraiss normal form.

If e_1, e_2, e_3 are the roots of the polynomial p in 2.23, then we can define the discriminant of p to be

$$\Delta_p = 16(e_1 - e_2)^2 (e_2 - e_3)^2 (e_3 - e_1)^2 \tag{2.24}$$

and these roots are distinct if and only if $\Delta_p \neq 0$.

Theorem 2.9.1. $\Delta_p = g_2^3 - 27g_3^2$.

Proof. putting

$$p(z) = 4(z - e_1)(z - e_2)(z - e_3), \tag{2.25}$$

and equating coefficients between this and 2.23, we have

$$\begin{aligned} e_1 + e_2 + e_3 &= 0 \\ e_1e_2 + e_2e_3 + e_3e_1 &= -\frac{g_2}{4} \\ e_1e_2e_3 &= \frac{g_3}{4}. \end{aligned} \tag{2.26}$$

The remaining symmetric functions of the roots may be obtained from 2.26:

$$e_1^2 + e_2^2 + e_3^2 = (e_1 + e_2 + e_3)^2 - 2(e_1e_2 + e_2e_3 + e_3e_1) = \frac{g_2}{2}$$

and

$$e_1e_2^2 + e_2e_3^2 + e_3e_1^2 = (e_1e_2 + e_2e_3 + e_3e_1)^2 - 2e_1e_2e_3(e_1 + e_2 + e_3) = \frac{g_2^2}{16}.$$

Differentiating 2.22 and 2.25 at $z = e_1$, we have:

$$4(e_1 - 4_e2)(e_1 - e_3) = p'(e_1) = 12e_1^2 - g_2,$$

with similar expressions for $p'(e_2)$ and $p'(e_3)$. Hence

$$\begin{aligned} \Delta_p &= -\frac{1}{4}p'(e_1)p'(e_2)p'(e_3) \\ &= \frac{1}{4}\prod_i(12e_i^2 - g_2) \\ &- \frac{1}{4}(1728(e_1e_2e_3)^2 - 144g_2(e_1e_2^2 + e_2e_3^2 + e_3e_1^2) \\ &+ 12g_2^2(e_1^2 + e_2^2 + e_3^2) - g_2^3) \\ &= \frac{1}{4}(108g_3^2 - 9g_2^3 + 6g_2^3 - g_2^3) \\ &= g_2^3 - 27g_3^2. \end{aligned}$$

□

Corollary 2.9.2. *p has distinct roots if and only if* $g_2^3 - 27g_3^2 \neq 0$.

Chapter 3

Lagrangian Mechanics

3.1 Calculus of variations

The calculus of variations is concerned with the extremals of functions whose domain is an infinite-dimensional space: the space of curves. Such functions are called *functionals*. An example of a functional is the length ϕ of a curve in the euclidean plane:
if $\gamma = \{(t, x) : x = x(t), t_0 \leq t \leq t_1\}$, then

$$\Phi = \int_{t_0}^{t_1} \sqrt{1 + \dot{x}^2}.$$

where $\dot{x} = dx/dt$.
In general, a functional is any mapping from the space of curves to the real numbers.
We consider an "approximation" γ' to γ, $\gamma' = \{(t, x) : x = x(t) + h(t)\}$. We will call it $\gamma' = \gamma + h$. Consider the increment of Φ, $\Phi(\gamma + h) - \Phi(\gamma)$.

Definition 3.1.1. Let $\gamma, h \in C^\infty([t_0, t_1], \mathbb{R}^2)$. A functional Φ is called *differentiable* if

$$\Phi(\gamma + h) - \Phi(\gamma) = F(h) + R(h, \gamma), \tag{3.1}$$

where F depends linearly on h (i.e. for fixed γ, $F(h_1 + h_2) = F(h_1) + F(h_2)$ and $F(ch) = cF(h)$), and $R(h, \gamma) = O(h^2)$, in the sense that if $|h|, \left|\frac{dh}{dt}\right| < \epsilon$, then $|h| < C\epsilon^2$
The linear part of the increment, $F(h)$, is called the *differential* or *variation*, and h is called the *variation of the curve*.

Theorem 3.1.1. *Let $L(u, v, w)$ be a C^∞ differentiable function in three variables. Then the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x(t), \dot{x}(t), t)dt$ is differentiable, and its derivative is given by the formula*

$$F(h) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) h dt + \left[\frac{\partial L}{\partial \dot{x}} h \right]_{t_0}^{t_1}, \tag{3.2}$$

with $h \in C^\infty([t_0, t_1], \mathbb{R}^2)$

Definition 3.1.2. An *extremal* of a differentiable functional $\Phi(\gamma)$ is a curve γ such that $\forall h \; F(h) = 0$.

Theorem 3.1.2. *The curve $\gamma : x = x(t)$ is an extremal of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x(t), \dot{x}(t), t)dt$ on the space of the smooth curves passing through the points $x(t_0) = x_0$ and $x(t_1) = x_1$ if and only if*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \text{along the curve } x(t). \tag{3.3}$$

Lemma 3.1.3. *If a continuous function $L(t)$, $t_o \le t \le t_1$ satisfies $\int_{t_0}^{t_1} L(t)h(t)dt =$ 0 for any continuous function $h(t)$, with $h(t_1) = h(t_0) = 0$, then $f(t) = 0$.*

Definition 3.1.3. The equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \tag{3.4}$$

is called the *Euler-Lagrange equation* for the functional

$$\Phi = \int_{t_0}^{t_1} L(x(t), \dot{x}(t), t)dt.$$

Written out explicitly (recall $L = L(x, \dot{x}, t)$), 3.4 is

$$\frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial \dot{x}\partial x}\dot{x} - \frac{\partial^2 L}{\partial \dot{x}^2}\ddot{x} - \frac{\partial^2 L}{\partial \dot{x}\partial t} = 0 \tag{3.5}$$

and it is a second order ordinary differential equation for the function $x(t)$ of the form

$$g_1(x, \dot{x}, t)\frac{d^2x}{dt^2} + g_2(x, \dot{x}, t)\frac{dx}{dt} + g_3(x, \dot{x}, t) = 0. \tag{3.6}$$

The two arbitrary constants in the general solution are fixed by the boundary conditions.

If $\mathbf{x}$ is a vector in the n-dimensional coordinate space $\mathbb{R}^n$, $\mathbf{x} \in C^\infty([t_0, t_1, \mathbb{R}^n])$ $\gamma = \{t, \mathbf{x} = \mathbf{x}(t), t_0 \le t \le t_1\}$ a curve in the $(n+1)$-dimensional space $\mathbb{R} \times \mathbb{R}^n$, and $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ a differentiable function of $2n + 1$ variables, as before we have:

Theorem 3.1.4. *The curve is an extremal of the functional* $\Phi = \int_{t_0}^{t_1} f(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), t)dt$ *on the space of curves joining* $(t_0, \boldsymbol{x}_0)$ *and* $(t_1, \boldsymbol{x}_1)$ *if and only if the Euler-Lagrange equation is satisfied along* γ*.*

This a system of n second order equations, and the solution depends on $2n$ arbitrary constants.
Given a solution of 3.4, the value of F is extremal but not necessary minimal.
There are two special cases of $f = f(x, \dot{x}, t))$:

i) f does not depend on x

$$f = f(\dot{x}, t). \tag{3.7}$$

In this case, since $\frac{\partial f}{\partial x} = 0$, the Euler - Lagrange equation implies

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0 \Longrightarrow \frac{\partial f}{\partial \dot{x}} = const. \tag{3.8}$$

ii) f does not depend on t explicitly

$$f = f(x, \dot{x}). \tag{3.9}$$

We can write

$$\dot{x} = \frac{dx}{dt} = \frac{1}{\frac{dt}{dx}} := \frac{1}{\dot{t}} \qquad (\text{note } \dot{t} = \frac{dt}{dx}). \tag{3.10}$$

So

$$\int f(x, \dot{x})dt = \int f(x, \frac{1}{\dot{t}})\dot{t}dx =: \int F(\dot{t}, x)dx. \tag{3.11}$$

Since F is independent of t, and using i) we put

$$\frac{\partial F}{\partial \dot{t}} = \frac{\partial}{\partial \dot{t}} f(x, \frac{1}{\dot{t}})\dot{t} = const \Longrightarrow f(x, \frac{1}{\dot{t}}) + \dot{t}\frac{\partial f}{\partial\left(\frac{1}{\dot{t}}\right)}\left(\frac{1}{-\dot{t}^2}\right) = const \tag{3.12}$$

which may be rewritten in terms of x and $\dot{x}$ as

$$f(x, \dot{x}) - \frac{\partial f}{\partial \dot{x}}\dot{x} = const. \tag{3.13}$$

Corollary 3.1.5. *The condition for a curve to be an extremal of a functional does not depend on the choice of coordinate system.*

3.2 Lagrange's equation

Lagrangian mechanics describes the motion of a mechanical system by means of the configuration space. The configuration space of a mechanical system has the structure of a differentiable manifold, on which its group of diffeomorphism acts.

One of the fundamental concepts of mechanics is that of *particle*. By this we mean a body whose dimensions may be neglected in describing its motion. The *position* of a particle in space is defined by a vector $\mathbf{x} = (x_1, x_2, x_3)$. The derivative

$$v = \frac{dx}{dt} \equiv \dot{\mathbf{x}} = (\dot{x_1}, \dot{x_2}, \dot{x_3})$$

of $\mathbf{x}$ with respect to the time t is called the *velocity* of the particle, and the second derivative

$$\frac{d^2\mathbf{x}}{dt^2} = (\ddot{x_2}, \ddot{x_2}, \ddot{x_3})$$

is its *acceleration*.
To define the position of a system of n particles in space, it is necessary to specify n vectors, i.e. $3n$ coordinates. The number of independent quantities which must be specified in order to define uniquely the position of any system is called the number of *degrees of freedom*; here, this number is $3n$. These quantities don't need to be the Cartesian coordinates of the particles, and the condition of the problem may render some other choice of coordinates more convenient. Any s quantities $q_1, q_2, \ldots, q_s$ which completely define the position of a system with s degrees of freedom are called *generalized coordinates* of the system, and the derivatives $\dot{q}_i$ are called *generalized velocities*. If all the co-ordinates and the velocities are simultaneously specified, the state of the system is completely determined and its motion can be calculated. The relations between the accelerations, velocities and coordinates are called the *equations of motion*. They are second order differential equations for the functions $q(t)$, and their integration makes possible the determination of these functions and so the path of the system. The set $(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_s) = (\mathbf{q}, \dot{\mathbf{q}})$ is called the *state variable* of the system.
Let's us consider the equations for the one dimensional motion of a particle in

a potential $V(\mathbf{q})$

$$m\ddot{\mathbf{q}} = \frac{d}{d\mathbf{q}}V(\mathbf{q}), \tag{3.14}$$

and let's express it in the Eulero-Lagrange equation. We want a function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ that satisfies the variational problem

$$\int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)dt = \text{extreme value}$$

such that the Euler-Lagrange differential equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{3.15}$$

takes the same form as the equation of the motion 3.14.
Since only the second term in the Euler-Lagrange equation can contain $\ddot{\mathbf{q}}$, by comparing it with Newton's equation, we immediately deduce the two equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = m\ddot{\mathbf{q}} \tag{3.16}$$

$$\frac{\partial L}{\partial \mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}. \tag{3.17}$$

The solution of 3.17 give

$$L = T(\dot{\mathbf{q}}) - V(\mathbf{q})$$

for some unknow function $T(\dot{\mathbf{q}})$. This function $T(\dot{\mathbf{q}})$ can be determined from 3.16, which, after inserting $L = T(\dot{\mathbf{q}}) - V(\mathbf{q})$, becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} = m\ddot{\mathbf{q}}.$$

Integrating over t yields

$$\frac{\partial T}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} \ (+const)$$

and further integration over $\dot{\mathbf{q}}$ shows that T is the kinetic energy of the particle

$$T = \frac{1}{2}m\dot{\mathbf{q}}^2 \ (c_1\dot{\mathbf{q}} + c_2).$$

Since the term $c_1\dot{\mathbf{q}} + c_2$ does not contribute to the Euler-Lagrange equations, we have $c_1 = c_2 = 0$. Hence, the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ becomes

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2}m\dot{\mathbf{q}}^2 - V(\mathbf{q}) = T - V. \tag{3.18}$$

Since L does not depend on time explicitely, according to 3.13 we have

$$L - \frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} = const. \tag{3.19}$$

Since

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = 2T \Longrightarrow \frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{1}{2}m\dot{\mathbf{q}}^2 - V(\mathbf{q})\right) = m\ddot{\mathbf{q}} = 2T$$

we have

$$L - \frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} = L - 2T = T - V - 2T = -(T + V) = -E = const.$$

So L is constant in time, and 3.19 are called a *first integrals* of the Euler-Lagrange equation.
It follows that the total energy is conserved

$$E = T + V = const$$

If $V = V(\mathbf{q}, t)$, i.e. also depends on the time, then the Lagrangian is $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T - V(\mathbf{q}, t)$ and the Euler-Lagrange equations take form 3.15. However in this case, the energy $T + V$ is not conserved anymore (since 3.19 is not a solution of the Euler-Lagrange equation).

Definition 3.2.1. The derivative of the Lagrangian 3.18 with respect to the generalized velocity $\dot{\mathbf{q}}$ is called *generalized momentum* $\mathbf{p}_q$

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} = \mathbf{p}_q. \tag{3.20}$$

Definition 3.2.2. A coordinate q is called *cyclic* if does not enter into the Lagrangian, i.e.

$$\frac{\partial L}{\partial q_i} = 0.$$

Theorem 3.2.1. *The generalized momentum corresponding to a cyclic coordinate is conserved: $p_i = const$.*

Proof.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \Longrightarrow \frac{d}{dt}p_i = 0 \Longrightarrow p_i = const.$$

□

3.3 Hamilton's Principle

We now recall the variational problem, which leads to the Lagrange equations. The most general formulation of the law governing the motion of mechanical systems is the principle of least action or Hamilton's principle, according to which every mechanical system is characterised by a definite function $L(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_s, t)$, or briefly $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, and the motion of the system is such that a certain condition is satisfied.

Let the system occupy, at the instants t_1 and t_2, positions defined by $\mathbf{q}^{(1)} = (q_1^1, q_2^1, \ldots q_s^1)$ and $\mathbf{q}^{(2)} = (q_1^2, q_2^2, \ldots q_s^2)$. Then the condiction is that the system moves between these positions in such a way that the integral

$$S = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt \tag{3.21}$$

takes the least possible values. The function L is called the *Lagrangian* of the system concerned, and the integral 3.21 is called *action*.

Let us now derive the differential equations which solve the problem of minimising the integral 3.21. For simplicity, we shall first assume that the system has only one degree of freedom, so that only the function $q(t)$ has to be determined. Let $q = q(t)$ be the function for which S is a minimum. A *variation* $\delta q(t)$ of the function $q(t)$ (we have called it $h(t)$ in section 3.1)

$$q(t) \longmapsto q(t) + \delta q(t), \qquad \delta q(t_1) = \delta q(t_2) = 0 \tag{3.22}$$

causes the increase of the action

$$S[q + \delta q] - S[q] = \int_{t_1}^{t_2} [L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t)] dt.$$

Expanding the difference under the integral in Taylor's series δq and $\delta \dot{q}$ in the integrand, the leading terms are of the first order:

$$S[q + \delta q] - S[q] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt + O(\|\delta q\|^2).$$

The necessary condition for S to have a minimum is that these terms (called the first variation, or simply the variation, of the integral) should be zero:

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0.$$

So, since $\delta\dot{q} = d\delta q/dt$, we obtain, integrating the second term by parts

$$S[q+\delta q] - S[q] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} + O(\|\delta q\|^2) = 0.$$

By 3.22, $\left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} = 0$. So $S[q+\delta q] - S[q]$ =0 only if

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

When the system has more than one degree of freedom, the s different functions $q_i(t)$ must be varied independently in the principle of least action. We then evidently obtain s equations of the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad (i = 1, \ldots, s). \tag{3.23}$$

These are the required differential equations, called *Lagrange's equations.* If the Lagrangian of a given mechanical system is known, the equations 3.4 give the relations between accelerations, velocities and co-ordinates, i.e.they are the equations of motion of the system.

Chapter 4

Hamiltonian mechanics

4.1 From Lagrangian's equations to Hamiltonian's equations

4.1.1 Legendre transformation

Let $y = f(x)$ be a convex function, $f''(x) > 0$. The *Legendre transformation* of the function f is a new function g of a new variable p, which is costructed in the following way:
we draw the graph of f in the x, y plane. Let p be a given number. Consider the straight line $y = px$. We take the point $x = x(p)$ at which the curve is farthest from the straight line in the vertical direction: for each p the function $px - f(x) = F(p, x)$ has a minimum with respect to x at the point $x(p)$. Now we define

$$g(p) = F(p, x(p)).$$

The point $x(p)$is defined by the extremal condiction $\dfrac{\partial F}{\partial x} = 0$, i.e. $f'(x) = p$. Since f is convex, the point $x(p)$ is unique.

Theorem 4.1.1. *The Legendre transformation is involutive, i.e. its square is the identity: if under the Legendre transformation f is taken to g, then the Legendre transformation of g will again be f.*

4.1.2 The case of many variables

Let $f(\mathbf{x})$ be a convex function of the vector variable $\mathbf{x} = (x_1, \ldots, x_n)$ (i.e. the quadratic form $((\partial^2 f/\partial \mathbf{x}^2)d\mathbf{x}, d\mathbf{x})$ is positive definite). Then the Legendre transform is the function $g(\mathbf{p})$ of the vector variable $\mathbf{p} = (p_1, \ldots, p_n)$ defined as above by the equalities $g(\mathbf{p}) = F(\mathbf{p}, \mathbf{x}(\mathbf{p}))$, where $F(\mathbf{p}, \mathbf{x}) = (\mathbf{p}, \mathbf{x}) - f(\mathbf{x})$ and $\mathbf{p} = \partial f/\partial x$.

Theorem 4.1.2. *Let $L \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ be a Lagrangian function convex with respect to $\dot{\boldsymbol{q}}$. We consider a system of Lagrangian's equations $\dot{\boldsymbol{p}} = \partial L/\partial \boldsymbol{q}$, where $\boldsymbol{p} = \partial L/\partial \dot{\boldsymbol{q}}$. Then the system of Lagrange's equations is equivalent to the system of 2n-first order equations, called the Hamilton's equations, with*

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}$$
$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}$$

where $H(\boldsymbol{p}, \boldsymbol{q}, t) = \boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ is the Legendre transform of the Lagrangian function viewed as a function of $\dot{\boldsymbol{q}}$.

So by means of a Legendre transformation, a Lagrangian system of second order differential equations is converted into a symmetrical system of $2n$ first-order equations called a *Hamiltonian system of equations* (or *canonical equations*).

Example

Let $L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q)$. Let's find the Hamiltonian. Let's apply the Legendre transform

$$F(p, \dot{q}) = p\dot{q} - f(\dot{q})$$

with

$$p = \frac{df}{d\dot{q}}(\dot{q}).$$

Then

$$H(p, q) = p\dot{q} - \frac{1}{2}m\dot{q}^2 + V(q).$$

The Hamiltonian equations are

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} = -\dfrac{\partial V}{\partial q}. \end{cases}$$

Then from 3.14 we deduce that $p = m\dot{q} \iff \dot{q} = \dfrac{p}{m}$, that is $f(\dot{q}) = \frac{1}{2}m\dot{q}^2$ and

$$H = \frac{1}{2}\frac{p^2}{m} + V(q).$$

4.1.3 Liouville's theorem

For simplicity we assume that the Hamiltonian function does not depend explicitly on the time: $H = H(\mathbf{p}, \mathbf{q})$. Moreover, the space M where H is defined M is a $2n$-dimensional smooth manifold.

Definition 4.1.1. The $2n$-dimensional space with coordinates $p_1, \ldots, p_n, q_1, \ldots q_n$ is called the *phase space.*

Proposition 4.1.3. *The right-hand sides of the Hamilton's equations give a vector field: at each point $(\boldsymbol{p}, \boldsymbol{q})$ of the phace space there is a $2n$-dimensional vector $(-\partial H/\partial \boldsymbol{q}, \partial H/\partial \boldsymbol{p})$.*

Definition 4.1.2. The *phase flow* is the one parameter group of transformations of phase space

$$g^t : (\mathbf{p}(0), \mathbf{q}(0)) \mapsto (\mathbf{p}(t), \mathbf{q}(t)),$$

where $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are solutions of Hamilton's system of equations.

Theorem 4.1.4 (Liouville's theorem)**.** *The phase flow preserves the volume: for any region $D \in M$ we have*

$$\textit{volume of } (g^t D) = \textit{volume of } (D)$$

4.2 Hamiltonian mechanical system

In this section we will define the *Hamiltonian mechanical system*, i.e. any triple (M, ω, H) where:

- M is an even- dimensional manifold is the phace space and has the structure of symplectic manifold, i.e a closed non-degenerated differential 2-form;
- ω is the symplectic structure and it is an integral invariant;
- $H \in C^\infty(M.\mathbb{R})$ is the hamiltonian function.

On a symplectic manifold (phase space), there is a natural isomorphism between vector field an 1-forms. A vector field on a symplectic manifold corresponding to the differential of a function is called Hamiltonian vector field. A vector field on a manifold determines a phase flow, i.e. a one-parameter group of symplectic diffeomorphisms that acts on the phace space and preserves the Hamiltonian function. Vector fields on a manifold form a Lie-algebra. The Hamiltonian vector fields on a symplectic manifold also form a Lie algebra, with an operation on it called the Poisson bracket.

4.2.1 Hamiltonian vector field

Definition 4.2.1. Let M^{2n} be an even-dimensional differentiable manifold. A *symplectic form* on M^{2n} is a closed non degenerate differential form ω^2 on M^{2n}.

The pair (M^{2n}, ω^2) is called *symplectic manifold.*
According to the Darboux's theorem, in a small neighborhood of each point of M ω^2 can be express in the standard (or *canonical*) form:

$$\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq_i$$

where $p_1, \ldots, p_n, q_1, \ldots, q_n$ are suitable coordinates.

Definition 4.2.2. To each vector ξ, tangent to a symplectic manifold (M^{2n}, ω^2) at the point x, we associate a 1-form ω^1_ξ on T_xM by the formula

$$\omega^1_\xi(\eta) = \omega^2(\eta, \xi).$$

We will denote by I the isomorphism

$$\begin{aligned} I = T_x^*M &\longrightarrow T_xM \\ \xi &\longrightarrow \omega_\xi^1(\eta). \end{aligned}$$

Definition 4.2.3. Let H be a function on a symplectic manifold M^{2n}. Then dH is a differential 1-form on M, and to every point there is a tangent vector to M associated to it. In this way we obtain a vector field IdH on M.
The vector field IdH is called a *Hamiltonian vector field*; H is called the *Hamiltonian function.*

If $M^{2n} = \mathbb{R}^{2n} = \big\{(p,q)\big\}$, then we obtain the phase velocity vector field of Hamilton's canonical equation:

$$\dot{x} = IdH(x) \Longleftrightarrow \dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \dot{q} = \frac{\partial H}{\partial p}.$$

4.2.2 Hamiltonian phase flow

Definition 4.2.4. Let (M^{2n}, ω^2) be a symplectic manifold and $H : M^{2n} \longrightarrow \mathbb{R}$ a function. Assume that the vector field IdH corresponding to H gives a 1-parameter group of diffeomorphisms $g^t : M^{2n} \longrightarrow M^{2n}$:

$$\frac{d}{dt} |_{t=0}\, g^t\mathbf{x} = IdH(\mathbf{x}).$$

The group g^t is called the Hamiltonian phase flow with Hamiltonian finction H.

Theorem 4.2.1. *A Hamiltonian phase flow preserves the symplectic structure:*

$$(g^t)^*\omega^2 = \omega^2.$$

In the case $n = 1$, $M^{2n} = \mathbb{R}^2$, this theorem says that the phase flow preserves the area (Liouville's theorem).

4.2.3 Canonical transformations

Definition 4.2.5. A differential k-form ω is called an *integral invariant* of the map $g : M \longrightarrow M$ if the integrals of ω on any k-chain c and on its image under g are the same.

Theorem 4.2.2. *The form ω^2 giving the symplectic structure in an integral invariant of hamiltonian phase flow.*

Corollary 4.2.3. *Each of the forms $(\omega^2)^2, (\omega^2)^3, (\omega^2)^4, \ldots$ is an integral invariant of the phase flow.*

Definition 4.2.6. A map $g : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is called *canonical* if it has ω^2 as an integral invariant. A canonical map is generally called a *canonical transformation.* Moreover g preserves the 2-form $\omega^2 = \sum dp_i \wedge dq_i$.

Corollary 4.2.4. *Canonical transformation preserve the volume element in phase space:*

the volume of gD is equal to the volume of D, for any region D.

In particular, let $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a canonical transformation of phase space taking a point with coordinates $(\mathbf{p}, \mathbf{q})$ to a point with coordinates $(\mathbf{P}, \mathbf{Q})$. The functions $\mathbf{P}(\mathbf{p}, \mathbf{q})$ and $\mathbf{Q}(\mathbf{p}, \mathbf{q})$ can be considered as new coordinates on phase space.

Proposition 4.2.5. *The 1-form $\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{P}d\boldsymbol{Q}$ is an exact differential:*

$$\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{P}d\boldsymbol{Q} = dS(\boldsymbol{p}, d\boldsymbol{q}). \tag{4.1}$$

Definition 4.2.7. The coordinates $(\mathbf{Q}, \mathbf{q})$ are called *free* if in a neighborhood of some point $(\mathbf{p}_0, \mathbf{q}_0)$ we can take $(\mathbf{Q}, \mathbf{q})$ as independent coordinates, i.e. at $(\mathbf{p}_0, \mathbf{q}_0)$

$$\det \frac{\partial(\mathbf{Q}, \mathbf{q})}{\partial(\mathbf{p}, \mathbf{q})} = \det \frac{\partial \mathbf{Q}}{\partial \mathbf{p}} \neq 0.$$

In this case the function S can be expresses locally in these coordinates:

$$S(\mathbf{p}, \mathbf{q}) = S_1(\mathbf{Q}, \mathbf{q})$$

where $S_1(\mathbf{Q}, \mathbf{q})$ is called *generating function* of the canonical transformation g.

4.2.4 The Lie Algebra of Hamiltonian functions

Hamilton's equations can be written in a simple form if we define the Poisson bracket of two smooth functions F and G on M. The Poisson bracket $\{F, G\}$

of F and G is the smooth function $\omega^2(IdG, IdH)$ and in the local coordinates $\mathbf{p}, \mathbf{q}$

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right). \tag{4.2}$$

It has the following properties:

- $\{F + G, H\} = \{F, H\} + \{G, H\}$ (bilinearity)
- $\{F, G\} = -\{G, H\}$ (skewsimmetry)
- $\{F_1 F_2, G\} = F_1 \{F_2, G\} + F_2 \{F_1, G\}$ (Leibnitz rule)
- $\big\{\{H, F\}, G\big\} + \big\{\{F, G\}, H\big\} + \big\{\{G, H\}, F\big\} = 0$ (Jacoby identity)
- if $x \in M$ is not a critical point of F, then there exists a smooth function G such that $\{F, G\}(x) \neq 0$ (nondegeneracy).

The Poisson bracket $\{F, G\}$ may also be calculated by the formula $dF(IdG)$, i.e. as the value of the covector dF on the vector IdG. Therefore, the derivative of function F in the direction of the Hamiltonian vector field IdH in in fact $\{F, H\}$.
Thus, the Hamilton's equation can be written in the equivalent form $\dot{F} = \{F, G\}$. Since the coordinate functions $p_1, \ldots p_n, q_1, \ldots q_n$ form a complete set of independent functions, the equations

$$\dot{p}_i = \{p_i, H\}$$
$$\dot{q}_i = \{q_i, H\}$$

with $1 \leq i \leq n$, form a closed system. They are called *Hamilton's canonical equations.*

4.2.5 The Arnold-Liouville's theorem on integrable systems

In the following sections, we will consider (M, ω) as a symplectic manifold, and ω^2 in the canonical form $\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq_i$.

Definition 4.2.8. A function F is a *first integral* of the phase flow with hamiltonian function H if and only if its Poisson bracket with H is identically zero: $\{H, F\} \equiv 0$,i.e.

$$\{H, F\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} \right) = 0.$$

Theorem 4.2.6. *The function H is a first integral of the hamiltonian phase flow with hamiltonian function H.*

Definition 4.2.9. Two functions F_1 and F_2 on the symplectic manifold are *in involution* if $\{F_1, F_2\} = 0$.

Theorem 4.2.7. *Arnold-Liouville's theorem*
If in a Hamiltonian system with n degrees of freedom (i.e., let (M, ω) a $2n$-dimensional phase space), $F_1, \ldots, F_n$, with $F_1 = H$, n independent first integrals in involution are known, then the system is integrable by quadratures.

Suppose there are given n functions in involution on a symplectic $2n$-dimensional manifold

$$F_1 \, , \ldots , \; F_n \qquad \{F_i, F_j\} \equiv 0 \, , \qquad i, j = 1, \ldots, n$$

and let $M_{\mathbf{f}}$ be a smooth manifold

$$M_{\mathbf{f}} = \{x : F_i(x) = f_i \; , \; i = 1, \ldots, n\} \; .$$

Assume that the n functions F_i are independent on $M_{\mathbf{f}}$ (i.e., the n 1-forms dF_i are linearly independent at each point of $M_{\mathbf{f}}$). Then

1. *$M_{\mathbf{f}}$ is invariant under the phase flow with hamiltonian function $H = F_1$.*

2. *If the manifold $M_{\mathbf{f}}$ is compact and connected, then it is diffeomorphic to the n-dimensional torus*

$$T^n = \{(\phi_1, \ldots, \phi_n) \mod 2\pi\} \; .$$

3. *The phase flow with hamiltonian function H determines a conditionally periodic motion on $M_{\mathbf{f}}$, i.e., in angular coordinates $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ we have*

$$\frac{d\boldsymbol{\phi}}{dt} = \boldsymbol{\omega} \; , \quad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{f}) \; .$$

4. *The canonical equations with hamiltonian function H can be integrated by quadratures.*

Corollary 4.2.8. *If, in a canonical system with two degrees of freedom, a first integral F is known which does not depend on the hamiltonian H, then the system is integrable by quadratures; a compact connected two-dimensional submanifold of the phase space $H = h$, $F = f$ is an invariant torus, and the motion on it is conditionally periodic.*

4.2.6 The action-angle variables

If M is compact and connected, we may choose angular coordinates ϕ_i on M so that the phase flow with hamiltonian function $H = F_1$ takes an especially simple form:

$$\frac{d\boldsymbol{\phi}}{dt} = \boldsymbol{\omega}(\mathbf{f}) \qquad \boldsymbol{\phi}(t) = \boldsymbol{\phi}(0) + \boldsymbol{\omega}t \ .$$

We will now look at a neighborhood of the n-dimensional manifold $M_{\mathbf{f}}$ in $2n$-dimensional space.

In the coordinates $(\mathbf{F}, \boldsymbol{\phi})$ the phase flow with hamiltonian function $H = F_1$ can be written in the form of the simple system of $2n$ ordinary differential equations

$$\frac{d\mathbf{F}}{dt} = 0 \qquad \frac{d\boldsymbol{\phi}}{dt} = \boldsymbol{\omega}(\mathbf{F}) \ , \tag{4.3}$$

which is easily integrated: $\mathbf{F}(t) = \mathbf{F}(0)$, $\boldsymbol{\phi}(t) = \boldsymbol{\phi}(0) + \boldsymbol{\omega}(\mathbf{F}(0))t$.
Thus, in order to integrate explicitly the original canonical system of differential equation, it is sufficient to find the variables $\boldsymbol{\phi}$ in explicit form. It turns out that this can be done using only quadratures. A construction of the variables $\boldsymbol{\phi}$ is given below.
We note that the variables $(\mathbf{F}, \boldsymbol{\phi})$ are not, in general, symplectic coordinates. It turns out that there are functions of $\mathbf{F}$, which we will denote by $\mathbf{I} = \mathbf{I}(\mathbf{F})$, $\mathbf{I} = (I_1, \dots, I_n)$, such that the variables $(\mathbf{I}, \boldsymbol{\phi})$ are symplectic coordinates: the original symplectic structure ω^2 is expressed in them by the usual formula

$$\omega^2 = \sum dI_i \wedge d\phi_i \ .$$

The variables $\mathbf{I}$ are called action variables; together with the angle variables $\boldsymbol{\phi}$ they form the *action-angle system of canonical coordinates* in a neighborhood of $M_{\mathbf{f}}$.

The quantities I_i are first integrals of the system with Hamiltonian function $H = F_1$, since they are functions of the first integrals F_j. In turn, the variables F_i can be expressed in terms of $\mathbf{I}$ and, in particular, $H = F_1 = H(\mathbf{I})$. In action-angle variables the differential equations of our flow (4.3) have the form

$$\frac{d\mathbf{I}}{dt} = 0 \qquad \frac{d\boldsymbol{\phi}}{dt} = \boldsymbol{\omega}(\mathbf{I}) \ .$$

Construction of action-angle variables in the case of one degree of freedom

A system with one degree of freedom in the phase plane (p, q) is given by the hamiltonian function $H(p, q)$.

In order to construct the action-angle variables, we will look for a canonical transformation $(p, q) \to (I, \phi)$ satisfying the two conditions:

$$\begin{aligned} &1. \quad I = I(H) \ , \\ &2. \quad \oint_{M_h} d\phi = 2\pi \ . \end{aligned} \tag{4.4}$$

In order to construct the canonical transformation $p, q \to I, \phi$ in the general case, we will look for its generating function $S(I, q)$:

$$p = \frac{\partial S(I,q)}{\partial q} \quad \phi = \frac{\partial S(I,q)}{\partial I} \quad H\left(\frac{\partial S(I,q)}{\partial q}\ , \ q\right) = h(I) \ . \tag{4.5}$$

We first assume that the function $h(I)$ is known and invertible, so that every curve M_h is determined by the value of I ($M_h = M_{h(i)}$). Then for a fixed value of I we have from (4.5)

$$\begin{gathered} dS = pdq + \phi dI \\ 0 = ddS = \frac{\partial p}{\partial I} dq \wedge dI + \frac{\partial \phi}{\partial q} dI \wedge dq \iff \frac{\partial \phi}{\partial q} = \frac{\partial p}{\partial I} \\ \implies d\phi = \frac{\partial p}{\partial I} dq + \frac{\partial \phi}{\partial I} dI. \end{gathered}$$

Since we are considering $I = const$, it follows

$$\begin{cases} d\phi\big|_I = \dfrac{\partial p}{\partial I} dq \\ dS\big|_I = pdq \end{cases} \implies \begin{cases} \phi = \int_{q_0}^q = \frac{\partial p}{\partial I} dq \\ S = \int_{q_0}^q pdq. \end{cases}$$

Let's call

$$\sqcap = \oint_{M_n} pdq \quad \text{(area)}$$
$$\Delta\phi = \oint_{M_n} \frac{\partial p}{\partial I} dq \quad \text{(period)}.$$

We have

$$\frac{d\sqcap}{dI} \oint_{M_n} \frac{\partial p}{\partial I} dq = \Delta\phi = 2\pi$$
$$\Longrightarrow \frac{dI}{d\sqcap} = \frac{1}{2\pi}.$$

In this way, we have obtained the *action variable*

$$I = \frac{1}{2\pi}\sqcap = \frac{1}{2\pi} \oint_{M_n} pdq$$

and the *angle variable*

$$\phi = \int_{q_0}^{q} \frac{\partial p}{\partial I} dq.$$

Definition 4.2.10. The *action variable* in the one-dimensional problem with hamiltonian function $H(p, q)$ is the quantity $I(h) = \frac{1}{2\pi}\Pi(h)$.

Finally, we arrive at the following conclusion. Let $\frac{d\Pi}{dh} \neq 0$. Then the inverse $I(h)$ of the function $h(I)$ is defined.

Theorem 4.2.9. *Set*

$$S(I, q) = \int_{q_0}^{q} p\, dq\big|_{H=h(I)}.$$

Then formulas (4.5) *give a canonical transformation* $p, q \to I, \phi$ *satisfying conditions* (4.4).

Thus, the action-angle variables in the one dimensional case are constructed.

4.3 Examples

Let us apply the theory of Hamiltonian systems and elliptic integrals to a simple example.

4.3.1 Example 1

Let

$$H(p,q) = \frac{1}{2}p^2 + V(q), \quad \text{where} \quad V(q) = -\frac{q^3}{3} + q$$

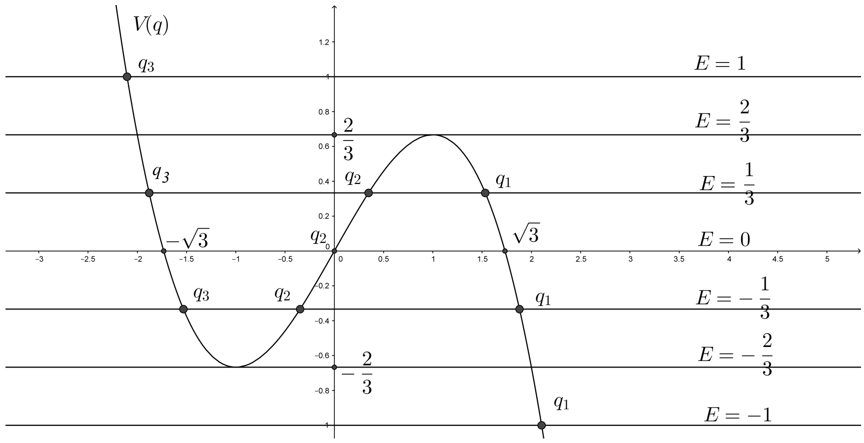


Figure 4.1: $V(q) = -\frac{q^3}{3} + q$

Since $p = \dfrac{dq}{dt}$ and $H(p,q) = E$

$$E = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 + V(q) \Longrightarrow$$
$$\frac{dq}{dt} = \pm\sqrt{2(E - V(q))} \Longrightarrow$$
$$dt = \pm\frac{dq}{\sqrt{2(E - V(q))}} \Longrightarrow$$
$$t - t_0 = \int_{q_0}^{q} \frac{dq}{\sqrt{2(E - V(q))}}.$$

where

$$2(E - V(q)) = 2(E - q + q^3/3) = \frac{2}{3}(q - q_1)(q - q_2)(q - q_3)$$

and q_1, q_2, q_3 are the turning point and are shown on the figure above
By a change of coordinate $q \longrightarrow ax + b$

$$t - t_0 = \pm\int_{x_0a+b}^{xa+b} \frac{adx}{\sqrt{2(E - V(ax+b))}} =$$

$$\pm\int_{x_0a+b}^{xa+b} \frac{adx}{a\sqrt{\frac{2}{3}ax^3 + 2bx^2 + (2b^2 - 2)\frac{x}{a} + (2E - 2b + \frac{2}{3}b^3)\frac{1}{a^2}}}$$

We want the Weierstrass form is $4x^3 - g_2x - g_3$, so

$$\frac{2}{3}a = 4 \Longrightarrow a = 6$$
$$2b = 0 \Longrightarrow b = 0$$

By substituting the values of a and b we obtain

$$t - t_0 = \pm \int_{6x_0}^{6x} \frac{dx}{\sqrt{4x^3 - \frac{x}{3} + \frac{E}{18}}}, \qquad \text{with } g_2 = \frac{1}{3}, g_3 = -\frac{E}{18}.$$

Now, $P(x) = 4x^3 - \frac{x}{3} + \frac{E}{18}$ has three distinct roots $\iff \Delta = g_2^3 - 27g_3^2 \neq 0$, i.e. $\iff E \neq \pm\frac{2}{3}$.

- Case $E = -1$ There is only one turnig point at q_1

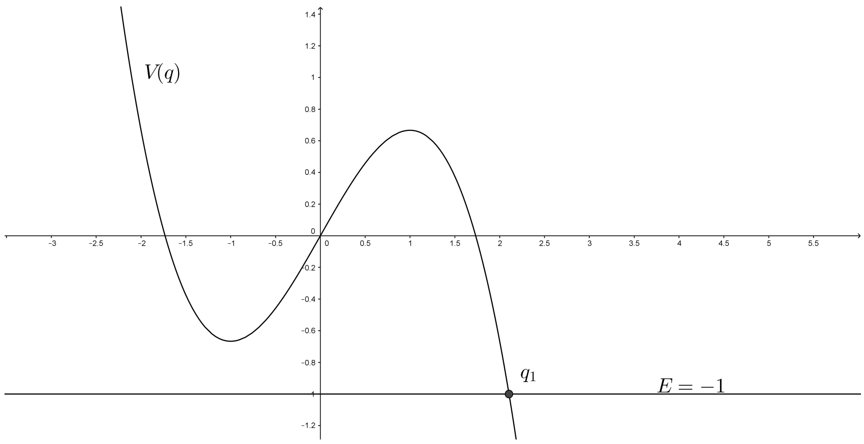


Figure 4.2: $V(q) < -1$

$$t - t_0 = \pm \int_{\frac{q_1}{6}}^{x} \frac{dx}{\sqrt{4x^3 - \frac{x}{3} + \frac{-1}{18}}} =$$
$$\Longrightarrow t = \wp^{-1}(x, g_2, g_3) - \underbrace{\wp^{-1}(\frac{q_1}{6}, g_2, g_3)}_{=C}$$
$$\Longrightarrow t + C = \wp^{-1}(x, g_2, g_3)$$
$$\Longrightarrow x = \wp(t + c, g_2, g_3)$$
$$\Longrightarrow \frac{q}{6} = \wp(t + c, g_2, g_3)$$
$$\Longrightarrow q = 6\wp(t + c, g_2, g_3)$$

where $g_2 = 1/3$, $g_3 = 1/18$

- Case $E = -1/3$

  In this case we have three distinct inversion points q_1, q_2, q_3.

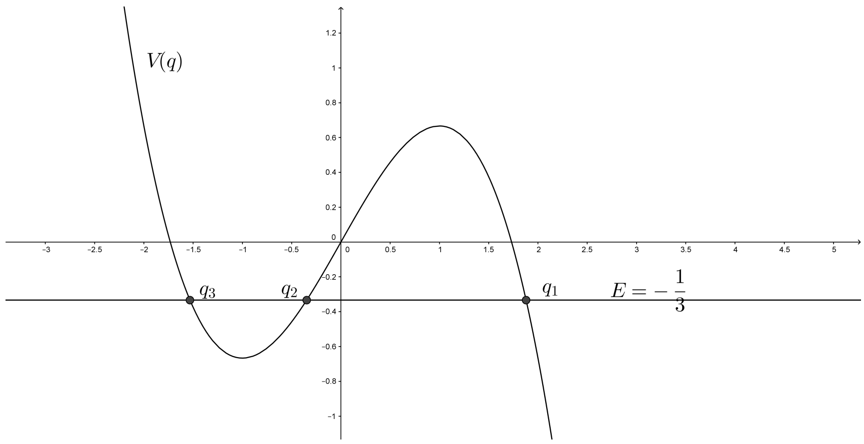


Figure 4.3: $V(q) < -\frac{1}{3}$

Case $q_3 < q < q_2$

We have a periodic motion between q_3 and q_2.

The period T is given by

$$T = 2\int_{\frac{q_2}{6}}^{\frac{q_3}{6}} \frac{dx}{\sqrt{4x^3 - \frac{x}{3} - \frac{1}{54}}} = 2\left[\wp^{-1}(\frac{q_3}{6}, g_2, g_3) - \wp^{-1}(\frac{q_2}{6}, g_2, g_3)\right]$$

Case $q > q_1$

$$t - t_0 = \int_{\frac{q_1}{6}}^{x} \frac{dx}{\sqrt{4x^3 - \frac{x}{3} - \frac{1}{54}}}$$

$$\Longrightarrow t = \wp^{-1}(x, g_2, g_3) - \underbrace{\wp^{-1}(\frac{q_1}{6}, g_2, g_3)}_{=C}$$

$$\Longrightarrow t + C = \wp^{-1}(x, g_2, g_3)$$

$$\Longrightarrow x = \wp(t + C, g_2, g_3)$$

$$\Longrightarrow \frac{q}{6} = \wp(t + C, g_2, g_3)$$

$$\Longrightarrow q = 6\wp(t + C, g_2, g_3)$$

with $g_2 = 1/3$, $g_3 = 1/54$, and where we have assumed the initial time $t_0 = 0$.

Action-angle variables

Let's calculate the action-angle variables. By a canonical transformation

$$(p, q) \longrightarrow (I, \phi)$$

where

$$\begin{cases} I = I(h) \\ \oint d\phi = 2\pi. \end{cases}$$

Let $S(I, q)$ be its generating function s.t.

$$p = \frac{\partial S(I,q)}{\partial q}, \quad \phi = \frac{\partial S(I,q)}{\partial I}, \quad , \quad H(\frac{\partial S(I,q)}{\partial q}, q) = h(I).$$

then

$$S(I, q) = \int_{q_3}^{q} pdq = \int_{q_3}^{q} \sqrt{2(E - V(y))}dy \tag{4.6}$$

and

$$I = \frac{1}{2\pi}\oint pdq = \frac{1}{2\pi}2\int_{q_2}^{q_3} \sqrt{2(E - V(y))}dy = \frac{1}{\pi}\int_{q_2}^{q_3} \sqrt{2(E - V(y))}dy$$

$$\phi = \frac{\partial S(I,q)}{\partial I} = \frac{1}{2}\int_{q_2}^{q_3} \frac{\frac{\partial E}{\partial I}}{\sqrt{2(E - V(y))}}dy \tag{4.7}$$

Imposing the condiction

$$\oint \phi = 2\pi \Longrightarrow 2\int_{q_2}^{q_3} \frac{\frac{\partial E}{\partial I}}{\sqrt{2(E - V(y))}}dy = 2\pi$$

$$\Longrightarrow \frac{\partial E}{\partial I} = \frac{\pi}{\int_{q_2}^{q_3} \frac{dy}{\sqrt{2(E-V(y))}}}$$

$$\Longrightarrow \frac{dI}{dE} = \frac{1}{\pi}\int_{q_2}^{q_3} \frac{dy}{2(E - V(y))}$$

$$\Longrightarrow \frac{dI}{dE} = \frac{1}{\pi}(\wp^{-1}(\frac{q_3}{6}, g_2, g_3) - \wp^{-1}(\frac{q_2}{6}, g_2, g_3))$$

$$\Longrightarrow \frac{dE}{dI} = \frac{\pi}{\wp^{-1}(\frac{q_3}{6}, g_2, g_3) - \wp^{-1}(\frac{q_2}{6}, g_2, g_3)}$$

substituting on 4.7 we can find ϕ.

4.3.2 Example 2

In this example we are going to show how to transform a polynomial of fourth degree into the Weierstrass form. The calculation were made by the help of Matlab, of which we report the codex in 5.7.
Let's assume $V(q) = q^4 - 1 = (q-1)(q+1)(q+i)(q-i)$.
$H(p,q) = \frac{1}{2}p^2 + V(q)$.

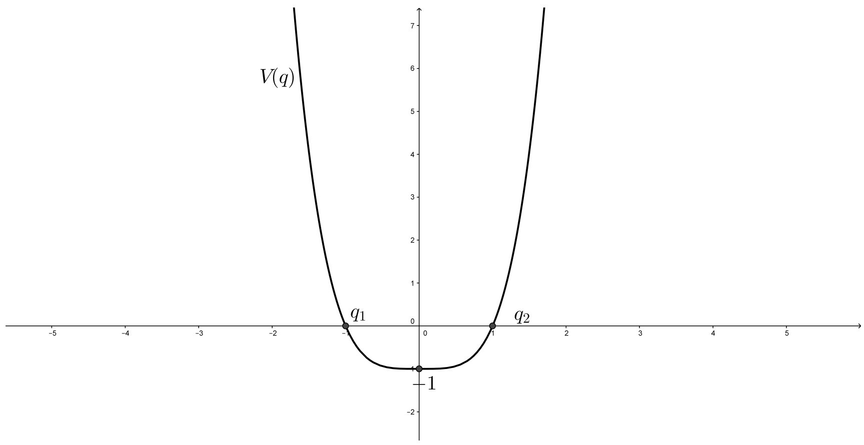


Figure 4.4

$$t - t_0 = \int_{q_1}^{q} \frac{dq}{\sqrt{-2x^4 + 2E + 2}} =$$
$$\int_{q_1}^{q} \frac{dq}{\sqrt{(q-a)(q-b)(q-c)(q-d)}},$$

where

$$a = \sqrt[4]{E+1}$$
$$b = -\sqrt[4]{E+1},$$
$$c = -1\sqrt[4]{E+1},$$
$$d = i\sqrt[4]{E+1}.$$

Let's apply the transformation $q \longrightarrow a + \frac{1}{t}$. Then $dq = -\frac{1}{t^2}$ and $t = \frac{1}{q-a}$. So our integral becomes

$$\int_{\frac{1}{q_1-a}}^{\frac{1}{q-a}} \frac{-\frac{dt}{t^2}}{\sqrt{-\frac{2}{t^4} - \frac{8(E+1)^{-1/4}}{t^3} - \frac{8(E+1)^{1/4}}{t} - \frac{12(E+1)^{1/4}}{t^2}}} =$$

$$\int_{\frac{1}{q_1-a}}^{\frac{1}{q-a}} \frac{-\frac{dt}{t^2}}{\frac{1}{t^2}\sqrt{-2 - 8(E+1)^{-1/4}t - 8(E+1)^{1/4}t^3 - 12(E+1)^{1/4}t^2}}$$

We apply the second transformation $t \longrightarrow -\alpha s + \beta$, with $dt = \alpha ds$ and $s = \frac{1}{\alpha}\left(\frac{-1}{q-a} + \beta\right)$ and the integral becomes

$$\int_{\frac{1}{\alpha}\left(\frac{-1}{q_0-a}+\beta\right)}^{\frac{1}{\alpha}\left(\frac{-1}{q-a}+\beta\right)} \frac{ds}{\sqrt{4s^3 + 4(E+1)s}},$$

where

$$\begin{cases} \alpha &= \dfrac{1}{2(E+1)^{3/4}} \\ \beta &= \dfrac{-1}{2(E+1)^{-1/4}} \end{cases}$$

This polynomial has distinct roots $\iff \Delta \neq 0$ i.e. $-4(E+1)^3 \neq 0 \iff E \neq -1$.

Let's assume $E = 0$.

The motion is periodic between $q_1 = -1$ and $q_2 = 1$. The period T is

$$T = 2\int_{\frac{1}{\alpha}\left(\frac{-1}{q_1-a}+\beta\right)}^{\frac{1}{\alpha}\left(\frac{-1}{q_2-a}+\beta\right)} \frac{ds}{\sqrt{4s^3+4s}}.$$

Let's calculate the motion

$$t - t_0 = \int_{\frac{1}{\alpha}\left(\frac{-1}{q_1-a}+\beta\right)}^{s} \frac{ds}{\sqrt{4s^3+4s}} = \int_{f(q_1)}^{f(q)} \frac{ds}{\sqrt{4s^3+4s}} \Longrightarrow$$

$$t = \wp^{-1}(s, g_2, g_3) - \underbrace{\wp^{-1}(\frac{1}{\alpha}\left(\frac{-1}{q_1-a}+\beta\right), g_2, g_3)}_{=C} \Longrightarrow$$

$$s = \wp(t + C, g_2, g_3).$$

Chapter 5

Geodesic flow on an ellipsoid and the mechanical problem of C. Neumann

5.1 Constrained Hamilton system

Consider a constrained Hamiltonian systems on M in $\mathbb{R}^{2n}$, given by

$$M = \left\{ x \in \mathbb{R}^{2n} | G_1(x) = 0, \ldots, G_{2r}(x) = 0 \right\}.$$

The dimension of M is $2n - 2r$ if dG_i are linearly independent on M. We will require more by assuming

$$\det(\left\{G_j, G_k\right\})_{j,k=1,\ldots,2r} \neq 0 \tag{5.1}$$

which makes M a symplectic manifold.
If a given system

$$\frac{dx}{dt} = JH_x$$

defines a vector field tangential to M, then there is no difficulty in restricting this system to M. The condictions for this to be so is

$$X_H G_j = -\left\{H, G_j\right\} = 0 \qquad \text{for } j = 1, \ldots, 2r \tag{5.2}$$

on M.
This can done in many ways, for example, using the mehod of Lagrange mul-

tipliers, by replacing X_H by

$$X_H - \sum_{j=1}^{2r} \lambda_j(x) X_{G_j}$$

where the multipliers λ_j are defined so that this vectorfield is tangential to M. This requires that

$$\{H, G_k\} - \sum_{j=1}^{2r} \left\{G_j, G_k\right\} = 0 \tag{5.3}$$

which by 5.1 defines the $\lambda_j = \lambda_j(x)$ uniquely on M.
If we set

$$H^* = H - \sum_{j=1}^{2r} \lambda_j G_j,$$

the *constrained vectorfield* is given by

$$X_{H^*} = X_H - \sum_{j=1}^{2r} \lambda_j X_{G_j}.$$

5.2 Geodesics on an Ellipsoid

Let's consider the geodesic flow on an ellipsoid

$$\left\{x \in \mathbb{R}^n, \left\langle A^{-1}x, x\right\rangle = 1\right\},$$

where $A = A^T$ is a positive definite symmetric matrix with distinct eigenvalues $0 < \alpha_1 < \cdots < \alpha_n$. For simplicity we assume, without loss of generality,

$$A = diag(\alpha_1, \cdots, \alpha_n) = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}.$$

The differential equations are given by

$$\frac{d^2x}{ds^2} = -vA^{-1}x \Rightarrow \begin{cases} y = \dfrac{dx}{ds} = x' \\ \dfrac{dy}{ds} = y' = -vA^{-1}x \end{cases}. \tag{5.4}$$

Proposition 5.2.1. *The multiplier v is determined so that*

$$\frac{1}{2}\left(\frac{d}{ds}\right)^2 \langle A^{-1}x, x\rangle = 0.$$

i.e.

$$v = |A^{-1}x|^{-2}\langle A^{-1}x, x\rangle .$$

Proof.

$$\frac{1}{2}\left(\frac{d}{ds}\right)^2 \langle A^{-1}x, x\rangle = \frac{1}{2}\left(\frac{d}{ds}\right)^2 \left(\sum_{i=1}^{n} \frac{x_i^2}{a_i}\right) = 0$$
$$\Rightarrow \frac{1}{2}\frac{d}{ds}\left(\sum_{i=1}^{n} 2\frac{x_i x_i'}{a_i}\right) = \sum_{i=1}^{n}\left(\frac{x_i' x_i'}{a_i} + \frac{x_i x_i''}{a_i}\right) =$$
$$\sum_{i=1}^{n}\left(\frac{(x_i')^2}{a_i} + \frac{x_i}{a_i}(-vA^{-1}x)\right) = \langle A^{-1}x, x\rangle - v\frac{x_i^2}{a_i^2} =$$
$$\langle A^{-1}x, x\rangle - v|A^{-1}x|^2 = 0$$
$$\Rightarrow v = \frac{\langle A^{-1}x, x\rangle}{|A^{-1}x|^2}.$$

□

We will show that this system is integrable and that the integrals can be written as quartic polynomials in x and x'.
It's usefull to represent this system by constraining the free Hamiltonian

$$H = \frac{1}{2}|y|^2 \tag{5.5}$$

to the tangent bundle of the ellipsoid

$$G_1 = \langle A^{-1}x, x\rangle - 1 = \sum_{i=1}^{n}\frac{x_i^2}{a_i} - 1,$$
$$G_2 = \langle A^{-1}x, y\rangle = \sum_{i=1}^{n}\frac{x_i y_i}{a_i}.$$

Since

$$\{G_1, G_2\} = \sum_{i=1}^{n} \frac{\partial G_1}{\partial x_i}\frac{\partial G_2}{\partial y_i} - \underbrace{\frac{\partial G_1}{\partial y_i}\frac{\partial G_2}{\partial x_i}}_{=0} =$$

$$= \sum_{i=1}^{n} 2\frac{x_i^2}{a_i^2} = 2|A^{-1}x|^2 \neq 0,$$

then $\det(\{G_1, G_2\}) \neq 0$, so the condiction 5.1 is satisfied.

$$X_H G_1 = -\{H, G_1\} = -\left(\sum_{i=1}^{n} \underbrace{\frac{\partial H}{\partial x_i}\frac{\partial G_1}{\partial y_i}}_{=0} - \frac{\partial H}{\partial y_i}\frac{\partial G_1}{\partial x_i}\right) =$$

$$= -\left(-2\sum_{i=1}^{n} y_i \frac{x_i}{a_i}\right) = 2\left\langle A^{-1}x, y\right\rangle;$$

$$X_H G_2 = -\{H, G_2\} = -\left(\sum_{i=1}^{n} \underbrace{\frac{\partial H}{\partial x_i}\frac{\partial G_2}{\partial y_i}}_{=0} - \frac{\partial H}{\partial y_i}\frac{\partial G_2}{\partial x_i}\right) =$$

$$= \sum_{i=1}^{n} \frac{y_i^2}{a_i} = \left\langle A^{-1}y, y\right\rangle;$$

$$\{H, G_1\} - \lambda_2 \{G_1, G_2\} = 0 \Rightarrow 2\left\langle A^{-1}x, y\right\rangle - \lambda_2(2|A^{-1}x|^2) = 0$$

$$\Rightarrow \lambda_2 = \frac{\left\langle A^{-1}x, y\right\rangle}{|A^{-1}x|^2};$$

$$\{H, G_2\} - \lambda_1 \{G_1, G_2\} = 0 \Rightarrow \left\langle A^{-1}y, y\right\rangle - \lambda_1(2|A^{-1}x|^2) = 0$$

$$\Rightarrow \lambda_1 = \frac{\left\langle A^{-1}y, y\right\rangle}{2|A^{-1}x|^2}.$$

So we have

$$H^* = \frac{1}{2}|y|^2 - \lambda_1(\left\langle A^{-1}x, x\right\rangle - 1) - \lambda_2\left\langle A^{-1}x, y\right\rangle$$

or

$$H^* = \frac{1}{2}|y|^2 + \frac{\mu}{2}\Phi_0(x, y) - \frac{\mu}{2}\left\langle A^{-1}x, y\right\rangle^2, \tag{5.6}$$

where

$$
\begin{gathered}
\mu = |A^{-1}x|^{-2}, \\
\phi_0 = (\langle A^{-1}x, x\rangle - 1)\langle A^{-1}y, y\rangle - \langle A^{-1}x, y\rangle^2 = \\
\left(\sum_{i=1}^{n} \frac{x_i^2}{\alpha_i} - 1\right) \sum_{j=1}^{n} \frac{y_j^2}{\alpha_j} - \left(\sum_{i=1}^{n} \frac{x_i y_i}{\alpha_i}\right)^2 .
\end{gathered}
$$

Proposition 5.2.2. *The constrained system is*

$$
\begin{cases}
\dfrac{dx}{ds} = H_y^* = y \\
\dfrac{dy}{ds} - H_x^* = -\mu \langle A^{-1}y, y\rangle A^{-1}x.
\end{cases}
$$

Proof.

$$
\begin{gathered}
H_y^* = \frac{\partial}{\partial y}\left(\frac{1}{2}|y|^2\right) + \frac{\mu}{2}\frac{\partial}{\partial y}(\Phi_0(x,y)) - \frac{\mu}{2}\frac{\partial}{\partial y}(\langle A^{-1}x, y\rangle^2) = \\
y + \frac{\mu}{2}(\underbrace{\langle A^{-1}x, x\rangle - 1}_{=0})\frac{\partial}{\partial y}\langle A^{-1}y, y\rangle - \frac{\mu}{2}[2\underbrace{\langle A^{-1}x, y\rangle}_{=0}\frac{\partial}{\partial y}\langle A^{-1}x, y\rangle] = y.
\end{gathered}
$$

$$
\begin{aligned}
H_x^* = \frac{1}{2}\frac{\partial\mu}{\partial x}\Phi_0(x,y) + \frac{1}{2}\mu\frac{\partial}{\partial x}\Phi_0(x,y) - \frac{1}{2}\frac{\partial\mu}{\partial x}\underbrace{\langle A^{-1}x, y\rangle^2}_{=0} - \frac{\mu}{2}\frac{\partial}{\partial x}(\underbrace{\langle A^{-1}x, y\rangle^2}_{=0}) = \\
= \frac{1}{2}\frac{\partial}{\partial x}\mu\,\underbrace{[(\underbrace{\langle A^{-1}x, x\rangle - 1}_{=0})\langle A^{-1}y, y\rangle - \underbrace{\langle A^{-1}x, y\rangle^2}_{=0}]}_{=0} + \\
\frac{1}{2}\mu[\frac{\partial}{\partial x}(\langle A^{-1}x, x\rangle)\langle A^{-1}y, y\rangle - \underbrace{\frac{\partial}{\partial x}(\langle A^{-1}x, y\rangle^2)}_{=0}] = \\
\frac{1}{2}\mu 2\sum_{i=1}^{n}\frac{x_i}{a_i}\langle A^{-1}y, y\rangle = \mu\underbrace{\sum_{i=1}^{n}\frac{x_i}{a_i}}_{A^{-1}x}\langle A^{-1}y, y\rangle = \mu A^{-1}x\langle A^{-1}y, y\rangle .
\end{aligned}
$$

□

Hence they agree with 5.4.

With this approach we extend the system from a flow on the tangent bundle of the ellipsoid to $\mathbb{R}^{2n}$, so we avoid the use of the local coordinates on the ellipsoid.

Also, the extended system X_{H*} has an interesting geometric interpretation. The cone

$$\left\{y \in \mathbb{R}^n | \phi_0(x,y) = 0\right\}$$

when traslated by x represents the cone of vectors throught the point the point x which are tangent to the ellipsoid.

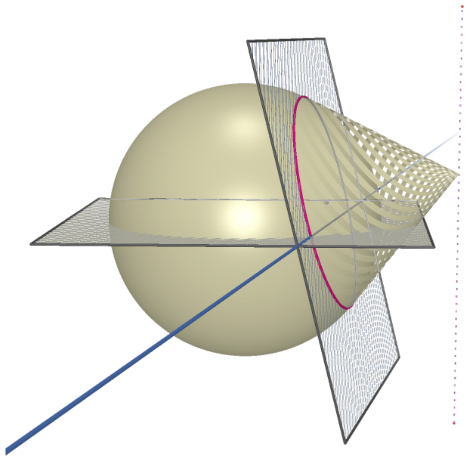


Figure 5.1: Tangent cone

Proposition 5.2.3. *We have that* $|y|^2$ *and* ϕ_0 *commute, i.e.*

$$\left\{|y|^2, \phi_0\right\} = 0.$$

Proof.

$$\left\{|y|^2, \phi_0\right\} = 2\sum_{i=1}^n \frac{\partial \phi_0}{\partial x_i} y_i =$$

$$2\sum_{i=1}^n \left[\left(2\frac{x_i}{\alpha_i}\sum_{j=1}^n \frac{y_j^2}{\alpha_j} - 2(\sum_{j=1}^n \frac{x_j y_i}{\alpha_j})\frac{y_i}{\alpha_i}\right) y_i\right] =$$

$$4\left[\left(\sum_{i=1}^n \frac{x_i y_i}{\alpha_i}\right)\left(\sum_{j=1}^n \frac{y_j^2}{\alpha_j}\right) - \left(\sum_{j=1}^n \frac{x_j y_i}{\alpha_j}\right)\left(\sum_{i=1}^n \frac{y_i^2}{\alpha_i}\right)\right] = 0.$$

□

So, because of the proposition above, in the vectorfield

$$X_{H^*} = \frac{1}{2} X_{|y|^2} + \frac{\mu}{2} X_{\phi_0}$$

also the two summands commute, we can discuss the two vectorfields separately. The first summand describes the free flow

$$(x, y) \Longrightarrow (x + sy, y)$$

and the second is given by

$$\begin{cases} \dot{x} = \mu\phi_{0y} \\ \dot{y} = -\mu\phi_{0x}. \end{cases}$$

5.3 Confocal quadrics, construction of integrals

Basic for the understing of the geodesics on the ellipsoid is the family of confocal quadrics

$$Q_z = \left\langle (zI - A)^{-1}x, x \right\rangle + 1 = 0, \qquad \text{with } z \in \mathbb{R}, \quad z \neq \alpha_k$$

where the $\alpha_k,\ k = 1, \dots n$ are the eigenvalues of the matrix A. Q_z contains the ellipsoid for $z = 0$.

For abbreviation we set

$$Q_z(x, y) = \left\langle (zI - A)^{-1}y, y \right\rangle; \qquad Q_z(x) = Q_z(x, x),$$

and we introduce

$$\phi_z(x, y) = (1 - Q_z(x))Q_z(y) - Q_z^2(x, y). \tag{5.7}$$

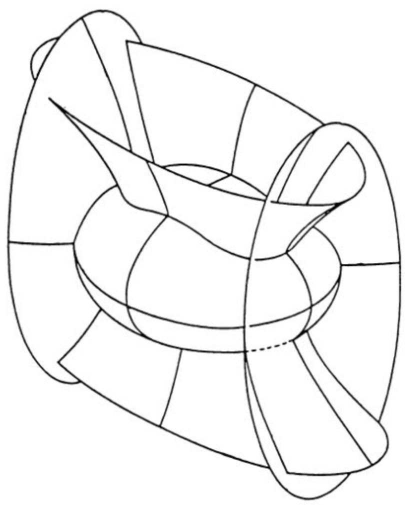


Figure 5.2: Confocal quadrics

The functions $\phi_z(x,y)$ are quartic polynomials as far as x and y are concerned, and rational functions of z with simple poles at the eigenvalues α_k of A.

$$\phi_z(x,y) = (1-Q_z(x))Q_z(y) - Q_z^2(x,y) =$$

$$Q_z(y) + \left(\sum_{i=1}^n \frac{x_i^2}{z-\alpha_i}\right)\left(\sum_{j=1}^n \frac{y_j^2}{z-\alpha_j}\right) - \left(\sum_{i=1}^n \frac{x_i y_i}{z-\alpha_i}\right)\left(\sum_{j=1}^n \frac{x_j y_j}{z-\alpha_j}\right) =$$

$$\sum_{i=1}^n \frac{y_i^2}{z-\alpha_i} + \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{x_i^2 y_j^2 - x_i y_j x_j y_i}{(z-\alpha_i)(z-\alpha_j)} =$$

$$\sum_{i=1}^n \frac{y_i^2}{z-\alpha_i} + \sum_{1\leq i<j\leq n} \frac{x_i^2 y_j^2 + x_j^2 y_i^2 - 2x_i y_i x_j y_j}{(z-\alpha_i)(z-\alpha_j)} =$$

$$\sum_{i=1}^n \frac{y_i^2}{z-\alpha_i} + \sum_{1\leq i<j\leq n} \frac{(x_i y_j - x_j y_i)^2}{(z-\alpha_i)(z-\alpha_j)} =$$

$$\sum_{i=1}^n \left(\frac{y_i^2}{z-\alpha_i} + \sum_{\substack{j=1\\ j\neq i}}^n \frac{(x_i y_j - x_j y_i)^2}{(z-\alpha_i)(z-\alpha_j)}\right)$$

By using in the second summation the partial fraction decomposition method, we obtain

$$\sum_{\substack{j=1\\ j\neq i}}^n \frac{(x_i y_j - x_j y_i)^2}{(z-\alpha_i)(z-\alpha_j)} = \sum_{\substack{j=1\\ j\neq i}}^n \frac{B_i}{z-\alpha_i}$$

where

$$B_i = \frac{(x_j y_i - x_j y_i)^2}{\alpha_j - \alpha_i}.$$

So we can write

$$\phi_z(x, y) = \sum_{i=1}^{n} \frac{F_i(x, y)}{z - \alpha_i}, \tag{5.8}$$

where

$$F_i(x, y) = y_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(x_j y_i - x_j y_i)^2}{\alpha_j - \alpha_i}. \tag{5.9}$$

Proposition 5.3.1. *For any two numbers z_1, z_2 one has for the functions ϕ_{z_1}, ϕ_{z_2} defined by 5.7 the identity*

$$\{\phi_{z_1}, \phi_{z_2}\} = 0$$

hence also

$$\left\{F_j, F_k\right\} = 0.$$

Proof.

$$\{\phi_{z_1}, \phi_{z_2}\} = \sum_{i=1}^{n} \left(\frac{\partial \phi_{z_1}}{\partial x_i} \frac{\partial \phi_{z_2}}{\partial y_i} - \frac{\partial \phi_{z_1}}{\partial y_i} \frac{\partial \phi_{z_2}}{\partial x_i} \right) =$$

$$\sum_{i=1}^{n} \left[\frac{1}{z_1 - \alpha_i} \frac{\partial F_i(x, y)}{\partial x_i} \frac{1}{z_2 - \alpha_i} \frac{\partial F_i(x, y)}{\partial y_i} - \frac{1}{z_1 - \alpha_i} \frac{\partial F_i(x, y)}{\partial y_i} \frac{1}{z_2 - \alpha_i} \frac{\partial F_i(x, y)}{\partial x_i} \right] = 0$$

Let's prove $\left\{F_j, F_k\right\} = 0$ in the case $n = 2$. We have

$$F_1 = y_1^2 + \frac{(x_2 y_1 - x_1 y_2)^2}{\alpha_2 - \alpha_1},$$

$$F_2 = y_2^2 + \frac{(x_1 y_2 - x_2 y_1)^2}{\alpha_1 - \alpha_2},$$

$$\{F_1, F_2\} = \sum_{i=1}^{2} \frac{\partial F_1}{\partial x_i} \frac{\partial F_2}{\partial y_i} - \frac{\partial F_1}{\partial y_i} \frac{\partial F_2}{\partial x_i} =$$

$$2\frac{(x_2 y_1 - x_1 y_2)}{\alpha_2 - \alpha_1}(-y_2) 2\frac{(x_1 y_2 - x_2 y_1)}{\alpha_1 - \alpha_2}(-x_2) - (2y_1 + 2\frac{(x_2 y_1 - x_1 y_2)}{\alpha_2 - \alpha_1} x_2) 2\frac{(x_1 y_2 - x_2 y_1)}{\alpha_1 - \alpha_2} y_2$$
$$+2\frac{(x_2 y_1 - x_1 y_2)}{\alpha_2 - \alpha_1} y_1 (2y_2 + 2\frac{(x_1 y_2 - x_2 y_1)}{\alpha_1 - \alpha_2} x_1) - 2\frac{(x_2 y_1 - x_1 y_2)}{\alpha_2 - \alpha_1}(-x_1) 2\frac{(x_1 y_2 - x_2 y_1)}{\alpha_1 - \alpha_2}(-y_1) = 0.$$

□

Since

$$\sum_{i=1}^{n} F_i = |y|^2$$

also commutes with the F_i, it follows that the F_i also are integrals for 5.6, and hence the restrictions of the F_i to the tangent bundle of the ellipsoid Q_0 are integrals of the geodesic problem. By proposition 5.3.1 they commute, and the dF_i are linearly independent on an open set of $\mathbb{R}^{2n}$. This shows that the geodesic problem is integrable on an open and dense set of the tangent bundle, and the integrals are given by the restriction of the functions 5.9.

5.4 Iso-spectral deformations

The system

$$\begin{cases} x_j' = \dfrac{\partial}{\partial y_j}\phi_0 \\ y_j' = -\dfrac{\partial}{\partial x_j}\phi_0 \end{cases} \tag{5.10}$$

can be interpreted as iso-spectral deformation. The difficulty is to guess the matrices L and B with whitch the above equation can be written in the form

$$\frac{dL}{dt} = [B, L] = BL - LB.$$

Following Moser [11], let

$$L = L(x, y) = P_y(A - x \otimes x)P_y, \qquad |y| > 0,$$

where $(x \otimes x) = x_i x_j$ is the tensor product and

$$P_y = \delta_{ij} - y_i y_j |y|^{-2}$$

is the projection onto the orthogonal complement of y.

Since

$$A - x \otimes x = \begin{pmatrix} \alpha_1 - x_1^2 & -x_1x_2 & \cdots & -x_1x_n \\ -x_2x_1 & \alpha_2 - x_2^2 & \cdots & -x_nx_2 \\ \vdots & \vdots & \ddots & \vdots \\ -x_nx_1 & -x_nx_2 & \cdots & \alpha_n - x_n^2 \end{pmatrix}$$

and

$$P_y = \begin{pmatrix} 1-\frac{y_1^2}{|y|^2} & \frac{y_1y_2}{|y|^2} & \cdots & \frac{y_1y_n}{|y|^2} \\ \frac{y_2y_1}{|y|^2} & 1-\frac{y_2^2}{|y|^2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{y_ny_1}{|y|^2} & \frac{y_ny_2}{|y|^2} & \cdots & 1-\frac{y_n^2}{|y|^2} \end{pmatrix}$$

are both symmetric matrices, we have

$$(P_yCP_y)^T = P_y^TC^TP_y^T = P_yCP_y, \text{ where } C = A - x\otimes x.$$

So $L = P_y(A - x\otimes x)P_y$ also is symmetric. Thanks to the Matlab codex 5.7, we calculate L in the case $n=2$

$$L = \begin{pmatrix} -y_2^2\frac{[(x_1y_2-x_2y_1)^2-\alpha_2y_1^2-a_1y_2^2]}{(y_1^2+y_2^2)} & y_1y_2\frac{[(x_1y_2-x_2y_1)^2-\alpha_2y_1^2-a_1y_2^2]}{(y_1^2+y_2^2)} \\ y_1y_2\frac{[(x_1y_2-x_2y_1)^2-\alpha_2y_1^2-a_1y_2^2]}{(y_1^2+y_2^2)} & y_1y_2\frac{[(x_1y_2-x_2y_1)^2-\alpha_2y_1^2-a_1y_2^2]}{(y_1^2+y_2^2)} \end{pmatrix} =$$

$$= \begin{pmatrix} -y_2^2 & y_1y_2 \\ y_1y_2 & y_2^2 \end{pmatrix}$$

and it's easy to verify that

$$Ly = \begin{pmatrix} -y_2^2 & y_1y_2 \\ y_1y_2 & y_2^2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$

i.e. y is eigenvector for the eigenvalue $\lambda = 0$.
Let B be a skewsymmetric matrix

$$B = -\left(\frac{x_iy_j - x_yy_i}{\alpha_i\alpha_j}\right) = \begin{pmatrix} 0 & -x_1y_2+x_2y_1 & \cdots & -x_1y_n+x_ny_1 \\ -x_2y_1+x_1y_2 & 0 & \cdots & -x_2y_n+x_ny_2 \\ \vdots & \vdots & \ddots & \vdots \\ -x_ny_1+x_1y_n & -x_ny_2+x_2y_n & \cdots & 0 \end{pmatrix}$$

the differential equation $L' = [B, L]$ agrees with 5.10, and so the eigenvalues of L are integrals for 5.10.

Proposition 5.4.1. *Since we have*

$$\frac{|y|^2}{2}\frac{\det(zI-L)}{\det(zI-A)} = \phi_z(x,y) = \sum_{i=1}^{n}\frac{F_i(x,y)}{z-\alpha_i}$$

then the eigenvalues of L are related to the polynomials F_k and ϕ_z.
It follows that the eigenvalues of L are integrals for 5.10.

Proof. We proved it by using matlab for the case $n=2$, see 5.7.1. □

Therefore the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, with $\lambda_n = 0$, and $|y|^2$ can be viewed as functions of the F_k and therefore commute also.

5.5 The mechanical problem of C. Neumann

The system in question describes the motion of a mass point on a sphere

$$S^{n-1} = \left\{q \in \mathbb{R}^n, |q| = 1\right\}$$

under the influence of the force $-Aq$, where A is a symmetric matrix with distinct eigenvalues. For simplicity we use $A = diag(\alpha_1, \dots, \alpha_n)$.
We can obtain the differential equations by constraining X_H with

$$H = \frac{1}{2}\langle Aq, q\rangle + \frac{1}{2}(|q|^2|p|^2 - \langle q,p\rangle^2) = \tag{5.11}$$

$$\frac{1}{2}\sum_{i=1}^{n}\alpha_i q_i^2 + \frac{1}{2}\left(\sum_{i=1}^{n} q_i^2 \sum_{j=1}^{n} p_j^2 - \left(\sum_{j=1}^{n} q_j q_j\right)^2\right) = \tag{5.12}$$

$$\frac{1}{2}\sum_{i=1}^{n}\alpha_i q_i^2 + \frac{1}{2}\sum_{\substack{i=1\\ i\neq j}}^{n}\left(q_i p_j - p_i q_j\right)^2, \qquad q,p \in \mathbb{R}^n. \tag{5.13}$$

In the following way.
There are n commuting integrals $F_1, \dots, F_n$ which are polynomials in q, p, of which

$$F_1 = \frac{1}{2}(|q|^2 - 1)$$

is one. We constrain this system to the tangent bundle of the sphere

$$2F_1 = |q|^2 - 1 = 0$$

$$G_1 = \langle p, q\rangle = 0.$$

Since

$$\{F_1, G_1\} = \sum_{i=1}^{n} \frac{\partial F_1}{q_i}\frac{\partial G_1}{p_i} - \underbrace{\frac{\partial F_1}{p_i}\frac{\partial G_1}{q_i}}_{=0} = \sum_{i=1}^{n} q_i^2 = 1 \neq 0$$

and

$$\{H, G_1\} = \sum_{i=1}^{n} \frac{\partial H}{q_i}\frac{\partial G_1}{p_i} - \frac{\partial H}{p_i}\frac{\partial G_1}{q_i} =$$

$$\sum_{i=1}^{n}\left[\left(\alpha_i q_i + \frac{1}{2}\left(2q_i p_i^2 - 2p_i^2 q_i\right)\right) q_i - \frac{1}{2}\left(2p_i q_i^2 - 2p_i q_i q_i\right) p_i\right] =$$

$$\sum_{i=1}^{n} \alpha_i q_i^2 = \langle Aq, q\rangle$$

from 5.3 we want

$$\{H, G_1\} - \lambda_1 \{F_1, G_1\} = 0 \Longrightarrow \lambda_1 = \langle Aq, q\rangle$$

so follows

$$\underbrace{\{H, G_1\}}_{=\langle Aq,q\rangle} - \lambda_1 \underbrace{\{F_1, G_1\}}_{=1} = 0 \Longrightarrow \lambda_1 = \langle Aq, q\rangle .$$

Then the constrained Hamiltonian system is

$$H^* = H - \lambda_1 F_1, \quad \text{with } \lambda_1 = \langle Aq, q\rangle .$$

Therefore the differetial equations becomes

$$\dot{q} = H_p^* = H_p - \underbrace{\lambda_1 F_{1_p}}_{=0} = \frac{\partial}{\partial p}\left(\frac{1}{2}\langle Aq, q\rangle + \frac{1}{2}(|q|^2|p|^2 - \underbrace{\langle q, p\rangle^2}_{=0})\right) =$$

$$\underbrace{\frac{\partial}{\partial p}\left(\frac{1}{2}\langle Aq, q\rangle\right)}_{=0} + \frac{\partial}{\partial p}\left(\frac{1}{2}(|q|^2|p|^2\right) = p;$$

$$\dot{p} = -H_q^* = -H_q + \lambda_1 F_{1_q} = -Aq - q|p|^2 + \lambda_1 q = -Aq + q(\lambda_1 - |p|^2)$$

where $|q|^1$ and $\langle q, p\rangle = 0$.
With $v = \lambda_1 - |p|^2 = \langle Aq, q\rangle - |\dot{q}|^2$, we can rewrite the differential equations as follows

$$\dot{p} = \ddot{q} = \frac{d^2 q}{dt^2} = -Aq + vq, \tag{5.14}$$

where vq is the force keeping y on the sphere.

Now we show that this system is integrable.

Expanding the rational function ϕ_z at $z=\infty$, we find from 5.7

$$\phi_z(x,y)=\frac{|y|^2}{2}+\frac{1}{z^2}\left\{\langle Ay,y\rangle+|x|^2|y|^2-\langle x,y\rangle^2\right\}+\dots$$

or

$$\phi_z(p.q)=\frac{|q|^2}{z}+\frac{2H(q,p)}{z^2}.$$

We have also

$$\frac{1}{2}\sum_{i=1}^{n}\alpha_i F_k(p,q)=\frac{1}{2}\sum_{i=1}^{n}\alpha_i q_i^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_i\frac{(q_jp_i-q_ip_j)^2}{\alpha_i-\alpha_j}$$

$$\frac{1}{2}\sum_{i=1}^{n}\alpha_i q_i^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_i(q_jp_i-q_ip_j)^2=H$$

so the functions $F_k(p,q)$ being defined in 5.9 with (x,y) replaced by (p,q) are the desired integrals of the system 5.11. Since $|q|^2=\sum_{i=1}^{n}F_i$ also commutes with the F_i, the constrained system is integrable.

5.6 The connection between the two systems

The geodesic flow on the ellipsoid $\langle A^{-1}x,x\rangle=1$ and the Neumann's problem are closely related. There is also another connection between these problems, found by Knorrer, that uses the Gauss map of the ellipsoid Q_0 onto the unit sphere in this way

$$\begin{aligned} Q_0 &\longrightarrow S^{n-1}\\ x &\longmapsto q=rA^{-1}x \end{aligned}\tag{5.15}$$

where $r=\left|A^{-1}x\right|^{-1}$.

The Gauss mapping take solutions of $\dfrac{d^x}{ds^2}=-vA^{-1}x$ into solution of $\dfrac{d^2q}{dt^2}=-Aq+vq$, where A is replaced by A^{-1}.

We will change the independent variable s into t via $s=\psi(t)$, so that $\dfrac{d^x}{ds^2}$

becomes

$$\dddot{x} = \frac{d^x(\psi(t))}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}x(\psi(t))\right) = \frac{d}{dt}\left(\frac{dx}{ds}(\psi(t))\underbrace{\frac{d\psi(t)}{dt}}_{=\dot{\psi}}\right) =$$
$$\frac{d^2x}{ds^2}(\psi(t))\dot{\psi}\dot{\psi} + \frac{dx}{ds}(\psi(t))\ddot{\psi} =$$
$$-vA^{-1}z\dot{\psi}^2 + \underbrace{\frac{dx}{ds}(\psi(t))}_{=\frac{\dot{x}}{\dot{\phi}}}\ddot{\psi} = -vA^{-1}x\dot{\psi}^2 + \dot{x}\frac{\ddot{\psi}}{\dot{\psi}}.$$

Choosing $\psi(t)$ so that $-v\dot{\psi}^2 = 1$ and writing $B = A^{-1}$, the system becomes

$$\ddot{x} = -Bx + b\dot{x}, \qquad \text{where } b = \frac{\ddot{\psi}}{\dot{\psi}}. \tag{5.16}$$

Since $\left\langle A^{-1}x, x\right\rangle = \langle Bx, x\rangle = 1$, by differentiating once

$$\frac{d}{dt}\left\langle Bx(t), x(t)\right\rangle = 0$$
$$\frac{d}{dt}\sum_{i=1}^{n} x_i^2 b_i = \sum_{i=1}^{n} 2x_i\dot{x}_i b_i = 2\langle Bx, \dot{x}\rangle = 0$$
$$\Longrightarrow \langle Bx, \dot{x}\rangle = 0$$

two times we obtain

$$\frac{d^2}{dt^2}\langle Bx, \dot{x}\rangle = \frac{d^2}{dt^2}\sum_{i=1}^{n} 2x_i\dot{x}_i b_i = 0$$
$$\Longrightarrow \sum_{i=1}^{n}(2\dot{x}_i\dot{x}_i b_i + 2x_i\dot{x}_i b_i) = 2\langle B\dot{x}, \dot{x}\rangle + 2\underbrace{\langle Bx, \dot{x}\rangle}_{=0} - 2x|Bx|^2 = 0$$
$$\Longrightarrow \frac{\langle B\dot{x}, \dot{x}\rangle}{|Bx|^2} = 1$$

and three times

$$\frac{d^3}{dt^3}\langle Bx, x\rangle = \frac{d}{dt}\left(\sum_{i=1}^{n} \dot{x}_i^2 b_i + (x_i b_i)^2\right) = 0$$

$$\Longrightarrow \sum_{i=1}^{n} 2\dot{x}_i\ddot{x}_i b_i + 2x_i b_i \dot{x}_i b_i = 0$$

$$\text{replacing } \ddot{x} = -Bx + b\dot{x}$$

$$\sum_{i=1}^{n} 2\dot{x}_i(-b_i x_i + b_i\dot{x}_i)b_i + 2\underbrace{\langle Bx, \dot{x}\rangle}_{=0} b_i = 0$$

$$-2\langle Bx, B\dot{x}\rangle + 2b\underbrace{\langle B\dot{x}, \dot{x}\rangle}_{=|Bx|^2} = 0$$

$$\Longrightarrow b = \frac{\langle Bx, B\dot{x}\rangle}{|Bx|^2}.$$

Theorem 5.6.1. *The Gauss map of $Q_0 \longrightarrow S^{n-1}$ takes the solutions of*

$$\ddot{x} - Bx + b\dot{x}, \qquad b = \frac{\ddot{\psi}}{\dot{\psi}}$$

satisfying

$$\langle Bx, x\rangle = 1, \quad \langle Bx, \dot{x}\rangle = 0, \quad \langle B\dot{x}, \dot{x}\rangle = |Bx|^2,$$

into the solution of the Neumann problem

$$\ddot{q} = -Bq + vq, \qquad v = \langle Bq, q\rangle - |\dot{q}|^2 \tag{5.17}$$

satisfying

$$|\dot{q}|^2 = 1, \qquad \langle \dot{q}, q\rangle = 0, \qquad \Psi_0(\dot{q}, q) = 0, \tag{5.18}$$

where $\Psi_z(x,y)$ is defined like $\phi_z(x,y)$ but with A replaced by $B = A^{-1}$.

Proof. Differentiating $q = rA^{-1}x = rBx$ we have

$$\dot{q} = B\dot{x}r + \dot{r}Bx = rB(\dot{x} + \frac{\dot{r}}{r}x), \tag{5.19}$$

where

$$\dot{r} = \frac{d}{dt}|Bx|^{-1} = -|Bx|^{-2} B\dot{x} = -\frac{B\dot{x}}{|Bx|^2}$$

$$\Longrightarrow \frac{\dot{r}}{r} = -\frac{B\dot{x}}{|Bx|^2|Bx|^{-1}} = -\frac{\langle Bx, B\dot{x}\rangle}{|Bx|^2},$$

and

$$\ddot{q} = B\ddot{x}r + B\dot{x}\dot{r} + \dot{r}B\dot{x} + \ddot{r}Bx = \ddot{x}Br + 2\dot{x}B\dot{r} + \ddot{r}Bx =$$
$$Br(-Bx + b\dot{x}) + 2\dot{r}B\dot{x} + \ddot{r}Bx = -BrBx + Brb\dot{x} + 2\dot{r}B\dot{x} + \ddot{r}Bxr =$$
$$-Bq + B\dot{x}r\underbrace{\left(b + 2\frac{\dot{r}}{r}\right)}_{=0} + \frac{\ddot{r}}{r}q = -Bq + \frac{\ddot{r}}{r}q.$$

□

This shows that the solutions of the geodesic problem correspond to the solutions of the Neumann problem.

Moreover, following Moser [11],

$$\Psi_0(\dot{q}, q) = \left(\frac{\langle B\dot{x}, \dot{x}\rangle}{|Bx|^2} - 1\right)\langle Aq, q\rangle .$$

Definition 5.6.1. A solution $q = q(t)$ of 5.17 is said *non degenerate* if

$$\Phi_z(\dot{q}, q) = \frac{\prod_{j=1}^{n-1}(z - \mu_j)}{\det(zI - B)}$$

has a zero, say μ_1, which is not an eigenvalue of B.

Replacing B by $B - \mu_1 I$

$$\Phi_z(\dot{q}, q) = \frac{\prod_{j=1}^{n-1}(z - \mu_j)}{det(zI - B + \mu_1 I)} =$$
$$\frac{\prod_{j=1}^{n-1}(z - \mu_j)}{\det((z + \mu_1)I - B)} = \frac{\prod_{j=1}^{n-1}(z - \mu_j)}{\prod_{j=1}^{n}\big((z + \mu_1) - b_i\big)}.$$

The condition

$$\Psi_{\mu_1}(\dot{q}, q) = \frac{\prod_{j=1}^{n-1}(\mu_1 - \mu_j)}{\prod_{j=1}^{n}\big((\mu_1 + \mu_1) - b_i\big)} = 0$$

becomes

$$\Psi_0(\dot q, q) = \frac{\prod_{j=1}^{n-1}(-\mu_j)}{\prod_{j=1}^{n}(\mu_1 - b_i)} = 0$$

and the above reduction become possible, then the two problems are essentially equivalent.

Theorem 5.6.2. *Let $\phi_z(x, y)$ be the integrals of the geodesic problem and $\Psi_z(\dot q, q)$ the integrals of the Neumann's problem. If*

$$\langle Bx, x\rangle = 1, \quad \langle Bx, \dot x\rangle = 0, \quad \langle B\dot x, \dot x\rangle = |Bx|^2, \tag{5.20}$$

holds and $(x, \dot x)$ are related to $(\dot q, q)$ by the Gauss map

$$\begin{cases} q = rBx \\ \dot q = rB\left(\dot x + \frac{\dot r}{r}x\right), \end{cases} \tag{5.21}$$

then

$$\phi_z(x, \dot x) = |Bx|^4\, \Psi_w(\dot q, q), \qquad \text{where } w = \frac{1}{z}. \tag{5.22}$$

Let's introduce first a proposition that we will use in the proof

Proposition 5.6.3. *We have the following identity*

$$(wI - B)^{-1} + B^{-1} = -(zI - A)^{-1}A^2. \tag{5.23}$$

Proof. Since

$$(-(zI - A)^{-1}A^2)^{-1} = -A^{-2}(zI - A) =$$
$$-A^{-2}zI + A^{-1} = A^{-1}zI(wI - A^{-1}) = BzI(wI - B)$$

i.e. $(-(zI - A)^{-1}A^2)^{-1} = BzI(wI - B)$,
then we should have

$$BzI(wI - B)\left[(wI - B)^{-1} + B^{-1}\right] = I$$
$$\Longrightarrow BzI - BzI(wI - B)B^{-1} = I$$
$$\Longrightarrow BzI + BB^{-1} - BzI = I \Longrightarrow I = I.$$

□

Proof. To verify the relation we introduce the abbreviations

$$
\begin{aligned}
&P_w(p,q) = \left\langle (w-B)^{-1}p, q \right\rangle \\
&P_w(q = P_w(q,q)) \\
&\varrho = -\frac{\dot r}{r} = \frac{\langle Bx, B\dot x\rangle}{|Bx|^2}.
\end{aligned}
$$

We find from 5.21

$$
\begin{cases}
q = rBx \\
\dot q = rB\dot x + rB(-\varrho)x = rB\dot x - q\varrho \Longrightarrow \dot q + q\varrho = rB\dot x
\end{cases}
$$

and from 5.20

$$
\begin{cases}
P_0(q) = -\langle Aq, q\rangle = -\langle ArBx, rBx\rangle = -r^2\langle Bx, x\rangle = -r^2 \\
P_0(q, \dot q + \varrho q) = -\langle Aq, \dot q + \varrho q\rangle = -\langle ArBx, rB\dot x\rangle = r^2\langle Bx, \dot x\rangle = 0 \\
P_0(\dot q + \varrho q) = -\langle ArB\dot x, rB\dot x\rangle = -r^2\langle B\dot x, \dot x\rangle = -|Bx|^{-2}|Bx|^2 = -1.
\end{cases}
$$

Moreover, the identity 5.23 gives

$$
\begin{aligned}
P_w(q) - P_0(q) = &\left\langle (w-B)^{-1}q, q\right\rangle + \langle Aq, q\rangle = \\
&\left\langle (-(z-A)^{-1}A^2 - A)q, q\right\rangle + \langle Aq, q\rangle = \\
&-\left\langle (z-A)^{-1}A^2 q, q\right\rangle - \langle Aq, q\rangle + spAq = \\
&-Q_z(Aq) = -Q_z(ArBx) = -r^2 Q_z(x);
\end{aligned}
$$

$$
\begin{aligned}
P_w(q, \dot q + \varrho q) - P_0(q, \dot q + \varrho q) = &\left\langle (w-B)^{-1}q, \dot q + \varrho q\right\rangle + \langle Aq, \dot q + \varrho q\rangle = \\
&\left\langle (-(z-A)^{-1}A^2 - A)q, \dot q + \varrho q\right\rangle + \langle Aq, \dot q + \varrho q\rangle = \\
&\left\langle (-(z-A)^{-1}A^2)q, \dot q + \varrho q\right\rangle - \langle Aq, \dot q + \varrho q\rangle + \langle Aq, \dot q + \varrho q\rangle = \\
&\left\langle (z-A)^{-1}A^2 rBx, rB\dot x\right\rangle = \\
&-r^2\left\langle (z-A)^{-1}x, \dot x\right\rangle = -r^2 Q_z(x, \dot x);
\end{aligned}
$$

$$
\begin{aligned}
P_w(\overbrace{\dot q + \varrho q}^{rB\dot x}) - P_0(\dot q + \varrho q) = &\left\langle (w-B)^{-1}rB\dot x, rB\dot x\right\rangle + r\langle B\dot x, \dot x\rangle = \\
&\left\langle (-(z-A)^{-1}A^2 - A)rB\dot x, rB\dot x\right\rangle + r\langle B\dot x, \dot x\rangle = \\
&\left\langle (-(z-A)^{-1}A^2)rB\dot x, rB\dot x\right\rangle - \langle ArB\dot x, rB\dot x\rangle + r\langle B\dot x, \dot x\rangle = \\
&-r^2 Q_z(\dot x) - r\langle B\dot x, \dot x\rangle + r\langle B\dot x, \dot x\rangle = -r^2 Q_2(\dot x).
\end{aligned}
$$

Hence

$$\Psi_w(q,\dot q) = P_w(q)(1+P_w(\dot q+\varrho q)) - (P_w(q,\dot a+\varrho q))^2 =$$
$$r^4\left[(1+Q_z(x))Q_z(\dot x) - Q_z(x,\dot x)^2\right] =$$
$$r^4\phi_z(x,\dot x) = \frac{\phi_z(x,\dot x)}{|Bx|^4}.$$

□

5.7 The Riemann surface

Consider the flow in the $n-1$ dimensional manifold

$$\mathcal{M} = \left\{q,\dot q|\,\langle q,\dot q\rangle = 0; F_k(\dot q,q) = c_k, k = 1,\dots n\right\},$$

where $c_1,\dots,c_n$ are given so that

$$|q|^2 = \sum_{i=1}^n c_i = 1.$$

We will consider only the general case where the rational function

$$\phi_z(\dot q,q) = \sum_{i=1}^n \frac{F_i(\dot q,q)}{z-\alpha_i} = \sum_{i=1}^n \frac{c_i}{z-\alpha_i}$$

has $\beta_1,\dots,\beta_{n-1}$ distinct roots, so $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{n-1}$ are distinct and real, so

$$\phi_z(\dot q,q) = \frac{\prod_{i=1}^{n-1}(z-\beta_i)}{\prod_{i=1}^{n}(z-\alpha_i)} = \frac{b(z)}{a(z)},$$

where

$$b(z) = \prod_{i=1}^{n-1}(z-\beta_i), \quad \text{and} \quad a(z) = \prod_{i=1}^{n}(z-\alpha_i) = \det(zI-A).$$

We prefer to use the elliptic coordinates as parameters on the manifold $\mathcal{M}$.

Definition 5.7.1. $\mu_1,\dots,\mu_{n-1}$ are called *elliptic coordinates* on the sphere if they are the zeros of

$$Q_z(q) = \sum_{i=1}^n \frac{q_i^2}{z-\alpha_i}.$$

Let's see how we can compute $q, \dot{q}$ from them.

Since $|q|^2 = 1$, we can write Q_z as

$$Q_z(q) = \sum_{i=1}^{n} \frac{q_i^2}{z-\alpha_i} = \frac{\sum_{i=1}^{n} q_i^2 \prod_{i=1}^{n-1}(z-\mu_i)}{\prod_{i=1}^{n}(z-\alpha_i)} = \frac{\prod_{i=1}^{n-1}(z-\mu_i)}{\prod_{i=1}^{n}(z-\alpha_i)} = \frac{m(z)}{a(z)}$$

with

$$m(z) = \prod_{i=1}^{n-1}(z-\mu_i).$$

With the residues method we recover the q_i up to sign from $\mu_1, .., \mu_{n-1}$:

$$q_i^2 = \frac{m(\alpha_i)}{a'(\alpha_i)}.$$

Let's now compute $\dot{q}$.
Since for $z = \mu_i$, $i = 1, \dots n-1$

$$\phi_z(\dot{q}, q) = (Q_z(\dot{q}) + 1)\underbrace{Q_z(q)}_{=0} - Q_z^2(\dot{q}, q)$$

$$\Longrightarrow Q_z(\dot{q}, q) = \sqrt{-\phi_z(\dot{q}, q)} = \sqrt{-\frac{b(z)}{a(z)}}$$

we have obtained $n-1$ linear equations for $\dot{q}$ and these, together to $\langle \dot{q}, q\rangle$, allow us to recover $\dot{q}$.
The differential equations theke the implicit form

$$\sum_{i=1}^{n-1} \frac{\mu_i^{n-j-1}\dot{\mu}_i}{2\sqrt{-R(\mu_i)}} = \delta_{i,1}, \quad \text{for } j = 1, \dots, n-1$$

where $R(z) = a(z)b(z)$. These formulas are related to the Jacobi map given by

$$\sum_{i=1}^{n-1} \int_{(u_0,w_0)}^{(\mu_i,w_i)} \frac{z^{n-j-1}dz}{2\sqrt{-R(z)}} = s_j$$

that takes the divisor class defined by $(\mu_i, 2\sqrt{-R(z)})$ into a point $s \in \mathbb{C}^{n-1} \setminus \Gamma$, where Γ denotes the period lattice of the differentials of the first kind.
The Riemann surface

$$w^2 = -4R(z)$$

is an hyperelliptic curve of genus $n-1$ with branch points at $\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_{n-1},$. Then the differential equation becomes

$$\dot{s}_i = \delta_{i,1}, \quad or \quad s_i = \delta_{i,1} t + s_i(0).$$

In the case $n = 2$ we are on an ellipse and

$$R(z) = a(z)b(z) = (z - \alpha_1)(z - \alpha_2)(z - b_1)$$

i.e it is a third degree polynomial, so we can easier calculate the differential equation since

$$\int_{(u_0,w_0)}^{(u,w)} \frac{dz}{\sqrt{-4R(z)}} = s$$

is an elliptic integral of the first kind that we studied in 2.6.

Appendix

Example 1

```
clc
clear
% To convert a polynomial of third degree into the Weierstrass form
syms x E a b t q
%V=input('inserire potenziale V(q)\')
% In our case
V= -q^3/3+q;
p=2*(E-V) % polinimio sotto radice
P1=subs(p,q,a*x+b);
P1=expand(P1);
P1=collect(P1,x);
P2=collect(expand(P1/(a^2)));
C = coeffs(P2,x);
a1=C(1,4);
a2=solve(a1==4,a)
b1=C(1,3);
b2=solve(b1==0,b)
P3=subs(P2,a,a2);
P3=subs(P3,b,b2)

p =

(2*q^3)/3 - 2*q + 2*E
```

```
a2 =

6


b2 =

0


P3 =

4*x^3 - x/3 + E/18
```

Example 2

```
clear
clc
r=[1 -1 i -i];
syms x a b t s E;
V=poly(r);
E=[0 0 0 0 E];
P=2*(E-V);
v=roots(P);
P=poly2sym(P,x);
P1=subs(P,x,v(1)+1/t);
P1=expand(P1);
P2=expand(t^4*P1);
P3=expand(subs(P2,t,-a*s+b));
P4=collect(P3,s);
p4=expand(P4/a^2);
C = coeffs(p4,s);
```

```
a1=C(1,4);
a2=solve(a1==4,a);
b1=C(1,3)
b2=solve(b1==0,b);
P5=subs(p4,a,a2);
P6=subs(P5,b,b2)
```

```
P6 =

4*s*(E + 1) + 4*s^3
```

Some chapter 5 proofs

- confocal quadrics
- Iso-spectral deformations

```
clc
clear
syms y1 y2 y a1 a2 x1 x2 z z1
a=sym('a',[1 2]);
x=sym('x',[1 2]);
y=sym('y',[1 2]);
A=[a1 0; 0 a2];
A1=inv(A);
```

confocal quadrics

We set

- $Qx = Q_z(x)$

- $Qy = Q_z(y)$
- $Qxy = Q_z(x, y)$
- $P_z = \phi_z(x, y)$

```
Qx=x*inv(z*eye(2,2)-A)*x.';
Qxy=x*inv(z*eye(2,2)-A)*y.';
Qy=y*inv(z*eye(2,2)-A)*y.';
format rat
P_z=(1+Qx)*Qy-Qxy^2
simplify(P_z)
```

Let's proof proposition 5.3.1, with $P_{z1} = \phi_{z_1}$

```
P_z1=subs(P_z,z,z1)
fprintf('vediamo che {P_z,P_z1}=0\n')
% Let's calculate the Poisson bracket Pb
Pb=0;
for j=1:2
Pb=Pb+diff(P_z,x(j))*diff(P_z1,y(j))-diff(P_z,y(j))*diff(P_z1,x(j));
end
Pb=simplify(Pb)
fprintf('è uguale a 0\n')
```

Iso-spectral deformations

Let's construct the matrix P_y

```
Y=[y1^2 y1*y2; y1*y2 y2^2];
Y=1/(y1^2+y2^2).*Y;
P=eye(2)-Y
X=[x1^2 x1*x2; x1*x2 x2^2];
```

```
X=A-X
L=P*X*P;
L=simplify(L)
```

We proof the symmetry of L by calculating the transpose matrix and then doing the difference. If the difference is 0, the matrix is symmetric;

```
L1=L.'; % L1 is the transpose
fprintf('L-L1=\n')
simplify(L-L1)
fprintf('L-L1=0, so L is symmetric\n');
fprintf('vediamo che Ly=0 )\n');
Ly=L*y.'
fprintf('Vediamo com''è fatta B\n');
B=[0 (x2*y1-x1*y2)/(a1*a2);(x1*y2-x2*y2)/(a1*a2) 0 ]
L2=simplify(B*L-L*B)
w=(-x1^2*y2^2+2*x1*x2*y1*y2-x2^2*y1^2+a2*y1^2+a1*y2^2)/(a1*a2*(y1^2+y2^2)^2);
L2=L2/w % we devide by w since matlab doesn't do it
P_0=(x*A1*x.'-1)*(y*A1*y.')-(x*A1*y.')^2
```

5.7.1 Proof Proposition 5.4.1.

```
fprintf('Let's proof 5.4.1\n')
N=((y1^2+y2^2)/z)*det(z*eye(2,2)-L)/det(z*eye(2,2)-A)
N=simplify(N)
simplify(P_z-N)
fprintf('P_z-N=0, so 5.4.1 is verified\n')
```

$\wp$ Weierstrass function

The following Matlab function enable us to calculate the Weierstrass function.

```
function [P] = weierstrassfunc(z,g2,g3)
C=zeros(1,20);
```

```
C(1)=g2/20;
C(2)=g3/28;
for j=3:20
b=zeros(1,j-1);
for m=1:j-2
b(m)=C(m)*C(j-1-m);
end
s=sum(b);
M=3/((2*(j+1)+1)*(j-2));
C(j)=s*M;
end
C;
cz=zeros(1,20);
for i=1:20
cz(i)=double(C(i)*z^(2*i)) ;
end
P=double(z^(-2)+ sum(cz));
end
```

Bibliography

[1] Abramowitz Milton, Stegun Irene, *Handbook of Mathematical Functions with Formulas, Graph, and Mathematical Tables*, Dover Publications, INC., New York, 1965

[2] Naum I. Akhiezer, *Elements of the Theory of Elliptic Functions*, American Mathematical Society, Providence, 1990

[3] Vladimir I. Arnold, *Mathematical Methods of Classical Mechanics*, Second Edition, Springer Science+Business Media, New York, 1989

[4] Antonio Fasano, Stefano Marmi, *Analytical Mechanics: An Introduction*, Oxford University Press, New York, 2006

[5] Harald Iro, *A Modern Approach to Classical Mechanics*, World Scientific, Singapore, 2002

[6] Gareth A. Jones, David Singerman, *Complex Functions: An Algebraic and Geometric Viewpoint*, Cambridge University Press, Cambridge, 1987

[7] Anthony W. Knapp, *Elliptic Curves*, Mathematical Notes 40, Princeton University Press, Princeton, 1992

[8] Lev D. Landau, Evgeny M. Lifshitz, *Course of Theoretical Physics: Volume 1 - Mechanics*, Third Edition, Butterworth-Heinemann, 1976

[9] Stephen T. Lovett, *Differential Geometry of Manifolds*, CRC Press, 2010

[10] Jerrold E. Marsden, Michael J. Hoffman, *Basic Complex Analysis*, Third Edition, W.H. Freeman and Company, New York, 1999

[11] Jürgen Moser, *Lezioni Fermiane - Integrable Hamiltonian Systems and Spectral Theory*, Edizioni della Normale, Pisa, 1981

[12] Carl L. Siegel, *Topics in Complex Function Theory: Volume III - Abelian Functions and Modular Functions of Several Variables*, Wiley Classic Edition, Toronto, 1989

[13] John Stillwell, *Mathematics and Its History*, Springer-Verlag, New York, 1989

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